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INTEGRAL REGION CHOICE PROBLEMS ON LINK DIAGRAMS

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Abstract
Shimizu introduced a region crossing change unknotting operation for knot diagrams. As extensions, two integral region choice problems were proposed and the existences of solutions of the problems were shown for all non-trivial knot diagrams by Ahara and Suzuki, and Harada. We relate both integral region choice problems with an Alexander numbering for regions of a link diagram,and give alternative proofs of the existences of solutions for knot diagrams. We also discuss the problems on link diagrams. For each of the problems on the diagram of a two-component link, we give a necessary and sufficient condition that there exists a solution.

1. Introduction

A *link* is a closed 1-manifold smoothly embedded in the 3-space $\mathbb{R}^3$ or in the 3-sphere S^3 and a *knot* is a link with one component. A link in the 3-space is presented as the natural projection image on the 2-plane $\mathbb{R}^2$ where the singular points are transverse double points with over/under information. This presentation is called a *link diagram* or a *diagram* of the link. A *diagram* of a link in the 3-sphere $S^3 = \mathbb{R}^3 \cup \{\infty\}$ is given on the 2-sphere $S^2 = \mathbb{R}^2 \cup \{\infty\}$ similarly. For each link diagram, a connected component of the complement of the projection image on $\mathbb{R}^2$ or S^2 is called a *region*.

In [10], Shimizu defined a *region crossing change* at a region for a diagram to be the crossing change at all the crossings on the boundary of the region as an unknotting operation for a knot diagram, which was proposed by Kengo Kishimoto. For example in Fig.1, the left diagram is changed to the right diagram, choosing the region marked with $*$ as illustrated on the middle and changing the three crossings on the boundary of the marked region. In [3, 4], Cheng and Gao gave a necessary and sufficient condition that a region crossing change is an unknotting operation on a link diagram.

It is known that a region crossing change can be interpreted as follows. We call a diagram ignored over/under information a *projection*. Let each crossing of the given projection be equipped with a score 0 or 1 modulo 2. We choose a region of the projection. Then the scores of all the crossings on its boundary are increased by 1 modulo 2. For example, the region crossing change illustrated on Fig.1 is interpreted as Fig.2. Shimizu showed that the scores of all the crossings on any knot diagram become 0 by some choices of regions. Cheng and Gao induced a $\mathbb{Z}_2$-homomorphism from region crossing changes on link diagrams. In [6, 7], Hashizume studied structures of their $\mathbb{Z}_2$-homomorphism.

As an extension of a region crossing change to an integral range, Ahara and Suzuki pro-

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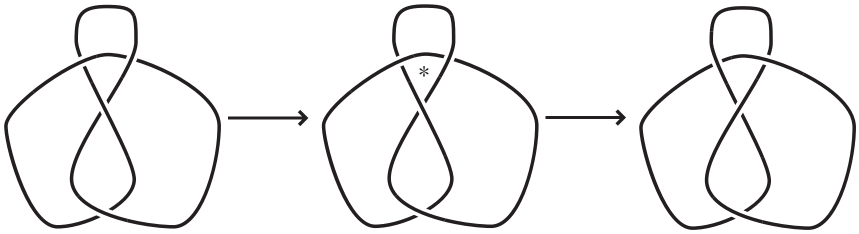


Fig. 1. An example of a region crossing change.

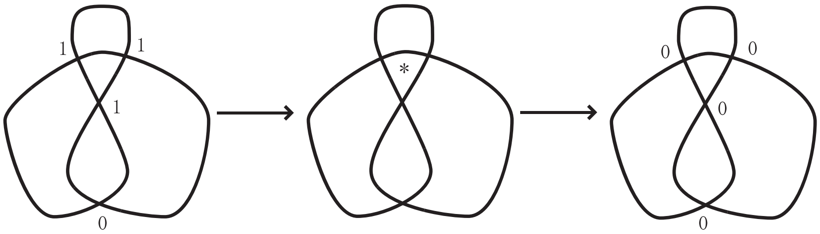


Fig. 2. An example of an interpreted region crossing change.

posed an integral region choice problem and showed the existence of a solution of this problem for all knot projections in [1]. Let each crossing of the given projection be equipped with an integral score. We choose a region of the projection and assign an integer u to it. Then the scores of all the crossings on its boundary are increased by u. For example in Fig.3, the scores of the crossings on the left projection are changed to the right, assigning integers to regions as the middle projection; $1 \mapsto 1+0+2+(-1)+(-2) = 0$, $-1 \mapsto -1+0+0+(-1)+2 = 0$, $3 \mapsto 3+0+0+(-2)+(-1) = 0$, and $2 \mapsto 2+0+0+0+(-2) = 0$. Ahara and Suzuki showed that the scores of all the crossings on any knot projection become 0 by some choices of regions and some assignments of the integers to them. We shall call their problem a *definite integral region choice problem*. In Section 3, we state their result exactly.

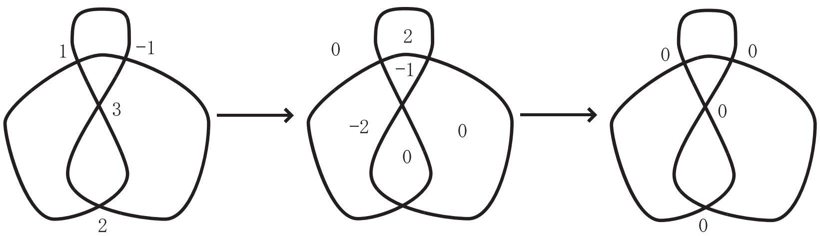


Fig. 3. An example of a definite integral region choice problem.

By an argument similar to that due to Ahara and Suzuki, Harada showed in his master thesis [5] that there exists a solution of an alternating integral region choice problem for all knot diagrams, which was suggested by Yasuyoshi Yonezawa. Let each crossing of the given diagram be equipped with an integral score. We choose a region of the projection and assign an integer u to it. Then the score of each crossing on its boundary is changed as follows. If the region lies on the right when we move along the underpass to the crossing, illustrated as  or , the score of this crossing is increased by u. If the region lies on the left when we move along the underpass to the crossing, illustrated as  or , the score of this crossing is decreased by u. For example in Fig.4, the scores of the crossings on the left diagram are changed to the right, assigning integers to regions as the middle diagram; $1 \mapsto 1+0-(-2)+(-1)-2 = 0$, $-1 \mapsto -1+0-0+(-1)-(-2) = 0$, $3 \mapsto 3-0+0-2+(-1) = 0$, and $2 \mapsto 2+0-0+0-2 = 0$. Harada showed that the scores of all the crossings on any knot diagram become 0 by some choices of regions and some assignments of the integers to them. We shall call this proposed problem as another extension of a region crossing change an *alternating integral region choice problem*. In Section 4, we state his result exactly.

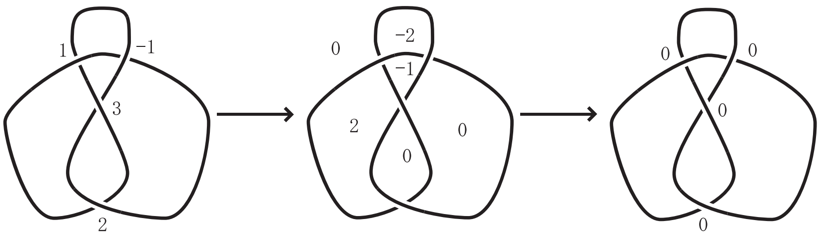


Fig. 4. An example of an alternating integral region choice problem.

