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# ON THE BLOW-UP SOLUTIONS FOR THE NONLINEAR RADIAL SCHRÖDINGER EQUATIONS WITH SPATIAL VARIABLE COEFFICIENTS

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**Abstract**

We study a generalized nonlinear Schrödinger equations with spatial variable coefficients, which models the remarkable inhomogeneous Schrö dinger maps (ISM). A new weighted Sobolev space $\mathcal{W}^{1,q}(\mathbb{R}^+)$ is introduced and the existence of blow-up solutions of this equations, including the integrable radial ISM, with the initial data in $\mathcal{W}^{1,2}(\mathbb{R}^+)$ is proved.

## 1. Introduction

In this paper, we consider the following nonlinear Schrödinger equation with spatial variable coefficients:

$$i\partial_t v + \mathrm{A}_\mu v = \lambda_1 r^{p_3}|v|^b v + \lambda_2 v \int_0^r (r')^{p_4}|v|^c dr', \tag{1.1}$$
$$v(r,0) = v_0(r), \quad v(0,t) = 0, \quad (r,t) \in \mathbb{R}^+ \times \mathbb{R},$$

where $v : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{C}$, $r = |x|$, $(x \in \mathbb{R}^n)$ is the radius, $\lambda_1, \lambda_2 \in \mathbb{R}$, $b, c \geq 1$ and the operator

$$\mathrm{A}_\mu := ar^{p_0}(\partial_{rr} + \frac{p_1}{r}\partial_r - \frac{p_2}{r^2}), \quad a < 0,$$

with the array $(p_0, p_1, p_2)$ satisfies the assumption

$$p_0 < \min\{p_1 + 1,\ 2\}, \quad p_1 > -1, \quad p_2 := (\frac{2 - p_0}{2}\mu)^2 - (\frac{p_1 - 1}{2})^2, \quad \mu \geq 0.$$

The elliptic operator $\mathrm{A}_\mu = ar^{p_0}(\partial_{rr} + \frac{p_1}{r}\partial_r - \frac{p_2}{r^2})$ plays a key role in searching the solution of (1.1). Schrödinger type equations with variable coefficients have been of considerable interest among both mathematicians and physicists, and some remarkable progress on the Cauchy problem have been made, see e.g. [10]-[13] for a detailed discussion. The mathematical interest in (1.1) comes mainly from the spatial variable coefficient $r^{p_0}$, which arises in a model for the inhomogeneous Schrödinger maps (ISM) with $\vec{S} \in \mathbb{S}^2 \subset \mathbb{R}^3$

$$\partial_t \vec{S}(x,t) = \varrho(x)(\vec{S} \times \Delta\vec{S}) + \nabla\varrho(x) \cdot (\vec{S} \times \nabla\vec{S}), \tag{1.2}$$

or, equivalently, the nonlinear Schrödinger equation

$$iv_t + \varrho(v_{rr} + \frac{n-1}{r}v_r - \frac{n-1}{r^2}v + 2|v|^2 v) + 2\varrho_r v_r \tag{1.3}$$

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$$+ [\varrho_{rr} + \frac{n-1}{r}\varrho_r + 2\int_0^r \varrho_{r'}|v|^2 dr' + 4(n-1)\int_0^r \frac{\varrho}{r'}|v|^2 dr']v = 0,$$

based on a known geometrical process [4, 12], where $\Delta$ is the Laplacian in $\mathbb{R}^n$, $\times$ denotes the cross product in $\mathbb{R}^3$, and

$$\nabla\varrho(x)\cdot(\vec{S}\times\nabla\vec{S}) = \sum_{j=1}^{n}\frac{\partial\varrho(x)}{\partial x_j}(\vec{S}\times\frac{\partial\vec{S}}{\partial x_j}).$$

Obviously, the factor $r^{p_0}$ in $\mathrm{A}_\mu$ corresponds to the inhomogeneity $\varrho(r)$ in (1.3). Noticing that (1.1) includes radial ISM (1.2) with $\varrho(r) = r^{p_0}$.

When $\varrho$ is a constant, the ISM (1.2) reduces to the well-known (homogeneous) Schrödinger maps

$$\partial_t\vec{S}(x,t) = \vec{S}\times\Delta\vec{S}, \tag{1.4}$$

of which global well-posedness problem has attracted a great deal of attention in past years. Local existence for smooth initial data goes back to [14], see also [8]. Some progress of small initial data existence results can be found in [3] and [1] for $n \geq 2$. Especially, the classical solution with small energy is global in time for the radial case [3]. For some special large initial data, the possibility of finite time blowup and the blowup rate have been proved [9, 11].

In the setting of the ISM (1.2), when the inhomogeneity $\varrho$ is chosen as

$$\varrho(r) = \varepsilon_1 r^{2(n-1)} + \varepsilon_2 r^{n-2}, \tag{1.5}$$

in which case (1.2) is completely integrable by means of the inverse scattering transform, Daniel et al. [4] present some soliton like solutions of (1.2) by using the equivalent Schrödinger equation (1.3). Based on the above equivalent relation, some further works about the possible blowup of the solutions, in the particular case where $\varrho(r) = r^{2-n}$, is made by the author [16] in an energy space $\mathcal{W}^{1,2}(\mathbb{R}^+)$ (see Definition 1.1).

In this paper, we concentrate on a nonintegrable case ($\varrho(r) = r^{p_0}$), and investigate the global behavior of the deduced equation (1.1), which is a generalized version of (1.3). For technical reasons, we require $p_3,\ p_4$ satisfy

$$\begin{cases} \max\{(p_1-1)b_0 + 2(p_0-2),\ -2d\} \leq 2p_3 \leq (d + \frac{p_0-2}{2}n)b_0, \\ \max\{-2d-2,\ 2p_0 - 6 + (p_1-1)c_0\} \leq 2p_4 \leq (d + \frac{p_0-2}{2}n)c_0 - 2, \end{cases} \tag{1.6}$$

with $b_0 := \frac{2(p_3-p_0+2)}{p_1-p_0+1}$, $c_0 := \frac{2(p_4-p_0+3)}{p_1-p_0+1}$, $d := p_1 - p_0 + 1$.

