Koike, N.
Osaka J. Math.
59 (2022), 905–950

THE BEHAVIOUR OF THE MEAN CURVATURE FLOW FOR PINCHED SUBMANIFOLDS IN RANK ONE SYMMETRIC SPACES

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(Received May 11, 2021, revised August 20, 2021)

Abstract

In this paper, we consider the mean curvature flow starting from closed submanifolds in rank one symmetric spaces satisfying some pinching condition for the norm of the second fundamental form. We prove that, under some additional condition, the closed submanifold satisfying the pinching condition collapses to a round point in finite time or converges to a totally geodesic submanifold in infinite time along the mean curvature flow.

1. Introduction

Let $f : M \hookrightarrow \widetilde{M}$ be a C^∞-immersion of a closed connected C^∞-manifold M into a C^∞-Riemannian manifold $\widetilde{M}$. Denote by h and H the second fundamental form and the mean curvature vector field of f, respectively. Let $\{f_t : M \hookrightarrow \widetilde{M}\}_{t\in[0,T)}$ the mean curvature flow starting from f, that is, the C^∞-family of C^∞-immersions satisfying

$$\frac{\partial F}{\partial t} = H_t \quad (0 \le t < T), \qquad f_0 = f,$$

where F is the map of $M \times [0,T)$ into $\widetilde{M}$ defined by $F(p,t) := f_t(p)$ $((p,t) \in M \times [0,T))$ and H_t is the mean curvature vector field of f_t and T is the maximal time of the flow. Set $M_t := f_t(M)$. If f_t's are embeddings, we call $\{M_t\}_{t\in[0,T)}$ the mean curvature flow starting from M_0.

In 1984, the study of the mean curvature flow treated as the evolution of immersions was originated by G. Huisken ([7]). He ([7]) proved that any closed convex hypersurface in Euclidean space collapses to a round point in finite time along the mean curvature flow. In 1986, he ([8]) proved that the same fact holds for the mean curvature flow starting from closed hypersurfaces in Riemannian manifolds (of bounded curvature) satisfying a stronger convexity condition, where this stronger convexity condition coincides with the usual convexity condition in the case where the ambient space is a Euclidean space.

Let f be an isometric immersion of m-dimensional Riemannian manifold into another Riemannian manifold, h and H be the second fundamental form and the mean curvature vector field of f, respectively. In general, the relation $\|h\|^2 \ge \frac{\|H\|^2}{m}$ holds between their norms and the equality in this inequality holds if and only if f is totally umbilic. Hence the following type of condition is interpreted as a pinching condition for the norm of the second fundamental form:

2020 Mathematics Subject Classification. Primary 53E10; Secondary 53C35.

$$(*_{a,b}) \qquad ||h||^2 \leq \frac{1}{a}||H||^2 + b,$$

where a is a positive constant and b is a constant. In 2010, B. Andrews and C. Baker ([1]) proved that, if a closed submanifold in a sphere satisfies the pinching condition $(*_{a,b})$ for suitably chosen a, b, then the submanifold collapses to a round point in finite time or converges to a totally geodesic submanifold in infinite time along the mean curvature flow in finite time or converges to a totally geodesic submanifold along the mean curvature flow. In 2011, K. Liu, H. Xu, F. Ye and E. Zhao ([13]) proved that the similar result holds for a closed submanifold satisfying the pinching condition $(*_{a,b})$ for suitably chosen a, b in a hyperbolic space. In 2012, K. Liu, H. Xu and E. Zhao ([14]) proved that the similar result holds for a closed submanifold satisfying the pinching condition $(*_{a,b})$ for suitably chosen a, b in a Riemannian manifold of some bounded curvature condition. In 2017, G. Pipoli and C. Sinestrari ([18]) proved that the similar result holds for a closed submanifold of low codimension satisfying the pinching condition $(*_{a,b})$ for suitably chosen a, b in a complex projective space. On the basis of the discussion in [18], Y. Mizumura ([16]) proved that the similar result holds for a closed submanifold of low codimension in a quaternionic projective space and N. Uenoyama ([19]) proved that the similar result holds for a closed submanifold of low codimension in a complex hyperbolic space.

We shall prepare to state results in this paper. Denote by $\mathbb{C}P^n(4c)$, $\mathbb{H}P^n(4c)$ and $\mathbb{O}P^2(4c)$ the complex projective space of constant holomorphic sectional curvature $4c$, the quaternionic projective space of constant quaternionic sectional curvature $4c$ and the Cayley plane of constant octonian sectional curvature $4c$, and by $\mathbb{C}H^n(-4c)$, $\mathbb{H}H^n(-4c)$ and $\mathbb{O}H^2(-4c)$ the complex hyperbolic space of constant holomorphic sectional curvature $-4c$, the quaternionic hyperbolic space of constant quaternionic sectional curvature $-4c$ and the Cayley hyperbolic plane of constant octonian sectional curvature $-4c$. Throughout this paper, $\mathbb{F}$ denotes the complex number field $\mathbb{C}$, the quaternionic algebra $\mathbb{H}$ or the Cayley algebra $\mathbb{O}$, $\mathbb{F}P^n(c)$ denotes one of rank one symmetric spaces of compact type:

$$\mathbb{C}P^n(4c),\ \mathbb{H}P^n(4c) \text{ or } \mathbb{O}P^2(4c)$$

and $\mathbb{F}H^n(c)$ denotes one of rank one symmetric spaces of non-compact type:

$$\mathbb{C}H^n(-4c),\ \mathbb{H}H^n(-4c) \text{ or } \mathbb{O}H^2(-4c).$$

Also, throughout this paper, $\widetilde{M}$ denotes $\mathbb{F}P^n(c)$ or $\mathbb{F}H^n(c)$. Set

$$d := \begin{cases} 2 & (\text{when } \widetilde{M} = \mathbb{C}P^n(4c), \mathbb{C}H^n(-4c)) \\ 4 & (\text{when } \widetilde{M} = \mathbb{H}H^n(-4c), \mathbb{H}H^n(-4c)) \\ 8 & (\text{when } \widetilde{M} = \mathbb{O}H^2(-4c), \mathbb{O}H^2(-4c)). \end{cases}$$

Let M be an m-dimensional closed submanifold in $\widetilde{M}$ and set $k := dn - m$. Set

$$b := \begin{cases} 2c & (\text{when } \widetilde{M} = \mathbb{F}P^n(4c) \text{ and } k = 1) \\ \dfrac{(m - 4(d-1)k - 3)c}{m} & (\text{when } \widetilde{M} = \mathbb{C}P^n(4c),\ \mathbb{H}P^n(4c) \text{ and } k \geq 2) \\ -8c & (\text{when } \widetilde{M} = \mathbb{F}H^n(-4c) \text{ and } k = 1) \\ -\dfrac{(8m + 4(d-1)k + 3)c}{m} & (\text{when } \widetilde{M} = \mathbb{C}H^n(-4c), \mathbb{H}H^n(-4c) \text{ and } k \geq 2). \end{cases}$$

We consider the following condition:

$$(*_{m-1,b}) \qquad ||h||^2 < \frac{1}{m-1}||H||^2 + b,$$

where h and H denote the second fundamental form and the mean curvature vector of M, respectively. In this paper, we prove the following facts for the mean curvature flows starting from closed submanifolds in rank one symmetric spaces $\widetilde{M}$ satisfying the above pinching condition $(*_{m-1,b})$.

Theorem 1.1. *Let M be a closed real hypersurface in $\mathbb{C}P^n(4c)$ $(n \geq 3)$, $\mathbb{H}P^n(4c)$ $(n \geq 2)$ or $\mathbb{O}P^2(4c)$, and $\{M_t\}_{t\in[0,T)}$ be the mean curvature flow starting from M. Assume that M satisfies the above pinching condition $(*_{m-1,b})$ (for $b = 2c$). Then the following statements* (i) *and* (ii) *hold:*

(i) *The condition $(*_{m-1,b})$ is preserved along the mean curvature flow $\{M_t\}_{t\in[0,T)}$;*

(ii) *$T < \infty$ and M_t collapses to a round point as $t \to T$.*

Theorem 1.2. *Let M be an m-dimensional closed submanifold of codimension greater than one in $\mathbb{C}P^n(4c)$ or $\mathbb{H}P^n(4c)$, and $\{M_t\}_{t\in[0,T)}$ be the mean curvature flow starting from M. Assume that $m \geq \max\{\frac{nd}{2}, \frac{3d}{2} + 5\}$, M satisfies the pinching condition $(*_{m-1,b})$. Then the following statements* (i) *and* (ii) *hold:*

(i) *The condition $(*_{m-1,b})$ is preserved along the mean curvature flow $\{M_t\}_{t\in[0,T)}$;*

(ii) *One of the followings holds:*

(ii-1) *$T < \infty$ and M_t collapses to a round point as $t \to T$;*

(ii-2) *$T = \infty$ and M_t converges to a totally geodesic submanifold as $t \to \infty$.*

Theorem 1.3. *Under the hypothesis of* Theorem 1.2*, if the diameter of M in $(\widetilde{M}, \widetilde{g})$ is smaller than $\frac{\pi}{6\sqrt{c}}$, then $T < \infty$ and M_t collapses to a round point as $t \to T$.*

Theorem 1.4. *Let M be a closed real hypersurface in $\mathbb{C}H^n(4c)$ $(n \geq 2)$, $\mathbb{H}H^n(4c)$ $(n \geq 2)$ or $\mathbb{O}H^2(4c)$, and $\{M_t\}_{t\in[0,T)}$ be the mean curvature flow starting from M. Assume that M satisfies the pinching condition $(*_{m-1,b})$. Then the following statements* (i) *and* (ii) *hold:*

(i) *The condition $(*_{m-1,b})$ is preserved along the mean curvature flow $\{M_t\}_{t\in[0,T)}$;*

(ii) *$T < \infty$ and M_t collapses to a round point as $t \to T$.*

Theorem 1.5. *Let M be an m-dimensional closed submanifold of codimension greater than one in $\mathbb{C}H^n(-4c)$ or $\mathbb{H}H^n(-4c)$, and $\{M_t\}_{t\in[0,T)}$ be the mean curvature flow starting from M. Assume that $m \geq \max\{\frac{nd}{2}, \frac{3d}{2} + 5\}$, M satisfies the pinching condition $(*_{m-1,b})$. Then the following statements* (i) *and* (ii) *hold:*

(i) *The condition $(*_{m-1,b})$ is preserved along the mean curvature flow $\{M_t\}_{t\in[0,T)}$;*

(ii) *$T < \infty$ and M_t collapses to a round point as $t \to T$.*

REMARK 1.1. (i) By comparing the above b with b_1 in the proof of Theorem 3.2 in [14], we have $-b_1 < b$. Hence Theorems 1.1–1.5 improve Theorem 3.2 in [14].

(ii) In the result in [18], a small codimension condition is imposed. In our results (Theorems 1.2 and 1.5), such a small codimension condition need not be imposed because we do not claim that the term b in our pinching condition $(*_{m-1,b})$ is positive. On the other hand, we need to impose the lower bound condition $m \geq \max\{\frac{nd}{2}, \frac{3d}{2} + 5\}$ for the dimension of the submanifold to prove the preservability of the condition $(*_{m-1,b})$ along the mean curvature flow. In fact, since we use an orthonormal frame of type (II) (as in Lemma 3.1) to prove the preservability of the condition $(*_{m-1,b})$, we need to impose $m \geq \frac{nd}{2}$. Also, according to the proof of Proposition 3.8, we need to impose $m \geq \frac{3d}{2} + 5$.

(iii) The condition $(*_{m-1,b})$ implies that

$$\|H\|^2 > \begin{cases} 8(dn-1)(dn-2)c & (\text{when } \widetilde{M} = \mathbb{F}H^n(-4c) \text{ and } k = 1) \\ (m-1)(8m+4(d-1)k+3)c & (\text{when } \widetilde{M} = \mathbb{C}H^n(-4c),\ \mathbb{H}H^n(-4c) \text{ and } k \geq 2) \end{cases}$$

Thus the conditions $(*_{m-1,b})$ in Theorems 1.4 and 1.5 imply that $\|H\|$ is rather big.

(iv) In our method of the proof, we cannot derive the result similar to Theorems 1.2 and 1.5 in the case of $\widetilde{M} = \mathbb{O}P^2(4c)$ or $\mathbb{O}H^2(-4c)$. For, in these cases, m must be larger than or equal to $\frac{3d}{2} + 5 = \frac{3\cdot 8}{2} + 5 = 17$ in order that the inequality (3.19) in Section 3 holds. However, this is impossible because $\dim \mathbb{O}P^2(4c) = \dim \mathbb{O}H^2(-4c) = 16$. Also, the constant $\alpha = \frac{(11-2d)m-19}{9m(m+2)}$ in (4.1) of Section 4 is negative in these cases. Hence the evolution inequality (4.2) for f_σ in Section 4 does not hold.

This paper is organized as follows. In Section 2, we recall some basic notions and facts. In Section 3, we prove the preservability of the above pinching condition $(*_{m-1,b})$ along the mean curvature flow. In Section 4, we study the behavior of the norm of the traceless part of the second fundamental form, which will be used to measure the improvement of the pinching as $t \to T$. In Sections 5–7, we prove Theorems 1.1–1.5.

2. Basic notions and facts

Set

$$\widetilde{\epsilon} := \begin{cases} 1 & (\text{when } \widetilde{M} = \mathbb{F}P^n(4c)) \\ -1 & (\text{when } \widetilde{M} = \mathbb{F}H^n(-4c)). \end{cases} \tag{2.1}$$

Denote by $\widetilde{g}$ and $\widetilde{R}$ the metric and the curvature tensor of $\widetilde{M}$, respectively. First we recall that $\widetilde{R}$ is given by

$$\begin{aligned} \widetilde{R}(X,Y,Z,W) = \widetilde{\epsilon}c\{&\widetilde{g}(Y,Z)\widetilde{g}(X,W) - \widetilde{g}(X,Z)\widetilde{g}(Y,W) \\ &+ \sum_{B=1}^{d-1}(\widetilde{g}(Y,J_BZ)\widetilde{g}(X,J_BW) - \widetilde{g}(X,J_BZ)\widetilde{g}(Y,J_BW) \\ &\qquad -2\widetilde{g}(X,J_BY)\widetilde{g}(Z,J_BW))\} \end{aligned} \tag{2.2}$$

for all tangent vector fields X, Y, Z, W of $\widetilde{M}$, where $(J_1, \ldots, J_{d-1})$ is the complex structure, a canonical local frame field of the quaternionic structure or the octonian structure of $\widetilde{M}$. Hence the sectional curvature $\widetilde{K}(X,Y)$ of the tangent plane spanned by orthonormal tangent system X, Y of $\widetilde{M}$ is given by

$$\widetilde{K}(X,Y) = \widetilde{R}(X,Y,Y,X) = \widetilde{\epsilon}c\left(1 + 3\sum_{B=1}^{d-1}\widetilde{g}(X,J_BY)^2\right), \tag{2.3}$$

that is, $c \leq \widetilde{\epsilon}\widetilde{K} \leq 4c$. Furthermore, $\widetilde{M}$ is a symmetric space (hence $\widetilde{\nabla}\widetilde{R} = 0$) and an Einstein manifold with Einstein constant $\widetilde{\epsilon}c(dn + 3d - 4)$, which is denoted by $\widetilde{r}$.

Let M be an m-dimensional closed submanifold in $\widetilde{M}$. Denote by g, ∇ and R the induced metric, the Levi-Civita connection and the curvature tensor of M, respectively. Denote by T_pM and N_pM the tangent and normal spaces of M at a point p, respectively. Set $k := n - m$.

Unless otherwise mentioned, Latin letters $i, j, l, \ldots$ run from 1 to m, Greek letters $\alpha, \beta, \gamma, \ldots$ run from $m+1$ to n. Unless necessary, we abbreviate S_p as S for a tensor field S on M. Let $(e_1, \ldots, e_n)$ be an orthonormal frame of $\widetilde{M}$ at a point of M, such that the first m vectors are tangent to M and the other ones are normal. With respect to this orthonormal frame, the second fundamental form h can be written as

$$h_{ij} = \sum_{\alpha} h^{\alpha}_{ij} e_{\alpha} \quad \left(h = \sum_{\alpha} h^{\alpha} \otimes e_{\alpha}\right)$$

for some symmetric $(0,2)$-tensor fields h^{α}. The mean curvature vector field H of M is written as

$$H = \sum_{\alpha} \mathrm{trace}_g h^{\alpha} e_{\alpha} = \sum_{\alpha} \sum_{i,j} g^{ij} h^{\alpha}_{ij} e_{\alpha}.$$

Set $H^{\alpha} := \mathrm{trace}_g h^{\alpha} (= \sum_{r,s} g^{rs} h^{\alpha}_{rs})$. Denote by $\mathring{h}$ the traceless part $h - \frac{1}{m} H \otimes g$ of the second fundamental form. Clearly we have $\|\mathring{h}\|^2 = \|h\|^2 - \frac{1}{m}\|H\|^2$. In the case where M is a hypersurface, the mean curvature vector field H is a multiple of the unit normal vector field ν and $H = -(\lambda_1 + \cdots + \lambda_m)\nu$ holds, where $\lambda_1 \leq \cdots \leq \lambda_m$ are the principal curvatures of M. In addition, we have $\|h\|^2 = \lambda_1^2 + \cdots + \lambda_m^2$ and

$$\|\mathring{h}\|^2 = \|h\|^2 - \frac{1}{m}\|H\|^2 = \frac{1}{m} \sum_{i<j} (\lambda_i - \lambda_j)^2. \tag{2.4}$$

Thus the smallness of $\|\mathring{h}\|^2$ implies that the principal curvatures are close to one another.

Let $\{M_t = f_t(M)\}_{t \in [0,T)}$ be the mean curvature flow starting from an m-dimensional closed submanifold M in $\widetilde{M}$. Denote by g_t, ∇^t, R_t, h_t, H_t, $d\mu_t$ the induced metric, the Levi-Civita connection, the curvature tensor, the second fundamental form, the mean curvature vector and the volume element of M_t, respectively. The evolution equations of the various geometric quantities along the mean curvature flow in a general Riemannian manifold were computed in [1] and [2]. In our case, they take a simpler form because the ambient space $\widetilde{M}$ is a locally symmetric space. In our case, the evolution equations of $\|H_t\|^2$, $\|h_t\|^2$ and $d\mu_t$ are as follows.

Lemma 2.1. *The quantities $\|H_t\|^2$, $\|h_t\|^2$ and $d\mu_t$ satisfy the following evolution equations:*

$$\frac{\partial}{\partial t}\|H_t\|^2 = \Delta\|H_t\|^2 - 2\|\nabla^t H_t\|^2 + 2\sum_{i,j}\left(\sum_{\alpha} H^{\alpha} h^{\alpha}_{ij}\right)^2 + 2\sum_{l,\alpha,\beta} \widetilde{R}_{l\alpha\beta l} H^{\alpha} H^{\beta}, \tag{2.5}$$

$$\begin{aligned}\frac{\partial}{\partial t}\|h_t\|^2 = {} & \Delta\|h_t\|^2 - 2\|\nabla^t h_t\|^2 + 2\sum_{\alpha,\beta}\left(\sum_{i,j} h^{\alpha}_{ij} h^{\beta}_{ij}\right)^2 + 2\sum_{i,j,\alpha,\beta}\left(\sum_{p} h^{\alpha}_{ip} h^{\beta}_{jp} - h^{\beta}_{ip} h^{\alpha}_{jp}\right)^2 \\ & + 4\sum_{i,j,p,q} \widetilde{R}_{ipqi}\left(\sum_{\alpha} h^{\alpha}_{pq} h^{\alpha}_{ij}\right) - 4\sum_{j,l,p} \widetilde{R}_{ljpl}\left(\sum_{i,\alpha} h^{\alpha}_{pi} h^{\alpha}_{ij}\right) \\ & + 2\sum_{l,\alpha,\beta} \widetilde{R}_{l\alpha\beta l}\left(\sum_{i,j} h^{\alpha}_{ij} h^{\alpha}_{ij}\right) - 8\sum_{j,p,\alpha,\beta} \widetilde{R}_{jp\beta\alpha}\left(\sum_{i} h^{\alpha}_{ip} h^{\beta}_{ij}\right),\end{aligned} \tag{2.6}$$

$$\frac{\partial}{\partial t} d\mu_t = -\|H_t\|^2 d\mu_t. \tag{2.7}$$

In the case where M is a hypersurface, these equations have the following simpler forms.

Lemma 2.2. *Assume that M is a hypersurface. Then we have*

$$\frac{\partial}{\partial t}\|H_t\|^2 = \Delta\|H_t\|^2 - 2\|\nabla^t H_t\|^2 + 2\|H_t\|^2(\|h_t\|^2 + \widetilde{Ric}(\nu_t,\nu_t)) \tag{2.8}$$

$$\begin{aligned}\frac{\partial}{\partial t}\|h_t\|^2 = &\Delta\|h_t\|^2 - 2\|\nabla^t h_t\|^2 + 2\|h_t\|^2(\|h_t\|^2 + \widetilde{Ric}(\nu_t,\nu_t))\\ &- 4\sum_{i,j,p,l}(h_{ij}h^p_j\widetilde{R}^l_{pli} - h^{ij}h^{lp}\widetilde{R}_{pilj}),\end{aligned} \tag{2.9}$$

where $\widetilde{Ric}$ is the Ricci tensor of $\widetilde{M}$.

3. The preservability of pinching condition

In this section, we prove that the pinching conditions in Theorems 1.1–1.5 are preserved along the mean curvature flow under the settings of Theorems 1.1–1.5, respectively. Let M be an m-dimensional closed submanifold in $\widetilde{M}$. Set $k := dn - m$. Denote by h and H the second fundamental form and the mean curvature vector of M, respectively.