In [1, 5], Ahara, Suzuki and Harada reduced the above integral region choice problems to systems of linear equations, as explained in Sections 3 and 4 in this article, and they showed the existences of solutions for non-trivial knot diagrams. We show that an Alexander numbering for regions of a link diagram is a solution of the system of homogeneous linear equations reduced from an alternating integral region choice problem in Section 5. By this result, we give alternative proofs of the existences of solutions of both alternating and definite integral region choice problems for all non-trivial knot diagrams in Sections 6 and 7.

In [10], Shimizu used checkerboard colorings to regions of knot diagrams for showing that a region crossing change is an unknotting operation. Cheng and Gao [4], and Hashizume [6, 7] also used checkerboard colorings for discussing region crossing changes on link diagrams. An Alexander numbering is an integral extension of a checkerboard coloring, as mentioned in Section 2. In this article, we use Alexander numberings to discuss the integral region choice problems on link diagrams, which are integral extensions of region crossing changes.

In Sections 8 and 9, we determine the ranks for the coefficient matrices of the systems of linear equations reduced from the integral region choice problems, applying the arguments in the original proofs of the solvability of integral region choice problems on knot diagrams in [1, 5] to link diagrams. Then we obtain an extension of the result about the incidence matrix due to Cheng and Gao [4].

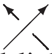
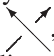
In Section 10, we give a basis of the space of solutions of the system of homogeneous linear equations reduced from each of integral region choice problems on link diagrams. In Section 11, we give necessary and sufficient conditions that there exist solutions of integral region choice problems on the connected diagram of a two-component link. These results are extensions of some of the results about region crossing changes on link diagrams due to Cheng and Gao [4], and Hashizume [6, 7].

2. Preliminary

By the Jordan curve theorem, any short arc without a crossing on a link diagram lies on the intersection of just two boundaries of regions. Each crossing is touched by at most four regions. If the number of the regions touching the fixed crossing is less than four, it must be three and the pair of the corners of the same region touching the crossing are not adjacent each other around the crossing. This fact is also shown from the Jordan curve theorem. In this case, such a crossing is called a *reducible* crossing. If a link diagram have a reducible crossing, it is called a *reducible* diagram. Otherwise, it is called an *irreducible* diagram.

Lemma 2.1 (cf. [1, 7]). *Let D be a link diagram or projection. If D has d connected components and n crossings, then it has $n + d + 1$ regions.*

Proof. It is shown by the Euler formula. □

On an oriented link diagram D, we say that we *splice* at a crossing x if we change the diagram D around the crossing x,  or , to  and obtain the new link diagram D_x. This local move between oriented link diagrams is called a *splicing* or *smoothing* at the crossing x. The change from D_x to D is called an *unsplicing* at x. In this article, the local moves  to  and  to  among oriented link projections are also called a *splicing* and an *unsplicing* respectively.

In [2], Alexander assigned an integer index to each region of an oriented link diagram or projection, so that for any oriented arc on the link diagram, an index of the left region adjacent to the arc is larger that of the right by one. Such an index is called an *Alexander index*, and this assignment of the indexes is called an *Alexander indexing* or an *Alexander numbering*. In [8], Kauffman also defined an Alexander indexing for an oriented link projection and show that there exist an Alexander indexing for any projection, though an index of the right region is assigned larger than that of the left by one for any oriented arc on the link projection. We note that these assignments may begin with an arbitrary region assigned arbitrary integer index for the given oriented link diagram or projection.

It is known that we can shade regions for any link projection so that each two regions adjacent by an arc on the projection are shaded and unshaded, and such shading is call a *checkerboard coloring*. For any oriented link diagram or projection, if we shade only the regions assigned odd number by an Alexander numbering, then we obtain a checkerboard coloring. If we reverse the orientation of some link components fixing a region and its index, we obtain a new Alexander numbering and the same checkerboard coloring. In this article, we shall call an Alexander numbering modulo 2 a *checkerboard coloring*.

3. A definite integral region choice problem

Let D be a link diagram or projection with d connected components and n crossings $x_1, \cdots, x_n$, $n \geq 1$. We note that d is not greater than the number of the link components. Let $R_1, \cdots, R_{n+d+1}$ be the regions of D. In [1], Ahara and Suzuki induced two region choice matrices $A_{d1}(D)$ and $A_{d2}(D)$ with n rows and $n + d + 1$ columns as follows, where they denoted them by $A_1(D)$ and $A_2(D)$. We determine each element $a_{ij}^{(d1)}$ by

$$a_{ij}^{(d1)} = \begin{cases} 1 & \text{if } x_i \in \partial R_j, \\ 0 & \text{if } x_i \notin \partial R_j. \end{cases}$$

The *region choice matrix of the single counting rule* for D is the matrix $A_{d1}(D)$ with the element $a_{ij}^{(d1)}$ on the i-th row and the j-th column. We determine each element $a_{ij}^{(d2)}$ by

$$a_{ij}^{(d2)} = \begin{cases} 2 & \text{if } R_j \text{ touches } x_i \text{ twice}, \\ a_{ij}^{(d1)} & \text{otherwise}. \end{cases}$$

The *region choice matrix of the double counting rule* for D is the matrix $A_{d2}(D)$ with the element $a_{ij}^{(d2)}$ on the i-th row and the j-th column. We shall call these two region choice matrices by the *definite region choice matrices*.

Using the definite region choice matrices, the definite integral region choice problem and the existence of solutions for it are stated as follows.

Theorem 3.1 ([1]). *Let D be a knot diagram or projection with n crossings $x_1, \cdots, x_n$, $n \geq 1$. Let $R_1, \cdots, R_{n+2}$ be the regions of D.*

(1) *Let $A_{d1}(D)$ be the definite region choice matrix of the single counting rule for D. For any $\mathbf{c} \in \mathbb{Z}^n$, there exists a solution $\mathbf{u} \in \mathbb{Z}^{n+2}$ such that $A_{d1}(D)\mathbf{u} + \mathbf{c} = \mathbf{0}$.*

(2) *Let $A_{d2}(D)$ be the definite region choice matrix of the double counting rule for D. For any $\mathbf{c} \in \mathbb{Z}^n$, there exists a solution $\mathbf{u} \in \mathbb{Z}^{n+2}$ such that $A_{d2}(D)\mathbf{u} + \mathbf{c} = \mathbf{0}$.*

Example 3.2. Let D be the knot projection given in Fig.3. Under certain orders of crossings and regions, we have

$$A_{d1}(D) = A_{d2}(D) = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

Fig.3 implies the equation

$$A_{di}(D) \begin{pmatrix} 2 \\ -1 \\ -2 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \\ 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

holds for $i = 1, 2$.

If we transpose the incidence matrices induced by Cheng and Gao [4] and Hashizume [6], it is the same as the definite region choice matrix of the single counting rule modulo 2 up to permutations of rows and columns.

4. An alternating integral region choice problem

Let D be a link diagram with d connected components and n crossings $x_1, \cdots, x_n$, $n \geq 1$. Let $R_1, \cdots, R_{n+d+1}$ be the regions of D. In [5], Harada induced two region choice matrices

$A_{a1}(D)$ and $A_{a2}(D)$ with n rows and $n + d + 1$ columns as follows, where he denoted them by $B_1(D)$ and $B_2(D)$.