We introduce the definition of a new weighted Sobolev spaces $\mathcal{W}^{1,p}(\mathbb{R}^+)$ and weighted space-time spaces $L^h(I; L^p_{\kappa,\sigma})$.

Definition 1.1 ([17]). For $1 \leq p,\ h \leq \infty$ and $\kappa := \frac{2-p_0}{2}\mu + \frac{1-p_1}{2}$, we define the **weighted Sobolev space** $\mathcal{W}^{1,p}(\mathbb{R}^+)$ by

$$\mathcal{W}^{1,p}(\mathbb{R}^+) = \{u \in L^p_{\kappa,\sigma}(\mathbb{R}^+) :\ D_r u \in L^p_{\kappa,\widetilde{\sigma}}(\mathbb{R}^+),\ D_r := r^{p_0-p_1}\partial_r\},$$

endowed with the norm $\|u\|_{\mathcal{W}^{1,p}(\mathbb{R}^+)} = \|u\|_{L^p_{\kappa,\sigma}(\mathbb{R}^+)} + \|D_r u\|_{L^p_{\kappa,\widetilde{\sigma}}(\mathbb{R}^+)}$, where the norm of the weighted Lebesgue space $L^p_{\kappa,\sigma}(\mathbb{R}^+)$ and space-time space $L^h(I; L^p_{\kappa,\sigma})$ of function $v$ are defined as

$$\|v\|_{L^p_{\kappa,\sigma}(\mathbb{R}^+)} = (\int_{\mathbb{R}^+} |v|^p r^{-\kappa p} d\sigma_r)^{\frac{1}{p}} < \infty,$$

$$\|v\|_{L^h(I;L^p_{\kappa,\sigma})} = (\int_I \|v\|^h_{L^p_{\kappa,\sigma}} dt)^{\frac{1}{h}},$$

with a usual modification when $p$ or $h$ is infinity, where $d\sigma_r = r^{2\kappa+p_1-p_0}dr, d\widetilde{\sigma}_r = r^{2\kappa+3p_1-2p_0}dr$ are the Lebesgue measures. For simplicity, $\|f\|_{L^p(\mathbb{R}^+)} = (\int_{\mathbb{R}^+} |f(r)|^p dr)^{\frac{1}{p}}$.

Moreover, we also define the function space $\mathcal{W}^{1,p}_0(\mathbb{R}^+)$ as the closure of $C^\infty_0(\mathbb{R}^+)$ in $\mathcal{W}^{1,p}(\mathbb{R}^+)$.

Thanks to Strichartz estimates, the Cauchy problem for (1.1) is locally well-posed in $\mathcal{W}^{1,q}(\mathbb{R}^+)$ (see [17], Theorem 1.4): for any $v_0 \in \mathcal{W}^{1,2}_0(\mathbb{R}^+)$, there exists $T \in (0,\infty)$ and a unique solution $v(t)$ of (1.1) with $v(r,0) = v_0$ such that

$$v \in X_q(I) := L^\infty(I;\mathcal{W}^{1,q}_0(\mathbb{R}^+)) \cap L^m(I;\mathcal{W}^{1,l}_0(\mathbb{R}^+)),$$

where $I = [0,T]$, the triplet $(m,l,q)$ is an $L^q_{\kappa,\sigma}$-admissible in the Strichartz's sense, if $1 < q \leq l < \frac{\gamma q}{\gamma-1}$ and satisfy

$$\text{(1.7)} \qquad \frac{1}{m} = \gamma(\frac{1}{q} - \frac{1}{l}), \qquad \gamma := \frac{2\kappa + p_1 - p_0 + 1}{2 - p_0}.$$

Let $v(r,t)$ be a solution of the equation (1.1), we define the following quantities:

$$\text{(1.8)} \qquad M(v(t)) = \|v(t)\|^2_{L^2_{\kappa,\sigma}(\mathbb{R}^+)},$$

$$\text{(1.9)} \qquad E(v(t)) = -\frac{a}{2}\int_{\mathbb{R}^+}(r^{p_1}|\partial_r v|^2 + p_2 r^{p_1-2}|v|^2)dr - \frac{\lambda_1}{b+2}\int_{\mathbb{R}^+}|v|^{b+2}r^{p_3+p_1-p_0}dr.$$

It is easy to prove that if $v$ is a solution of (1.1), then $M(v(t)) = M(v_0)$ and

$$\text{(1.10)} \qquad \frac{d}{dt}E(v(t)) = -a\lambda_2 \mathrm{Im}\int_{\mathbb{R}^+}|v|^c v\overline{\partial_r v} r^{p_4+p_1}dr, \quad t \in [0,T].$$

We remark the energy $E(v)$ defined above is no anymore conserved along the flow of (1.1) (unless the nonlocal term of the equation (1.1) vanishes, i.e. $\lambda_2 = 0$), which is a key challenge what we faced to develop the global behavior of (1.1).