To obtain the desired estimates, it is important to perform the computations by using a special orthonormal frame with suitable properties. Let p be a point of M with $\|H_p\| \neq 0$. A first kind of orthonormal frame is an orthonormal frame of $T_p\widetilde{M}$ satisfying

$$e_{m+1} = \frac{H_p}{\|H_p\|}. \tag{3.1}$$

Then we can choose $e_{m+2},\dots,e_{dn}$ such that $(e_{m+1},\dots,e_{dn})$ is an orthonormal frame of N_pM and choose any orthonormal frame $(e_1,\dots,e_m)$ of T_pM. An orthonormal frame obtained in this way will be said to be *of type (I)*. For the components of the second fundamental form h and its traceless part $\mathring{h}$ with respect to an orthonormal frame $(e_1,\dots,e_n)$ of type (I), the following relations hold:

$$\begin{cases}\mathrm{trace}_g h^{m+1} = \|H\|, & \\ \mathrm{trace}_g h^{\alpha} = 0, & \alpha \geq m+2\end{cases}$$

and

$$\begin{cases}\mathring{h}^{m+1} = h^{m+1} - \dfrac{\|H\|}{m}g, & \\ \mathring{h}^{\alpha} = h^{\alpha}, & \alpha \geq m+2.\end{cases}$$

With respect to an orthonormal frame $(e_1,\dots,e_n)$ of type (I), we adopt the following notation:

$$\begin{aligned}&\|h_1\|^2 := \|h^{m+1}\|^2, \quad \|\mathring{h}_1\|^2 := \|\mathring{h}^{m+1}\|^2,\\ &\|h_-\|^2 = \|\mathring{h}_-\|^2 := \sum_{\alpha=m+2}^{n}\|\mathring{h}^{\alpha}\|^2.\end{aligned} \tag{3.2}$$

Next we define a second kind of orthonormal frame, which is useful to calculate explicitly the components of the curvature tensor of $\widetilde{M}$. Let $(J_1, \dots, J_{d-1})$ be the complex structure, a canonical local frame field of the quaternionic structure or the octonian structure of $\widetilde{M}$.

Lemma 3.1. *If $k \le m$, then for each $p \in M$ and each $B \in \{1, \dots, d-1\}$, there exists an orthonormal frame $(e_1^B, \dots, e_m^B)$ of T_pM and an orthonormal frame $(e_{m+1}^B, \dots, e_{dn}^B)$ of N_pM satisfying the following conditions:*

(i) *For every $r \le \left[\frac{k}{2}\right]$, we have*

$$\begin{cases} J_B e_{m+2r-1}^B = \tau_r^B e_{2r-1}^B + \nu_r^B e_{m+2r}^B, \\ J_B e_{m+2r}^B = \tau_r^B e_{2r}^B - \nu_r^B e_{m+2r-1}^B, \end{cases} \tag{3.3}$$

where $B \in \{1, \dots, d-1\}$, $\tau_r^B, \nu_r^B \in [0,1]$, $(\tau_r^B)^2 + (\nu_r^B)^2 = 1$ and $[\bullet]$ denotes the floor function of $\bullet$;

(ii) *If k is odd, then $J_B e_{m+k}^B = e_k^B$;*

(iii) *The remaining vectors satisfy*

$$e_{k+1}^B,\ J_B e_{k+1}^B = e_{k+2}^B,\ J_B e_{k+3}^B = e_{k+4}^B,\ \dots,\ J_B e_{m-1}^B = e_m^B.$$

See the proof of Lemma 3.1 in [18] about the proof of this lemma. An orthonormal frame satisfying the properties of this lemma will be said to be *of type (II)*. Since $J_B^2 = -id$, from (3.3) it follows easily that such an orthonormal frame also satisfies

$$\begin{cases} J_B e_{2r-1}^B = -\nu_r^B e_{2r}^B - \tau_r^B e_{m+2r-1}^B, \\ J_B e_{2r}^B = \nu_r^B e_{2r-1}^B - \tau_r^B e_{m+2r}^B. \end{cases} \tag{3.4}$$

If k is odd, it is convenient to define $\tau_r^B = 1$, $\nu_r^B = 0$ for $r = \frac{k+1}{2}$. In this way, the first equations in (3.3) and in (3.4) hold also for this value of r.

In general, the requirements for orthonormal frames of types (I) and (II) are incompatible. In case of $k \ge 2$, we introduce the following notations in [1]

$$R_1 := \sum_{\alpha,\beta} \left(\sum_{i,j} h_{ij}^\alpha h_{ij}^\beta \right)^2 + \sum_{i,j,\alpha,\beta} \left(\sum_p h_{ip}^\alpha h_{jp}^\beta - h_{ip}^\beta h_{jp}^\alpha \right)^2,$$

$$R_2 := \sum_{i,j} \left(\sum_\alpha H^\alpha h_{ij}^\alpha \right)^2.$$

If we use an orthonormal frame of type (I), it is easy to check that

$$R_2 = \begin{cases} \|h_1\|^2\|H\|^2 = \|\mathring{h}_1\|^2\|H\|^2 + \dfrac{1}{m}\|H\|^4 & \text{(when } H \neq 0\text{)} \\ 0 & \text{(when } H = 0\text{).} \end{cases} \tag{3.5}$$

The following result, which was proved in Section 3 of [1] and in Section 5.2 of [2], is useful in the estimation of the reaction term occurring in the evolution equations of Lemma 2.1. In the proof, only the algebraic properties of R_1 and R_2 are used.

Lemma 3.2. *At a point where $H \neq 0$ we have*

$$2R_1 - 2aR_2 \le 2\|\mathring{h}_1\|^4 - 2\left(a - \frac{2}{m}\right)\|\mathring{h}_1\|^2\|H\|^2 - \frac{2}{m}\left(a - \frac{1}{m}\right)\|H\|^4 + 8\|\mathring{h}_1\|^2\|\mathring{h}_-\|^2 + 3\|\mathring{h}_-\|^4$$

for any $a \in \mathbb{R}$. In addition, if $a > \frac{1}{m}$ and if $||h||^2 = a||H||^2 + b$ holds for some $b \in \mathbb{R}$, we have

$$2R_1 - 2aR_2 \leq \left(6 - \frac{2}{ma-1}\right)||\mathring{h}||^2||\mathring{h}_-||^2 - 3||\mathring{h}_-||^4 + \frac{2mab}{ma-1}||\mathring{h}_1||^2 + \frac{4b}{ma-1}||\mathring{h}_-||^2 - \frac{2b^2}{ma-1}.$$

Now we shall derive a sharp estimate on the gradient terms appearing in the evolution equations for $||h||^2$ and $||H||^2$, which will be used many times in the rest of this paper. Observe that the results are independent of the property of the flow. Our starting point is the following inequality, which was originally proved by Huisken (see Lemma 2.2 of [8]) in the case of hypersurfaces, and later extended to general codimension by Liu, Xu and Zhao (see Lemma 3.2 of [14]).

Lemma 3.3. *Let M be an m-dimensional submanifold in $\widetilde{M}$. Then*

$$||\nabla h||^2 \geq \left(\frac{3}{m+2} - \eta\right)||\nabla H||^2 - \frac{2}{m+2}\left(\frac{2}{(m+2)\eta} - \frac{m}{m-1}\right)||\omega||^2 \tag{3.6}$$

holds for any $\eta > 0$. Here $\omega = \sum_{i,j,\alpha} \widetilde{R}_{\alpha jji} e_i \otimes \omega_\alpha$, where ω_α is the dual frame to e_α. In particular, if $\widetilde{M}$ is $\mathbb{H}P^n(4)$ (in more general, Einstein) and if M is a hypersurface, then $\omega = 0$ and as $\eta \to 0$ in (3.6), we find

$$||\nabla h||^2 \geq \frac{3}{m+2}||\nabla H||^2. \tag{3.7}$$

For submanifolds of higher codimension, ω is in general nonzero. For any tangent vector field X on M, we write $J_B X = P_B X + F_B X$, where $P_B X$ and $F_B X$ are the tangent and normal components of $J_B X$, respectively. Similarly, for any normal vector field V, we write $J_B V = t_B V + f_B V$, where $t_B V$ and $f_B V$ are tangent and normal components of $J_B V$, respectively. Let P and Q be elements of $\Gamma(T^*M \otimes TM)$, $\Gamma(T^*M \otimes T^\perp M)$, $\Gamma((T^\perp M)^* \otimes TM)$ and $\Gamma((T^\perp M)^* \otimes T^\perp M)$, where TM (resp. $T^\perp M$) denotes the tangent (resp. normal) bundle of M, $(\bullet)^*$ denotes the dual bundle of $(\bullet)$ and $\Gamma(\bullet)$ denotes the space of all sections of the vector bundle $(\bullet)$. Define $\langle P, Q\rangle$ by

$$\langle P, Q\rangle := \begin{cases} \sum_i g(Pe_i, Qe_i) & (P, Q \in \Gamma(T^*M \otimes TM)) \\ \sum_i \widetilde{g}(Pe_i, Qe_i) & (P, Q \in \Gamma(T^*M \otimes T^\perp M)) \\ \sum_\alpha g(Pe_\alpha, Qe_\alpha) & (P, Q \in \Gamma((T^\perp M)^* \otimes TM)) \\ \sum_\alpha \widetilde{g}(Pe_\alpha, Qe_\alpha) & (P, Q \in \Gamma((T^\perp M)^* \otimes T^\perp M)), \end{cases}$$

where (e_i) is an orthonormal tangent frame of M (with respect to g) and (e_α) is an orthonormal normal frame of M (with respect to $\widetilde{g}$). Set $||P|| := \sqrt{\langle P, P\rangle}$.

Now we shall derive a relation among $||P_B||$, $||F_B||$ and $||P_B F_B||$.

Lemma 3.4. *For $||P_B||$, $||F_B||$ and $||F_B P_B||$, the following relation holds:*

$$||P_B||^2 \cdot ||F_B||^2 \geq m||F_B P_B||^2. \tag{3.8}$$

Proof. We discuss in the cases where k is even and where k is odd separately. First we consider the case of $k = 2k'$ (even). By using the relations (3.3) and (3.4), we can derive

$$\|P_B\|^2 = (m-k) + 2\sum_{r\le k'}(\nu_r^B)^2 = m - 2\sum_{r\le k'}(\tau_r^B)^2, \quad \|t_B\|^2 = 2\sum_{r\le k'}(\tau_r^B)^2.$$

Therefore, by using $(\tau_r^B)^2 + (\nu_r^B)^2 = 1$ and $k \le m$, we find

$$\begin{aligned}\|P_B\|^2\|t_B\|^2 &= 2m\sum_{r\le k'}(\tau_r^B)^2 - 4\sum_{r,s\le k'}(\tau_r^B)^2(\tau_s^B)^2\\ &\ge 2m\sum_{r\le k'}(\tau_r^B)^2 - 2\sum_{r,s\le k'}\left((\tau_r^B)^4 + (\tau_s^B)^4\right)\\ &= 2m\sum_{r\le k'}(\tau_r^B)^2 - 2k\sum_{r\le k'}(\tau_r^B)^4\\ &\ge 2m\sum_{r\le k'}\left((\tau_r^B)^2 - (\tau_r^B)^4\right) = 2m\sum_{r\le k'}(\tau_r^B\nu_r^B)^2.\end{aligned}$$

Similarly, in the case of $k = 2k'+1$ (odd), we can derive

$$\|P_B\|^2 = (m-k) + 2\sum_{r\le k'}(\nu_r^B)^2 = (m-1) - 2\sum_{r\le k'}(\tau_r^B)^2, \quad \|t_B\|^2 = 1 + 2\sum_{r\le k'}(\tau_r^B)^2,$$

and hence

$$\begin{aligned}\|P_B\|^2\|t_B\|^2 &\ge (m-1) + 2(m-2)\sum_{r\le k'}(\tau_r^B)^2 - 2(k-1)\sum_{r\le k'}(\tau_r^B)^4\\ &\ge (m-1) + 2(m-2)\sum_{r\le k'}(\tau_r^B)^2(\nu_r^B)^2.\end{aligned}$$

For any r, we have $(\tau_r^B)^2(\nu_r^B)^2 \le \frac{1}{4}$ by $(\tau_r^B)^2 + (\nu_r^B)^2 = 1$. Therefore, by using $m-1 \ge k-1 = 2k'$, we can derive

$$\begin{aligned}\|P_B\|^2\|t_B\|^2 &\ge 2k' + 2(m-2)\sum_{r\le k'}(\tau_r^B\nu_r^B)^2\\ &\ge 8\sum_{r\le k'}(\tau_r^B\nu_r^B)^2 + 2(m-2)\sum_{r\le k'}(\tau_r^B\nu_r^B)^2\\ &= 2(m+2)\sum_{r\le k'}(\tau_r^B\nu_r^B)^2.\end{aligned}$$

On the other hand, we have

$$\|F_BP_B\|^2 = 2\sum_{r\le[k/2]}(\tau_r^B\nu_r^B)^2 \tag{3.9}$$

in both cases where k is even and odd. Hence we obtain

$$\|P_B\|^2\cdot\|F_B\|^2 \ge m\|F_BP_B\|^2$$

in both cases where k is even and odd. □

Lemma 3.5. *Let M be an m-dimensional submanifold in $\widetilde{M}$. If $k \le m$, then, at any point of M, we have*

$$\|\nabla h\|^2 \geq \frac{2(m+1)}{9(d-1)^2}\|\omega\|^2.$$

Proof. We first compute $\|\omega\|^2$ by using an orthonormal frame of type (II). Define $(0,4)$-tensor field $\rho_{(B)}$ $(B=0,1,2,3)$ on $\widetilde{M}$ by

$$\rho_{(0)}(X,Y,Z,W) := \widetilde{\epsilon}c\{\widetilde{g}(Y,Z)\widetilde{g}(X,W) - \widetilde{g}(X,Z)\widetilde{g}(Y,W)\}$$

and

$$\rho_{(B)}(X,Y,Z,W) := \widetilde{\epsilon}c\{\widetilde{g}(Y,J_BZ)\widetilde{g}(X,J_BW) - \widetilde{g}(X,J_BZ)\widetilde{g}(Y,J_BW) - 2\widetilde{g}(X,J_BY)\widetilde{g}(Z,J_BW)\} \qquad (B=1,2,3),$$

for $X,Y,Z,W \in T\widetilde{M}$. By using (2.2) and $(\rho_{(0)})_{\alpha jji} = 0$, we have

$$\|\omega\|^2 = \sum_{\alpha,j,i}\left(\sum_{B=1}^{d-1}(\rho_{(B)})_{\alpha jji}\right)^2 \leq (d-1)\sum_{B=1}^{d-1}\sum_{\alpha,j,i}\left((\rho_{(B)})_{\alpha jji}\right)^2.$$

On the other hand, by using (3.4), we have

$$\sum_{\alpha,j,i}\left((\rho_{(B)})_{\alpha jji}\right)^2 = 18c^2\sum_{r\leq[k/2]}(\tau_r^B\nu_r^B)^2 = 9c^2\|F_BP_B\|^2.$$

Hence we can derive

$$\|\omega\|^2 \leq 9(d-1)c^2\sum_{B=1}^{d-1}\|F_BP_B\|^2, \tag{3.10}$$

where we use also (3.9). Define a $(0,3)$-tensor field T on M by

$$T(X,Y,Z) := (\nabla_X h)(Y,Z) + \frac{\widetilde{\epsilon}c}{d-1}\sum_{B=1}^{d-1}(g(P_BX,Y)F_BZ + g(P_BX,Z)F_BY) \qquad (X,Y,Z \in TM).$$

Then we have

$$\|T\|^2 = \|\nabla h\|^2 + \frac{4\widetilde{\epsilon}c}{d-1}\sum_{B=1}^{d-1}\sum_{i,j}\widetilde{g}((\nabla_{e_i}h)(P_Be_i,e_j),\ F_Be_j) + \frac{2c^2}{d-1}\sum_{B=1}^{d-1}(\|P_B\|^2\cdot\|F_B\|^2 + \|F_BP_B\|^2). \tag{3.11}$$

By using the Codazzi equation, we have

$$\begin{aligned}\widetilde{\epsilon}\sum_{B=1}^{d-1}\sum_{i,j}&\widetilde{g}((\nabla_{e_i}h)(P_Be_i,e_j),\ F_Be_j)\\ &= \sum_{B=1}^{d-1}\sum_{i,j}\widetilde{g}((\nabla_{e_i}h)(e_j,P_Be_i),F_Be_j)\end{aligned} \tag{3.12}$$

$$
\begin{aligned}
&= \widetilde{\epsilon}\sum_{B=1}^{d-1}\sum_{i,j}\widetilde{g}((\nabla_{e_j}h)(P_Be_i,e_i),F_Be_j) + c\sum_{B=1}^{d-1}\langle P_B^2, t_BF_B\rangle \\
&\quad - c\sum_{B=1}^{d-1}\|P_B\|^2\cdot\|F_B\|^2 - 2c\sum_{B=1}^{d-1}\|F_BP_B\|^2 \\
&\leq -c\sum_{B=1}^{d-1}\left(\|P_B\|^2\cdot\|F_B\|^2 + \|F_BP_B\|^2\right),
\end{aligned}
$$

where we use the fact that $(\nabla_{e_j}h)(P_Be_i,e_i)$ vanishes because $\nabla_{e_j}h$ is symmetric and P_B is skew-symmetric. From (3.8), (3.11) and (3.12), we obtain

$$
\begin{aligned}
\|T\|^2 &\leq \|\nabla h\|^2 - \frac{2c^2}{d-1}\sum_{B=1}^{d-1}(\|P_B\|^2\cdot\|F_B\|^2 + \|F_BP_B\|^2) \\
&\leq \|\nabla h\|^2 - \frac{2(m+1)c^2}{d-1}\sum_{B=1}^{d-1}\|F_BP_B\|^2
\end{aligned}
$$

and hence

$$
\|\nabla h\|^2 \geq \frac{2(m+1)c^2}{d-1}\sum_{B=1}^{d-1}\|F_BP_B\|^2. \tag{3.13}
$$

From (3.10) and (3.13), we obtain the desired inequality. □

Lemma 3.6. *Let M be an m-dimensional submanifold in $\widetilde{M}$. If*

$$
m \geq \begin{cases} 8 & (\text{when } \widetilde{M} = \mathbb{C}P^n(4c),\ \mathbb{C}H^n(-4c)) \\ 11 & (\text{when } \widetilde{M} = \mathbb{H}P^n(4c),\ \mathbb{H}H^n(-4c)) \\ 1 & (\text{when } \widetilde{M} = \mathbb{O}P^2(4c),\ \mathbb{O}H^2(-4c)), \end{cases}
$$

then we have

$$
\|\nabla h\|^2 \geq \frac{2(10-d)}{9(m+2)}\|\nabla H\|^2.
$$

Proof. If the codimension is one, then the result follows directly from (3.7). In the case of higher codimension, it follows from Lemmas 3.3 and 3.5 that

$$
\begin{aligned}
3\|\nabla h\|^2 &= 2\|\nabla h\|^2 + \|\nabla h\|^2 \\
&\geq 2\left(\frac{3}{m+2} - \eta\right)\|\nabla H\|^2 + \left(\frac{2(m+1)}{9(d-1)^2} - \frac{4}{m+2}\left(\frac{2}{(m+2)\eta} - \frac{m}{m-1}\right)\right)\|\omega\|^2.
\end{aligned}
$$

We take $\frac{d-1}{3(m+2)}$ as η. Then we obtain

$$
3\|\nabla h\|^2 \geq \frac{2(10-d)}{3(m+2)}\|\nabla H\|^2 + \frac{2}{9(d-1)^2}\left(m+1-\frac{(d-1)\{18(7-d)m-108\}}{(m-1)(m+2)}\right)\|\omega\|^2.
$$

Then the coefficient of $\|\omega\|^2$ in the right-hand side of this inequality is positive when m is as in the statement of this lemma. Hence we can derive the desired inequality. □

For the real number b as in the introduction and a sufficiently small positive number ε, define a real number b_ε by

$$b_\varepsilon := \begin{cases} (1-\varepsilon)b & (\text{when } \widetilde{M} = \mathbb{F}P^n(4c)) \\ (1+\varepsilon)b & (\text{when } \widetilde{M} = \mathbb{F}H^n(-4c)). \end{cases}$$

For simplicity, set $a_\varepsilon := \frac{1}{m-1+\varepsilon}$. We consider the following pinching condition:

$$(*_{m-1+\varepsilon,b_\varepsilon}) \qquad ||h||^2 \leq \frac{1}{m-1+\varepsilon}||H||^2 + b_\varepsilon.$$

Now we shall prove the preservability of the pinching condition in Theorems 1.1 and 1.4.

Proof of (i) of Theorems 1.1 and 1.4. Since M satisfies the condition $(*_{m-1,b})$ and it is closed, it satisfies the condition $(*_{m-1+\varepsilon,b_\varepsilon})$ for a sufficiently small positive number ε. Define Q_ε by $Q_\varepsilon := ||h||^2 - a_\varepsilon||H||^2 - b_\varepsilon$. From Lemma 2.2, we obtain

$$\begin{aligned}(3.14)\qquad \frac{\partial}{\partial t}Q_\varepsilon - \Delta Q_\varepsilon &= -2(||\nabla h||^2 - a_\varepsilon||\nabla H||^2) + 2(||h||^2 - a_\varepsilon||H||^2)(||h||^2 + \widetilde{r}) \\ &\quad - 4\sum_{i,j,p,l}(h^{ij}h_j^p\widetilde{R}^l_{pli} - h^{ij}h^{lp}\widetilde{R}_{pilj}) \\ &= -2(||\nabla h||^2 - a_\varepsilon||\nabla H||^2) + 2Q_\varepsilon(||h||^2 + \widetilde{r}) + 2b_\varepsilon(||h||^2 + \widetilde{r}) \\ &\quad - 4\sum_{i,j,p,l}(h^{ij}h_j^p\widetilde{R}^l_{pli} - h^{ij}h^{lp}\widetilde{R}_{pilj}),\end{aligned}$$

where $\widetilde{r}$ denotes the Einstein constant $\widetilde{\epsilon}cd(n+1)$. Also, it follows from (3.7) that

$$(3.15)\qquad ||\nabla h||^2 - a_\varepsilon||\nabla H||^2 \geq \left(\frac{3}{m+2} - \frac{1}{m-1+\varepsilon}\right)||\nabla H||^2 \geq 0$$

because $m \geq 3$. Thus the gradient term in the evolution equation (3.14) is non-positive. Next we shall investigate the reaction term of (3.14). Fix an orthonormal tangent frame $(e_1, \ldots, e_m)$ of M_t consisting of eigenvectors of the shape operator A_t of M_t. Let λ_i be the eigenvalue corresponding to e_i. First we consider the case of Theorem 1.1. From $c \leq \widetilde{K}_{ij} \leq 4c$, we can derive

$$\begin{aligned}(3.16)\qquad -4\sum_{i,j,p,l}(h^{ij}h_j^p\widetilde{R}^l_{pli} - h^{ij}h^{lp}\widetilde{R}_{pilj}) &= -4\sum_{i<p}\widetilde{K}_{ip}(\lambda_i^\alpha - \lambda_p^\alpha)^2 \\ &\leq -4mc\left(||h||^2 - \frac{1}{m}||H||^2\right) \leq -4mc||\mathring{h}||^2.\end{aligned}$$

From the assumption for n, we have $n \geq 1 + \frac{4}{d}$ and hence $\frac{2c}{a_\varepsilon} \geq \widetilde{r}$. Hence we obtain

$$2b_\varepsilon(||h||^2 + \bar{r}) - 4\sum_{i,j,p,l}(h^{ij}h_j^p\widetilde{R}^l_{pli} - h^{ij}h^{lp}\widetilde{R}_{pilj}) \leq -\frac{4c}{a_\varepsilon}Q_\varepsilon.$$

From (3.14), (3.15) and this inequality, we can derive

$$\frac{\partial}{\partial t}Q_\varepsilon \leq \Delta Q_\varepsilon + 2Q_\varepsilon\left(||h||^2 + \widetilde{r} - \frac{2c}{a_\varepsilon}\right).$$

Therefore, by the maximum principle, the condition $(*_{m-1+\varepsilon,b_\varepsilon})$ is preserved along the mean curvature flow.