We determine each elements $a^{(a1)}_{ij}$ as follows. We define $a^{(a1)}_{ij} = 1$, if the region R_j lies on the right when we move along the underpass to the crossing x_i, illustrated as  or . We define $a^{(a1)}_{ij} = -1$, if R_j lies on the left when we move along the underpass to x_i, illustrated as  or . If x_i does not lie on ∂R_j, we define $a^{(a1)}_{ij} = 0$. The *alternating region choice matrix of the single counting rule* for D is the matrix $A_{a1}(D)$ with the element $a^{(a1)}_{ij}$ on the i-th row and the j-th column. We determine each element $a^{(a2)}_{ij}$ by

$$a^{(a2)}_{ij} = \begin{cases} 2a^{(a1)}_{ij} & \text{if } R_j \text{ touches } x_i \text{ twice as} \end{cases}$$  or ,

$$\qquad\qquad a^{(a1)}_{ij} \quad \text{otherwise.}$$

The *alternating region choice matrix of the double counting rule* for D is the matrix $A_{a2}(D)$ with the element $a^{(a2)}_{ij}$ on the i-th row and the j-th column.

We compare the definitions of $a^{(d1)}_{ij}, a^{(d2)}_{ij}, a^{(a1)}_{ij}, a^{(a2)}_{ij}$ on Table 1, where the region R_j includes the corners marked with $*$ but does not include the unmarked corners around the crossing x_i.

Table 1. $a^{(d1)}_{ij}, a^{(d2)}_{ij}, a^{(a1)}_{ij}, a^{(a2)}_{ij}$.

x_i and Rj		 or 	 or 		otherwise
$a^{(d1)}_{ij}$	1	1	1	1	0
$a^{(d2)}_{ij}$	2	1	1	2	0
$a^{(a1)}_{ij}$	1	1	−1	−1	0
$a^{(a2)}_{ij}$	2	1	−1	−2	0

Using the alternating region choice matrices, the alternating integral region choice problem and the existence of solutions for it are stated as follows.

Theorem 4.1 ([5]). *Let D be a knot diagram with n crossings $x_1, \cdots, x_n$, $n \geq 1$. Let $R_1, \cdots, R_{n+2}$ be the regions of D.*

(1) *Let $A_{a1}(D)$ be the alternating region choice matrix of the single counting rule for D. For any $\mathbf{c} \in \mathbb{Z}^n$, there exists a solution $\mathbf{u} \in \mathbb{Z}^{n+2}$ such that $A_{a1}(D)\mathbf{u} + \mathbf{c} = \mathbf{0}$.*

(2) *Let $A_{a2}(D)$ be the alternating region choice matrix of the double counting rule for D. For any $\mathbf{c} \in \mathbb{Z}^n$, there exists a solution $\mathbf{u} \in \mathbb{Z}^{n+2}$ such that $A_{a2}(D)\mathbf{u} + \mathbf{c} = \mathbf{0}$.*

Example 4.2. Let D be the knot diagram given in Fig.4. Under certain orders of crossings and regions, we have

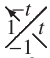
$$A_{a1}(D) = A_{a2}(D) = \begin{pmatrix} -1 & 1 & -1 & 0 & 0 & 1 \\ -1 & 1 & 0 & 0 & -1 & 1 \\ 0 & 1 & -1 & 1 & -1 & 0 \\ 0 & 0 & -1 & 1 & -1 & 1 \end{pmatrix}.$$

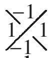
Fig.4 implies the equation

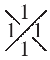
$$A_{ai}(D)\begin{pmatrix}-2\\-1\\2\\0\\0\\0\end{pmatrix}+\begin{pmatrix}1\\-1\\3\\2\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\end{pmatrix}$$

holds for $i = 1, 2$.

REMARK 4.3. If we transpose the incidence matrix induced by Cheng and Gao [4] and Hashizume [6], it is the same as the alternating region choice matrix of the single counting rule modulo 2 up to permutations of rows and columns.

REMARK 4.4. In this article, we reverse signs of the elements in the alternating region choice matrices defined by Harada [5], since our alternating region choice matrix of the double counting rule coincides with the Alexander matrix defined in [2] if we substitute 1 for the variable. In [8], Kauffman illustrated the definition of the Alexander matrix as a crossing with labeled corners 

. In his terms, our alternating region choice matrix and the definite region choice matrix of the double counting rule are denoted by 

and 

respectively. In [9], Kawauchi indicated that the transposed incidence matrix is the same as the Alexander matrix substituted 1 modulo 2, and that the solvability of the original region choice problem on knot diagrams is induced by the fact the Alexander polynomial substituted 1 becomes 1 for any knot. This fact also implies that Theorem 4.1 (2).

We give more examples to compare definite and alternating region choice matrices of the single counting rule and of the double counting rule.

EXAMPLE 4.5. Let D be the link diagram given as the split sum of the l copies of the knot diagram with only one crossing such that one region touches all crossings twice. The diagram D represents a trivial l-component link. Under certain orders of crossings and regions, we obtain

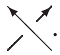
$$A_{d1}(D)=\begin{pmatrix}1&1&1&0&0&0&\dots&0\\1&0&0&1&1&0&\dots&0\\\vdots&&&&&\ddots&&\\1&0&0&0&0&\dots&1&1\end{pmatrix},$$

$$A_{d2}(D)=\begin{pmatrix}2&1&1&0&0&0&\dots&0\\2&0&0&1&1&0&\dots&0\\\vdots&&&&&\ddots&&\\2&0&0&0&0&\dots&1&1\end{pmatrix},$$

and

$$A_{a1}(D) = \begin{pmatrix} -\varepsilon_1 & \varepsilon_1 & \varepsilon_1 & 0 & 0 & 0 & \dots & 0 \\ -\varepsilon_2 & 0 & 0 & \varepsilon_2 & \varepsilon_2 & 0 & \dots & 0 \\ \vdots & & & & & \ddots & & \\ -\varepsilon_l & 0 & 0 & 0 & 0 & \dots & \varepsilon_l & \varepsilon_l \end{pmatrix},$$

$$A_{a2}(D) = \begin{pmatrix} -2\varepsilon_1 & \varepsilon_1 & \varepsilon_1 & 0 & 0 & 0 & \dots & 0 \\ -2\varepsilon_2 & 0 & 0 & \varepsilon_2 & \varepsilon_2 & 0 & \dots & 0 \\ \vdots & & & & & \ddots & & \\ -2\varepsilon_l & 0 & 0 & 0 & 0 & \dots & \varepsilon_l & \varepsilon_l \end{pmatrix},$$

where $\varepsilon_i = 1$ if the i-th crossing is positive , $\varepsilon_i = -1$ if it is negative . Each of these matrices has l rows and $2l + 1$ columns.

EXAMPLE 4.6. Let D be the link diagram given as the split sum of the knot diagram with only one crossing and the $l - 1$ copies of the trivial knot daigaram ○. The diagram D represents a trivial l-component link. On D with certain orders of regions, we obtain

$$A_{d1}(D) = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & \dots & 0 \end{pmatrix},$$

$$A_{d2}(D) = \begin{pmatrix} 2 & 1 & 1 & 0 & 0 & 0 & \dots & 0 \end{pmatrix},$$

and

$$A_{a1}(D) = \begin{pmatrix} -\varepsilon & \varepsilon & \varepsilon & 0 & 0 & 0 & \dots & 0 \end{pmatrix},$$

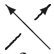
$$A_{a2}(D) = \begin{pmatrix} -2\varepsilon & \varepsilon & \varepsilon & 0 & 0 & 0 & \dots & 0 \end{pmatrix},$$

where $\varepsilon = 1$ if the crossing is positive, otherwise $\varepsilon = -1$, and the number of 0 appearing on each matrix is $l - 1$.