In order to overcome it, we need to refine the variance defined by

$$\mathcal{V}(t) := \frac{1}{(2-p_0)^2}\|r^{\frac{2-p_0}{2}}v(t)\|^2_{L^2_{\kappa,\sigma}(\mathbb{R}^+)},$$

in the spirit of the seminal work of R. Glassey [5], which relies heavily on the conservation of the energy. The main blow up result of (1.1) is stated as follows:

**Theorem 1.2.** *Let $p_2 \geq 0$, and $p_3$, $p_4$ satisfy* (1.6). *Assume that* (1.1) *admits a local solution $v \in X_2([0,T])$ with Schwartz initial data $v_0 \in \mathcal{W}^{1,2}_0(\mathbb{R}^+)$ where $T$ is the maximal existence time. If $\mathcal{V}(0)$ and*

$$\text{(1.11)} \qquad \mathcal{V}'(0) := \frac{2a}{2-p_0} Im \int_{\mathbb{R}^+} \overline{v}_0 \partial_r v_0 r^{p_1-p_0+1}dr > 0$$

*are finite, then blow up occurs in each of the following cases:*

(1) $\lambda_1 > 0,\ b = b_0$ *with* $M(v_0)^{\frac{b}{2}} \leq \frac{-a(b+2)}{2\lambda_1 C_b^{b+2}}$ :

$$\begin{cases} \lambda_2 < 0: & c = c_0,\ M(v_0)^{\frac{c}{2}} \leq \dfrac{2-p_0}{\lambda_2 C_c^{c+2}}[a + \dfrac{2\lambda_1 C_b^{b+2}}{b+2} M(v_0)^{\frac{b}{2}}], \\ \lambda_2 \geq 0: & \textit{for all } c. \end{cases} \tag{1.12}$$

(2) $\lambda_1 \leq 0,\ b \geq b_0$ :

$$\begin{cases} \lambda_2 < 0: & c = c_0,\ M(v_0)^{\frac{c}{2}} \leq \dfrac{(p_1 - p_0 + 3)a}{\lambda_2 C_c^{c+2}}, \\ \lambda_2 \geq 0: & \textit{for all } c \textit{ and } M(v_0). \end{cases} \tag{1.13}$$

*Moreover, let*

$$\widetilde{C} = \begin{cases} \left( \dfrac{8a}{a + \frac{2\lambda_1}{b+2} C_b^{b+2} M(v_0)^{\frac{b}{2}} - \frac{\lambda_2}{2-p_0} C_c^{c+2} M(v_0)^{\frac{c}{2}}} \right)^{\frac{1}{2}}, & \lambda_1 > 0, b = b_0, (1.12), \\ \left( \dfrac{8a(2-p_0)}{(2-p_0)a - \lambda_2 C_c^{c+2} M(v_0)^{\frac{c}{2}}} \right)^{\frac{1}{2}}, & \lambda_1 \leq 0, b \geq b_0, (1.13). \end{cases}$$

*the solution $v$ of* (1.1) *blows up at finite time provided* $\widetilde{C} < 1$.

REMARK. Here are some comments on Theorem 1.2.

(i) When $b < \frac{2(p_3 - p_0 + 2)}{p_1 - p_0 + 1}$ or $\lambda_2 < 0, c \neq \frac{2(p_4 - p_0 + 3)}{p_1 - p_0 + 1}$, the global behavior of the solutions to (1.1) has not been proved yet. This restriction is due to the absence of energy monotonicity inequality about (1.1).

(ii) This theorem is stronger than the result in [16] and generalize the range of both the nonlinear power and the spatial variable coefficient.

## 2. Preliminaries

In this section, we give some identities which will be used in the proof of Theorem 1.2. The Cauchy problem to be considered is the following:

$$i\partial_t v + ar^{p_0}(\partial_{rr} + \frac{p_1}{r}\partial_r)v = U(r)v + W(r)|v|^b v + H(v)v, \tag{2.1}$$

where the functions $U,\ W : \mathbb{R}^+ \to \mathbb{R}$ and $H : \mathbb{C} \to \mathbb{R}$. In particular, (2.1) includes the equation (1.1) with

$$U(r) = ap_2 r^{p_0 - 2}, \quad W(r) = \lambda_1 r^{p_3}, \quad H(v) = \lambda_2 \int_0^r (r')^{p_4} |v|^c dr'.$$

We begin with a lemma giving a sufficient condition for the energy quantity (1.10):

**Lemma 2.1.** *If $v$ is the solution to the equation* (2.1) *with the initial data $v_0(r)$, then the solution $v$ satisfies*

$$\frac{d}{dt}\mathcal{E}(v(t)) = -aIm \int_{\mathbb{R}^+} r^{p_1} \overline{\partial_r v} \partial_r (H(v)v) dr, \tag{2.2}$$

*where the energy*

$$\mathcal{E}(v) := -\frac{a}{2}\int_{\mathbb{R}^+} r^{p_1}|\partial_r v|^2 dr - \frac{1}{2}\int_{\mathbb{R}^+} U(r)|v|^2 r^{p_1-p_0} dr - \frac{1}{b+2}\int_{\mathbb{R}^+} W(r)|v|^{b+2} r^{p_1-p_0} dr.$$

Proof. We multiply the equation (2.1) by $\overline{v}_t r^{p_1-p_0}$, integrate over $\mathbb{R}^+$ and take the real part of the result to obtain

$$\begin{aligned}(2.3)\qquad &\frac{d}{dt}\Big(-\frac{a}{2}\int_{\mathbb{R}^+} r^{p_1}|\partial_r v|^2 dr - \frac{1}{2}\int_{\mathbb{R}^+} U(r)|v|^2 r^{p_1-p_0} dr - \frac{1}{b+2}\int_{\mathbb{R}^+} W(r)|v|^{b+2} r^{p_1-p_0} dr\Big)\\ &= \mathrm{Re}\int_{\mathbb{R}^+} H(v) v\overline{v}_t r^{p_1-p_0} dr.\end{aligned}$$

For the right hand side of (2.3), it is easy to derive from (2.1)