Next we consider the case of Theorem 1.4. From $-4c \leq \widetilde{K}_{ij} \leq -c$, we can derive

$$-4\sum_{i,j,p,l}(h^{ij}h_j^p\widetilde{R}^l_{pli}-h^{ij}h^{lp}\widetilde{R}_{pilj})=-4\sum_{i<p}\widetilde{K}_{ip}(\lambda_i^\alpha-\lambda_p^\alpha)^2 \tag{3.17}$$
$$\leq 16mc\left(\|h\|^2-\frac{1}{m}\|H\|^2\right)\leq 16mc\|\mathring{h}\|^2.$$

Since $\frac{8c}{a_\varepsilon}\geq -\widetilde{r}$ by $n\geq 2>\frac{d+16}{7d}$, we obtain

$$2b_\varepsilon(\|h\|^2+\bar{r})-4\sum_{i,j,p,l}(h^{ij}h_j^p\widetilde{R}^l_{pli}-h^{ij}h^{lp}\widetilde{R}_{pilj})\leq\frac{16c}{a_\varepsilon}Q_\varepsilon.$$

From (3.14), (3.15) and this inequality, we can derive

$$\frac{\partial}{\partial t}Q_\varepsilon\leq\Delta Q_\varepsilon+2Q_\varepsilon\left(\|h\|^2+\widetilde{r}+\frac{8c}{a_\varepsilon}\right).$$

Therefore, by the maximum principle, the condition $(*_{m-1+\varepsilon,b_\varepsilon})$ is preserved along the mean curvature flow. □

Now we shall prove the preservability of the pinching condition of Theorem 1.2.

Proof of (i) of Theorem 1.2. Since M satisfies the condition $(*_{m-1,b})$ and it is closed, it satisfies the condition $(*_{m-1+\varepsilon,b_\varepsilon})$ for a sufficiently small positive number ε. Define Q_ε by $Q_\varepsilon:=\|h\|^2-a_\varepsilon\|H\|^2-b_\varepsilon$. From Lemma 2.1, we can derive

$$\frac{\partial}{\partial t}Q_\varepsilon=\Delta Q_\varepsilon-2(\|\nabla h\|^2-a_\varepsilon\|\nabla H\|^2)+2R_1-2a_\varepsilon R_2+P_{a_\varepsilon}. \tag{3.18}$$

Here $P_{a_\varepsilon}:=P_I+P_{II,a_\varepsilon}+P_{III}$, where

$$P_I:=4\sum_{i,j,p,q}\widetilde{R}_{ipqj}\left(\sum_\alpha h^\alpha_{pq}h^\alpha_{ij}\right)-4\sum_{j,l,p}\widetilde{R}_{ljpl}\left(\sum_{i,\alpha}h^\alpha_{pi}h^\alpha_{ij}\right),$$
$$P_{II,a_\varepsilon}:=2\sum_{l,\alpha,\beta}\widetilde{R}_{l\alpha\beta l}\left(\sum_{i,j}h^\alpha_{ij}h^\alpha_{ij}\right)-2a_\varepsilon\sum_{l,\alpha,\beta}\widetilde{R}_{l\alpha\beta l}H^\alpha H^\beta,$$
$$P_{III}:=-8\sum_{j,p,\alpha,\beta}\widetilde{R}_{jp\beta\alpha}\left(\sum_i h^\alpha_{ip}h^\beta_{ij}\right).$$

By Lemma 3.6 and the assumption for m in Theorem 1.2, we obtain

$$\|\nabla h\|^2-a_\varepsilon\|\nabla H\|^2\geq\left(\frac{2(10-d)}{9(m+2)}-\frac{1}{m-1+\varepsilon}\right)\|\nabla H\|^2\geq 0. \tag{3.19}$$

Thus the gradient terms in the evolution equation (3.18) are non-positive.

Assume that there exists $t_0\in[0,T)$ and $p_0\in M_{t_0}$ with $((Q_\varepsilon)_{t_0})_{p_0}=0$, where we take t_0 as small as possible. We shall investigate the reaction term of (3.18) at (p_0,t_0). Take any orthonormal normal frame $(e_{m+1},\ldots,e_{dn})$ of M_{t_0} at p_0 and, for arbitrarily fixed $\alpha\in\{m+1,\ldots,dn\}$, take an orthonormal tangent frame $(e_1,\ldots,e_m)$ of $T_{p_0}M_{t_0}$ consisting of eigenvectors of the shape operator $(A_{t_0})_{e_\alpha}$, which is not necessarily that of type (I) or (II). Let λ_i be the eigenvalue of A_{e_α} corresponding to e_i. Similarly to (3.16), we have

$$4\sum_{i,j,p,q}\widetilde{R}_{ipjq}h^{\alpha}_{pq}h^{\alpha}_{ij}-4\sum_{j,l,p}\widetilde{R}_{ljlp}\left(\sum_i h^{\alpha}_{pi}h^{\alpha}_{ij}\right)$$
$$=-4\sum_{i,p}\widetilde{R}_{ipip}((\lambda^{\alpha}_i)^2-\lambda^{\alpha}_i\lambda^{\alpha}_p)$$
$$=-4\sum_{i<p}\widetilde{K}_{ip}(\lambda^{\alpha}_i-\lambda^{\alpha}_p)^2\leq -4mc\|\mathring{h}^{\alpha}\|^2.$$

Hence we can derive

$$P_I\leq -4mc\|\mathring{h}\|^2 \tag{3.20}$$

at (p_0,t_0). Next we shall estimate the terms P_{II,a_ε} and P_{III} at (p_0,t_0). We shall use an orthonormal frame of type (II) to estimating these terms at (p_0,t_0). Take an orthonormal frame $(e^B_1,\ldots,e^B_{dn})$ $(B=1,\ldots,d-1)$ of type (II) at $p_0\in M_{t_0}$. Set $\widetilde{K}^B_{s\alpha}:=\widetilde{K}(e^B_s,e^B_\alpha)$. From (2.3) and (3.3), we have

$$\widetilde{K}^B_{s\alpha}=c\left(1+3\widetilde{g}(e^B_s,J_Be^B_\alpha)\right)^2=c\left(1+3\delta_{s,\alpha-m}\tau^B_{[\frac{s+1}{2}]}\right)^2\leq c(1+3\delta_{s,\alpha-m}).$$

On the other hand, it follows from $((Q_\varepsilon)_{t_0})_{p_0}=0$ that $\|h\|^2=a_\varepsilon\|H\|^2+b_\varepsilon$, that is, $(a_\varepsilon-\frac{1}{m})\|H\|^2=\|\mathring{h}\|^2-b_\varepsilon$ holds at (p_0,t_0). Hence, by noticing $a_\varepsilon\geq\frac{1}{m}$, we can derive

$$\begin{aligned}P_{II,a_\varepsilon}&=2\sum_{s,\alpha}\widetilde{K}^B_{s\alpha}(\|h^{\alpha}\|^2-a_\varepsilon\|H^{\alpha}\|^2)\\&=2\sum_{s,\alpha}\widetilde{K}^B_{s\alpha}\left(\|\mathring{h}^{\alpha}\|^2-\left(a_\varepsilon-\frac{1}{m}\right)\|H^{\alpha}\|^2\right)\\&\leq 2\sum_{s,\alpha}\widetilde{K}^B_{s,\alpha}\|\mathring{h}^{\alpha}\|^2\\&\leq 2c\sum_{s,\alpha}(1+3\delta_{s,\alpha-m})\|\mathring{h}^{\alpha}\|^2\\&\leq 2(m+3)c\|\mathring{h}\|^2.\end{aligned}\tag{3.21}$$

By using the notations $\rho_{(B)}$ $(B=0,1,\cdots,d-1)$ in the proof of Lemma 3.5, part P_{III} may be written as

$$P_{III}:=-8\sum_{B=1}^{d-1}\sum_{j,p,\alpha,\beta}(\rho_{(B)})_{jp\beta\alpha}\left(\sum_i h^{\alpha}_{ip}h^{\beta}_{ij}\right),$$

where we note that $(\rho_{(0)})_{jp\beta\alpha}=0$. We shall estimate

$$(P_{III})_{(B)}:=-8\sum_{j,p,\alpha,\beta}(\rho_{(B)})_{jp\beta\alpha}\left(\sum_i h^{\alpha}_{ip}h^{\beta}_{ij}\right)\quad(B=1,\ldots,d-1).$$

By the same calculation as the estimate of the part (III) in the proof of Proposition 3.6 in [18], we can derive

$$(P_{III})_{(B)}\leq 8kc\|\mathring{h}\|^2\qquad(B=1,\ldots,d-1).\tag{3.22}$$

Hence we obtain

$$P_{III} \leq 8(d-1)kc\|\mathring{h}\|^2. \tag{3.23}$$

By (3.20), (3.21) and (3.23), we obtain

$$P_{a_\varepsilon} \leq -2(m-4(d-1)k-3)c\|\mathring{h}\|^2. \tag{3.24}$$

Set $R := 2R_1 - 2a_\varepsilon R_2 + P_{a_\varepsilon}$. First we consider the case of $(H_{t_0})_{p_0} \neq 0$. We use an orthonormal frame of type (I) at $p_0 \in M_{t_0}$. Then it follows from Lemma 3.2 that

$$\begin{aligned} R &= 2R_1 - 2a_\varepsilon R_2 + P_{a_\varepsilon} \\ &\leq \left(6 - \frac{2}{ma_\varepsilon - 1}\right)\|\mathring{h}\|^2\|\mathring{h}_-\|^2 - 3\|\mathring{h}_-\|^4 \\ &\quad + \left\{\frac{2ma_\varepsilon b_\varepsilon}{ma_\varepsilon - 1} - 2(m-4(d-1)k-3)c\right\}\|\mathring{h}_1\|^2 \\ &\quad + \left\{\frac{4b_\varepsilon}{ma_\varepsilon - 1} - 2(m-4(d-1)k-3)c\right\}\|\mathring{h}_-\|^2 - \frac{2b_\varepsilon^2}{ma_\varepsilon - 1}. \end{aligned}$$

By the assumption $m \geq \frac{3d}{2} + 5$ in Theorem 1.2, the coefficient of $\|\mathring{h}\|^2\|\mathring{h}_-\|^2$ is negative. It is easy to show that the coefficient of $\|\mathring{h}_1\|^2$ vanishes. Also, it follows from $((Q_\varepsilon)_{t_0})_{p_0} = 0$ that $\|\mathring{h}\|^2 \geq b_\varepsilon$ holds at (p_0, t_0). These facts imply

$$\begin{aligned} R &\leq -3\|\mathring{h}_-\|^4 + \left\{\left(6 - \frac{2}{ma_\varepsilon - 1}\right)b_\varepsilon + \frac{4b_\varepsilon}{ma_\varepsilon - 1} - 2(m-4(d-1)k-3)c\right\}\|\mathring{h}_-\|^2 \\ &\quad - \frac{2b_\varepsilon^2}{ma_\varepsilon - 1} \\ &\leq -3\|\mathring{h}_-\|^4 + 4b_\varepsilon\|\mathring{h}_-\|^2 + 2b_\varepsilon(b_\varepsilon - (m-4(d-1)k-3)c). \end{aligned}$$

Furthermore, from $4b_\varepsilon\|\mathring{h}\|^2 \leq 3\|\mathring{h}\|^4 + \frac{4}{3}b_\varepsilon^2$, we can derive

$$R \leq 2b_\varepsilon\left\{\frac{5}{3}b_\varepsilon - (m-4(d-1)k-3)c\right\}.$$

The right-hand side of this inequality is negative by the assumption $m \geq \frac{3d}{2} + 5$ in Theorem 1.2. Hence we obtain $R < 0$ at (p_0, t_0). Next we consider the case of $(H_{t_0})_{p_0} = 0$. Then we have $\|h\|^2 = \|\mathring{h}\|^2 = b_\varepsilon$ and $R_2 = 0$. Furthermore, by using Theorem 1 in [12], we find $2R_1 \leq 3\|h\|^4 = 3b_\varepsilon^2$. These together with (3.24) imply

$$R \leq 3b_\varepsilon^2 - 2(m-4(d-1)k-3)b_\varepsilon.$$

The right-hand side of this inequality is negative by the assumption $m \geq \frac{3d}{2} + 5$ in Theorem 1.2. Hence we obtain $R < 0$ at (p_0, t_0). Therefore, since $R < 0$ at (p_0, t_0) in both cases, it is shown that the condition $(*_{m-1+\varepsilon, b_\varepsilon})$ is preserved along the mean curvature flow by using the maximum principle. Hence the statement of this proposition follows from the arbitrariness of ε. □

Now we shall prove the preservability of the pinching condition of Theorem 1.5.

Proof of (i) of Theorem 1.5. Since M satisfies the condition $(*_{m-1,b})$ and it is closed, it satisfies the condition $(*_{m-1+\varepsilon, b_\varepsilon})$ for a sufficiently small positive number ε. Define Q_ε by

$Q_\varepsilon := \|h\|^2 - a_\varepsilon\|H\|^2 - b_\varepsilon$. From Lemma 2.1, we can derive the evolution equation (3.18) for Q_ε. By Lemma 3.6 and the assumption for m in Theorem 1.5, we obtain the inequality (3.19). Thus the gradient terms in the evolution equation (3.18) are non-positive.

Assume that there exists $t_0 \in [0,T)$ and $p_0 \in M_{t_0}$ with $((Q_\varepsilon)_{t_0})_{p_0} = 0$, where we take t_0 as small as possible. We shall investigate the reaction term of (3.24) at (p_0,t_0). Take any orthonormal normal frame $(e_{m+1},\dots,e_{dn})$ of M_{t_0} at p_0 and, for arbitrarily fixed $\alpha \in \{m+1,\dots,dn\}$, take an orthonormal tangent frame $(e_1,\dots,e_m)$ of $T_{p_0}M_{t_0}$ consisting of eigenvectors of the shape operator $(A_{t_0})_{e_\alpha}$, which is not necessarily that of type (I) or (II). Let λ_i $(1 \le i \le m)$ be the eigenvalue corresponding to $\tilde{e}_i$, that is, In similar to (3.17), we have

$$\begin{aligned}
&4\sum_{i,j,p,q}\widetilde{R}_{ipjq}h^\alpha_{pq}h^\alpha_{ij} - 4\sum_{j,l,p}\widetilde{R}_{ljlp}\left(\sum_i h^\alpha_{pi}h^\alpha_{ij}\right)\\
&= -4\sum_{i,p}\widetilde{R}_{ipip}((\lambda^\alpha_i)^2 - \lambda^\alpha_i\lambda^\alpha_p)\\
&= -2\sum_{i<p}\widetilde{K}_{ip}(\lambda^\alpha_i - \lambda^\alpha_p)^2 \le 16mc\|\mathring{h}^\alpha\|^2.
\end{aligned}$$

Hence we can derive

$$P_I \le 16mc\|\mathring{h}\|^2 \tag{3.25}$$

at (p_0,t_0).

Next we shall estimate the terms P_{II,a_ε} and P_{III} at (p_0,t_0). We shall use an orthonormal frame of type (II) to estimating these terms at (p_0,t_0). Take an orthonormal frame $(e^B_1,\dots,e^B_{dn})$ $(B = 1,\dots,d-1)$ of type (II) at $p_0 \in M_{t_0}$. Set $\widetilde{K}^B_{s\alpha} := \widetilde{K}(e^B_s, e^B_\alpha)$. From (2.3) and (3.3), we have

$$\widetilde{K}^B_{s\alpha} = -c\left(1 + 3\widetilde{g}(e^B_s, J_B e^B_\alpha)\right)^2 = -c\left(1 + 3\delta_{s,\alpha-m}\tau^B_{[\frac{s+1}{2}]}\right)^2 \le -c.$$

On the other hand, it follows from $((Q_\varepsilon)_{t_0})_{p_0} = 0$ that $\|h\|^2 = a_\varepsilon\|H\|^2 + b_\varepsilon$, that is, $(a_\varepsilon - \frac{1}{m})\|H\|^2 = \|\mathring{h}\|^2 - b_\varepsilon$ holds at (p_0,t_0). Therefore, by noticing $a_\varepsilon \ge \frac{1}{m}$, we can derive

$$\begin{aligned}
P_{II,a_\varepsilon} &= 2\sum_{s,\alpha}\widetilde{K}^B_{s\alpha}(\|h^\alpha\|^2 - a_\varepsilon\|H^\alpha\|^2)\\
&= 2\sum_{s,\alpha}\widetilde{K}^B_{s\alpha}\left(\|\mathring{h}^\alpha\|^2 - \left(a_\varepsilon - \frac{1}{m}\right)\|H^\alpha\|^2\right)\\
&\le -2c\sum_{s,\alpha}\|\mathring{h}^\alpha\|^2 + 2c\sum_{s,\alpha}(1 + 3\delta_{s,\alpha-m})\left(a_\varepsilon - \frac{1}{m}\right)\|H^\alpha\|^2\\
&= -2mc\|\mathring{h}\|^2 + 2(m+3)c\left(a_\varepsilon - \frac{1}{m}\right)\|H\|^2\\
&= -2(m+3)cb_\varepsilon + 6c\|\mathring{h}\|^2.
\end{aligned} \tag{3.26}$$

As in (3.22), we can derive

$$P_{III} \le 8(d-1)kc\|\mathring{h}\|^2. \tag{3.27}$$

By (3.25), (3.26) and (3.27), we obtain

$$P_{a_\varepsilon} \leq 2(8m + 4(d-1)k + 3)c\|\mathring{h}\|^2 - 2(m+3)b_\varepsilon c. \tag{3.28}$$

Set $R := 2R_1 - 2a_\varepsilon R_2 + P_{a_\varepsilon}$. Note that $\|(H_{t_0})_{p_0}\| > 0$ because $b_\varepsilon < 0$. We use an orthonormal frame of type (I) at $p_0 \in M_{t_0}$. It follows from Lemma 3.2 that

$$\begin{aligned} R \leq& \left(6 - \frac{2}{ma_\varepsilon - 1}\right)\|\mathring{h}\|^2\|\mathring{h}_-\|^2 - 3\|\mathring{h}_-\|^4 \\ &+ \left(\frac{2ma_\varepsilon b_\varepsilon}{ma_\varepsilon - 1} + (8m + 4(d-1)k + 3)c\right)\|\mathring{h}_1\|^2 \\ &+ \left(\frac{4b_\varepsilon}{ma_\varepsilon - 1} + 2(8m + 4(d-1)k + 3)c\right)\|\mathring{h}_-\|^2 \\ &- \frac{2b_\varepsilon^2}{ma_\varepsilon - 1} - 2(m+3)b_\varepsilon c. \end{aligned}$$

By the assumption $m \geq \frac{3d}{2} + 5$ in Theorem 1.5, the coefficient of $\|\mathring{h}\|^2\|\mathring{h}_-\|^2$ is negative. It is easy to show that the coefficient of $\|\mathring{h}_1\|^2$ vanishes and that the coefficient of $\|\mathring{h}_-\|^2$ is negative. Hence we have

$$R \leq -\frac{2b_\varepsilon^2}{ma_\varepsilon - 1} - 2(m+3)b_\varepsilon < 0$$

by the assumption $m \geq \frac{3d}{2} + 5$ in Theorem 1.5. Therefore, since $R < 0$ at (p_0, t_0), it is shown that the condition $(*_{m-1+\varepsilon, b_\varepsilon})$ is preserved along the mean curvature flow by using the maximum principle. Hence the statement of this proposition follows from the arbitrariness of ε. □

4. Evolution of the traceless second fundamental form

Let M be a closed submanifold in $\widetilde{M}$ as in Theorems 1.1–1.5 and $\{M_t\}_{t\in[0,T)}$ be the mean curvature flow starting from M. Following to the discussion in [5, 7, 18], we shall analyze the traceless part of the second fundamental form and show that it becomes small in a suitable sense if the extrinsic curvature becomes unbounded. Since the initial manifold M satisfies the condition $(*_{m-1,b})$, it satisfies the condition $(*_{m-1+\varepsilon, b_\varepsilon})$ for some $\varepsilon \in [0, 1)$. Hence it follows from Propositions 3.7–3.9 that this condition is preserved along the mean curvature flow. So, as in [9, 2, 18], set

$$W := \alpha\|H\|^2 + \beta \quad \text{and} \quad f_\sigma := \frac{\|\mathring{h}\|^2}{W^{1-\sigma}},$$

where σ is a suitably small non-negative constant, $\beta := b$ and

$$
\alpha := \begin{cases} \dfrac{2c}{(m-1+\varepsilon)(\bar{r}+2c(1-\varepsilon))} & (\widetilde{M} = \mathbb{F}P^n(4c),\ k=1) \\ \dfrac{(11-2d)m-19}{9m(m+2)} & (\widetilde{M} = \mathbb{C}P^n(4c), \mathbb{H}P^n(4c),\ k \geq 2) \\ \dfrac{-8c}{(m-1+\varepsilon)(\bar{r}-8c(1+\varepsilon))} & (\widetilde{M} = \mathbb{F}H^n(-4c),\ k=1) \\ \dfrac{(11-2d)m-19}{9m(m+2)} & (\widetilde{M} = \mathbb{C}H^n(-4c), \mathbb{H}H^n(-4c),\ k \geq 2). \end{cases} \tag{4.1}
$$

By using Lemmas 2.1, 2.2, 3.2, 3.6, Propositions 3.7–3.9, we can derive the following result by the same discussion as the proof of Proposition 4.1 in [18].

Proposition 4.1. *For any $\sigma \in [0, \frac{1}{4}]$, the following inequality*

$$
\begin{aligned} \frac{\partial}{\partial t} f_\sigma &\leq \Delta f_\sigma + \frac{2\alpha(1-\sigma)}{W}\langle \nabla f_\sigma, \nabla \|H\|^2\rangle - 2C_1 W^{\sigma-1}\|\nabla H\|^2 \\ &\quad + 2\sigma\|h\|^2 f_\sigma + 2C_2 f_\sigma + 2C_3 W^{\sigma-1} \end{aligned} \tag{4.2}
$$

holds for some constants $C_1 > 0$, C_2 and C_3 depending only on m and M.