5. Kernel solutions from Alexander numberings

Let D be a link diagram with d connected components and n crossings $x_1, \cdots, x_n$, $n \geq 1$. Let $R_1, \cdots, R_{n+d+1}$ be the regions of D. Let all crossings be equipped with 0. Then the integral region choice problems induce $\mathbb{Z}$-homomorphisms. We denote by $\Phi_{di}(D) : \mathbb{Z}^{n+d+1} \to \mathbb{Z}^n$ and $\Phi_{ai}(D) : \mathbb{Z}^{n+d+1} \to \mathbb{Z}^n$ the induced homomorphisms with representation matrices $A_{di}(D)$ and $A_{ai}(D)$ respectively, $i = 1, 2$. We call a vector $\mathbf{u} \in \mathbb{Z}^{n+d+1}$ with $A_{d1}(D)\mathbf{u} = \mathbf{0}$ (resp. $A_{d2}(D)\mathbf{u} = \mathbf{0}$) a *kernel solution* for the definite region choice matrix of the single (resp. double) counting rule, similarly to that defined to knot projections in [1]. We call a vector $\mathbf{u} \in \mathbb{Z}^{n+d+1}$ with $A_{a1}(D)\mathbf{u} = \mathbf{0}$ (resp. $A_{a2}(D)\mathbf{u} = \mathbf{0}$) a *kernel solution* for the alternating region choice matrix of the single (resp. double) counting rule, similarly to that defined to knot diagrams in [5].

Lemma 5.1. *On any link diagram with at least one crossing, an Alexander numbering for an arbitrary orientation gives a kernel solution for an alternating region choice matrix of the double counting rule.*

Proof. On the given oriented link diagram D, we fix an Alexander numbering for it. We take an arbitrary crossing x of D. We may assume that x lies as  or  in D. We suppose that the index of the right region of x is $p \in \mathbb{Z}$. Then the index of the left region of x is $p+2$ and the rest regions touching x are $p+1$. We have $p-(p+1)+(p+2)-(p-1)=0$ and $-p+(p+1)-(p+2)+(p-1)=0$. Then the alternating region choice obtained from the Alexander numbering does not change the scores of the crossings. □

Let D be an oriented link diagram with ordered link components, and D_i be a sub-diagram of D representing i-th link component, $i=1,\cdots,l$. We fix a sub-diagram D_i. We ignore the diagrams of link components other D_i, and take an Alexander numbering. Each region R of the diagram D is a subset of one region S of the diagram D_i. Let a_S be the integer assigned to S by this Alexander numbering. We assign the integer a_S to the region R and denote it by u_R. We call this assignment of the integers to the region $\{u_R\}_R$ a *componentwise Alexander numbering associated with D_i*. Fig.5 gives an example of a pair of componentwise Alexander numberings on a 2-component link diagram.

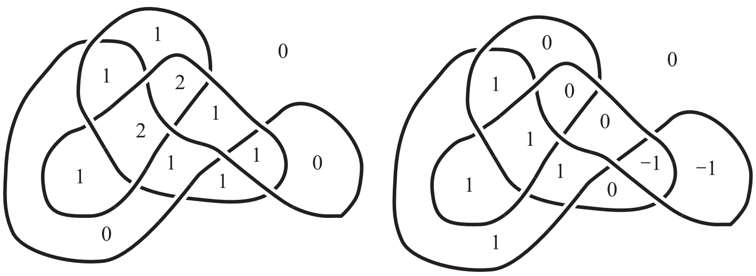


Fig.5. Componentwise Alexander numberings.

Lemma 5.2. *On any oriented link diagram with at least one crossing, each componentwise Alexander numbering gives a kernel solution for an alternating region choice matrix of the double counting rule.*

Proof. Let D be an oriented link diagram with at least one crossing and ordered link components, and D_i be a sub-diagram of D representing i-th link component, $i=1,\cdots,l$. We fix a sub-diagram D_i. We take a componentwise Alexander numbering associated with D_i. Let q be a crossing of D and we denote the four corners touching q by $C_q^1, C_q^2, C_q^3, C_q^4$ clockwise, and the regions on D including C_q^j by R_q^j, $j=1,2,3,4$. If q is a crossing of D_i, the regions $R_q^1, R_q^2, R_q^3, R_q^4$ are assigned integers r_1, r_2, r_3, r_4 with $r_1-r_2+r_3-r_4=0$ by Lemma 5.1. If q is a crossing of an arc of D_i and an arc of other components, we may assume that R_q^1 and R_q^2 are subsets of a region S of the diagram D_i, and that R_q^3 and R_q^4 are subsets of a region S' of the diagram D_i. Then the regions $R_q^1, R_q^2, R_q^3, R_q^4$ are assigned integers r_1, r_1, r_3, r_3 with $r_1-r_3=\pm 1$, and we have $r_1-r_1+r_3-r_3=0$. If q is a crossing not included in D_i, the regions $R_q^1, R_q^2, R_q^3, R_q^4$ are subsets of a region of the diagram D_i. Then they are assigned same integer r_1, and we have $r_1-r_1+r_1-r_1=0$.

Therefore the componentwise Alexander numbering associated with D_i becomes a kernel solution for the alternating integral region choice problem of double counting rule. □

We can obtain kernel solutions for the definite region choice matrix from kernel solutions for the alternating region choice matrix and a fixed checkerboard coloring.

Lemma 5.3. *For a given link diagram, we fix a checkerboard coloring. We take a kernel solution for an alternating region choice matrix of the double counting rule. For each region R, let c_R and u_R be the integers assigned by the checkerboard coloring and the kernel solution respectively. Assigning the integer $(-1)^{c_R}u_R$ to each region R, we obtain a kernel solution for a definite region choice matrix of the double counting rule.*

Proof. For a crossing x of the diagram, We denote the four corners touching x by C_1, C_2, C_3, C_4 clockwise, and the regions including C_j by R_j, $j = 1, 2, 3, 4$. Then we have $\pm(u_{R_1} - u_{R_2} + u_{R_3} - u_{R_4}) = 0$. We may assume $c_{R_1} = 0$. Then the equalities $c_{R_2} = 1, c_{R_3} = 0, c_{R_4} = 1$ hold. Hence we have

$$\begin{aligned} & (-1)^{c_{R_1}} u_{R_1} + (-1)^{c_{R_2}} u_{R_2} + (-1)^{c_{R_3}} u_{R_3} + (-1)^{c_{R_4}} u_{R_4} \\ = \; & u_{R_1} - u_{R_2} + u_{R_3} - u_{R_4} \\ = \; & 0. \end{aligned}$$

□

Lemma 5.4. *For a given link diagram, we fix a checkerboard coloring. We take a kernel solution for an alternating region choice matrix of the single counting rule. For each region R, let c_R and u_R be the integers assigned by the checkerboard coloring and the kernel solution respectively. Assigning the integer $(-1)^{c_R}u_R$ to each region R, we obtain a kernel solution for a definite region choice matrix of the single counting rule.*

Proof. For a crossing x of the diagram, We denote the four corners touching x by C_1, C_2, C_3, C_4 clockwise, and the regions including C_j by R_j, $j = 1, 2, 3, 4$. We may assume $c_{R_1} = 0$. Then the equalitiess $c_{R_2} = 1, c_{R_3} = 0, c_{R_4} = 1$ hold.

If x is not reducible, then R_j's are different each other and we have $\pm(u_{R_1} - u_{R_2} + u_{R_3} - u_{R_4}) = 0$. Hence we have

$$\begin{aligned} & (-1)^{c_{R_1}} u_{R_1} + (-1)^{c_{R_2}} u_{R_2} + (-1)^{c_{R_3}} u_{R_3} + (-1)^{c_{R_4}} u_{R_4} \\ = \; & u_{R_1} - u_{R_2} + u_{R_3} - u_{R_4} \\ = \; & 0. \end{aligned}$$

We suppose that x is reducible. Then there exists just one pair of R_j's coinciding each other. If R_1 coincides with R_3, the equality $\pm(u_{R_1} - u_{R_2} - u_{R_4}) = 0$ holds. Hence we have

$$(-1)^{c_{R_1}} u_{R_1} + (-1)^{c_{R_2}} u_{R_2} + (-1)^{c_{R_4}} u_{R_4} = u_{R_1} - u_{R_2} - u_{R_4} = 0.$$

Otherwise, R_2 coincides with R_4 and we have $\pm(u_{R_1} - u_{R_2} + u_{R_3}) = 0$. Hence we have

$$(-1)^{c_{R_1}} u_{R_1} + (-1)^{c_{R_2}} u_{R_2} + (-1)^{c_{R_3}} u_{R_3} = u_{R_1} - u_{R_2} + u_{R_3} = 0.$$

□

Similarly, we can obtain kernel solutions for the alternating region choice matrix from kernel solutions for the definite region choice matrix and a fixed checkerboard coloring.