(2.4)

$$\mathrm{Re}\int_{\mathbb{R}^+} H(v)v\overline{v}_t r^{p_1-p_0} dr = a\mathrm{Im}\int_{\mathbb{R}^+} r^{p_1}(\partial_{rr} + \frac{p_1}{r}\partial_r)\overline{v}H(v)v dr = -a\mathrm{Im}\int_{\mathbb{R}^+} r^{p_1}\overline{\partial_r v}\partial_r(H(v)v) dr,$$

which together with (2.3) yield that

$$\frac{d}{dt}\mathcal{E}(v) = -a\mathrm{Im}\int_{\mathbb{R}^+} r^{p_1}\overline{\partial_r v}\partial_r(H(v)v) dr. \qquad \square$$

Given a real valued function $\psi(r)$, we consider

$$\mathcal{V}_\psi(t) := \int_{\mathbb{R}^+} \psi(r)|v(t)|^2 r^{p_1-p_0} dr.$$

An important preliminary step in this analysis is the following virial identity:

**Lemma 2.2.** *If $v$ is a (sufficiently smooth and decaying) solution to the equation* (2.1)*, and let $\phi(r), \psi(r) \in C(\mathbb{R}^+)$ be real-valued functions with compact support that satisfy*

$$\partial_r\psi = \frac{\partial_r\phi}{r^{p_0+p_1}},\ \forall\, r \in \mathbb{R}^+,$$

*then*

$$\begin{aligned}(2.5)\qquad \mathcal{V}'_\psi(t) &= 2aIm\int_{\mathbb{R}^+} \overline{v}\partial_r v\partial_r\psi r^{p_1} dr,\\ (2.6)\qquad \mathcal{V}''_\psi(t) &= 2a^2\int_{\mathbb{R}^+} \varpi_1(\phi)|\partial_r v|^2 dr - a^2\int_{\mathbb{R}^+} \varpi_2(\phi)|v|^2 dr + \frac{2a}{b+2}\int_{\mathbb{R}^+} \varpi_3(\phi)|v|^{b+2} dr,\end{aligned}$$

*where $\bar{v}$ denotes the conjugate of $v$, and*

$$\begin{aligned}\varpi_1(\phi) &= 2\partial_r^2\phi - \frac{p_0+2p_1}{r}\partial_r\phi,\\ \varpi_2(\phi) &= \partial_r^4\phi + \frac{Q_3}{r}\partial_r^3\phi + \frac{Q_2}{r^2}\partial_r^2\phi + \frac{Q_1}{r^3}\partial_r\phi + \frac{2}{a}(\partial_r U + \partial_r H(v))r^{-p_0}\partial_r\phi,\\ \varpi_3(\phi) &= br^{-p_0}[(\partial_r^2\phi - \frac{p_0}{r}\partial_r\phi)W(r) - \frac{2}{b}\partial_r\phi\partial_r W],\end{aligned}$$

*with $Q_3 := -p_1 - p_0,\ Q_2 := p_1 + p_0(p_1+2),\ Q_1 := -2p_0(p_1+1)$.*

Proof. (1) Multiplying the equation (2.1) by $\overline{v}r^{p_1-p_0}$ and taking the imaginary part of the result, we get

$$\frac{\partial}{\partial t}(|v|^2 r^{p_1-p_0}) = -2a\mathrm{Im}[\bar{v}\partial_r(r^{p_1}\partial_r v)]. \tag{2.7}$$

We multiply (2.7) by $\psi(r)$ and integrate over $\mathbb{R}^+$ to get

$$\frac{d}{dt}\int_{\mathbb{R}^+}\psi(r)|v|^2 r^{p_1-p_0}dr = 2a\mathrm{Im}\int_{\mathbb{R}^+}\bar{v}\partial_r v\partial_r\psi r^{p_1}dr.$$

(2) For $\partial_r\psi = \frac{\partial_r\phi}{r^{p_0+p_1}}$, by writing

$$\mathcal{M}_\psi(t) := 2a\mathrm{Im}\int_{\mathbb{R}^+}\bar{v}\partial_r v r^{-p_0}\partial_r\phi dr, \tag{2.8}$$

we claim that

$$\mathcal{M}'_\psi(t) = -2a\mathrm{Im}\int_{\mathbb{R}^+}v_t[2(r^{-p_0}\partial_r\phi)\overline{\partial_r v} + \partial_r(r^{-p_0}\partial_r\phi)\bar{v}]dr. \tag{2.9}$$

Indeed, we deduce from (2.8) that

$$\mathcal{M}'_\psi(t) = -2a\mathrm{Im}\int_{\mathbb{R}^+}v_t\overline{\partial_r v}r^{-p_0}\partial_r\phi dr - 2a\mathrm{Im}\int_{\mathbb{R}^+}v\overline{\partial_r v_t}r^{-p_0}\partial_r\phi dr,$$

and (2.9) follows by integration by parts, since

$$v\overline{\partial_r v_t}r^{-p_0}\partial_r\phi = \partial_r[v\bar{v}_t r^{-p_0}\partial_r\phi] - \bar{v}_t\partial_r(vr^{-p_0}\partial_r\phi),$$

which proves the claim.