Proof. By straightforward calculations, we can derive

$$
\begin{aligned} &\frac{\partial}{\partial t} f_\sigma - \Delta f_\sigma \\ &= W^{\sigma-1}\left(\frac{\partial}{\partial t}\|\mathring{h}\|^2 - \Delta\|\mathring{h}\|^2\right) - \alpha(1-\sigma)\frac{f_\sigma}{W}\left(\frac{\partial}{\partial t}\|H\|^2 - \Delta\|H\|^2\right) \\ &\quad + \frac{2\alpha(1-\sigma)}{W}\langle \nabla f_\sigma, \nabla\|H\|^2\rangle - \alpha^2(1-\sigma)\sigma\frac{f_\sigma}{W^2}\|\nabla\|H\|^2\|^2. \end{aligned} \tag{4.3}
$$

First we consider the case $\widetilde{M} = \mathbb{F}P^n(4c)$ or $\mathbb{F}H^n(-4c)$ and $k = 1$ (i.e., Theorems 1.1 and 1.4-case). By using the evolution equations in Lemma 2.2 and neglecting the negative $\|\nabla\|H\|^2\|^2$ term, we have

$$
\begin{aligned} &\frac{\partial}{\partial t} f_\sigma - \Delta f_\sigma \\ &\leq \frac{2\alpha(1-\sigma)}{W}\langle \nabla f_\sigma, \nabla\|H\|^2\rangle - 2W^{\sigma-1}\|\nabla h\|^2 \\ &\quad + 2W^{\sigma-1}\left(\frac{1}{m} + f_0(1-\sigma)\alpha\right)\|\nabla H\|^2 + 2\beta\frac{1-\sigma}{W} f_\sigma(\|h\|^2 + \bar{r}) \\ &\quad + 2\sigma f_\sigma(\|h\|^2 + \bar{r}) - 4W^{\sigma-1}(h^{ij}h^p_j\widetilde{R}^l_{pli} - h^{ij}h^{lp}\widetilde{R}_{pili}). \end{aligned} \tag{4.4}
$$

Our choice of α and β gives $0 \leq f_0 < 1$. Hence, from the inequality (3.7) in Lemma 3.3, we have

$$
-\|\nabla h\|^2 + \left(\frac{1}{m} + f_0(1-\sigma)\alpha\right)\|\nabla H\|^2 \leq -\|\nabla h\|^2 + \left(\frac{1}{m} + \alpha\right)\|\nabla H\|^2 \leq -C_1\|\nabla H\|^2, \tag{4.5}
$$

where $C_1 := \frac{3}{m+2} - \frac{1}{m} - \alpha$. We have $C_1 > 0$ by our choice of α and m $(= dn - 1)$. Denote by R the reaction term in (4.4), that is,

$$
R := 2\beta\frac{(1-\sigma)}{W} f_\sigma(\|h\|^2 + \bar{r}) + 2\sigma f_\sigma(\|h\|^2 + \bar{r}) - 4W^{\sigma-1}(h^{ij}h^p_j\widetilde{R}^l_{pli} - h^{ij}h^{lp}\widetilde{R}_{pili}).
$$

By using inequalities (3.16) and (3.17), we can derive

$$R \leq \begin{cases} 2f_\sigma\left(\beta(1-\sigma)\dfrac{(\|h\|^2+\bar{r})}{W} + \sigma(\|h\|^2+\bar{r}) - 2mc\right) & (\text{when } \widetilde{M} = \mathbb{F}P^n(4c)) \\ 2f_\sigma\left(\beta(1-\sigma)\dfrac{(\|h\|^2+\bar{r})}{W} + \sigma(\|h\|^2+\bar{r}) + 8mc\right) & (\text{when } \widetilde{M} = \mathbb{F}H^n(-4c)). \end{cases}$$

Easily we have

$$\|h\|^2 + \bar{r} \leq \begin{cases} \dfrac{1}{m-1+\varepsilon}\|H\|^2 + 2c(1-\varepsilon) + \bar{r} = \dfrac{\bar{r}+2c(1-\varepsilon)}{\beta}W & (\text{when } \mathbb{F}P^n(4c)) \\ \dfrac{1}{m-1+\varepsilon}\|H\|^2 - 8c(1+\varepsilon) + \bar{r} = \dfrac{\bar{r}-8c(1+\varepsilon)}{\beta}W & (\text{when } \mathbb{F}H^n(-4c)). \end{cases}$$

Hence we can derive

$$R \leq \begin{cases} 2f_\sigma(-m+3d-1-2\varepsilon-2\sigma(1-\varepsilon))c + 2\sigma f_\sigma\|h\|^2 & (\text{when } \widetilde{M} = \mathbb{F}P^n(4c)) \\ 2f_\sigma(7m-3d-5-8\varepsilon-8\sigma(1+\varepsilon))c + 2\sigma f_\sigma\|h\|^2 & (\text{when } \widetilde{M} = \mathbb{F}H^n(-4c)) \end{cases}$$

and hence

$$R \leq \begin{cases} 2f_\sigma(-m+3d-1)c + 2\sigma f_\sigma\|h\|^2 & (\text{when } \widetilde{M} = \mathbb{F}P^n(4c)) \\ 2f_\sigma(7m-3d-5)c + 2\sigma f_\sigma\|h\|^2 & (\text{when } \widetilde{M} = \mathbb{F}H^n(-4c)). \end{cases}$$

This together with (4.4) and (4.5) implies the statement of this proposition with $C_3 = 0$ and

$$C_2 = \begin{cases} (-m+3d-1)c & (\text{when } \widetilde{M} = \mathbb{F}P^n(4c)) \\ (7m-3d-5)c & (\text{when } \widetilde{M} = \mathbb{F}H^n(-4c)). \end{cases}$$

Next we consider the case of $\widetilde{M} = \mathbb{C}P^n(4c)$ or $\mathbb{H}P^n(4c)$ and $k \geq 2$ (i.e., Theorem 1.2-case). By straightforward calculations, we can derive (4.3). By using Lemma 2.1 and the properties of the curvature tensor $\widetilde{R}$, we can derive

$$\begin{aligned} \frac{\partial}{\partial t}\|H\|^2 &= \Delta\|H\|^2 - 2\|\nabla H\|^2 + 2R_2 + 2\sum_{s,\alpha}\widetilde{K}_{s\alpha}\|H^\alpha\|^2 \\ &\geq \Delta\|H\|^2 - 2\|\nabla H\|^2 + 2R_2 + 2mc\|H\|^2 \end{aligned} \tag{4.6}$$

and

$$\frac{\partial}{\partial t}\|\mathring{h}\|^2 = \Delta\|\mathring{h}\|^2 - 2\left(\|\nabla h\|^2 - \frac{1}{m}\|\nabla H\|^2\right) + 2\left(R_1 - \frac{1}{m}R_2\right) + P_{\frac{1}{m}}, \tag{4.7}$$

These together with (3.24) (which holds also for $a_\varepsilon = \frac{1}{m}$) implies

$$\begin{aligned} \frac{\partial}{\partial t}f_\sigma &= W^{\sigma-1}\frac{\partial}{\partial t}\|\mathring{h}\|^2 - \alpha(1-\sigma)\frac{f_\sigma}{W}\frac{\partial}{\partial t}\|H\|^2 \\ &\leq W^{\sigma-1}\left\{\Delta\|\mathring{h}\|^2 - 2\left(\|\nabla h\|^2 - \frac{1}{m}\|\nabla H\|^2\right)\right. \\ &\qquad \left. +2\left(R_1 - \frac{1}{m}R_2\right) - 2(m-4(d-1)k-3)c\|\mathring{h}\|^2\right\} \\ &\quad - \alpha(1-\sigma)\frac{f_\sigma}{W}\left(\Delta\|H\|^2 - 2\|\nabla H\|^2 + 2R_2 + 2mc\|H\|^2\right). \end{aligned} \tag{4.8}$$

On the other hand, we have

$$W^{\sigma-1}\Delta\|\mathring{h}\|^2-\alpha(1-\sigma)\frac{f_\sigma}{W}\Delta\|H\|^2\leq\Delta f_\sigma+\frac{2\alpha(1-\sigma)}{W}\langle\nabla f_\sigma,\nabla\|H\|^2\rangle. \tag{4.9}$$

From these inequalities, we can estimate the evolution of f_σ as follows:

$$\begin{aligned}\frac{\partial}{\partial t}f_\sigma\leq{}&\Delta f_\sigma+\frac{2\alpha(1-\sigma)}{W}\langle\nabla f_\sigma,\nabla\|H\|^2\rangle-2W^{\sigma-1}\|\nabla h\|^2\\&+2W^{\sigma-1}\left(\frac{1}{m}+\alpha(1-\sigma)f_0\right)\|\nabla H\|^2+2W^{\sigma-1}\left(R_1-\frac{1}{m}R_2\right)\\&-2\alpha(1-\sigma)\frac{f_\sigma}{W}R_2-2m\alpha(1-\sigma)c\frac{f_\sigma}{W}\|H\|^2\\&-2(m-4(d-1)k-3)cW^{\sigma-1}\|\mathring{h}\|^2.\end{aligned} \tag{4.10}$$

Now we shall estimate the gradient terms in the right-hand side of this evolution inequality. By using Lemma 3.6 and $0\leq f_0<1$, we can derive

$$\begin{aligned}&-\|\nabla h\|^2+\left(\frac{1}{m}+f_0(1-\sigma)\alpha\right)\|\nabla H\|^2\\&\leq\left(-\frac{2(10-d)}{9(m+2)}+\frac{1}{m}+\alpha\right)\|\nabla H\|^2\\&\leq -C_1\|\nabla H\|^2,\end{aligned} \tag{4.11}$$

where

$$C_1:=-\frac{2(10-d)}{9(m+2)}+\frac{1}{m}+\alpha=\frac{(11-2d)m-18}{9m(m+2)}-\frac{(11-2d)m-19}{9m(m+2)}=\frac{1}{9m(m+2)}\ (>0).$$

Next we shall analyze the reaction term of (4.8). Denote by R the reaction term. We can write R as

$$R=2W^{\sigma-2}R'+2\alpha\sigma\frac{f_\sigma}{W}R_2,$$

where

$$\begin{aligned}R':={}&\left(R_1-\frac{1}{m}R_2\right)W-\alpha\|\mathring{h}\|^2R_2-m\alpha(1-\sigma)c\|\mathring{h}\|^2\|H\|^2\\&-(m-4(d-1)k-3)c\|\mathring{h}\|^2W.\end{aligned}$$

Easily we can show

$$\alpha\sigma\frac{f_\sigma}{W}R_2\leq\sigma f_\sigma\|h\|^2. \tag{4.12}$$

We take σ as $0\leq\sigma<\frac{1}{4}$. Then, by using Lemma 3.2, $\|\mathring{h}\|^2=\|\mathring{h}_1\|^2+\|\mathring{h}_-\|^2$, $R_2=\|\mathring{h}_1\|^2\|H\|^2+\frac{1}{m}\|H\|^4$ and the pinching condition $(*_{m-1,b})$ (which holds for all time by Proposition 3.8), we can derive

$$
\begin{aligned}
R' \le & -\alpha\frac{m-4}{m(m-1)}\|\mathring{h}_-\|^2\|H\|^4 \\
& + \left(\frac{\beta(m+1)}{m(m-1)} - \alpha(m(2-\sigma) - 4(d-1)k - 3)c\right)\|\mathring{h}_1\|^2\|H\|^2 \\
& + \left(\frac{2\beta}{m(m-1)} - \alpha(m(2-\sigma)c - 4(d-1)kc - 3\beta)\right)\|\mathring{h}_-\|^2\|H\|^2 \\
& + \beta(2\beta - (m - 4(d-1)k - 3)c)(\|\mathring{h}_1\|^2 + \|\mathring{h}_-\|^2).
\end{aligned}
$$

Furthermore, by using $m \ge \frac{3d}{2} + 5$, $0 < \sigma < \frac{1}{4}$ and $k \ge 2$, we can derive

$$
\begin{aligned}
R' &\le -(m-2)\beta^2\|\mathring{h}\|^2 + \left\{\frac{2\beta}{\frac{3d}{2}+5} - \frac{3m\alpha c}{4}\right\}\|\mathring{h}\|^2 \cdot \|H\|^2 \\
&\le -(m-2)\beta^2\|\mathring{h}\|^2 + \left\{\frac{2(m-4(d-1)k-3)}{(\frac{3d}{2}+5)m} - \frac{3m\alpha}{4}\right\}c\|\mathring{h}\|^2 \cdot \|H\|^2 \\
&\le -\left\{\left(\frac{3m\alpha}{4} - \frac{2(m-4(d-1)k-3)}{(\frac{3d}{2}+5)m}\right)c\|H\|^2 + (m-2)\beta^2\right\}\|\mathring{h}\|^2.
\end{aligned}
$$

Easily we can show $\frac{3m\alpha}{4} - \frac{2(m-4(d-1)k-3)}{(3d/2+5)m} > 0$. Hence we can derive that

$$R' \le C_2\|\mathring{h}\|^2 W$$

holds for some negative constant C_2 depending only on m and d. This together with (4.12) implies that

$$R \le 2\sigma f_\sigma\|h\|^2 + 2C_2 f_\sigma. \tag{4.13}$$

From (4.10), (4.11) and (4.13), we can derive the desired inequality.

Next we consider the case of $\widetilde{M} = \mathbb{C}H^n(-4c)$ or $\mathbb{H}H^n(-4c)$ and $k \ge 2$. (i.e., Theorem 1.5-case). By straightforward calculations, we can derive (4.3). By using Lemma 2.1 and the properties of the curvature tensor $\widetilde{R}$, we can derive

$$\frac{\partial}{\partial t}\|H\|^2 \ge \Delta\|H\|^2 - 2\|\nabla H\|^2 + 2R_2 - 8mc\|H\|^2 \tag{4.14}$$

and (4.7). Since (3.28) holds for any $\varepsilon \in [0,1)$, as $\varepsilon = 0$, we have

$$P_{\frac{1}{m}} \le 2(8m + 4(d-1)k + 3)c\|\mathring{h}\|^2 + \frac{2(m+3)(8m+4(d-1)k+3)c^2}{m}. \tag{4.15}$$

From these relations, we can derive

$$
\begin{aligned}
\frac{\partial}{\partial t}f_\sigma \le & W^{\sigma-1}\left(\Delta\|\mathring{h}\|^2 - 2\left(\|\nabla h\|^2 - \frac{1}{m}\|\nabla H\|^2\right)\right) \\
& + 2W^{\sigma-1}\left\{\left(R_1 - \frac{1}{m}R_2\right) + (8m + 4(d-1)k + 3)c\|\mathring{h}\|^2 \right. \\
& \left. + \frac{(m+3)(8m+4(d-1)k+3)c^2}{m}\right\} \\
& - \alpha(1-\sigma)\frac{f_\sigma}{W}\left(\Delta\|H\|^2 - 2\|\nabla H\|^2 + 2R_2 - 8mc\|H\|^2\right).
\end{aligned} \tag{4.16}
$$

Furthermore, by using (4.9), we can estimate the evolution of f_σ as follows:

$$
\begin{aligned}
(4.17)\qquad & \frac{\partial}{\partial t} f_\sigma - \Delta f_\sigma \\
& \le \frac{2\alpha(1-\sigma)}{W}\langle \nabla f_\sigma, \nabla \|H\|^2\rangle - 2W^{\sigma-1}\|\nabla h\|^2 \\
& \quad + 2W^{\sigma-1}\left(\frac{1}{m} + \alpha(1-\sigma) f_0\right)\|\nabla H\|^2 + 2W^{\sigma-1}\left(R_1 - \frac{1}{m}R_2\right) \\
& \quad - 2\alpha(1-\sigma)\frac{f_\sigma}{W}R_2 + 8m\alpha(1-\sigma)c\frac{f_\sigma}{W}\|H\|^2 \\
& \quad + cW^{\sigma-1}\left(2(8m+4(d-1)k+3)\|\mathring{h}\|^2 + \frac{2(m+3)(8m+4(d-1)k+3)c}{m}\right).
\end{aligned}
$$

By using Lemma 3.6 and $0 \le f_0 < 1$, we can derive the estimate (4.11) of the gradient term in the right-hand side of this evolution inequality. We shall analyze the reaction term of (4.17). Denote by R the reaction term. We can write R as

$$R = 2W^{\sigma-2}R' + 2\alpha\sigma\frac{f_\sigma}{W}R_2,$$

where

$$
\begin{aligned}
R' := & \left(R_1 - \frac{1}{m}R_2\right)W - \alpha\|\mathring{h}\|^2 R_2 + 4m\alpha(1-\sigma)c\|\mathring{h}\|^2\|H\|^2 \\
& + (8m+4(d-1)k+3)\|\mathring{h}\|^2 cW + \frac{(m+3)(8m+4(d-1)k+3)c^2W}{m}.
\end{aligned}
$$

We take σ as $0 \le \sigma < \frac{1}{4}$. Then, by using Lemma 3.2, $\|\mathring{h}\|^2 = \|\mathring{h}_1\|^2 + \|\mathring{h}_-\|^2$, $R_2 = \|\mathring{h}_1\|^2\|H\|^2 + \frac{1}{m}\|H\|^4$ and the pinching condition $(*_{m-1,b})$ (which holds for all time by Proposition 3.9), we can derive

$$
\begin{aligned}
R' \le & -\frac{4\alpha}{m(m-1)}\|\mathring{h}_-\|^2\|H\|^4 \\
& + \left(\frac{\beta(m+1)}{m(m-1)} + \alpha(4m(3-\sigma) + 4(d-1)k + 3)c\right)\|\mathring{h}_1\|^2\|H\|^2 \\
& + \left(\frac{2\beta}{m(m-1)} + \alpha(4m(3-\sigma)c + 4(d-1)kc + 3c + 3\beta)\right)\|\mathring{h}_-\|^2\|H\|^2 \\
& + \beta(2\beta + (8m+4(d-1)k+3)c)\|\mathring{h}\|^2 + \frac{(m+3)(8m+4(d-1)k+3)c^2W}{m}.
\end{aligned}
$$

Furthermore, by using $\sigma > 0$, $\alpha > 0$ and $\beta < 0$, we can derive

$$
\begin{aligned}
(4.18)\qquad R' \le & \left\{\left(\alpha - \frac{2}{m^2(m-1)}\right)(8m+4(d-1)k+3) + 4\alpha m\right\} c\|\mathring{h}\|^2 \cdot \|H\|^2 \\
& + \beta(2\beta + (8m+4(d-1)k+3)c)\|\mathring{h}\|^2 + \frac{(m+3)(8m+4(d-1)k+3)c^2W}{m}.
\end{aligned}
$$

Since $\alpha - \frac{2}{m^2(m-1)} > 0$ by $m \ge \frac{3d}{2} + 5$, we see that the coefficient of $\|\mathring{h}\|^2 \cdot \|H\|^2$ in the right-hand side of (4.18) is positive. Also, we see that the coefficient of $\|\mathring{h}\|^2$ in the right-hand side of (4.18) is negative. Hence we can derive that

$$R' \leq C_2 \|\mathring{h}\|^2 W + C_3 W$$

for some positive constants C_2 and C_3 depending only on m and d. This together with (4.12) (which holds also in this case) implies that

$$R \leq 2\sigma f_\sigma \|h\|^2 + 2C_2 f_\sigma + 2C_3 W^{\sigma-1}. \tag{4.19}$$

From (4.11), (4.17) and (4.19), we can derive the desired inequality. □

By using Lemmas 2.1, 2.2, 3.3 and 3.6, we can derive the following evolution inequalities by the same calculation as the proof of Lemma 4.2 in [18].

Lemma 4.2. *In the case of* $\widetilde{M} = \mathbb{F}P^n(4c)$ *(i.e.,* Theorems 1.1 *and* 1.2*-cases), we have*
(i) $\frac{\partial}{\partial t}\|\mathring{h}\|^2 \leq \Delta\|\mathring{h}\|^2 - 2C_4\|\nabla h\|^2 + 4\|h\|^2\|\mathring{h}\|^2$ *for some* $C_4 > 0$ *only depending on m,*
(ii) $\frac{\partial}{\partial t}\|H\|^4 \geq \Delta\|H\|^4 - 12\|H\|^2\|\nabla H\|^2 + \frac{4}{m}\|H\|^6$.

Lemma 4.3. *In the case of* $\widetilde{M} = \mathbb{F}H^n(-4c)$ *(i.e.,* Theorems 1.4 *and* 1.5*-cases), we have*
(i) $\frac{\partial}{\partial t}\|\mathring{h}\|^2 \leq \Delta\|\mathring{h}\|^2 - 2C_4\|\nabla h\|^2 + 4\|h\|^2\|\mathring{h}\|^2 + 2(7m + 4(d-1)k)c\|\mathring{h}\|^2$ *for some* $C_4 > 0$ *depending only on m,*
(ii) $\frac{\partial}{\partial t}\|H\|^4 \geq \Delta\|H\|^4 - 12\|H\|^2\|\nabla H\|^2 + \frac{4}{m}\|H\|^6 - 16mc\|H\|^4$.