Lemma 5.5. *For a given link diagram, we fix a checkerboard coloring. We take a kernel solution for a definite region choice matrix of the double (resp. single) counting rule. For each region R, let c_R and u_R be the integers assigned by the checkerboard coloring and the kernel solution respectively. Assigning the integer $(-1)^{c_R} u_R$ to each region R, we obtain a kernel solution for an alternating region choice matrix of the double (resp. single) counting rule.*

6. Solutions of the alternating integral region choice problem on knot diagrams

In this section, we give an alternative proof of Theorem 4.1.

First, we observe the alternating integral region choice problem of the double counting rule.

Lemma 6.1. *Let D be a link diagram with n crossings, $n \geq 1$. We fix an arc γ in the link diagram D, and let R and R' be two regions which are the both sides of the arc γ. Then there exists a kernel solution $\mathbf{u}$ for $A_{a2}(D)$ such that the components of $\mathbf{u}$ corresponding to R and R' are 0 and 1 respectively.*

Proof. We orient the given diagram D arbitrarily. We take the Alexander numbering $\mathbf{u}$ for the oriented link diagram D, such that the region R is assigned 0. Then the index of the region R' is 1 or -1. By Lemma 5.1, this assignment $\mathbf{u}$ gives a kernel solution for $A_{a2}(D)$. If R' is assigned -1, we multiply all components of the kernel solution $\mathbf{u}$ by -1. □

Alternative proof of Lemma 6.1. We orient the given diagram D arbitrarily. Let D_γ be a sub-diagram of D including the arc γ and representing a link component. We take the componentwise Alexander numbering associated with D_γ, $\mathbf{u}_\gamma$, such that the region R is assigned 0. Then the index of the region R' is 1 or -1. By Lemma 5.2, this assignment $\mathbf{u}_\gamma$ gives a kernel solution for $A_{a2}(D)$. If R' is assigned -1, we multiply all components of the kernel solution $\mathbf{u}_\gamma$ by -1. □

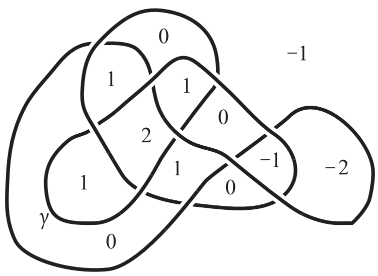


Fig. 6. A kernel solution for an alternating region choice matrix.

Fig.6 gives an example of a link diagram with a kernel solution for an alternating region choice matrix such that two regions adjacent to the arc γ are assigned 0 and 1. This kernel solution is obtained from an Alexander numbering. The componentwise Alexander numbering illustrated on the left of Fig.5 also gives a kernel solution such that two regions adjacent to the arc γ are assigned 0 and 1.

Remark 6.2. In [5], Harada proved Lemma 6.1 for a knot diagram, showing that Reidemeister moves and crossing changes preserve the existence of the kernel solution, and that the knot diagram with only one crossing has a kernel solution. His argument is similar to that due to Ahara and Suzuki [1] for Lemma 7.1.

The following theorem also has been proved by Harada [5] for a knot diagram using Lemma 6.1. We give a proof using Lemma 5.2 instead.

Theorem 6.3. *Let D be a link diagram with d connected components and n crossings, $n \geq 1$. We suppose that there exists a crossing x of D of arcs in same link component. Then there exist $\mathbf{v}_x \in \mathbb{Z}^{n+d+1}$ such that any components of $A_{a2}(D)\mathbf{v}_x$ are 0 but the component of $A_{a2}(D)\mathbf{v}_x$ to x is 1.*

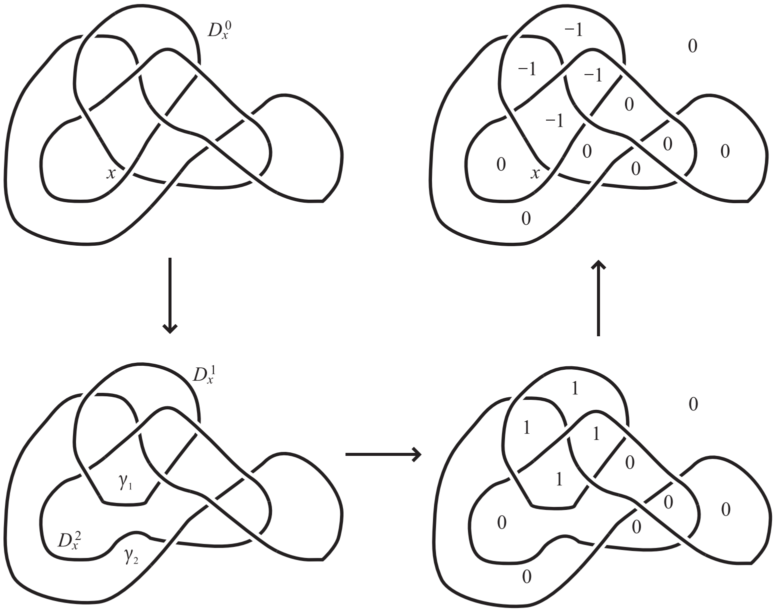


Fig.7. Finding $\mathbf{v}_x$ such that any components of $A_{a2}(D)\mathbf{v}_x$ are 0 but the component of $A_{a2}(D)\mathbf{v}_x$ to x is 1.

Proof. The argument is similar to that for knot diagrams due to Harada [5]. We orient D arbitrarily. We splice D at x. On Fig.7, this splicing is illustrated as the transformation from top left to bottom left. The sub-diagram D_x^0 of the link component including x splits to the diagrams of two link components D_x^1 and D_x^2. We note D_x^1 and D_x^2 may intersects each other as link projections. Let γ_i be an oriented arc in D_x^i appearing after the splice at x for each $i = 1, 2$. We may assume that γ_1 lies on the left of γ_2. For the diagram $(D \setminus D_x^0) \cup D_x^1 \cup D_x^2$, we take the componentwise Alexander numbering associated with D_x^1 such that the right and left regions of γ_1 are assigned 0 and 1 respectively. We denote this assignment of the indexes by $\mathbf{u}'$. On Fig.7, $\mathbf{u}'$ is illustrated on bottom right. By Lemma 5.2, $\mathbf{u}'$ gives a kernel solution of $A_{a2}((D \setminus D_x^0) \cup D_x^1 \cup D_x^2)$ if the spliced diagram has at least one crossing. We unsplice $(D \setminus D_x^0) \cup D_x^1 \cup D_x^2$ to D at x. Let $\varepsilon = 1$ if x is a positive crossing, otherwise $\varepsilon = -1$. We assign the same integers to all regions of D as the components of $\varepsilon\mathbf{u}'$, where the integer assigned to the region between γ_1 and γ_2 is assigned to the two regions splitting at x. On Fig.7, this unsplicing is illustrated as the transformation from bottom right to top right, where the crossing x is negative and we have $\varepsilon = -1$. Then we obtain the desired $\mathbf{v}_x \in \mathbb{Z}^{n+d+1}$. □

Theorem 6.3 implies Theorem 4.1 (2), that is the existence of a solution of an alternating integral region choice problem of the double counting rule for a knot diagram, by the same argument as that due to Harada [5].