Now using the equation (2.1) with $N(v) := U(r)v + W(r)|v|^b v + H(v)v$, we see that $\mathrm{Im}\, v_t = -\mathrm{Re}\,[-ar^{p_0-p_1}\partial_r(r^{p_1}\partial_r v) + N(v)]$, and

$$\begin{aligned}\mathcal{M}'_\psi(t) &= 2a\mathrm{Re}\int_{\mathbb{R}^+}[-ar^{p_0-p_1}\partial_r(r^{p_1}\partial_r v) + N(v)][2(r^{-p_0}\partial_r\phi)\overline{\partial_r v} + \partial_r(r^{-p_0}\partial_r\phi)\bar{v}]dr \\ &:= (B_L^1 + B_L^2) + (B_N^1 + B_N^2).\end{aligned} \tag{2.10}$$

Next, an elementary calculation shows that

$$\begin{aligned}B_L^1 &= 2a^2\mathrm{Re}\int_{\mathbb{R}^+}r^{p_1}\partial_r v\partial_r[2(r^{-p_1}\partial_r\phi)\overline{\partial_r v}]dr \\ &= 4a^2\int_{\mathbb{R}^+}r^{p_1}|\partial_r v|^2\partial_r(r^{-p_1}\partial_r\phi)dr - 2a^2\int_{\mathbb{R}^+}|\partial_r v|^2\partial_r^2\phi dr \\ &= 2a^2\int_{\mathbb{R}^+}[2r^{p_1}\partial_r(r^{-p_1}\partial_r\phi) - \partial_r^2\phi)]|\partial_r v|^2 dr,\end{aligned} \tag{2.11}$$

and

$$\begin{aligned}B_L^2 &= 2a^2\mathrm{Re}\int_{\mathbb{R}^+}r^{p_1}\partial_r v\partial_r[r^{p_0-p_1}\bar{v}\partial_r(r^{-p_0}\partial_r\phi)]dr \\ &= 2a^2\int_{\mathbb{R}^+}r^{p_0}|\partial_r v|^2\partial_r(r^{-p_0}\partial_r\phi)dr + 2a^2\int_{\mathbb{R}^+}r^{p_1} \\ &\quad\times\partial_r[r^{p_0-p_1}\partial_r(r^{-p_0}\partial_r\phi)]\mathrm{Re}(\bar{v}\partial_r v)dr.\end{aligned} \tag{2.12}$$

We now calculate the various terms corresponding to $N(v)$. The first term

$$B_N^1 = 4a\mathrm{Re}\int_{\mathbb{R}^+}[U(r)v + W(r)|v|^b v + H(v)v]\overline{\partial_r v}(r^{-p_0}\partial_r\phi)dr \tag{2.13}$$

$$= -2a \int_{\mathbb{R}^+} |v|^2 \partial_r[(U(r) + H(v))(r^{-p_0}\partial_r\phi)]dr - \frac{4a}{b+2}\int_{\mathbb{R}^+} |v|^{b+2}\partial_r[W(r)(r^{-p_0}\partial_r\phi)]dr,$$

and the second term

$$B_N^2 = 2a \int_{\mathbb{R}^+} [U(r)|v|^2 + W(r)|v|^{b+2} + H(v)|v|^2]\partial_r(r^{-p_0}\partial_r\phi)dr. \tag{2.14}$$

Finally, combining (2.11), (2.12), (2.13) with (2.14), we deduce from (2.10) that

$$\begin{aligned} \mathcal{M}'_\varepsilon(t) &= 2a^2 \int_{\mathbb{R}^+} [2r^{p_1}\partial_r(r^{-p_0}\partial_r\phi) - \partial_r^2\phi + r^{p_0}\partial_r(r^{-p_0}\partial_r\phi)]|\partial_r v|^2 dr \\ &\quad +2a^2 \int_{\mathbb{R}^+} r^{p_1}\partial_r[r^{p_0-p_1}\partial_r(r^{-p_0}\partial_r\phi)]\mathrm{Re}(\overline{v}\partial_r v)dr \\ &\quad -2a \int_{\mathbb{R}^+} |v|^2\partial_r[(U(r) + H(v))(r^{-p_0}\partial_r\phi)]dr \\ &\quad -\frac{4a}{b+2}\int_{\mathbb{R}^+} |v|^{b+2}\partial_r[W(r)(r^{-p_0}\partial_r\phi)]dr \\ &\quad +2a \int_{\mathbb{R}^+} [U(r)|v|^2 + W(r)|v|^{b+2} + H(v)|v|^2]\partial_r(r^{-p_0}\partial_r\phi)dr \\ &:= 2a^2 \int_{\mathbb{R}^+} \varpi_1(\phi)|\partial_r v|^2 dr - a^2 \int_{\mathbb{R}^+} \varpi_2(\phi)|v|^2 dr + \frac{2a}{b+2}\int_{\mathbb{R}^+} |v|^{b+2}\varpi_3(\phi)dr, \end{aligned} \tag{2.15}$$

where

$$\begin{aligned} \varpi_1(\phi) &= 2\partial_r^2\phi - \frac{p_0 + 2p_1}{r}\partial_r\phi, \\ \varpi_2(\phi) &= \partial_r^4\phi + \frac{Q_3}{r}\partial_r^3\phi + \frac{Q_2}{r^2}\partial_r^2\phi + \frac{Q_1}{r^3}\partial_r\phi + \frac{2}{a}(\partial_r U + \partial_r H(v))r^{-p_0}\partial_r\phi, \\ \varpi_3(\phi) &= br^{-p_0}[(\partial_r^2\phi - \frac{p_0}{r}\partial_r\phi)W(r) - \frac{2}{b}\partial_r\phi\partial_r W], \end{aligned}$$

with $Q_3 := -p_1 - p_0,\ Q_2 := p_1 + p_0(p_1 + 2),\ Q_1 := -2p_0(p_1 + 1)$. The proof of Lemma 2.2 is completed. □

As a consequence, we can prove a variant of [15, Corollary 5.1] related to (1.1).