Proof of Lemmas 4.2 and 4.3. First we consider the case of $k = 1$ (i.e., Theorems 1.1 and 1.4-cases). From Lemma 2.2, (3.7) (in Lemma 3.3), (3.16) and (3.17), we obtain

$$\begin{aligned}
&\frac{\partial}{\partial t}\|\mathring{h}\|^2 - \Delta\|\mathring{h}\|^2 \\
&= -2\left(\|\nabla h\|^2 - \frac{1}{m}\|\nabla H\|^2\right) + 2\|\mathring{h}\|^2(\|h\|^2 + \bar{r}) - 4\sum_{i,j,p,l}(h_{ij}h_j^p\widetilde{R}^l_{pli} - h^{ij}h^{lp}\widetilde{R}_{pilj}) \\
&\leq \begin{cases} -\dfrac{4(m-1)}{3m}\|\nabla h\|^2 + 2\|\mathring{h}\|^2(\|h\|^2 + \bar{r}) - 4mc\|\mathring{h}\|^2 & (\text{when } \widetilde{M} = \mathbb{F}P^n(4c)) \\ -\dfrac{4(m-1)}{3m}\|\nabla h\|^2 + 2\|\mathring{h}\|^2(\|h\|^2 + \bar{r}) + 16mc\|\mathring{h}\|^2 & (\text{when } \widetilde{M} = \mathbb{F}H^n(-4c)) \end{cases}
\end{aligned}$$

and hence

$$\begin{aligned}
&\frac{\partial}{\partial t}\|\mathring{h}\|^2 - \Delta\|\mathring{h}\|^2 \\
&\leq \begin{cases} -\dfrac{4(m-1)}{3m}\|\nabla h\|^2 + 2\|\mathring{h}\|^2\|h\|^2 - (4m - 6d + 6)c\|\mathring{h}\|^2 & (\text{when } \widetilde{M} = \mathbb{F}P^n(4c)) \\ -\dfrac{4(m-1)}{3m}\|\nabla h\|^2 + 2\|\mathring{h}\|^2\|h\|^2 + 2(7m - 3d + 3)c\|\mathring{h}\|^2 & (\text{when } \widetilde{M} = \mathbb{F}H^n(-4c)). \end{cases}
\end{aligned}$$

Therefore we obtain the evolution inequalities in (i) of Lemmas 4.2 and 4.3. Also, from Lemma 2.2, we obtain

$$\begin{aligned}
\frac{\partial}{\partial t}\|H\|^4 &= 2\|H\|^2(\Delta\|H\|^2 - 2\|\nabla H\|^2 + 2\|H\|^2(\|h\|^2 + \bar{r})) \\
&= \Delta\|H\|^4 - 2\|\nabla\|H\|^2\|^2 - 4\|H\|^2\|\nabla H\|^2 + 4\|H\|^4(\|h\|^2 + \bar{r}) \\
&\geq \Delta\|H\|^4 - 12\|H\|^2\|\nabla H\|^2 + 4\|H\|^4(\|h\|^2 + \bar{r}) \\
&\geq \Delta\|H\|^4 - 12\|H\|^2\|\nabla H\|^2 + \frac{4}{m}\|H\|^6 + 4\bar{r}\|H\|^4.
\end{aligned}$$

Since $\widetilde{r} = \widetilde{\epsilon}c(dn + 3d - 4) = \widetilde{\epsilon}c(m + 3d - 3)$, we have

$$4\widetilde{r} > \begin{cases} 0 & (\text{when } \widetilde{M} = \mathbb{F}P^n(4c)) \\ -16mc & (\text{when } \widetilde{M} = \mathbb{F}H^n(-4c)). \end{cases}$$

Therefore we obtain the evolution inequalities in (ii) of Lemmas 4.2 and 4.3.

Next we consider the case of $k \geq 2$ (i.e., Theorems 1.2 and 1.5-cases). From Lemma 2.1, we have

$$\frac{\partial}{\partial t}\|\mathring{h}\|^2 \leq \Delta\|\mathring{h}\|^2 - 2\left(\|\nabla h\|^2 - \frac{1}{m}\|\nabla H\|^2\right) + 2\left(R_1 - \frac{1}{m}R_2\right) + P_{\frac{1}{m}},$$

where R_1, R_2 and $P_{\frac{1}{m}}$ are as in the previous section. From Lemma 3.6, we have

$$\|\nabla h\|^2 - \frac{1}{m}\|\nabla H\|^2 \geq \frac{(11-2d)m - 18}{9m(m+2)}\|\nabla h\|^2.$$

Furthermore, from Lemma 3.2, we obtain

$$\begin{aligned} R_1 - \frac{1}{m}R_2 &\leq \|\mathring{h}_1\|^4 + 4\|\mathring{h}_1\|^2\|\mathring{h}_-\|^2 + \frac{3}{2}\|\mathring{h}_-\|^4 + \frac{1}{m}\|\mathring{h}_1\|^2\|H\|^2 \\ &\leq 2\left(\|\mathring{h}_1\|^2 + \|\mathring{h}_-\|^2\right)^2 + \frac{2}{m}\|H\|^2\left(\|\mathring{h}_1\|^2 + \|\mathring{h}_-\|^2\right) = 2\|\mathring{h}\|^2\|h\|^2. \end{aligned}$$

By simple calculations, we have

$$P_{II,\frac{1}{m}} \leq \begin{cases} 2(m+3)c\|\mathring{h}\|^2 & (\text{when } \widetilde{M} = \mathbb{F}P^n(4c)) \\ -2mc\|\mathring{h}\|^2 & (\text{when } \widetilde{M} = \mathbb{F}H^n(-4c)), \end{cases}$$

where $P_{II,\frac{1}{m}}$ is as in the previous section. This together with (3.20), (3.23), (3.25), and (3.27) implies that

$$P_{\frac{1}{m}} \leq \begin{cases} -2(m - 4(d-1)k - 3)c\|\mathring{h}\|^2 < 0 & (\text{when } \widetilde{M} = \mathbb{F}P^n(4c)) \\ 2(7m + 4(d-1)k)c\|\mathring{h}\|^2 & (\text{when } \widetilde{M} = \mathbb{F}H^n(-4c)). \end{cases}$$

Therefore we obtain the evolution inequalities (with $C_4 = \frac{(11-2d)m-18}{9m(m+2)}$) in (i) of Lemmas 4.2 and 4.3. Next we shall derive the evolution inequality in (ii) of Lemmas 4.2 and 4.3. From Lemma 2.1, we have

$$\frac{\partial}{\partial t}\|H\|^4 = \Delta\|H\|^4 - 2\|\nabla\|H\|^2\|^2 - 4\|H\|^2\|\nabla H\|^2 + 2\|H\|^2\left(2R_2 + 2\sum_{s,\alpha}\bar{K}_{s\alpha}\|H^\alpha\|^2\right).$$

Also, we have

$$2R_2 = 2\|H\|^2\left(\|\mathring{h}_1\|^2 + \frac{1}{m}\|H\|^2\right) \geq \frac{2}{m}\|H\|^4, \qquad \|\nabla\|H\|^2\|^2 = 4\|H\|^2\|\nabla H\|^2$$

and

$$\sum_{s,\alpha}\bar{K}_{s\alpha}\|H^\alpha\|^2 \geq \begin{cases} 0 & (\text{when } \widetilde{M} = \mathbb{F}P^n(4c)) \\ -4mc\|H\|^2 & (\text{when } \widetilde{M} = \mathbb{F}H^n(-4c)). \end{cases}$$

From these relations, we obtain the evolution inequalities in (ii) of Lemmas 4.2 and 4.3. □

Finally, we give the evolution inequality for $\|\nabla H\|^2$. By the same discussion as the proof of Corollary 5.10 in [2], we can derive the following result.

Lemma 4.4. *There exists a constant C_5 depending only on M such that*

$$\frac{\partial}{\partial t}\|\nabla H\|^2 \leq \Delta\|\nabla H\|^2 + C_5(\|H\|^2 + 1)\|\nabla h\|^2.$$

5. Finiteness of maximal time

In this section, we shall prove the finiteness of the maximal time in the statements of Theorems 1.3 and 1.5. First we shall consider the case of Theorem 1.5. Denote by $S_p(a)$ the geodesic sphere of radius a centered at p in $\mathbb{F}H^n(-4c)$, and by $h^{p,a}$ and $H^{p,a}$ the second fundamental form and the mean curvature vector of $S_p(a)$. Let M be an m-dimensional closed submanifold in $\mathbb{F}H^n(-4c)$ and $\{M_t = f_t(M)\}_{t\in[0,T)}$ be the mean curvature flow starting from M. Take a geodesic sphere $S_{p_0}(a)$ surrounding M. Denote by $\overline{r} : \mathbb{F}H^n(-4c) \to \mathbb{R}$ the (Riemannian) distance function from p_0 and set $r_t := \overline{r} \circ f_t$. Then we can show

$$(\widetilde{\Delta}\overline{r})_{f_t(p)} = \|H^{p_0,r_t(p)}\| \tag{5.1}$$

and

$$(\Delta_t r_t)_p = (\widetilde{\Delta}\overline{r})_{f_t(p)} + \|h^{p_0,r_t(p)}(((\nu_t)_p)_{TS}, ((\nu_t)_p)_{TS})\| + d\overline{r}((H_t)_p) \tag{5.2}$$

for any $p \in M$, where $\widetilde{\Delta}$ denotes the Laplace operator of the Levi-Civita connection $\widetilde{\nabla}$ of $\mathbb{F}H^n(-4c)$ and $((\nu_t)_p)_{TS}$ denotes the $T_{f_t(p)}S_{p_0}(r_t(p))$-component of the unit normal vector $(\nu_t)_p$ of M_t at $f_t(p)$. Set

$$\mathcal{D}_{1,p} := \mathrm{Span}\left\{ J_B\left(\frac{\partial}{\partial \overline{r}}\right)_{f_t(p)} \,\middle|\, B = 1, \cdots, d-1 \right\}^{\perp}$$

and

$$\mathcal{D}_{2,p} := \mathrm{Span}\left\{ J_B\left(\frac{\partial}{\partial \overline{r}}\right)_{f_t(p)} \,\middle|\, B = 1, \cdots, d-1 \right\},$$

where $(\bullet)^{\perp}$ denotes the orthogonal complement of $(\bullet)$ in $T_pS_{p_0}(r_t(p))$. Then we can show

$$\|h^{p_0,a}(X,X)\| = \begin{cases} \dfrac{\|X\|^2}{\tanh(\sqrt{-c}a)} & (X \in \mathcal{D}_{1,p}) \\[2ex] \dfrac{2\|X\|^2}{\tanh(2\sqrt{-c}a)} & (X \in \mathcal{D}_{2,p}) \end{cases} \tag{5.3}$$

and hence

$$\|H^{p_0,a}\| = \frac{(n-1)d}{\tanh(\sqrt{-c}a)} + \frac{2(d-1)}{\tanh(2\sqrt{-c}a)}. \tag{5.4}$$

From (5.1)–(5.4), we obtain

$$\begin{aligned}(\Delta_t r_t)_p = &\frac{2(d-1)}{\tanh(2\sqrt{-c}r_t(p))} + \frac{(n-1)d}{\tanh(\sqrt{-c}r_t(p))} + d\overline{r}((H_t)_p) \\ &+ \frac{2}{\tanh(2\sqrt{-c}r_t(p))}\|(((\nu_t)_p)_{TS})_{(2)}\|^2 + \frac{1}{\tanh(\sqrt{-c}r_t(p))}\|(((\nu_t)_p)_{TS})_{(1)}\|^2,\end{aligned} \tag{5.5}$$

where $(((\nu_t)_p)_{TS})_{(i)}$ $(i = 1, 2)$ denotes the $\mathcal{D}_{i,p}$-component of $((\nu_t)_p)_{TS}$. On the other hand, we have $\frac{\partial r_t}{\partial t} = d\overline{r}((H_t)_p)$. Hence we obtain

$$(5.6)\qquad \left(\frac{\partial r_t}{\partial t}\right)_p = (\Delta_t r_t)_p - \frac{2(d-1)}{\tanh(2\sqrt{-c}r_t(p))} - \frac{(n-1)d}{\tanh(\sqrt{-c}r_t(p))} - \frac{2}{\tanh(2\sqrt{-c}r_t(p))}\|(((\nu_t)_p)_{TS})_{(2)}\|^2 - \frac{1}{\tanh(\sqrt{-c}r_t(p))}\|(((\nu_t)_p)_{TS})_{(1)}\|^2 \leq (\Delta_t r_t)_p - ((n+1)d-2).$$

Therefore, by the maximum principle, we can derive $\max r_t \leq \max r_0 - ((n+1)d-2)t$ for all time t. This implies that $T \leq \frac{\max r_0}{(n+1)d-2} (< \infty)$.

Next, we consider the case of Theorem 1.3. Denote by $S_p(a)$ the geodesic sphere of radius a centered at p in $\mathbb{F}P^n(4c)$, and by $h^{p,a}$ and $H^{p,a}$ the second fundamental form and the mean curvature vector of $S_p(a)$. Let $\{M_t = f_t(M)\}_{t\in[0,T)}$ be the mean curvature flow starting from M. Take a geodesic sphere $S_{p_0}(a)$ surrounding M. Since the diameter of M in $(\widetilde{M},\widetilde{g})$ is smaller than $\frac{\pi}{6\sqrt{c}}$ by the assumption, we may assume that $a < \frac{\pi}{4\sqrt{c}}$ by taking p_0 as the midpoint of the geodesic of maximum length connecting two points of M. Denote by $\overline{r} : \mathbb{F}P^n(4c) \setminus C_{p_0} \to \mathbb{R}$ the (Riemannian) distance function from p_0 and set $r_t := \overline{r} \circ f_t$, where C_{p_0} is the cut locus of p_0. Then we can show

$$(5.7)\qquad (\overline{\Delta}\overline{r})_{f_t(p)} = \|H^{p_0,r_t(p)}\|$$

and

$$(5.8)\qquad (\Delta_t r_t)_p = (\overline{\Delta}\overline{r})_{f_t(p)} + \|h^{p_0,r_t(p)}(((\nu_t)_p)_{TS}, ((\nu_t)_p)_{TS})\| + d\overline{r}((H_t)_p)$$

for any $p \in M$, where $\overline{\Delta}$ denotes the Laplace operator of the Levi-Civita connection $\widetilde{\nabla}$ of $\mathbb{F}p^n(4c)$ and $((\nu_t)_p)_{TS}$ denotes the $T_{f_t(p)}S_{p_0}(r_t(p))$-component of the unit normal vector $(\nu_t)_p$ of M_t at $f_t(p)$. Set

$$\mathcal{D}_{1,p} := \mathrm{Span}\left\{ J_B\left(\frac{\partial}{\partial \overline{r}}\right)_{f_t(p)} \middle| B = 1, \cdots, d-1 \right\}^{\perp}$$

and

$$\mathcal{D}_{2,p} := \mathrm{Span}\left\{ J_B\left(\frac{\partial}{\partial \overline{r}}\right)_{f_t(p)} \middle| B = 1, \cdots, d-1 \right\},$$

where $(\bullet)^{\perp}$ denotes the orthogonal complement of $(\bullet)$ in $T_pS_{p_0}(r_t(p))$. Then we can show

$$(5.9)\qquad \|h^{p_0,a}(X,X)\| = \begin{cases} \dfrac{\|X\|^2}{\tan(\sqrt{c}a)} & (X \in \mathcal{D}_{1,p}) \\ \dfrac{2\|X\|^2}{\tan(2\sqrt{c}a)} & (X \in \mathcal{D}_{2,p}) \end{cases}$$

and hence

$$(5.10)\qquad \|H^{p_0,a}\| = \frac{(n-1)d}{\tan(\sqrt{c}a)} + \frac{2(d-1)}{\tan(2\sqrt{-c}a)},$$

where we note that $0 < \tan(2\sqrt{-c}a) < \infty$ because of $a < \frac{\pi}{4\sqrt{c}}$. From (5.7)–(5.10), we obtain

$$(5.11)\qquad (\Delta_t r_t)_p = \frac{2(d-1)}{\tan(2\sqrt{c}r_t(p))} + \frac{(n-1)d}{\tan(\sqrt{c}r_t(p))} + d\overline{r}((H_t)_p)$$

$$+\frac{2}{\tan(2\sqrt{c}r_t(p))}\|(((\nu_t)_p)_{TS})_{(2)}\|^2+\frac{1}{\tan(\sqrt{c}r_t(p))}\|(((\nu_t)_p)_{TS})_{(1)}\|^2,$$

where $(((\nu_t)_p)_{TS})_{(i)}$ $(i=1,2)$ denotes the $\mathcal{D}_{i,p}$-component of $((\nu_t)_p)_{TS}$. On the other hand, we have $\frac{\partial r_t}{\partial t}=d\overline{r}((H_t)_p)$. Hence we obtain

$$\begin{aligned}(5.12)\qquad \left(\frac{\partial r_t}{\partial t}\right)_p &= (\Delta_t r_t)_p-\frac{2(d-1)}{\tan(2\sqrt{c}r_t(p))}-\frac{(n-1)d}{\tan(\sqrt{c}r_t(p))}\\ &\quad -\frac{2}{\tan(2\sqrt{c}r_t(p))}\|(((\nu_t)_p)_{TS})_{(2)}\|^2-\frac{1}{\tan(\sqrt{c}r_t(p))}\|(((\nu_t)_p)_{TS})_{(1)}\|^2\\ &\le (\Delta_t r_t)_p-\frac{(n+1)d-2}{\tan(2\sqrt{c}a)}.\end{aligned}$$

Therefore, by the maximum principle, we can derive $\max r_t \le \max r_0-\frac{((n+1)d-2)t}{\tan(2\sqrt{c}a)}$ for all time t. This implies that $T\le\frac{\max r_0\cdot\tan(2\sqrt{c}a)}{(n+1)d-2}$ $(<\infty)$.

6. Proof of the collapse to a round point

In this section, we shall prove the collapse to a round point in the statements of Theorems 1.1–1.5. Throughout this section, let M be as in Theorems 1.1–1.5. Since M satisfies the condition $(*_{m-1,b})$ and it is compact, it satisfies the condition $(*_{m-1+\varepsilon,b_\varepsilon})$ for a sufficiently small positive number ε. By Propositions 3.7–3.9, the condition $(*_{m-1+\varepsilon,b_\varepsilon})$ is preserved along the mean curvature flow. As in the previous section, set $W=\alpha\|H\|^2+\beta$, where $\beta=b$ and α and β is as in (4.1).

Theorem 6.1. *Let M be as above. Then there exist positive constants C_0 and σ_0 depending only on the initial submanifold M such that, for all $t\in[0,T)$, the following inequality holds:*

$$\|\mathring{h}\|^2\le C_0(\|H\|^2+1)^{1-\sigma_0}.$$

Since there exists the positive term $2\sigma\|h\|^2 f_\sigma$ among the reaction term of the evolution inequality (4.2) in Proposition 4.1, we cannot use the maximum principle to show the uniform boundedness of $\{(f_\sigma)_t\}_{t\in[0,T)}$. So, as in Huisken [9], Baker [2] and Pipoli–Sinestrari [18], we shall estimate the L^p-norm of f_σ from above by exploiting the good negative term of $\|\nabla H\|^2$. By using this L^p-estimate, the Sobolev's inequality for submanifolds and the Stampacchia's iteration lemma, we shall derive the uniform boundedness of $\{(f_\sigma)_t\}_{t\in[0,T)}$.

For a function $\overline{\rho}$ over $M\times[0,T)$, we denote $\int_M\overline{\rho}(\cdot,t)d\mu_t$ by $\int_{M_t}\overline{\rho}d\mu$ for the simplicity. By the same discussion as the proof of Proposition 5.4 in [18], we shall derive the following Poincaré-type inequality for f_σ.

Proposition 6.2. *There exists a positive constant C_6 depending only on m,k and the initial submanifold M such that, for any $p\ge 2, 0<\sigma<\frac{1}{4}$ and $\eta>0$, we have*

$$\begin{aligned}\frac{\varepsilon p}{2}\int_{M_t} f_\sigma^p W d\mu &\le (\eta(p+1)+5)\int_{M_t} W^{\sigma-1}f_\sigma^{p-1}\|\nabla H\|^2\,d\mu\\ &\quad +\frac{p+1}{\eta}\int_{M_t} f_\sigma^{p-2}\|\nabla f_\sigma\|^2\,d\mu+\overline{\epsilon}2^{\overline{\epsilon}+1}mb\int_{M_t} f_\sigma^p d\mu+\frac{1}{p}C_6^p.\end{aligned}$$

First we show the following fact in the same method as the proof of Proposition 5.2 in [18].

Lemma 6.3. *In the case of* $k = 1$ *(i.e.,* Theorems 1.1 *and* 1.4*-cases), there exists a positive constant* C_7 *depending only on* m *such that the intrinsic sectional curvature* $K(: G_2(M_t) \to \mathbb{R})$ *of* M_t *satisfies*

$$K > \varepsilon C_7 W,$$

where $G_2(M_t)$ *denotes the Grassmann bundle of* M_t *consisting of the* 2*-planes.*

Proof. Let $(e_1, \ldots, e_m)$ be an orthonormal tangent frame consisting of eigenvectors of the shape operator A_t of M_t. Let $A_t e_i = \lambda_i e_i$ $(i = 1, \ldots, n)$. For any $i \neq j$, the Gauss equation gives

$$K_{ij} = \overline{K}_{ij} + \lambda_i \lambda_j.$$

Like in [9], we can use the following algebraic property: for any $i \neq j$

$$\|h\|^2 - \frac{1}{m-1}\|H\|^2 = -2\lambda_i\lambda_j + \left(\lambda_i + \lambda_j - \frac{\|H\|^2}{m-1}\right) + \sum_{l \neq i,j}\left(\lambda_l - \frac{\|H\|^2}{m-1}\right) \geq -2\lambda_i\lambda_j.$$

In the case of $\widetilde{M} = \mathbb{F}P^n(4c)$, we have

$$\begin{aligned} 2K_{ij} &\geq 2c - \|h\|^2 + \frac{1}{m-1}\|H\|^2 \\ &\geq \left(\frac{1}{m-1} - a_\varepsilon\right)\|H\|^2 + 2c - b_\varepsilon \\ &= \frac{\varepsilon}{(m-1)(m-1+\varepsilon)}\|H\|^2 + 2c\varepsilon. \end{aligned}$$

In the case of $\widetilde{M} = \mathbb{F}H^n(-4c)$, we have

$$\begin{aligned} 2K_{ij} &\geq -8c - \|h\|^2 + \frac{1}{m-1}\|H\|^2 \\ &\geq \left(\frac{1}{m-1} - a_\varepsilon\right)\|H\|^2 - 8c - b_\varepsilon \\ &= \frac{\varepsilon}{(m-1)(m-1+\varepsilon)}\|H\|^2 + 8c\varepsilon. \end{aligned}$$

Thus, in both cases, we see that

$$K_{ij} > \varepsilon C_7 W$$

for a suitable positive constant C_7 depending only on m. □

By using (23) in [1], we obtain

$$\Delta\|\mathring{h}\|^2 \geq 2\|\nabla\mathring{h}\|^2 + 2\langle\mathring{h}, \nabla\nabla H\rangle + 2Z - C\|h\|^2,$$

where C is a suitable positive constant depending only on m, k and Z is given by

$$Z = \sum_{i,j,p,\alpha,\beta} H^\alpha h^\alpha_{ip} h^\beta_{pj} h^\beta_{ij} - \sum_{\alpha,\beta}\left(\sum_{i,j} h^\alpha_{ij} h^\beta_{ij}\right)^2 - \sum_{i,j,\alpha,\beta}\left(\sum_p (h^\alpha_{ip} h^\beta_{pj} - h^\alpha_{jp} h^\beta_{ip})\right)^2.$$

Since the condition $(*_{m-1+\varepsilon,b_\varepsilon})$ is preserved along the mean curvature flow, we can derive

$$\Delta\|\mathring{h}\|^2 \geq 2\|\nabla\mathring{h}\|^2 + 2\langle\mathring{h}, \nabla\nabla H\rangle + 2Z - \gamma W, \tag{6.1}$$

where γ is a suitable positive constant depending only on m and k.

By using Lemma 6.3 and noticing $1 \leq \overline{\epsilon K} \leq 4$, we can derive the following fact in the same method as the proof of Lemma 5.3 in [18].