Proof of Theorem 4.1 (2). Applying Theorem 6.3 for each crossing x_i, there exist $\mathbf{v}_i \in \mathbb{Z}^{n+d+1}$ such that any components of $A_{a2}(D)\mathbf{v}_i$ are 0 but the i-th component of $A_{a2}(D)\mathbf{v}_i$ is 1, $i = 1, 2, \cdots, n$. Let c_i be the i-th component of $\mathbf{c}$. If we take $\mathbf{u} = -\sum_{i=1}^{n} c_i\mathbf{v}_i$, then we have $A_{a2}(D)\mathbf{u} + \mathbf{c} = \mathbf{0}$. □

Next, we observe the alternating integral region choice problem of the single counting rule. The following lemma has been proved by Harada [5] for knot diagrams.

Lemma 6.4. *Let D be a link diagram with n crossings, $n \geq 1$. We fix an arc γ in the link diagram D, and let R and R' be two regions which are the both sides of the arc γ. We take two arbitrary integers a and b. Then there exists a kernel solution $\mathbf{u}$ for $A_{a1}(D)$ such that the components of $\mathbf{u}$ corresponding to R and R' are a and b respectively.*

Proof. The argument is the same as that for knot diagrams due to Harada [5]. His argument is similar to that due to Ahara and Suzuki [1] for Lemma 7.4.

We use an induction on the number of reducible crossings.

If the given link diagram D is irreducible, the matrices $A_{a1}(D)$ and $A_{a2}(D)$ coincide. We apply Lemma 6.1 to the pairs R, R' and R', R in order to a kernel solutions $\mathbf{u}'$ and $\mathbf{u}''$ respectively. Then the components of $\mathbf{u}'$ corresponding to R and R' are 0 and 1 respectively, and the components of $\mathbf{u}''$ corresponding to R and R' are 1 and 0 respectively. Therefore $\mathbf{u} = a\mathbf{u}'' + b\mathbf{u}'$ is the desired kernel solution for $A_{a1}(D)$ on the irreducible diagram D.

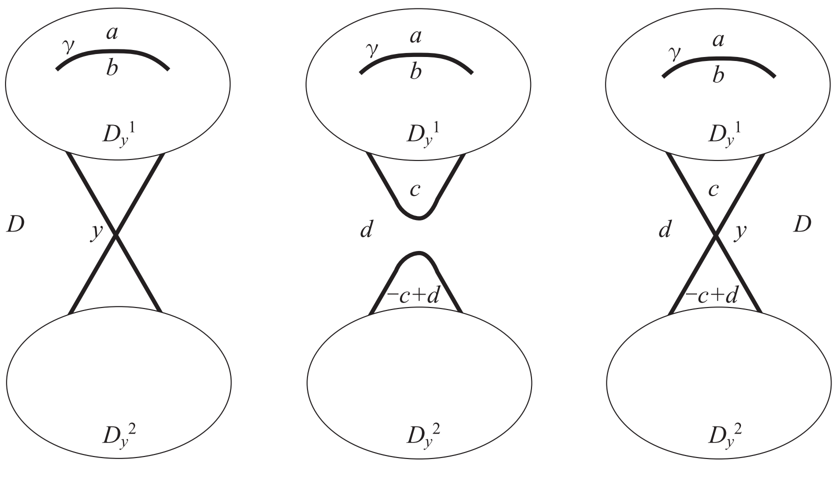


Fig. 8. Splicing at y and obtaining a kernel solution.

We assume that there exists a desired kernel solution if the number of reducible crossings is less than k. We suppose that the link diagram D has k reducible crossings. We take a reducible crossing y, and orient D arbitrarily. We splice D at y. The diagram D splits to the disjoint link diagrams D^1_y and D^2_y. Each of them has less reducible crossings than k. We may assume that the given arc γ lies on D^1_y. On Fig.8, we obtain the middle diagram splicing the reducible crossing y of the left diagram, where we omit over/under information for y and the orientation of D. Let γ_i be an arc in D^i_y appearing after the splice at y for each $i = 1, 2$. We denote the region between the arcs γ_1 and γ_2 by R^0_y, and another region adjacent to γ_j by R^j_y, $j = 1, 2$. We ignore D^2_y. If D^1_y has no crossing, we assign a and b to the regions including R and R' respectively, and 0 to other regions of D^1_y where we may assign arbitrary integers. If D^1_y has at least one crossing, we apply the assumption of induction to D^1_y and γ. Then we obtain a kernel solution for $A_{a1}(D^1_y)$ whose components corresponding to the regions including R and R' are a and b respectively. Let $c \in \mathbb{Z}$ assigned to R^1_y and $d \in \mathbb{Z}$ to the region of D^1_y including R^0_y. On the middle of Fig.8, we write c and d, though we omit γ_i and R^j_y. We assign the integer $-c + d$ to the region R^2_y on $D^1_y \cup D^2_y$ as the middle of Fig.8. We ignore D^1_y. If the diagram D^2_y has no crossing, we assign 0 to the regions of D^2_y including neither R^0_y nor R^2_y, though we may assign arbitrary integers. Otherwise we apply the assumption of the induction to D^2_y and γ_2, then we obtain a kernel solution for $A_{a1}(D^2_y)$ whose components corresponding to R^2_y and the region including R^0_y are $-c + d$ and d respectively. Let $\tilde{R}$ be a region of the diagram $D^1_y \cup D^2_y$. If $\tilde{R}$ is R^0_y, we assign d to $\tilde{R} = R^0_y$. Otherwise $\tilde{R}$ coincides with one of regions of D^1_y or D^2_y, then we assign to $\tilde{R}$ same integer as the region of D^1_y or D^2_y. Therefore we obtain a kernel solution $\mathbf{u}'$ of $A_{a1}(D^1_y \cup D^2_y)$. We unsplice $D^1_y \cup D^2_y$ at y. We assign the same integer as either $\mathbf{u}'$ to all regions of D, where the region touching y twice is assigned d, as illustrated on the right of Fig.8. Then we obtain the desired kernel solution for $A_{a1}(D)$ since we have $c - d + (-c + d) = 0$. □

The following lemma also has been proved by Harada [5] for knot diagrams. He proved it as a corollary to Lemma 6.4: the region R^2_y in the proof of Lemma 6.4 is assigned $-c + d + \varepsilon$ instead of $-c + d$, where $\varepsilon = 1$ if y is positive, otherwise $\varepsilon = -1$. We give an alternative proof.