**Corollary 2.3.** *Let $v$ be a local solution to the Cauchy problem* (1.1) *in* $C([0,T);\mathcal{W}_0^{1,2}(\mathbb{R}^+))$*, and let* $\partial_r\psi = \frac{r^{1-p_0}}{2-p_0}$*, then for* $t \in [0,T)$*,*

$$\begin{aligned} \mathcal{V}''(t) &= -4aE(v) + D_0 \int_{\mathbb{R}^+} |v|^2 r^{p_1-2}dr + D_1 \int_{\mathbb{R}^+} |v|^{b+2} r^{p_3+p_1-p_0}dr \\ &\quad +D_2 \int_{\mathbb{R}^+} |v|^{c+2} r^{p_4+p_1-p_0+1}dr, \end{aligned} \tag{2.16}$$

*with* $D_0 := \frac{2a^2p_2(\sigma-2)}{2-p_0}, D_1 := \frac{2a\lambda_1[b(p_1-p_0+1)-2(p_3-p_0+2)]}{(b+2)(2-p_0)}, D_2 := -\frac{2a\lambda_2}{2-p_0}$.

Proof. As in the previous Lemma 2.2, for $R > 0$, let $\partial_r\phi_R(r) \in \mathbb{C}_0^\infty(\mathbb{R}^+)$ satisfies

$$\partial_r\phi_R(r) := \begin{cases} r, & \text{if } r \le R, \\ 0, & \text{if } r \ge 2R, \end{cases} \tag{2.17}$$

and $\partial_r\psi_R = \frac{\partial_r\phi_R}{r^{p_0+p_1}}$, Lemma 2.2 implies

$$\mathcal{V}''_{\psi_R}(t) = 2a^2\int_{\mathbb{R}^+}\varpi_1(\phi_R)|\partial_r v|^2 dr - a^2\int_{\mathbb{R}^+}\varpi_2(\phi_R)|v|^2 dr + \frac{2a}{b+2}\int_{\mathbb{R}^+}\varpi_3(\phi_R)|v|^{b+2}dr. \tag{2.18}$$

Noting that as $R \to \infty$, the right side converges to

$$\begin{aligned}(2.19)\qquad & 2a^2\int_{\mathbb{R}^+} r^{p_1}|\partial_r v|^2 dr + 2a^2 p_2\int_{\mathbb{R}^+}|v|^2 r^{p_1-2}dr \\ &+C_0\int_{\mathbb{R}^+}|v|^{b+2}r^{p_1-p_0+p_3}dr - \frac{2a\lambda_2}{2-p_0}\int_{\mathbb{R}^+}|v|^{c+2}r^{p_1-p_0+p_4+1}dr,\end{aligned}$$

with $C_0 := \frac{2a\lambda_1(b(p_1-p_0+1)-2p_3)}{(b+2)(2-p_0)}$, that is bounded for $v$ to be a local solution to the Cauchy problem (1.1) in $C([0,T);\mathcal{W}_0^{1,2}(\mathbb{R}^+))$. So from (2.19) and Lemma 2.2, we obtain

$$\begin{aligned}\mathcal{V}''(t) &= 2a^2\int_{\mathbb{R}^+} r^{p_1}|\partial_r v|^2 dr + 2a^2 p_2\int_{\mathbb{R}^+}|v|^2 r^{p_1-2}dr \\ &\quad +C_0\int_{\mathbb{R}^+}|v|^{b+2}r^{p_1-p_0+p_3}dr - \frac{2a\lambda_2}{2-p_0}\int_{\mathbb{R}^+}|v|^{c+2}r^{p_1-p_0+p_4+1}dr,\end{aligned}$$

which together with (1.9) imply the desired result. □

At the end of this section, we recall the known Caffarelli-Kohn-Nirenberg inequality:

**Lemma 2.4** ([2]). *If $p,\ q \geq 1,\ l > 0,\ \alpha,\ \beta,\ \gamma$ satisfy $\gamma = a\sigma + (1-a)\beta,\ 0 \leq a \leq 1$ and $\frac{1}{p}+\frac{\alpha}{n},\ \frac{1}{q}+\frac{\beta}{n},\ \frac{1}{l}+\frac{\gamma}{n} > 0$, then there exists a positive constant C such that the following inequality holds for all $u \in C_0^\infty(\mathbb{R}^n)$*

$$\||x|^\gamma u\|_{L^l(\mathbb{R}^n)} \leq C\||x|^\alpha \nabla u\|^a_{L^p(\mathbb{R}^n)}\||x|^\beta u\|^{1-a}_{L^q(\mathbb{R}^n)} \tag{2.20}$$

*if and only if the following relations hold:*

$$\frac{1}{l}+\frac{\gamma}{n} = a\left(\frac{1}{p}+\frac{\alpha-1}{n}\right)+(1-a)\left(\frac{1}{q}+\frac{\beta}{n}\right), \tag{2.21}$$

$$\alpha-\sigma \geq 0 \quad \textit{if } a > 0.$$

*and*

$$\alpha-\sigma \leq 1 \quad \textit{if } a > 0,\ \frac{1}{p}+\frac{\alpha-1}{n} = \frac{1}{l}+\frac{\gamma}{n}.$$

## 3. Blowup Results

In this section, we prove the theorem 1.2 using the virial method developed in Section 2.

Proof. Assume the Schwartz initial data $v_0 \in \mathcal{W}_0^{1,2}(\mathbb{R}^+)$, we prove the result by contradiction. Suppose the maximal existence time $T$ of the solution $v$ to (1.1) is infinity.

Whenever $v$ exists we put $\mathcal{V}(t) = \frac{1}{(2-p_0)^2}\int_{\mathbb{R}^+}|v|^2 r^{p_1-2p_0+2}dr$. From (2.16) in Corollary 2.3, we have

$$\mathcal{V}''(t) \geq -4aE(v) + D_1\int_{\mathbb{R}^+}|v|^{b+2}r^{p_3+p_1-p_0}dr + D_2\int_{\mathbb{R}^+}|v|^{c+2}r^{p_1-p_0+1+p_4}dr, \tag{3.1}$$

where $D_1 = \frac{2a\lambda_1[b(p_1-p_0+1)-2(p_3-p_0+2)]}{(b+2)(2-p_0)}$, $D_2 = -\frac{2a\lambda_2}{2-p_0}$.