Lemma 6.4. (i) *In the case of $\widetilde{M} = \mathbb{F}P^n(4c)$, there exists a positive constant ρ depending only on m and k satisfying*

$$Z + 2mb\|\mathring{h}\|^2 \geq \rho\varepsilon\|\mathring{h}\|^2 W.$$

(ii) *In the case of $\mathbb{F}H^n(-4c)$, there exists a constant ρ depending only on m and k satisfying*

$$Z - \frac{mb}{2}\|\mathring{h}\|^2 \geq \rho\varepsilon\|\mathring{h}\|^2 W.$$

Proof. First we consider the case of $k = 1$. Take an orthonormal frame such that diagonalizes the shape operator. By using the Gauss equation, Lemma 6.3 and $1 \leq \overline{\epsilon K} \leq 4$, we have

$$\begin{aligned}
Z &= \left(\sum_i \lambda_i\right)\left(\sum_i \lambda_i^3\right) - \left(\sum_i \lambda_i^2\right)^2 \\
&= \sum_{i<j} \lambda_i\lambda_j(\lambda_i - \lambda_j)^2 \\
&= \sum_{i<j} K_{ij}(\lambda_i - \lambda_j)^2 - \sum_{i<j} \overline{K}_{ij}(\lambda_i - \lambda_j)^2 \\
&\geq \begin{cases} \varepsilon cmW\|\mathring{h}\|^2 - 2mb\|\mathring{h}\|^2 & (\text{when } \mathbb{F}P^n(4c)) \\ \varepsilon cmW\|\mathring{h}\|^2 - \dfrac{mb}{8}\|\mathring{h}\|^2 & (\text{when } \mathbb{F}H^n(-4c)). \end{cases}
\end{aligned}$$

Thus the statements (i) and (ii) of this lemma follows.

Next we consider the case of $k \geq 2$. Take any $(p, t) \in M \times [0, T)$. We need to distinguish into the cases where $H \neq 0$ and $H = 0$ at (p, t). First we consider the case where $H \neq 0$ at (p, t). In this case, by using the estimate in page 384 in [1], we have

$$Z \geq -\frac{m}{2}\|\mathring{h}_1\|^4 - \frac{3}{2}\|\mathring{h}_-\|^4 - \frac{m+2}{2}\|\mathring{h}_1\|^2\|\mathring{h}_-\|^2 + \frac{1}{2(m-1)}\left(\|\mathring{h}_1\|^2 + \|\mathring{h}_-\|^2\right)\|H\|^2.$$

Since $(*_{m-1+\varepsilon,b_\varepsilon})$ is preserved along the mean curvature flow, we have

$$\|H\|^2 > \frac{m(m-1+\varepsilon)}{1-\varepsilon}\left(\|\mathring{h}_1\|^2 + \|\mathring{h}_-\|^2 - b_\varepsilon\right) > \frac{m(m-1+\varepsilon)}{1-\varepsilon}\left(\|\mathring{h}_1\|^2 + \|\mathring{h}_-\|^2 - b\right).$$

Therefore we obtain

$$\begin{aligned}
Z &\geq -\frac{m}{2}\|\mathring{h}_1\|^4 - \frac{3}{2}\|\mathring{h}_-\|^4 - \frac{m+2}{2}\|\mathring{h}_1\|^2\|\mathring{h}_-\|^2 + \frac{m}{2(1-\varepsilon)}\left(\|\mathring{h}_1\|^2 + \|\mathring{h}_-\|^2\right)\left(\|\mathring{h}_1\|^2 + \|\mathring{h}_-\|^2 - b\right) \\
&\geq \frac{\varepsilon m}{2(1-\varepsilon)}\|\mathring{h}_1\|^4 + \frac{m-3+3\varepsilon}{2(1-\varepsilon)}\|\mathring{h}_-\|^4 + \frac{m-2+\varepsilon(m+2)}{2(1-\varepsilon)}\|\mathring{h}_1\|^2\|\mathring{h}_-\|^2 - \frac{m}{2(1-\varepsilon)}b\|\mathring{h}\|^2.
\end{aligned}$$

From this estimate, it follows that there exists a positive constant μ_1 depending only on m satisfying

$$Z+\frac{mb\overline{\epsilon}}{2(1-\varepsilon)}\|\mathring{h}\|^2\geq Z+\frac{m}{2(1-\varepsilon)}b\|\mathring{h}\|^2\geq\varepsilon\mu_1\|\mathring{h}\|^4.$$

On the other hand, by using the definition of Z and estimating various terms by the Peter–Paul inequality, we can derive

$$Z\geq\mu_2\|\mathring{h}\|^2\|H\|^2-\mu_3\|\mathring{h}\|^4$$

for some positive constants μ_2 and μ_3 depending on m. Hence we obtain

$$Z+\frac{mb\overline{\epsilon}}{2(1-\varepsilon)}\|\mathring{h}\|^2\geq\widehat{C}\left(\mu_2\|\mathring{h}\|^2\|H\|^2-\mu_3\|\mathring{h}\|^4+\frac{mb\overline{\epsilon}}{2(1-\varepsilon)}\|\mathring{h}\|^2\right)+(1-\widehat{C})\varepsilon\mu_1\|\mathring{h}\|^4$$

for any $\widehat{C}\in[0,1]$. Choose $\frac{\varepsilon\mu_1}{\varepsilon\mu_1+\mu_3}$ as $\widehat{C}$. Then, we have

$$Z+\frac{mb\overline{\epsilon}}{2}\|\mathring{h}\|^2\geq\frac{\varepsilon\mu_1}{\varepsilon\mu_1+\mu_3}\left(\mu_2\|H\|^2+\frac{mb\overline{\epsilon}}{2}\right)\|\mathring{h}\|^2.$$

From this inequality, we can derive the statements (i) and (ii) of this lemma. Next we consider the case where $H=0$ at (p,t). Then, since $(*_{m-1+\varepsilon,b_\varepsilon})$ holds in all time, this case cannot happen in the case of $\widetilde{M}=\mathbb{F}H^n(-4c)$. Hence we may assume that $\widetilde{M}=\mathbb{F}P^n(4c)$. Then we have $\|h\|^2=\|\mathring{h}\|^2\leq b$ and $W=\beta=b$ because $(*_{m-1+\varepsilon,b_\varepsilon})$ holds in all time. Hence by using Theorem 1.1 in [12], we can derive

$$Z\geq-\frac{3}{2}\|h\|^4\geq-\frac{3}{2}b\|\mathring{h}\|^2.$$

Therefore we obtain

$$Z+2mb\|\mathring{h}\|^2\geq\left(2m-\frac{3}{2}\right)\|\mathring{h}\|^2W.$$

Thus we can derive the statements (i) and (ii) of this lemma. □

Proof of Proposition 6.2. By using (6.1), we have

$$\begin{aligned}\Delta f_\sigma\geq\,&2W^{\sigma-1}\|\nabla\mathring{h}\|^2+2W^{\sigma-1}\langle\mathring{h},\nabla\nabla H\rangle+2W^{\sigma-1}Z-\gamma W^\sigma-\alpha(1-\sigma)\frac{f_\sigma}{W}\Delta\|H\|^2\\&-\frac{2\alpha(1-\sigma)}{W}\langle\nabla f_\sigma,\nabla\|H\|^2\rangle+\alpha^2\sigma(1-\sigma)\frac{f_\sigma}{W^2}\|\nabla\|H\|^2\,\|^2.\end{aligned}$$

Since the term $2W^{\sigma-1}\|\nabla\mathring{h}\|^2$ and $\alpha^2\sigma(1-\sigma)\frac{f_\sigma}{W^2}\|\nabla\|H\|^2\,\|^2$ are positive, we can omit them. By using Lemma 6.4, we have

$$\begin{aligned}\Delta f_\sigma\geq\,&2W^{\sigma-1}\langle\mathring{h},\nabla\nabla H\rangle-\alpha(1-\sigma)\frac{f_\sigma}{W}\Delta\|H\|^2-\frac{2\alpha(1-\sigma)}{W}\langle\nabla f_\sigma,\nabla\|H\|^2\rangle\\&-\overline{\epsilon}2^{\overline{\epsilon}+1}mbf_\sigma+2\rho\varepsilon f_\sigma W-\gamma W^\sigma.\end{aligned}$$

By multiplying f_σ^{p-1} to this inequality and integrating on M_t with respect to $d\mu_t$, we can derive

$$\begin{aligned}2\rho\varepsilon\int_{M_t}f_\sigma^pW\,d\mu\leq\,&(\eta(p+1)+5)\int_{M_t}W^{\sigma-1}f_\sigma^{p-1}\|\nabla H\|^2\,d\mu+\frac{p+1}{\eta}\int_{M_t}f_\sigma^{p-2}\|\nabla f_\sigma\|^2\,d\mu\\&+\overline{\epsilon}2^{\overline{\epsilon}+1}mb\int_{M_t}f_\sigma^p\,d\mu+\gamma\int_{M_t}f_\sigma^{p-1}W^\sigma\,d\mu.\end{aligned}$$

By Young's inequality, we obtain the following estimate with respect to the last term of the right-hand side of this inequality:

$$\gamma f_\sigma^{p-1} W^\sigma \leq \gamma W\left(\frac{r^p}{p} W^{(\sigma-1)p} + \frac{p-1}{p} r^{-\frac{p}{p-1}} f_\sigma^p\right),$$

where $r := \left(\frac{(p-1)\gamma}{\varepsilon\rho p}\right)^{\frac{p-1}{p}}$. Note that $r \leq \frac{\gamma}{\varepsilon\rho}$. From $0 < \sigma < \frac{1}{4}$ and $p \geq 2$, we have $(\sigma-1)p+1 < 0$ and hence $W^{(\sigma-1)p+1} \leq \beta^{(\sigma-1)p+1}$. Therefore we have

$$\frac{1}{p}\gamma r^p \int_{M_t} W^{(\sigma-1)p+1}\, d\mu \leq \frac{1}{p}\gamma r^p \beta^{(\sigma-1)p+1}\mathrm{vol}(M_t) \leq \gamma r^p \beta^{(\sigma-1)p+1}\mathrm{vol}(M_0).$$

Set $C_6 := \left(\gamma r^p \beta^{(\alpha-1)p+1}\mathrm{vol}(M_0)\right)^{1/p}$. Then we obtain the desired inequality. □

Set

$$C_8 := \begin{cases} 1 & (\text{when } \widetilde{M} = \mathbb{F}P^n(4c)) \\ \dfrac{(dn)^2 - 3dn + 2}{(dn)^2 - 4dn + 4 - \varepsilon} & (\text{when } \widetilde{M} = \mathbb{F}H^n(-4c),\ k = 1) \\ \dfrac{m(m-1)}{m^2 - 2m + 1 - \varepsilon} & (\text{when } \widetilde{M} = \mathbb{F}H^n(-4c),\ k \geq 2). \end{cases}$$

From Propositions 4.1 and 6.2, we can derive the following result for the estimate of the L^p-norm of f_σ by the same discussion as the proof of Proposition 5.5 in [18].

Proposition 6.5. *There exists a constant C_9 $(= C_9(\sigma, p))$ depending only on σ, p, m, k, ε, ρ, β, $\mathrm{Vol}(M_0)$ and T such that, if $p \geq \frac{8C_8}{C_1} + 1$ and $\sigma < \frac{\varepsilon\sqrt{C_1}\rho}{2^7 m\sqrt{p}}$, then the inequality $\left(\int_{M_t} f_\sigma^p\, d\mu\right)^{\frac{1}{p}} \leq C_9$ holds for all $t \in [0, T)$.*

Proof. By multiplying pf_σ^{p-1} to the inequality (4.2) in Proposition 4.1 and integrating on M with respect to $d\mu_t$, we obtain

$$\begin{aligned}(6.2)\quad & \frac{d}{dt}\int_{M_t} f_\sigma^p\, d\mu + p(p-1)\int_{M_t} f_\sigma^{p-2}\|\nabla f_\sigma\|^2\, d\mu + 2C_1 p\int_{M_t}\|\nabla H\|^2 W^{\sigma-1} f_\sigma^{p-1}\, d\mu \\ &\leq 4p\alpha\int_{M_t}\|H\|W^{-1}\|\nabla H\|\|\nabla f_\sigma\| f_\sigma^{p-1}\, d\mu + 2\sigma p\int_{M_t}\|h\|^2 f_\sigma^p\, d\mu \\ &\quad + 2C_2 p\int_{M_t} f_\sigma^p\, d\mu + 2C_3 p\int_{M_t} W^{\sigma-1} f_\sigma^{p-1}\, d\mu.\end{aligned}$$

Also, we have

$$\alpha\|H\| \leq \begin{cases} \sqrt{W} & (\text{when } \widetilde{M} = \mathbb{F}P^n(4c)) \\ \sqrt{W - \beta} & (\text{when } \widetilde{M} = \mathbb{F}H^n(-4c)). \end{cases}$$

By (i) of Remark 1.1, Propositions 3.7–3.9, we have $-\beta \leq (C_8 - 1)W$. Hence we obtain $\alpha\|H\| \leq \sqrt{C_8 W}$. Also, we have $f_\sigma \leq W^\sigma$. By using these inequalities and the Young's inequality, we obtain

$$(6.3)\quad 4p\alpha\int_{M_t}\|H\|W^{-1}\|\nabla H\|\|\nabla f_\sigma\| f_\sigma^{p-1}\, d\mu$$

$$= 2p\cdot 2\int_{M_t}\left\{\frac{\sqrt{p-1}}{2C_8^{1/4}}\left(\frac{\alpha\|H\|}{W}\cdot f_\sigma^{p-1}\cdot\|\nabla f_\sigma\|^2\cdot W^{\frac{1}{2}-\sigma}\right)^{\frac{1}{2}}\right.$$
$$\left.\times\frac{2C_8^{1/4}}{\sqrt{p-1}}\left(\frac{\alpha\|H\|}{W}\cdot f_\sigma^{p-1}\cdot W^{\sigma-\frac{1}{2}}\cdot\|\nabla H\|^2\right)^{\frac{1}{2}}\right\}d\mu$$
$$\leq\frac{p(p-1)}{2\sqrt{C_8}}\int_{M_t}\frac{\alpha\|H\|}{W}\cdot f_\sigma^{p-1}\cdot\|\nabla f_\sigma\|^2\cdot W^{\frac{1}{2}-\sigma}\,d\mu$$
$$+\frac{8p\sqrt{C_8}}{p-1}\int_{M_t}\frac{\alpha\|H\|}{W}\cdot f_\sigma^{p-1}\cdot W^{\sigma-\frac{1}{2}}\cdot\|\nabla H\|^2\,d\mu$$
$$\leq\frac{p(p-1)}{2}\int_{M_t}f_\sigma^{p-2}\cdot\|\nabla f_\sigma\|^2\,d\mu+\frac{8pC_8}{p-1}\int_{M_t}W^{\sigma-1}\cdot f_\sigma^{p-1}\cdot\|\nabla H\|^2\,d\mu.$$

From our choice of p, we have $C_1p\leq 2C_1p-\frac{8pC_8}{p-1}$. Also, we have

$$2mW\geq a_\varepsilon\|H\|^2+b_\varepsilon\geq\|h\|^2,$$

which holds by our assumption for m. From (6.2), (6.3) and these inequalities, we obtain

$$\frac{d}{dt}\int_{M_t}f_\sigma^p\,d\mu+\frac{p(p-1)}{2}\int_{M_t}f_\sigma^{p-2}\|\nabla f_\sigma\|^2\,d\mu+C_1p\int_{M_t}\|\nabla H\|^2W^{\sigma-1}f_\sigma^{p-1}\,d\mu \tag{6.4}$$
$$\leq 2\sigma p\int_{M_t}\|h\|^2f_\sigma^p\,d\mu+2C_2p\int_{M_t}f_\sigma^p\,d\mu+2C_3p\int_{M_t}W^{\sigma-1}f_\sigma^{p-1}\,d\mu$$
$$\leq 4\sigma pm\int_{M_t}Wf_\sigma^p\,d\mu+2C_2p\int_{M_t}f_\sigma^p\,d\mu+2C_3p\int_{M_t}W^{\sigma-1}f_\sigma^{p-1}\,d\mu,$$

which together with Proposition 6.2 derives

$$\frac{d}{dt}\int_{M_t}f_\sigma^p\,d\mu+\frac{p(p-1)}{2}\int_{M_t}f_\sigma^{p-2}\|\nabla f_\sigma\|^2\,d\mu+C_1p\int_{M_t}\|\nabla H\|^2W^{\sigma-1}f_\sigma^{p-1}\,d\mu \tag{6.5}$$
$$\leq\frac{8\sigma pm}{\varepsilon\rho}\left\{(\eta(p+1)+5)\int_{M_t}W^{\sigma-1}f_\sigma^{p-1}\|\nabla H\|^2\,d\mu\right.$$
$$\left.+\frac{p+1}{\eta}\int_{M_t}f_\sigma^{p-2}\|\nabla f_\sigma\|^2\,d\mu+\overline{\epsilon}2^{\overline{\epsilon}-1}mb\int_{M_t}f_\sigma^p\,d\mu+\frac{C_6^p}{p}\right\}$$
$$+2C_2p\int_{M_t}f_\sigma^p\,d\mu+2C_3p\int_{M_t}W^{\sigma-1}f_\sigma^{p-1}\,d\mu.$$

Let $\eta=\frac{\sqrt{C_1}}{4\sqrt{p}}$. Then, by using the assumptions for p and σ, we have

$$\frac{8\sigma pm}{\varepsilon\rho}(\eta(p+1)+5)\leq C_1p\quad\text{and}\quad\frac{8\sigma p(p+1)m}{\varepsilon\rho\eta}\leq\frac{p(p-1)}{2}.$$

From (6.5) and these inequalities, we obtain

$$\frac{d}{dt}\int_{M_t}f_\sigma^p\,d\mu \tag{6.6}$$
$$\leq\left(\frac{8\overline{\epsilon}2^{\overline{\epsilon}-1}\sigma pbm^2}{\varepsilon\rho}+2C_2p\right)\int_{M_t}f_\sigma^p\,d\mu+\frac{8\sigma m}{\varepsilon\rho}\cdot C_6^p+2C_3p\int_{M_t}W^{\sigma-1}f_\sigma^{p-1}\,d\mu.$$

By the Young's inequality, we have

$$W^{\sigma-1}\cdot f_\sigma^{p-1}\le\frac{W^{(\sigma-1)p}}{p}+\frac{p-1}{p}\cdot f_\sigma^p.$$

From $0<\sigma<\frac{1}{4}$, we have $(\sigma-1)p<0$ and hence $W^{(\sigma-1)p}\le\beta^{(\sigma-1)p}$. Hence we obtain

$$\int_{M_t}W^{\sigma-1}f_\sigma^{p-1}\,d\mu\le\frac{1}{p}\beta^{(\sigma-1)p}\cdot\mathrm{Vol}(M_0)+\frac{p-1}{p}\int_{M_t}f_\sigma^p\,d\mu. \tag{6.7}$$

From (6.6) and (6.7), we can derive

$$\frac{d}{dt}\int_{M_t}f_\sigma^p\,d\mu\le\widehat{C_1}\int_{M_t}f_\sigma^p\,d\mu+\widehat{C_2}$$

for some positive constants $\widehat{C_1}$ and $\widehat{C_2}$ depending only on $p,\sigma,m,k,\varepsilon,\rho,\beta$ and $\mathrm{Vol}(M_0)$. Therefore, since T is finite, we obtain the assertion for a constant C_9 depending only on $p,\ \sigma,m,\ k,\ \varepsilon,\ \rho,\ \beta,\ \mathrm{Vol}(M_0)$ and T. □

From this proposition, we can derive the following result.

Corollary 6.6. *Assume that $T<\infty$. Then the following statements* (i) *and* (ii) *hold:*

(i) *Let r be any positive number. For any $p>\frac{8C_8}{C_1r}+\frac{1}{r}+1$ and any $\sigma<\frac{\varepsilon\sqrt{C_1}\rho\sqrt{p-1}}{2^7mp\sqrt{r}}+\frac{1}{p}$, we have*

$$\left(\int_{M_t}W^{(\sigma-1)r}f_\sigma^{(p-1)r}\,d\mu\right)^{\frac{1}{(p-1)r}}\le C_9\left(\sigma-\frac{1-\sigma}{p-1},(p-1)r\right)\qquad(t\in[0,T)).$$

(ii) *For any $p>\frac{8C_8}{C_1}+1$ and any $\sigma<\frac{\varepsilon\sqrt{C_1}\rho}{2^7m\sqrt{p}}-\frac{q}{p}$, we have*

$$\left(\int_{M_t}\|h\|^{2q}f_\sigma^p d\mu\right)^{\frac{1}{p}}\le(2m)^{\frac{q}{p}}C_9\left(\sigma+\frac{q}{p},p\right)\qquad(t\in[0,T)).$$

Proof. First we shall show the statement (i). Easily we have $W^{(\sigma-1)r}f_\sigma^{(p-1)r}=f_{\sigma-\frac{1-\sigma}{p-1}}^{(p-1)r}$. Hence it follows from Proposition 6.5 that the desired inequality holds for any p and σ as in the statement (i). Next we shall show the statement (ii). From $\|h\|^2\le 2mW$ and Proposition 6.5, we obtain

$$\begin{aligned}\left(\int_{M_t}\|h\|^{2q}f_\sigma^p d\mu\right)^{\frac{1}{p}}&\le(2m)^{\frac{q}{p}}\left(\int_{M_t}W^qf_\sigma^p d\mu\right)^{\frac{1}{p}}=(2m)^{\frac{q}{p}}\left(\int_{M_t}f_{\sigma+\frac{q}{p}}^p d\mu\right)^{\frac{1}{p}}\\&\le(2m)^{\frac{q}{p}}C_9\left(\sigma+\frac{q}{p},p\right).\end{aligned}$$ □

Here we recall the Stampacchia's iteration lemma.

Lemma 6.7. *Let $\phi:[s_0,\infty)\to\mathbb{R}$ be a non-negative and non-increasing function satisfying*

$$\phi(s_2)\le\frac{C}{(s_2-s_1)^p}\|\phi(s_1)\|^\gamma$$

for any s_1,s_2 with $s_0<s_1<s_2$, where C, p are positive constants and γ is a constant with $\gamma>1$. Then $\phi(s_0+d_0)=0$ holds, where

$$d_0 = \left(C\|\phi(s_0)\|^{\gamma-1}2^{p\gamma/(\gamma-1)}\right)^{1/p}.$$

Also, we recall the Sobolev inequality for submanifolds.