Lemma 6.5. *Let D be a link diagram with d connected components and n crossings, $n \geq 1$. We suppose that there exists a reducible crossing y of D. Then there exist $\mathbf{v}_y \in \mathbb{Z}^{n+d+1}$ such that any components of $A_{a1}(D)\mathbf{v}_y$ are 0 but the component of $A_{a1}(D)\mathbf{v}_y$ to y is 1.*

Proof. We orient D arbitrarily. We splice D at the reducible crossing y. The diagram D splits to the two disjoint link diagrams D^1_y and D^2_y. Let γ_i be an arc in D^i_y appearing after the splice at y for each $i = 1, 2$. We denote the region between the arcs γ_1 and γ_2 by R^0_y, and another region adjacent to γ_j by R^j_y, $j = 1, 2$. We assign integers $0, 0, 1$ to R^0_y, R^1_y, R^2_y respectively, as illustrated on the middle of Fig.9 or Fig.10. If D^2_y has at least one crossing, we apply Lemma 6.4 to D^2_y and γ_2, in order to obtain a kernel solution for $A_{a1}(D^2_y)$ such that the region including R^0_y is assigned 0 and R^2_y is assigned 1. If D^2_y has no crossing, we assign 0 to the regions of D^2_y including neither R^0_y nor R^2_y, though we may assign arbitrary integers. We assign 0 to all regions of D^1_y: it gives the trivial kernel solution for $A_{a1}(D^1_y)$ if D^1_y has at least one crossing. Each of the other regions of $D^1_y \cup D^2_y$ than R^0_y, R^1_y, R^2_y is a region of D^1_y or D^2_y, then it has been assigned an integer. Therefore we obtain a kernel solution

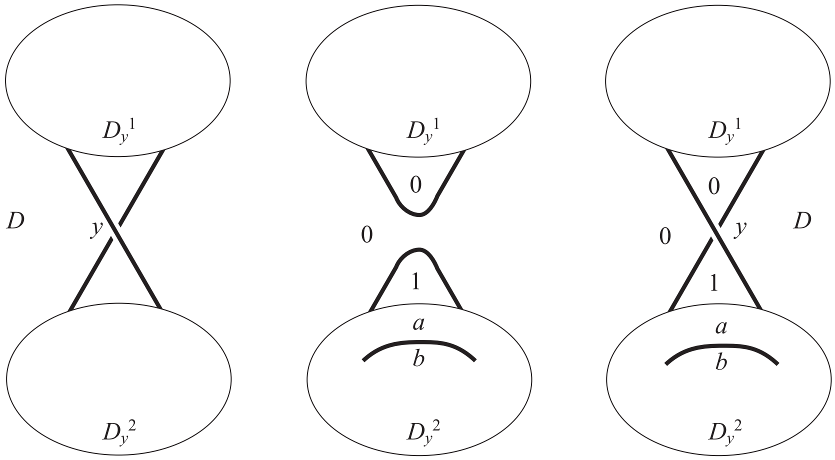


Fig. 9. Obtaining $\mathbf{v}_y$ for the reducible and positive crossing y.

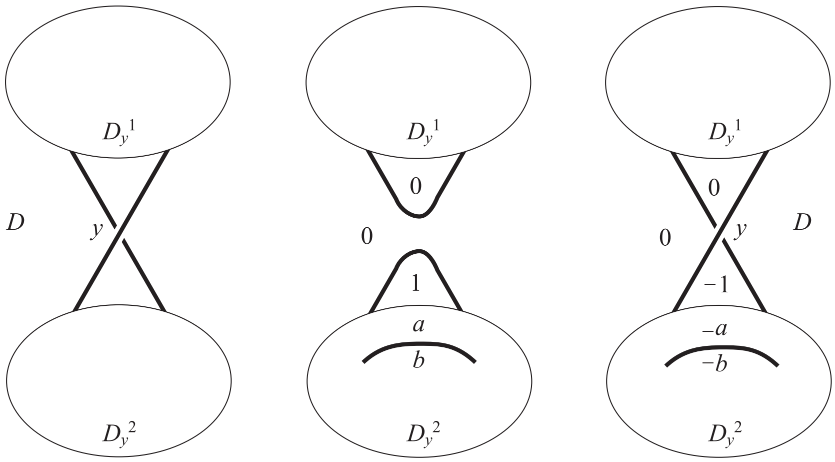


Fig. 10. Obtaining $\mathbf{v}_y$ for the reducible and negative crossing y.

$\mathbf{u}'$ for $A_{a1}(D_y^1 \cup D_y^2)$ such that the components of $\mathbf{u}'$ corresponding to R_y^0, R_y^1, R_y^2 are $0, 0, 1$ respectively. Let $\varepsilon = 1$ if y is a positive crossing, otherwise $\varepsilon = -1$. We unsplice at y and assign the same integers to all regions of D as $\varepsilon\mathbf{u}'$, where the region touching y twice is assigned 0. On the right of Fig.9, the crossing y is positive and regions of D are assigned the components of $\mathbf{u}'$. On the right of Fig.10, the crossing y is negative and regions of D are assigned the components of $-\mathbf{u}'$. Then we obtain the desired $\mathbf{v}_y \in \mathbb{Z}^{n+d+1}$. □

Combining Lemma 6.5 with Theorem 4.1 (2), we obtain the proof of Theorem 4.1 (1), that is the existence of a solution of an alternating integral region choice problem of the single counting rule for a knot diagram, by the same argument as that due to Harada [5].

Proof of Theorem 4.1 (1). If the given knot diagram D is irreducible, then we have $A_{a1}(D) = A_{a2}(D)$ and a solution of the double counting rule, which has been obtained, is also a solution of the single counting rule.

We suppose that the knot diagram D has at least one reducible crossing. For a region R_j, let $\mathcal{X}_j$ be the set of reducible crossings touched by R_j twice, $j = 1, \cdots, n+2$. A set $\mathcal{X}_j$ might be empty. Applying Lemma 6.5 for each reducible crossing $y \in \mathcal{X}_j$, we obtain $\mathbf{v}_y \in \mathbb{Z}^{n+2}$ such that any components of $A_{a1}(D)\mathbf{v}_y$ are 0 but the component of $A_{a1}(D)\mathbf{v}_y$ to y is 1. We take $\mathbf{r}_j \in \mathbb{Z}^{n+2}$ such that any components of $\mathbf{r}_j$ are 0 but the j-th component is 1. Choosing R_j once corresponds with $\mathbf{r}_j$. By the definitions of the alternating region choice matrices, we have

$$A_{a2}(D)\mathbf{r}_j - A_{a1}(D)\mathbf{r}_j = \sum_{y \in \mathcal{X}_j} A_{a1}(D)\mathbf{v}_y.$$

Applying Theorem 4.1 (2), which has been proved, we obtain a solution of double counting rule, $\mathbf{w} \in \mathbb{Z}^{n+2}$ with $A_{a2}(D)\mathbf{w}+\mathbf{c} = \mathbf{0}$. Let w_j be the j-th component of $\mathbf{w}$, $j = 1, \cdots, n+2$. We note $\mathbf{w} = \sum_j w_j \mathbf{r}_j$. We take $\mathbf{u} = \mathbf{w} + \sum_j w_j \sum_{y \in \mathcal{X}_j} \mathbf{v}_y$. Then we have

$$\begin{aligned} A_{a1}(D)\mathbf{u} &= A_{a1}(D)\mathbf{w} + \sum_j w_j \sum_{y \in \mathcal{X}_j} A_{a1}(D)\mathbf{v}_y \\ &= \sum_j w_j \left(A_{a1}(D)\mathbf{r}_j + \sum_{y \in \mathcal{X}_j} A_{a1}(D)\mathbf{v}_y \right) \\ &= \sum_j w_j A_{a2}(D)\mathbf{r}_j \\ &= A_{a2}(D)\mathbf{w} \\ &= -\mathbf{c}. \end{aligned}$$

Therefore we have $A_{a1}(D)\mathbf{u} + \mathbf{c} = \mathbf{0}$. □

Remark 6.6. In Section 9, we prove that the first and second results of Theorem 4.1 are equivalent.

The proof of Lemma 6.5 for knot diagrams due to Harada [5] implies the following fact. We give an alternative proof.