For $p_0 < \min\{p_1 + 1,\ 2\}$, $p_1 > -1$, and the hypothesis (1.6), set $b = b_0$, we invoke the Caffarelli-Kohn-Nirenberg inequality (2.20) to obtain

$$\|r^{\frac{p_3+p_1-p_0}{b+2}} v\|_{L^{b+2}(\mathbb{R}^+)} \le C_b \|r^{\frac{p_1}{2}}\partial_r v\|^{\frac{2}{b+2}}_{L^2(\mathbb{R}^+)} \|r^{\frac{p_1-p_0}{2}} v\|^{\frac{b}{b+2}}_{L^2(\mathbb{R}^+)}. \tag{3.2}$$

Notice that (1.8), (1.9) give

$$\begin{aligned} E(v(t)) &\ge -\frac{a}{2}\int_{\mathbb{R}^+} r^{p_1}|\partial_r v|^2 dr - \frac{\lambda_1}{b+2}\int_{\mathbb{R}^+} |v|^{b+2} r^{p_3+p_1-p_0} dr, \\ &\ge \begin{cases} [-\frac{a}{2} - \frac{\lambda_1}{b+2} C_b^{b+2} M(v_0)^{\frac{b}{2}}] \int_{\mathbb{R}^+} r^{p_1}|\partial_r v|^2 dr,\ \lambda_1 > 0,\ b = b_0, \\ -\frac{a}{2}\int_{\mathbb{R}^+} r^{p_1}|\partial_r v|^2 dr,\ \lambda_1 \le 0, \end{cases} \end{aligned} \tag{3.3}$$

where the last inequality are deduced from (3.2). Thus we divide it into two steps as follows:

**Case 1.** $\lambda_1 > 0$ and $b = b_0$ :

From (3.3) with $M(v_0)^{\frac{b}{2}} \le \frac{-a(b+2)}{2\lambda_1 C_b^{b+2}}$, we deduce that

$$\int_{\mathbb{R}^+} r^{p_1}|\partial_r v|^2 dr \le \frac{E(v)}{-\frac{a}{2} - \frac{\lambda_1}{b+2} C_b^{b+2} M(v_0)^{\frac{b}{2}}}. \tag{3.4}$$

(i) When $\lambda_2 < 0$, since $D_1 \equiv 0$ in case of $b = b_0$, applying the following Caffarelli-Kohn-Nirenberg inequality:

$$\|r^{\frac{p_4+1+p_1-p_0}{c+2}} v\|_{L^{c+2}(\mathbb{R}^+)} \le C_c \|r^{\frac{p_1}{2}}\partial_r v\|^{\frac{2}{c+2}}_{L^2(\mathbb{R}^+)} \|r^{\frac{p_1-p_0}{2}} v\|^{\frac{c}{c+2}}_{L^2(\mathbb{R}^+)}, \tag{3.5}$$

to the inequality (3.1) under the assumption of $c = c_0$ and (1.6), we infer that

$$\mathcal{V}''(t) \ \ge\ -4aE(v) + D_2 C_c^{c+2} M(v_0)^{\frac{c}{2}} \int_{\mathbb{R}^+} r^{p_1}|\partial_r v|^2 dr.$$

Now using (3.4), we further have

$$\mathcal{V}''(t) \ge [-4a + \frac{D_2 C_c^{c+2} M(v_0)^{\frac{c}{2}}}{-\frac{a}{2} - \frac{\lambda_1}{b+2} C_b^{b+2} M(v_0)^{\frac{b}{2}}}] E(v), \tag{3.6}$$

which implies that $\mathcal{V}''(t) \ge 0$ for $\frac{\lambda_2}{2-p_0} C_c^{c+2} M(v_0)^{\frac{c}{2}} \ge a + \frac{2\lambda_1}{b+2} C_b^{b+2} M(v_0)^{\frac{b}{2}}$.

(ii) When $\lambda_2 \ge 0$, it is obvious that for $M(v_0)^{\frac{b}{2}} \le \frac{-a(b+2)}{2\lambda_1 C_b^{b+2}}$,

$$\mathcal{V}''(t) \ge -4aE(v) \ge 0.$$

Hence, for $\lambda_1 > 0$ and $b = b_0$, we conclude that

$$\mathcal{V}''(t) \ge [-4a + \frac{D_2 C_c^{c+2} M(v_0)^{\frac{c}{2}}}{-\frac{a}{2} - \frac{\lambda_1}{b+2} C_b^{b+2} M(v_0)^{\frac{b}{2}}}] E(v) \ge 0, \tag{3.7}$$

provided $M(v_0)^{\frac{b}{2}} \le \frac{-a(b+2)}{2\lambda_1 C_b^{b+2}}$ and

$$\begin{cases} \lambda_2 < 0 : c = c_0,\ M(v_0)^{\frac{c}{2}} \le \frac{2-p_0}{\lambda_2 C_c^{c+2}} [a + \frac{2\lambda_1 C_b^{b+2}}{b+2} M(v_0)^{\frac{b}{2}}], \\ \lambda_2 \ge 0 : \quad \text{for all } c. \end{cases} \tag{3.8}$$

**Case 2.** $\lambda_1 \le 0$ and $b \ge b_0$ :

From (3.3), we have

$$\int_{\mathbb{R}^+} r^{p_1}|\partial_r v|^2 dr \le -\frac{2}{a}E(v). \tag{3.9}$$

(i) When $\lambda_2 < 0$.