Theorem 6.8 ([6]). *Let M be an m-dimensional submanifold in a Riemannian manifold $(\widetilde{M},\overline{g})$, where M may have the boundary. Denote by H the mean curvature vector field of M, $\bar{K}$ the maximal sectional curvature of $\widetilde{M}$, $\bar{R}(M)$ the injective radius of $\widetilde{M}$ restricted to M and ω_m the volume of the unit ball in the Euclidean space $\mathbb{R}^m$. Let b be a positive real number or a purely imaginary one satisfying $b^2 \geq \bar{K}$ and ψ a non-negative C^1 function on M vanishing on ∂M. Then the following inequality holds:*

$$\left(\int_M \psi^{\frac{m}{m-1}}d\mu\right)^{\frac{m-1}{m}} \leq \widehat{C}(m)\int_M (\|\nabla\psi\| + \psi\|H\|)\,d\mu \tag{6.8}$$

provided

$$b^2(1-\alpha)^{-\frac{2}{m}}\left(\omega_m^{-1}\cdot \mathrm{Vol}(\mathrm{supp}\,\psi)\right)^{\frac{2}{m}} \leq 1 \qquad \text{and} \qquad 2\rho_0 \leq \bar{R}(M), \tag{6.9}$$

where

$$\rho_0 := \begin{cases} b^{-1}\sin^{-1} b\cdot(1-\alpha)^{-\frac{1}{m}}\cdot\left(\omega_m^{-1}\cdot \mathrm{Vol}(\mathrm{supp}\,\psi)\right)^{\frac{1}{m}} & (\text{for } b \text{ is real}),\\ (1-\alpha)^{-\frac{1}{m}}\left(\omega_m^{-1}\cdot \mathrm{Vol}(\mathrm{supp}\,\psi)\right)^{\frac{1}{m}} & (\text{for } b \text{ is purely imaginary}). \end{cases}$$

Here α is a free parameter with $0 < \alpha < 1$, and

$$\widehat{C}(m) = \widehat{C}(m,\alpha) := \frac{\pi}{2}\cdot 2^{m-2}\alpha^{-1}(1-\alpha)^{-\frac{1}{m}}\frac{m}{m-1}\omega_m^{-\frac{1}{m}}.$$

Now we shall prove Theorem 6.1.

Proof of Theorem 6.1. Define a function $f_{\sigma,l} : M\times[0,T) \to \mathbb{R}$ by $f_{\sigma,l}(x,t) := \max\{f_\sigma(x,t)-l, 0\}$, where l is any positive number with $l \geq l_0 := \max_{x\in M} f_\sigma(x,0)$. Set $A_t(l) := \{x \in M \mid f_\sigma(x,t) \geq l\}$. For a function v over $M\times[0,T)$, we denote $\int_{A_t(l)} v(\cdot,t)\,d\mu_t$ by $\int_{A_t(l)} v\,d\mu$ for the simplicity. In similar to (6.4), we can derive the following evolution equation:

$$\begin{aligned}\frac{\partial}{\partial t}\int_{A_t(l)} f_{\sigma,l}^p d\mu \leq & -\frac{p(p-1)}{2}\int_{A_t(l)} f_{\sigma,l}^{p-2}\|\nabla f_{\sigma,l}\|^2 d\mu\\ & - C_1 p\int_{A_t(l)} W^{\sigma-1} f_{\sigma,l}^{p-1}\|\nabla H\|^2 d\mu\\ & + 2\sigma p\int_{A_t(l)}\|h\|^2 f_{\sigma,l}^p\,d\mu - 2C_2 p\int_{A_t(l)} f_{\sigma,l}^p d\mu\\ & + 2C_3 p\int_{A_t(l)} W^{\sigma-1}\cdot f_{\sigma,l}^{p-1}\,d\mu.\end{aligned} \tag{6.10}$$

For $p \geq 2$, we have the following estimate:

$$\frac{p(p-1)}{2}f_{\sigma,l}^{p-2}(\cdot,t)\|\nabla f_{\sigma,l}(\cdot,t)\|^2 \geq \|\nabla f_{\sigma,l}^{\frac{p}{2}}(\cdot,t)\|^2$$

on $A_t(l)$. Set $v_l := f_{\sigma,l}^{\frac{p}{2}}$. By using this estimate and discarding some terms in the right-hand side of (6.10), we have

$$\frac{\partial}{\partial t}\int_{A_t(l)} v_l^2\,d\mu + \int_{A_t(l)} \|\nabla v_l\|^2\,d\mu$$
$$\leq 2\sigma p\int_{A_t(l)} \|h\|^2 v_l^2\,d\mu + 2C_2 p\int_{A_t(l)} v_l^2\,d\mu + 2C_3 p\int_{A_t(l)} W^{\sigma-1} v_l^{\frac{2(p-1)}{p}}\,d\mu.$$

By integrating both sides of this inequality from 0 to any $t_0 \in [0, T)$, we have

$$\int_{A_{t_0}(l)} v_l^2\,d\mu + \int_0^{t_0}\left(\int_{A_t(l)} \|\nabla v_l\|^2\,d\mu\right)dt$$
$$\leq 2\sigma p\int_0^{t_0}\left(\int_{A_t(l)} \|h\|^2 v_l^2\,d\mu\right)dt + 2C_2 p\int_0^{t_0}\left(\int_{A_t(l)} v_l^2\,d\mu\right)dt$$
$$+ 2C_3 p\int_0^{t_0}\left(\int_{A_t(l)} W^{\sigma-1} v_l^{\frac{2(p-1)}{p}}\,d\mu\right)dt,$$

where we use $l \geq l_0$. By the arbitrariness of t_0, we have

$$\sup_{t\in[0,T)}\int_{A_t(l)} v_l^2 d\mu_t + \int_0^T\left(\int_{A_t(l)} \|\nabla v_l\|^2 d\mu\right)dt \tag{6.11}$$
$$\leq 4\sigma p\int_0^T\left(\int_{A_t(l)} \|h\|^2 v_l^2\,d\mu\right)dt + 4C_2 p\int_0^T\left(\int_{A_t(l)} v_l^2\,d\mu\right)dt$$
$$+ 4C_3 p\int_0^T\left(\int_{A_t(l)} W^{\sigma-1} v_l^{\frac{2(p-1)}{p}}\,d\mu\right)dt.$$

By applying the Sobolev inequality (6.8) to v_l and using the Hölder inequality, we can derive

$$\left(\int_{M_t} v_l^{2q} d\mu\right)^{\frac{1}{2q}} \leq \widehat{C}(m)\mathrm{Vol}(M_0)\left(\int_{M_t} \|\nabla v_l\|^2 d\mu\right)^{\frac{1}{2}} + \widehat{C}(m)\left(\int_{M_t} \|H\|^m d\mu\right)^{\frac{1}{m}}\left(\int_{M_t} v_l^{2q} d\mu\right)^{\frac{1}{2q}},$$

where $q := \frac{m}{2(m-1)}$. We want to take advantage of the good gradient term in the left-hand side of (6.11). By squaring both sides of this inequality and using $(a+b)^2 \leq 2(a^2+b^2)$, we obtain

$$\left(\int_M v_l^{2q} d\mu\right)^{\frac{1}{q}} \leq C_{10}\int_M \|\nabla v_l\|^2 d\mu + C_{11}\left(\int_M \|H\|^m d\mu\right)^{\frac{2}{m}}\left(\int_M v_l^{2q} d\mu\right)^{\frac{1}{q}},$$

where $C_{10} = 2\widehat{C}(m)^2\mathrm{Vol}(M_0)^2$ and $C_{11} = 2\widehat{C}(m)^2$. Since $f_\sigma(\cdot, t) \geq l$ on $A_t(l)$, it follows from Corollary 6.6 that

$$\left(\int_{A_t(l)} \|H\|^m\,d\mu\right)^{\frac{2}{m}} \leq m\left(\int_{A_t(l)} \|h\|^m \frac{f_\sigma^p}{l^p}\,d\mu\right)^{\frac{2}{m}} \tag{6.12}$$
$$= m\cdot l^{-\frac{2p}{m}}\left(\int_{A_t(l)} \|h\|^m f_\sigma^p\,d\mu\right)^{\frac{2}{m}}$$
$$\leq 2m^2\left(\frac{C_9(\sigma+\frac{m}{2p}, p)}{l}\right)^{\frac{2p}{m}}.$$

Fix $l_1 > l_0 > 0$, where we take l_1 as a sufficiently large number satisfying $2m^2 C_{11}(C_9(\sigma + \frac{m}{2p}, p)/l_1)^{\frac{2p}{m}} < 1$. In the sequel, let $l \geq l_1$. Then by absorbing the second term in the right-hand side of (6.12) into the left-hand side, we obtain

$$\hat{C}\left(\int_{M_t} v_l^{2q} d\mu\right)^{\frac{1}{q}} \leq \int_{M_t} \|\nabla v_l\|^2 d\mu, \tag{6.13}$$

where $\hat{C} := \frac{1-2m^2C_{11}(C_9(\sigma+m/2p,p)/l)^{2p/m}}{C_{10}}$. From (6.11) and (6.14), we obtain

$$\begin{aligned}
&\sup_{t\in[0,T)} \int_{A_t(l)} v_l^2 d\mu + \hat{C} \int_0^T \left(\int_{A_t(l)} v_l^{2q} d\mu\right)^{\frac{1}{q}} dt \\
&\leq 4\sigma p \int_0^T \left(\int_{A_t(l)} \|h\|^2 v_l^2\, d\mu\right) dt + 4C_2 p \int_0^T \left(\int_{A_t(l)} v_l^2\, d\mu\right) dt \\
&\quad + 4C_3 p \int_0^T \left(\int_{A_t(l)} W^{\sigma-1} v_l^{\frac{2(p-1)}{p}}\, d\mu\right) dt.
\end{aligned} \tag{6.14}$$

We need to estimate the second term of the left-hand side. According to the interpolation inequality for the L^p spaces, we have

$$\||\cdot\||_{L^{q_0}} \leq \||\cdot\||_{L^1}^{1-\theta} \||\cdot\||_{L^q}^{\theta},$$

where $q_0 := 2 - \frac{1}{q}$ and $\theta := \frac{q}{2q-1}$. By this interpolation inequality, we obtain

$$\left(\int_{A_t(l)} v_l^{2q_0} d\mu\right)^{\frac{1}{q_0}} \leq \left(\int_{A_t(l)} v_l^2 d\mu\right)^{\frac{q_0-1}{q_0}} \left\{\left(\int_{A_t(l)} v_l^{2q} d\mu\right)^{\frac{1}{q}}\right\}^{\frac{1}{q_0}},$$

that is,

$$\int_{A_t(l)} v_l^{2q_0} d\mu \leq \left(\int_{A_t(l)} v_l^2 d\mu\right)^{q_0-1} \left(\int_{A_t(l)} v_l^{2q} d\mu\right)^{\frac{1}{q}}.$$

By using this inequality and the Young's inequality, we can derive

$$\begin{aligned}
&\left(\int_0^T \left(\int_{A_t(l)} v_l^{2q_0} d\mu\right) dt\right)^{\frac{1}{q_0}} \\
&\leq \left(\int_0^T \left(\left(\int_{A_t(l)} v_l^2 d\mu\right)^{q_0-1} \left(\int_{A_t(l)} v_l^{2q} d\mu\right)^{\frac{1}{q}}\right) dt\right)^{\frac{1}{q_0}} \\
&\leq \sup_{t\in[0,T)} \left(\int_{A_t(l)} v_l^2 d\mu\right)^{\frac{q_0-1}{q_0}} \times \left(\int_0^T \left(\int_{A_t(l)} v_l^{2q} d\mu\right)^{\frac{1}{q}} dt\right)^{\frac{1}{q_0}} \\
&\leq \frac{q_0-1}{q_0} \cdot \sup_{t\in[0,T)} \left(\int_{A_t(l)} v_l^2 d\mu\right) + \frac{1}{q_0} \int_0^T \left(\int_{A_t(l)} v_l^{2q} d\mu\right)^{\frac{1}{q}} dt \\
&\leq \frac{q_0-1}{q_0} \left\{\sup_{t\in[0,T)} \left(\int_{A_t(l)} v_l^2 d\mu\right) + \int_0^T \left(\int_{A_t(l)} v_l^{2q} d\mu\right)^{\frac{1}{q}} dt\right\}.
\end{aligned} \tag{6.15}$$

We may assume that $\hat{C} < 1$ by taking l_1 as a larger positive number if necessary. From (6.15), (6.16) and $\hat{C} < 1$, we have

$$\frac{q_0\hat{C}}{q_0-1} \left(\int_0^T \left(\int_{A_t(l)} v_l^{2q_0} d\mu\right) dt\right)^{\frac{1}{q_0}} \tag{6.16}$$

$$\leq 4\sigma p \int_0^T \left(\int_{A_t(l)} \|h\|^2 v_l^2 \, d\mu \right) dt + 4C_2 p \int_0^T \left(\int_{A_t(l)} v_l^2 \, d\mu \right) dt$$
$$+ 4C_3 p \int_0^T \left(\int_{A_t(l)} W^{\sigma-1} v_l^{\frac{2(p-1)}{p}} \, d\mu \right) dt.$$

Set $|||A(l)||| := \int_0^T \left(\int_{A_t(l)} d\mu \right) dt$. By the Hölder inequality, we have

$$\left(\int_0^T \left(\int_{A_t(l)} v_l^{2q_0} d\mu \right) dt \right)^{\frac{1}{q_0}} \geq \left(\int_0^T \left(\int_{A_t(l)} v_l^2 d\mu \right) dt \right) \cdot |||A(l)|||^{\frac{1-q_0}{q_0}}. \tag{6.17}$$

By using (6.17), (6.18) and the Hölder inequality again, we obtain

$$\begin{aligned} &\int_0^T \left(\int_{A_t(l)} v_l^2 d\mu \right) dt \\ &\leq \frac{q_0 - 1}{q_0 \hat{C}} |||A(l)|||^{2-\frac{1}{q_0}-\frac{1}{r}} \\ &\times \left\{ 4\sigma p \left(\int_0^T \left(\int_{A_t(l)} \|h\|^{2r} v_l^{2r} \, d\mu \right) dt \right)^{\frac{1}{r}} + 4C_2 p \left(\int_0^T \left(\int_{A_t(l)} v_l^{2r} \, d\mu \right) dt \right)^{\frac{1}{r}} \right. \\ &\left. + 4C_3 p \left(\int_0^T \left(\int_{A_t(l)} W^{(\sigma-1)r} v_l^{\frac{2(p-1)r}{p}} \, d\mu \right) dt \right)^{\frac{1}{r}} \right\}, \end{aligned} \tag{6.18}$$

where r is a sufficiently large positive number so that $\gamma := 2 - \frac{1}{q_0} - \frac{1}{r} > 1$. According to Proposition 6.5 and Corollary 6.6, the second factor $\{\cdots\}$ of the right-hand side of this inequality can be bounded by a positive constant. Take any positive constants s_1 and s_2 with $s_2 > s_1 \geq l_1$. Clearly we have

$$\int_0^T \left(\int_{A_t(s_1)} v_{s_1}^2 d\mu \right) dt \geq \int_0^T \left(\int_{A_t(s_1)} (f_{\sigma,s_1} - f_{\sigma,s_2})^p d\mu \right) dt \geq (s_2 - s_1)^p |||A(s_2)|||.$$

This together with (6.19) derives

$$(s_2 - s_1)^p |||A(s_2)||| \leq \overline{C} |||A(s_1)|||^\gamma,$$

which holds for all $s_2 > s_1 \geq l_1$, where $\overline{C}$ is a positive constant which is independent of the choices of s_1 and s_2. It follows from Lemma 6.7 that $|||A(l_1 + d_0)||| = 0$, where $d_0 = \left(\overline{C} 2^{\frac{p\gamma}{\gamma-1}} |||A(l_1)|||^{\gamma-1} \right)^{1/p}$. This implies that

$$\sup_{t\in[0,T)} \max_M f_\sigma(\cdot, t) \leq l_1 + d_0.$$

This together with $f_\sigma \geq \frac{1}{\max\{\alpha, \|\beta\|\}^{1-\sigma}} \cdot \frac{\|\mathring{h}\|^2}{(\|H\|^2+1)^{1-\sigma}}$ implies that

$$\sup_{t\in[0,T)} \max_M \frac{\|\mathring{h}\|^2}{(\|H\| + 1)^{1-\sigma}} \leq (l_1 + d_0) \cdot \max\{\alpha, \|\beta\|\}^{1-\sigma}.$$

Thus the statement of Theorem 6.1 follows. □

Next we shall derive a gradient estimate for the mean curvature. This estimate is required to compare with the mean curvature oneself. First we prepare some technical inequalities. By the discussion similar to the proof of Lemma 5.6 in [18], we can derive the following technical inequality by using Lemmas 2.1, 4.2, 4.3 and Theorem 6.1.

Lemma 6.9. *The family* $\{||H_t||^2||\mathring{h}_t||^2\}_{t\in[0,T)}$ *satisfies*

$$\frac{\partial}{\partial t}\left(||H||^2||\mathring{h}||^2\right) \leq \Delta(||H||^2||\mathring{h}||^2) - C_4||H||^2||\nabla h||^2 + C_{12}||\nabla h||^2 + 2||H||^2||\mathring{h}||^2\{6||h||^2 - (\overline{\epsilon}\cdot 1 - 1)(7m + 4(d-1)k)c\}$$

for some positive constant C_{12}.

Proof. By Lemmas 2.1, 4.2 and 4.3,

$$\begin{aligned}(6.19)\quad &\frac{\partial}{\partial t}\left(||H||^2||\mathring{h}||^2\right)\\ &= \left(\frac{\partial}{\partial t}||H||^2\right)\cdot||\mathring{h}||^2 + ||H||^2\cdot\frac{\partial}{\partial t}||\mathring{h}||^2\\ &\leq \left(\Delta||H||^2 - 2||\nabla H||^2 + 2||H||^2(||h||^2 + \overline{r})\right)||\mathring{h}||^2\\ &\quad + ||H||^2\{\Delta||\mathring{h}||^2 - 2C_4||\nabla h||^2 + 4||h||^2||\mathring{h}||^2 - (\overline{\epsilon}\cdot 1 - 1)(7m + 4(d-1)k)c||\mathring{h}||^2\}\\ &= \Delta(||H||^2||\mathring{h}||^2) - 2\langle\nabla||H||^2, \nabla||\mathring{h}||^2\rangle - 2C_4||H||^2||\nabla h||^2 - 2||\mathring{h}||^2||\nabla H||^2\\ &\quad + ||\mathring{h}||^2||H||^2\{6||h||^2 - (\overline{\epsilon}\cdot 1 - 1)(7m + 4(d-1)k)c\}.\end{aligned}$$

Furthermore, by using Lemma 3.6, we have

$$\begin{aligned}&-2\left\langle\nabla||H||^2, \nabla||\mathring{h}||^2\right\rangle\\ &= -8||H||\cdot||\mathring{h}||\left\langle\nabla||H||, \nabla||\mathring{h}||\right\rangle\\ &\leq 8||H||\cdot||\mathring{h}||\cdot||\nabla H||\cdot||\nabla h||\\ &\leq 24\left(\frac{m+2}{2(10-2d)}\right)^{\frac{1}{2}}||H||\cdot||\mathring{h}||\cdot||\nabla h||^2.\end{aligned}$$

By using Theorem 6.1 and Young inequality, we can show that there exists a positive constant C_{12} satisfying

$$\begin{aligned}24\left(\frac{m+2}{2(10-2d)}\right)^{\frac{1}{2}}||H||||\nabla h||^2||\mathring{h}|| &\leq 24\left(\frac{m+2}{2(10-2d)}\right)^{\frac{1}{2}}||H||\cdot||\nabla h||^2\sqrt{C_0}(||H||^2+1)^{\frac{1-\sigma_0}{2}}\\ &\leq C_4||H||^2||\nabla h||^2 + C_{12}||\nabla h||^2.\end{aligned}$$

These relations together with (6.20) implies the desired inequality. □

Define a function g by

$$g := ||H||^2||\mathring{h}||^2 + \frac{1}{2}\left(\frac{C_{12}}{C_4} + 1\right)||\mathring{h}||^2. \tag{6.20}$$

By using Lemma 4.2, 4.3, 6.9 and $||H||^2 \leq m||h||^2$, we obtain

$$\frac{\partial}{\partial t}g \leq \Delta g - C_4(||H||^2+1)||\nabla h||^2 + 2||h||^2\cdot||\mathring{h}||^2(6m||h||^2 + C_{13}) + C_{14}||\mathring{h}||^2, \tag{6.21}$$

where $C_{13} := \frac{C_{12}}{C_4} + 1 - m(\overline{\epsilon} \cdot 1 - 1)(7m + 4(d-1)k)c$ and $C_{14} := -\frac{1}{2}(\frac{C_{12}}{C_4} + 1)(\overline{\epsilon} \cdot 1 - 1)(7m + 4(d-1)k)c$.

Proposition 6.10. *For any sufficiently small positive number η, there exists a constant $C_\eta > 0$ depending only on η such that the inequality*

$$\|\nabla H\|^2 \leq \eta \|H\|^4 + C_\eta$$

holds for all $t \in [0, T)$.

Proof. Set $f := \|\nabla H\|^2 + \frac{1}{C_4}(C_5 + 1)g - \eta \|H\|^4$, where η is a sufficiently small positive number. From Lemmas 4.2 – 4.4 and (6.22), we can derive

$$\begin{aligned}(6.22) \qquad & \frac{\partial}{\partial t} f - \Delta f \\ & \leq -(\|H\|^2 + 1)\|\nabla h\|^2 + \frac{C_{14}}{C_4}(C_5 + 1)\|\mathring{h}\|^2 + \frac{2}{C_4}(C_5 + 1)\|\mathring{h}\|^2 \|h\|^2 (6m\|h\|^2 + C_{13}) \\ & \quad - \eta \left(\frac{4}{m} \|H\|^6 - 12\|H\|^2 \|\nabla H\|^2 + 8(\overline{\epsilon} \cdot 1 - 1)mc\|H\|^4 \right).\end{aligned}$$

Since $\|\nabla h\|^2 \geq \frac{2(10-d)}{9(m+2)} \|\nabla H\|^2$ by Lemma 3.6, we have

$$-(\|H\|^2 + 1)\|\nabla h\|^2 + 12\eta \|H\|^2 \|\nabla H\|^2 \leq \left(-\|H\|^2 - 1 + \frac{108(m+2)}{2(10-2d)} \eta \|H\|^2 \right) \|\nabla h\|^2.$$

Hence we have

$$-(\|H\|^2 + 1)\|\nabla h\|^2 + 12\eta \|H\|^2 \|\nabla H\|^2 < 0$$

for a sufficiently small positive number η. Denote by R the reaction terms in (6.23), that is,

$$\begin{aligned}R := & \frac{2}{C_4}(C_5 + 1)\|\mathring{h}\|^2 \|h\|^2 (6m\|h\|^2 + C_{13}) + \frac{C_{14}}{C_4}(C_5 + 1)\|\mathring{h}\|^2 \\ & - \eta \left(\frac{4}{m} \|H\|^6 + 8(\overline{\epsilon} \cdot 1 - 1)mc\|H\|^4 \right).\end{aligned}$$

By using the pinching condition $(*_{m-1+\varepsilon, b_\varepsilon})$, we have

$$\begin{aligned}R \leq & \frac{2}{C_4}(C_5 + 1)\|\mathring{h}\|^2 (a_\varepsilon \|H\|^2 + b_\varepsilon)(6ma_\varepsilon \|H\|^2 + C_{13}) \\ & + \frac{C_{14}}{C_4}(C_5 + 1)\|\mathring{h}\|^2 - \frac{4\eta}{m}\|H\|^6 - 8(\overline{\epsilon} \cdot 1 - 1)mc\eta \|H\|^4.\end{aligned}$$

Hence, from Theorem 6.1 and the Young inequality, we obtain

$$\begin{aligned}R \leq & \frac{2}{C_4}(C_5 + 1)C_0(\|H\|^2 + 1)^{1-\sigma}(a_\varepsilon \|H\|^2 + b_\varepsilon)(6ma_\varepsilon \|H\|^2 + C_{13}) \\ & + \frac{C_{14}}{C_4}(C_5 + 1)C_0(\|H\|^2 + 1)^{1-\sigma} - \frac{4\eta}{m}\|H\|^6 - 8(\overline{\epsilon} \cdot 1 - 1)mc\eta \|H\|^4,\end{aligned}$$

where μ is any positive constant. Thus we have

$$(6.23) \qquad R \leq \left(-\frac{4\eta}{m} + \widehat{C_6}(\eta, \mu) \right) \|H\|^6 + \widehat{C_4}(\eta, \mu)\|H\|^4 + \widehat{C_2}(\eta, \mu)\|H\|^2 + \widehat{C_0}(\eta, \mu),$$

where $\widehat{C_i}(\eta,\mu)$ ($i = 0,2,4,6$) are constants depending only on η and μ. Since $\widehat{C_6}(\eta,\mu) \to 0$ as $\mu \to 0$, we can find such positive number μ_η as $\widehat{C_6}(\eta,\mu_\eta) < \frac{4\eta}{m}$. Set $\widehat{C_6}(\eta) := \widehat{C_6}(\eta,\mu_\eta)$. Then the coefficient $(-\frac{4\eta}{m} + \widehat{C_6}(\eta))$ of the term of the highest degree in the right-hand side (which is regarded as a polynomial with variable $\|H\|$) of (6.24) is negative if we take $\eta > 0$ sufficiently small. Hence, if $\|H\|$ is sufficiently large, then we have $R < 0$. Therefore, we can find a positive constant $C_{15}(\eta)$ depending only on η such that $R < C_{15}(\eta)$ always holds even if $\|H\|$ take any value. Hence we have

$$\frac{\partial}{\partial t} f \le \Delta f + C_{15}(\eta).$$

This together with $T < \infty$ implies that there exists a constant C_η depending only on η such that $f \le C_\eta$. Then, from the definition of f, we obtain

$$\|\nabla H\|^2 \le \|\nabla H\|^2 + \frac{1}{C_4}(C_5+1)g \le \eta\|H\|^4 + C_\eta. \qquad \square$$

Next we recall the Myers theorem.

Theorem 6.11 ([17]). *Let (M,g) be an m-dimensional complete connected Riemannian manifold. If its Ricci curvature Ric satisfies*

$$Ric \ge (m-1)\kappa g$$

for some positive constant κ, then the diameter of (M,g) is smaller than or equal to $\frac{\pi}{\sqrt{\kappa}}$.

By using Theorem 6.1 and Proposition 6.10, we shall prove that, if time is sufficiently close to T, then the sectional curvature $K_t(: G_2(M_t) \to \mathbb{R})$ of M_t is positive.

Proposition 6.12. *For any $\mu \in (0, \min\{\frac{1}{2\alpha m(m-1)}, \frac{1}{\beta}\})$ and any positive constant $\hat{b}$, there exists a constant $\theta(\mu,\hat{b}) \in [0,T)$ satisfying the following two conditions:*
(I) *for all $t \in [\theta(\mu,\hat{b}),T)$, $K_t > \mu W_t$ holds;*
(II) *for all $t \in [\theta(\mu,\hat{b}),T)$, $\|h_t\|^2 < \frac{1}{m-1}\|H_t\|^2 - \hat{b}$ holds.*

Proof. Fix an orthonormal basis of type (I) with the additional condition that $A_{e_{m+1}}(e_i) = \lambda_i e_i$ ($i = 1,\cdots,m$), where A $(= A_t)$ denotes the shape operator of M_t and $\lambda_1 \le \cdots \le \lambda_m$. According to the Gauss equation, we have

$$K_{ij} = \overline{K}_{ij} + \sum_{\alpha=m+1}^{dn}\left(h^\alpha_{ii}h^\alpha_{jj} - (h^\alpha_{ij})^2\right), \tag{6.24}$$

where K_{ij} denotes the sectional curvature $K_t(e_i,e_j)$ of M_t for the plane spanned by the orthonormal system (e_i,e_j), and $\overline{K}_{ij}$ is the sectional curvature of $\widetilde{M}$ for the same plane, which is regarded as an element of the Grassmann bundle $G_2(\widetilde{M})$ of $\widetilde{M}$ consisting of the 2-planes.

First we consider the case of $\widetilde{M} = \mathbb{F}P^n(4c)$. From (6.25) and $\overline{K} \ge 1$, we have

$$\begin{aligned} K_{ij} &\ge 1 + \lambda_i\lambda_j + \sum_{\alpha=m+2}^{dn}(\mathring{h}^\alpha_{ii}\mathring{h}^\alpha_{jj} - (\mathring{h}^\alpha_{ij})^2) \\ &\ge 1 + \frac{1}{2}\left(\frac{1}{m-1}\|H\|^2 - \|h_1\|^2\right) - \|\mathring{h}_-\|^2 \end{aligned}$$

$$\geq 1 + \frac{1}{2(m-1)}\|H\|^2 - \frac{1}{2}\left(\|\mathring{h}_1\|^2 + \frac{1}{m}\|H\|^2\right) - \|\mathring{h}_-\|^2$$
$$\geq 1 + \frac{1}{2m(m-1)}\|H\|^2 - \|\mathring{h}\|^2.$$

Furthermore, it follows from Theorem 6.1 that

$$K_{ij} \geq 1 + \frac{1}{2m(m-1)}\|H\|^2 - C_0(\|H\|^2 + 1)^{1-\sigma}. \tag{6.25}$$

Also, it follows from Theorem 6.1 that

$$\|h\|^2 - \frac{1}{m-1}\|H\|^2 + \hat{b} \leq \|\mathring{h}\|^2 - \frac{1}{m(m-1)}\|H\|^2 + \hat{b} \leq C_0(\|H\|^2 + 1)^{1-\sigma} - \frac{1}{m(m-1)}\|H\|^2 + \hat{b}. \tag{6.26}$$

On the other hand, it is shown that there exists a positive constant $C_*(\mu, \hat{b})$ depending only on μ and $\hat{b}$ such that, if $\|H\| \geq C_*(\mu, \hat{b})$, then

$$1 + \frac{1}{2m(m-1)}\|H\|^2 - C_0(\|H\|^2 + 1)^{1-\sigma} - \mu W > 0 \tag{6.27}$$

and

$$C_0(\|H\| + 1)^{1-\sigma} - \frac{1}{m(m-1)}\|H\|^2 + \hat{b} < 0 \tag{6.28}$$

because the coefficient $\frac{1}{2m(m-1)} - \mu\alpha$ (resp. $-\frac{1}{m(m-1)}$) of the term of the highest degree (with respect to $\|H\|$) of the right-hand side of (6.28) (resp. (6.29)) is positive (resp. negative). Hence, if $\|H\| \geq C_*(\mu, \hat{b})$, then we have $K > \mu W$ and $\|h\|^2 < \frac{1}{m-1}\|H\|^2 - \hat{b}$. According to Proposition 6.10, there exists a constant C_η with $\|\nabla H\|^2 \leq \eta\|H\|^4 + C_\eta$. Set $\|H\|_{\max}(t) := \max_M \|H_t\|$. Since $T < \infty$, we have $\lim_{t\to T} \|H\|_{\max}(t) = \infty$. Hence there exists a positive constant $\theta(\mu, \hat{b})$ such that, for all $t \in [\theta(\mu, \hat{b}), T)$, $\|H\|_{\max}(t) \geq \max\left\{\left(\frac{C_\eta}{\eta}\right)^{\frac{1}{4}}, 2C^*(\mu, \hat{b})\right\}$ holds. By using Proposition 6.10, we can show that $\|\nabla^t\|H_t\|\| \leq \|\nabla^t H_t\| \leq \sqrt{2\eta}\|H\|_{\max}(t)^2$ holds on M_t for all $t \in [\theta(\mu, \hat{b}), T)$. Fix $t_0 \in [\theta(\eta), T)$ and let x_0 be a point of M_{t_0} attaining the maximum $\|H\|_{\max}(t_0)$. Then, along any geodesic γ in M_{t_0} starting from x_0, we have

$$\|(H_{t_0})_{\gamma(s)}\| \geq \|H\|_{\max}(t_0) - \sqrt{2\eta}\|H\|_{\max}(t_0)^2 s \geq \frac{1}{2}\|H\|_{\max}(t_0)$$

for all $s \in [0, (2\sqrt{2\eta}\|H\|_{\max}(t_0))^{-1})$. For the simplicity, set $r_{t_0} := (2\sqrt{2\eta}\|H\|_{\max}(t_0))^{-1}$. Then we have $\|H_{t_0}\| > \frac{1}{2}\|H\|_{\max}(t_0) \geq C_*(\mu, \hat{b})$ holds on the geodesic ball $B_{x_0}(r_{t_0})$ of radius r_{t_0} centered at x_0 in M_{t_0}. Therefore, $K_{t_0} > \mu W_{t_0}$ and $\|h_{t_0}\|^2 < \frac{1}{m-1}\|H_{t_0}\|^2 - \hat{b}$ hold on $B_{x_0}(r_{t_0})$. Furthermore, it follows that

$$K_{t_0} > \mu W_{t_0} > \mu\alpha\|H_{t_0}\|^2 \geq \frac{\mu\alpha}{4}\|H\|_{\max}(t_0)^2 \tag{6.29}$$

holds on $B_{x_0}(r_{t_0})$. Hence we see that

$$Ric_{t_0} \geq (m-1)\frac{\mu\alpha}{4} \cdot \|H\|_{\max}(t_0)^2 g_{t_0} \tag{6.30}$$

holds on $B_{x_0}(r_{t_0})$. Hence, by using Myers theorem, we obtain that the diameter of $B_{x_0}(r_{t_0})$

is smaller than or equal to $\frac{2\pi}{\sqrt{\mu\alpha}\|H\|_{\max}(t_0)}$. Here we note that, even if $B_{x_0}(r_{t_0})$ is not complete, we can apply Myers theorem to $B_{x_0}(r_{t_0})$ according to its proof. By taking η as a sufficiently small positive number, we may assume $\frac{2\pi}{\sqrt{\mu\alpha}\|H\|_{\max}(t_0)} < r_{t_0}$. This implies that $M_{t_0} = B_{x_0}(r_{t_0})$. Thus $K_{t_0} > \mu W_{t_0}$ and $\|h_{t_0}\|^2 < \frac{1}{m-1}\|H_{t_0}\|^2 - \hat{b}$ hold on M_{t_0}. Therefore the statement of this proposition follows from the arbitrariness of t_0.

Next we consider the case of $\widetilde{M} = \mathbb{F}H^n(-4c)$. From (6.25), $\overline{K} \geq -4$ and Theorem 6.1, we can derive

$$K_{ij} \geq -4 + \frac{1}{2m(m-1)}\|H\|^2 - C_0(\|H\|^2 + 1)^{1-\sigma}$$

and (6.27). On the other hand, it is shown that there exists a positive constant $C_*(\mu, \hat{b})$ depending only on μ and $\hat{b}$ such that, if $\|H\| \geq C_*(\mu, \hat{b})$, then

$$-4 + \frac{1}{2m(m-1)}\|H\|^2 - C_0(\|H\|^2 + 1)^{1-\sigma} - \mu W > 0 \tag{6.31}$$

and (6.29) hold. Hence, if $\|H\| \geq C_*(\mu, \hat{b})$, then we have $K > \mu W$ and $\|h\|^2 < \frac{1}{m-1}\|H_t\|^2 - \hat{b}$. Hence we can derive the statement of this proposition by using Myers theorem as in the above proof of the case of $\widetilde{M} = \mathbb{F}P^n(4c)$. □

Next we shall recall the main result of [14].

Theorem 6.13. *For any Riemannian manifold with bounded curvature (for example, Riemannian homogeneous spaces), there exists a positive constant b_0 such that, if an m-dimensional submanifold in the Riemannian manifold satisfies*

$$\|h\|^2 < \frac{1}{m-1}\|H\|^2 - b_0, \tag{6.32}$$

then the submanifold collapses to a round point in finite time along the mean curvature flow.

By using these results, we prove the collapse to a round point in the statement of Theorems 1.1–1.5.

Proof of the collapse in Theorems 1.1–1.5. The pinching conditions $(*_{m-1,b})$ in Theorems 1.1–1.5 are weaker than (6.33), but it follows from Proposition 6.12 that (6.33) holds for all t sufficiently close to T. Therefore the collapse to a round point in the statements of Theorems 1.1–1.5 is derived from Theorem 6.13. □

7. Proof of the convergence to a totally geodesic submanifold

In this section, we shall prove the convergence to a totally geodesic submanifold ($T = \infty$-case) in the statement of Theorem 1.2 and the finiteness of the maximal time in the statement of Theorem 1.1. Throughout this section, we assume that $T = \infty$.

Proposition 7.1. *There exist positive constants C_0 and δ_0 depending only on the initial manifold M such that*

$$\|\mathring{h}_t\|^2 \leq C_0(\|H_t\|^2 + 1)e^{-\delta_0 t}$$

holds for any time $t \in [0, \infty)$.

Proof. According to Proposition 4.1 with $\sigma = 0$, we have

$$\frac{\partial f_0}{\partial t} \leq \Delta f_0 + \frac{2\alpha}{W}\langle \nabla f_0, \nabla \|H\|^2\rangle - 2C_1\frac{\|\nabla H\|^2}{W} + 2C_2 f_0 + \frac{2C_3}{W}.$$

Since $\overline{M} = \mathbb{F}P^n(4c)$, we have $C_2 < 0$ and $C_3 = 0$. Also, we have $C_1 > 0$. Hence we have

$$\frac{\partial f_0}{\partial t} \leq \Delta f_0 + \frac{2\alpha}{W}\langle \nabla f_0, \nabla \|H\|^2\rangle + 2C_2 f_0.$$

From this evolution inequality, we can derive $f_0(\cdot, t) \leq \widehat{C}e^{2C_2 t}$ $(0 \leq t < \infty)$ for some $\widehat{C}$ depending only on M_0. Since $C_2 < 0$, the statement of this proposition follows. □

From this estimate, we can prove that the intrinsic sectional curvature K_t of the evolving submanifold M_t is positive for sufficiently large time as in the case of finite maximal time.

Proposition 7.2. *There exist positive constants μ and θ such that, for any time $t \in [\theta, \infty)$, $K_t > \mu W_t$ (> 0) holds.*

Proof. As stated in the proof of Proposition 6.12, we have

$$K_{ij} \geq 1 + \frac{1}{2m(m-1)}\|H\|^2 - \|\mathring{h}\|^2.$$

Furthermore, according to Proposition 7.1, we have

$$K_{ij} \geq 1 + \frac{1}{2m(m-1)}\|H_t\|^2 - C_0(\|H_t\|^2 + 1)e^{-\delta_0 t}.$$

From this inequality, we can derive the statement of this proposition by the discussion similar to the proof of Proposition 6.12. □

According to Lemma 6.9, we have

$$\frac{\partial}{\partial t}\|H\|^2\|\mathring{h}\|^2 \leq \Delta(\|H\|^2\|\mathring{h}\|^2) - C_4\|H\|^2\|\nabla h\|^2 + C_{12}\|\nabla h\|^2 + 12\|H\|^2\|\mathring{h}\|^2\|h\|^2. \tag{7.1}$$

Now we consider the function g defined in (6.21). By using Lemma 4.2 and (7.1), we can repeat the computations of the previous sections to conclude that the inequality (6.22) holds also in this case. We shall give a gradient estimate for the mean curvature.

Proposition 7.3. *For any sufficiently small positive constant η, there exists a positive constant C_η depending only on η such that the inequality*

$$\|\nabla^t H_t\|^2 \leq (\eta\|H_t\|^4 + C_\eta)e^{-\delta_0 t/2}$$

holds for all $t \in [0, \infty)$.

Proof. Define f by

$$f = e^{\frac{\delta_0 t}{2}}\left(\|\nabla^t H_t\|^2 + \frac{1}{C_4}(C_5 + \delta_0 m)g\right) - \eta\|H_t\|^4,$$

where η is a sufficiently small positive number. Then, by the same discussion as the proof of Proposition 6.10, it follows from Lemma 3.6, 4.2, 4.4, Proposition 7.1 and (6.22) that $\frac{\partial}{\partial t}f \leq \Delta f + C_{16}(\eta)e^{-\frac{\delta_0 t}{4}}$ holds for some positive constant $C_{16}(\eta)$ depending only on η. From this evolution inequality, we can derive that there exists a constant C_η depending only on η such that $f \leq C_\eta$ holds for all time. From the definition of f, we obtain the desired inequality. □

Next we shall show the uniform boundedness of the mean curvature.

Lemma 7.4. *If $T = \infty$, then $\{||H_t||^2\}_{t\in[0,\infty)}$ is uniform bounded.*

Proof. Let b_0 be the positive constant in Theorem 6.13. From Proposition 7.1, we have

$$\begin{aligned} ||h_t||^2 - \frac{1}{m-1}||H_t||^2 + b_0 &= ||\mathring{h}_t||^2 - \frac{1}{m(m-1)}||H_t||^2 + b_0 \\ &\leq C_0(||H_t||^2 + 1)e^{-\delta_0 t} - \frac{1}{m(m-1)}||H_t||^2 + b_0. \end{aligned}$$

Notice that the right-hand side is negative if t and $||H_t||^2$ are sufficiently large. Suppose that $\{||H_t||^2\}_{t\in[0,\infty)}$ is not uniform bounded. Then there exists a sequence $\{t_i\}_{i=1}^\infty$ satisfying $\lim_{i\to\infty} t_i = \infty$ and $\lim_{i\to\infty} ||H||_{\max}(t_i) = \infty$. By using Propositions 7.1, 7.3 and Myers theorem as in the proof of Proposition 6.12, we can show that there exists i_0 such that $||h_{t_{i_0}}||^2 - \frac{1}{m-1}||H_{t_{i_0}}||^2 + b_0 < 0$ holds on the whole of $M_{t_{i_0}}$. According to Theorem 6.13, the mean curvature flow starting from $M_{t_{i_0}}$ collapses to a round point in finite time. Thus, so does the mean curvature flow starting from M_0. This contradicts $T = \infty$. Therefore $\{||H_t||^2\}_{t\in[0,\infty)}$ is uniform bounded. □

Proof of $T < \infty$ in Theorem 1.1 and the convergence in Theorem 1.2. Let M be as in Theorem 1.1 or 1.2. We assume that $T = \infty$. In this case, since $\{||H_t||^2\}_{t\in[0,\infty)}$ is uniform bounded by Lemma 7.4, it follows from Propositions 7.1 and 7.3 that there exists a positive constant C satisfying

$$||\mathring{h}||^2 \leq Ce^{-\delta_0 t} \quad \text{and} \quad ||\nabla H||^2 \leq Ce^{-\frac{\delta_0 t}{2}}.$$

As in the proof of Proposition 6.12, it follows from Proposition 7.2 that there exists a positive constant $\overline{C}$ such that $\mathrm{Ric}_t \geq \overline{C} g_t$ holds for all $t \in [0,\infty)$. Hence, by using Myers theorem, we can derive $\sup_{t\in[0,\infty)} d_t < \infty$, where d_t is the diameter of M_t. Set $d^* := \sup_{t\in[0,\infty)} d_t$. By using this fact and integrating the above second estimate along geodesics, we obtain

$$||H||_{\max}(t) - ||H||_{\min}(t) \leq d^* \sqrt{C} e^{-\frac{\delta_0 t}{4}}. \tag{7.2}$$

Suppose that $||H||_{\min}(t_1) \neq 0$ for some time t_1. Then, from (2.5) and (3.5), we obtain

$$\frac{\partial ||H||^2}{\partial t} \geq \Delta ||H||^2 - 2||\nabla H||^2 + \frac{1}{m}||H||^4.$$

From this evolution inequality, we can show that $||H||^2$ blows up in finite time by a standard comparison argument. This contradicts $T = \infty$. Hence we know that $||H||_{\min}(t) = 0$ for all time t. Therefore, from (7.2), we obtain

$$||H||_{\max}(t) \leq d^* \sqrt{C} e^{-\frac{\delta_0 t}{4}} \quad (0 \leq t < \infty).$$

This implies

$$\|h_t\|^2 = \|\mathring{h}_t\|^2 + \frac{1}{m}\|H_t\|^2 \leq \widehat{C}e^{-\frac{\delta_0 t}{2}} \quad (0 \leq t < \infty)$$

for some positive constant $\widehat{C}$. Furthermore, since the induced metrics g_t on M_t satisfies the evolution equation $\frac{\partial g}{\partial t} = -2\|H\|h$, we have

$$\begin{aligned}\int_0^\infty \left\|\frac{\partial g}{\partial t}\right\| dt &\leq 2\int_0^\infty \|H\|\|h\| dt \leq 2\sqrt{m}\int_0^\infty \|h\|^2 dt \\ &\leq 2\sqrt{m}\widehat{C}\int_0^\infty e^{-\frac{\delta_0 t}{2}} dt = \frac{4\sqrt{m}\widehat{C}}{\delta_0}.\end{aligned}$$

So we can apply a result by Hamilton [5, Lemma 14.2] to show that g_t converges uniformly to a continuous metric g_∞ as $t \to \infty$. By using the interpolation inequalities as in Section 10 of [7], we can show that the exponential decay for $\|h\|^2$ gives the exponential decay for the norms $\|\nabla^k h\|$ of k-th covariant derivatives of h for any k. From this fact, we can derive that the flow M_t converges to a (C^∞) totally geodesic submanifold M_∞ in the C^∞-topology as $t \to \infty$. However, if M is as in Theorem 1.1 (hence M is a hypersurface), then this case cannot happen because there exists no totally geodesic hypersurface in $\mathbb{F}P^n(4c)$. □

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