Applying (3.5) to the inequality (3.1), similar to the procedure of (i) in Case 1, we have

$$\begin{aligned}\mathcal{V}''(t) &\ge -4aE(v) + D_2 C_c^{c+2} M(v_0)^{\frac{c}{2}} \int_{\mathbb{R}^+} r^{p_1}|\partial_r v|^2 dr \\ &\ge [-4a - \frac{2D_2 C_c^{c+2} M(v_0)^{\frac{c}{2}}}{a}]E(v),\end{aligned}$$

for $c = c_0$, which yields that $\mathcal{V}''(t) \ge 0$ with $\frac{\lambda_2}{2-p_0} C_c^{c+2} M(v_0)^{\frac{c}{2}} \ge a$.

(ii) When $\lambda_2 \ge 0$, we have

$$\mathcal{V}''(t) \ge -4aE(v) \ge 0.$$

Hence, for $\lambda_1 \le 0$ and $b \ge b_0$, we conclude that

$$\mathcal{V}''(t) \ge [-4a - \frac{2D_2 C_c^{c+2} M(v_0)^{\frac{c}{2}}}{a}]E(v) \ge 0, \tag{3.10}$$

provided

$$\begin{cases} \lambda_2 < 0 : c = c_0, \ M(v_0)^{\frac{c}{2}} \le \dfrac{(2-p_0)a}{\lambda_2 C_c^{c+2}}, \\ \lambda_2 \ge 0 : \quad \text{for all } c \text{ and } M(v_0). \end{cases} \tag{3.11}$$

On the one hand, from (2.5) in Lemma 2.2, we notice that

$$\begin{aligned}\mathcal{V}'(t) &\le \frac{-4a}{2-p_0}(\int_{\mathbb{R}^+} r^{p_1}|\partial_r v|^2 dr)^{\frac{1}{2}}(\int_{\mathbb{R}^+} |v|^2 r^{p_1-2p_0+2} dr)^{\frac{1}{2}} \\ &= -4a\mathcal{V}(t)^{\frac{1}{2}}(\int_{\mathbb{R}^+} r^{p_1}|\partial_r v|^2 dr)^{\frac{1}{2}}.\end{aligned} \tag{3.12}$$

According to the above analysis, the integral term in (3.12) can be bounded by the following:

(1) For $\lambda_1 \le 0$, from (3.9) in Case 2 and (3.10), we have

$$(\int_{\mathbb{R}^+} r^{p_1}|\partial_r v|^2 dr)^{\frac{1}{2}} \le \left(-\frac{2}{a}\frac{\mathcal{V}''(t)}{[-4a - \frac{2D_2 C_c^{c+2} M(v_0)^{\frac{c}{2}}}{a}]}\right)^{\frac{1}{2}}. \tag{3.13}$$

(2) For $\lambda_1 > 0$, from (3.4) in Case 1 and (3.7), we have

$$(\int_{\mathbb{R}^+} r^{p_1}|\partial_r v|^2 dr)^{\frac{1}{2}} \le \left(\frac{1}{2a^2 + \frac{4a\lambda_1}{b+2} C_b^{b+2} M(v_0)^{\frac{b}{2}} - \frac{2a\lambda_2}{2-p_0} C_c^{c+2} M(v_0)^{\frac{c}{2}}}\right)^{\frac{1}{2}} \mathcal{V}''(t)^{\frac{1}{2}}. \tag{3.14}$$

Substituting (3.13), (3.14) into (3.12), we obtain the exact estimate of $\mathcal{V}'(t)$ as follows:

$$\mathcal{V}'(t) \le \widetilde{C}\mathcal{V}(t)^{\frac{1}{2}}\mathcal{V}''(t)^{\frac{1}{2}},$$

where

$$\widetilde{C}=\begin{cases}\left(\dfrac{8a}{a+\frac{2\lambda_1}{b+2}C_b^{b+2}M(v_0)^{\frac{b}{2}}-\frac{\lambda_2}{2-p_0}C_c^{c+2}M(v_0)^{\frac{c}{2}}}\right)^{\frac{1}{2}}, & \text{for } \lambda_1>0, b=b_0, \text{and (3.8)},\\ \left(\dfrac{8a}{a-\frac{\lambda_2 C_c^{c+2}M(v_0)^{\frac{c}{2}}}{2-p_0}}\right)^{\frac{1}{2}}, & \text{for } \lambda_1\le 0, b\ge b_0, \text{and (3.11)}.\end{cases}$$

Since $\mathcal{V}'(0)>0$, then from the continuity, at least for $t$ small enough, we have $\mathcal{V}'(t)>0$ and

$$\frac{\mathcal{V}'(t)}{\mathcal{V}'(0)}\ge\left(\frac{\mathcal{V}(t)}{\mathcal{V}(0)}\right)^{\frac{1}{\widetilde{C}^2}},\quad \text{for all } t>0. \tag{3.15}$$

In case of $\widetilde{C}<1$, we discover that $\mathcal{V}(t)$ blows up in finite time.

On the other hand, we deduce from (3.7) and (3.10) that

$$\mathcal{V}(t)\ge\mathcal{V}(0)+\mathcal{V}'(0)t,\quad \forall\, t>0. \tag{3.16}$$

Noting that (3.2) implies that the energy $E(v(t))$ is well defined for $v(\cdot,t)\in\mathcal{W}_0^{1,2}(\mathbb{R}^+)$. Furthermore since

$$\mathcal{V}(0)=\frac{1}{(2-p_0)^2}\int_{\mathbb{R}^+}|v_0|^2r^{p_1-2p_0+2}dr,\ \mathcal{V}'(0)=\frac{2a}{2-p_0}\mathrm{Im}\int_{\mathbb{R}^+}\overline{v}_0\partial_r v_0 r^{p_1-p_0+1}dr$$

are finite, (3.1) implies that $\mathcal{V}(t)$ is finite for $t>0$. Since $\mathcal{V}'(0)>0$, then $\mathcal{V}(t)\to\infty$ as $t\to\infty$. Hence $v$ blows up. □

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