Lemma 6.7. *Let D be a link diagram with d connected components and n crossings, $n \geq 1$. We suppose that there exists a reducible crossing y of D. We fix an arc γ in the link diagram D, and let R and R' be two regions which are the both sides of the arc γ. We take two arbitrary integers a and b. Then there exist $\mathbf{v}_y \in \mathbb{Z}^{n+d+1}$ such that any components of $A_{a1}(D)\mathbf{v}_y$ are 0 but the component of $A_{a1}(D)\mathbf{v}_y$ to y is 1, and such that the components of $\mathbf{v}_y$ corresponding to R and R' are a and b respectively.*

Proof. Applying Lemma 6.5, we obtain $\mathbf{v}'_y \in \mathbb{Z}^{n+d+1}$ such that any components of $A_{a1}(D)\mathbf{v}'_y$ are 0 but the component of $A_{a1}(D)\mathbf{v}'_y$ to y is 1. We denote the components of $\mathbf{v}'_y$ corresponding to R and R' by a' and b' respectively. Applying Lemma 6.4, there exists a kernel solution $\mathbf{u}$ for $A_{a1}(D)$ such that the components of $\mathbf{u}$ corresponding to R and R' are $a - a'$ and $b - b'$ respectively. Let $\mathbf{v}_y = \mathbf{u} + \mathbf{v}'_y$. Then we obtain the desired $\mathbf{v}_y$. □

We note that we use Lemma 6.5 but does not use Lemma 6.7 to prove Theorem 4.1 (1) in this article.

REMARK 6.8. In [10], Shimizu took checkerboard colorings to show that a region crossing change is an unknotting operation on a knot diagram. In the above argument for Theorem 4.1, we take Alexander numberings instead of checkerboard colorings. Then an extension of her argument is given.

7. Solutions of the definite integral region choice problem on knot diagrams

In this section, we give an alternative proof of Theorem 3.1.

First, we observe the definite integral region choice problem of the double counting rule.

Lemma 7.1. *Let D be a link diagram or projection with n crossings, $n \geq 1$. We fix an arc γ in the link diagram D, and let R and R' be two regions which are the both sides of the arc γ. Then there exists a kernel solution for $A_{d2}(D)$ such that the components of $\mathbf{u}$ corresponding to R and R' are* 0 *and* 1 *respectively.*

Proof. If D is a link projection, we arbitrarily add over/under information to crossings. Then we may assume that D is a link diagram. By Lemma 6.1, there exists a kernel solution $\tilde{\mathbf{u}}$ for $A_{a2}(D)$ such that the components of $\tilde{\mathbf{u}}$ corresponding to R and R' are 0 and 1 respectively. We fix a checkerboard coloring such that the region R' is assigned 0. Applying Lemma 5.3 to $\tilde{\mathbf{u}}$, we obtain a kernel solution $\mathbf{u}$ for $A_{d2}(D)$ such that the components of $\mathbf{u}$ corresponding to R and R' are 0 and 1 respectively. □

Fig.11 gives an example of a link diagram with a kernel solution for a definite region choice matrix such that two regions adjacent to the arc γ are assigned 0 and 1. This kernel solution is obtained from Fig.6 applying Lemma 5.3.

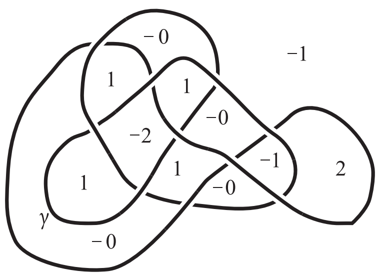


Fig. 11. A kernel solution for a definite region choice matrix.

REMARK 7.2. In [1], Ahara and Suzuki proved Lemma 7.1 for a knot diagram, showing that Reidemeister moves preserve the existence of the kernel solution, and that the knot diagram with only one crossing has a kernel solution.

The following theorem also has been proved by Ahara and Suzuki [1] for a knot diagram splicing at the given crossing and applying Lemma 7.1. We give an alternative proof below.

Theorem 7.3. *Let D be a link diagram or projection with d connected components and n crossings, $n \geq 1$. We suppose that there exists a crossing x of D of arcs in same link component. Then there exist $\mathbf{v}_x \in \mathbb{Z}^{n+d+1}$ such that any components of $A_{d2}(D)\mathbf{v}_x$ are* 0 *but the component of $A_{d2}(D)\mathbf{v}_x$ to x is* 1.

Proof. If D is a link projection, we arbitrarily add over/under information to crossings. Then we may assume that D is a link diagram. By Theorem 6.3, there exist $\mathbf{w}_x \in \mathbb{Z}^{n+d+1}$ such that any components of $A_{a2}(D)\mathbf{w}_x$ are 0 but the component of $A_{a2}(D)\mathbf{w}_x$ to x is 1. For each region R, let w_R be the component of $\mathbf{w}_x$ to R. We denote the four corners touching x by C_1, C_2, C_3, C_4 clockwise, and the regions including C_j by R_j, $j = 1, 2, 3, 4$. Then we have $w_{R_1} - w_{R_2} + w_{R_3} - w_{R_4} = 1$. We fix the checkerboard coloring such that the region R_1 is assigned 0. For each region R, let c_R be the integer assigned by this checkerboard coloring. Let $\mathbf{v}_x$ be the vector in $\mathbb{Z}^{n+d+1}$ such that the component to R is $(-1)^{c_R} w_R$. By similar argument to the proof of Lemma 5.3, it is shown that any components of $A_{d2}(D)\mathbf{v}_x$ are 0 but the component of $A_{d2}(D)\mathbf{v}_x$ to x is 1. Fig.12 gives an example of the above process from $\mathbf{w}_x$ to $\mathbf{v}_x$, where $\mathbf{w}_x$ is a solution illustrated on top right of Fig.7. □

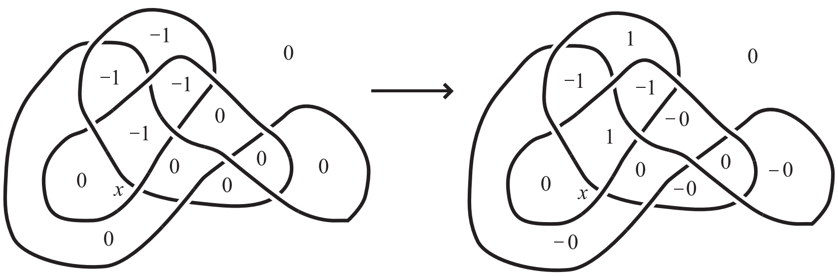


Fig. 12. Finding $\mathbf{v}_x$ such that any components of $A_{d2}(D)\mathbf{v}_x$ are 0 but the component of $A_{d2}(D)\mathbf{v}_x$ to x is 1.

In the above proof, we do not use Lemma 7.1 immediately.

Theorem 7.3 implies Theorem 3.1 (2), that is the existence of a solution of a definite integral region choice problem of the double counting rule for a knot diagram, by the same argument as that due to Ahara and Suzuki [1].

Proof of Theorem 3.1 (2). Applying Theorem 7.3 for each crossing x_i, there exist $\mathbf{v}_i \in \mathbb{Z}^{n+d+1}$ such that any components of $A_{d2}(D)\mathbf{v}_i$ are 0 but the i-th component of $A_{a2}(D)\mathbf{v}_i$ is 1, $i = 1, 2, \cdots, n$. Let c_i be the i-th component of $\mathbf{c}$. If we take $\mathbf{u} = -\sum_{i=1}^{n} c_i \mathbf{v}_i$, then we have $A_{d2}(D)\mathbf{u} + \mathbf{c} = \mathbf{0}$. □

Next, we observe the definite integral region choice problem of the single counting rule. The following lemma has been proved by Ahara and Suzuki [1] for knot diagrams. We give an alternative proof.

Lemma 7.4. *Let D be a link diagram or projection with n crossings, $n \geq 1$. We fix an arc γ in the link diagram D, and let R and R' be two regions which are the both sides of the arc γ. We take two arbitrary integers a and b. Then there exists a kernel solution $\mathbf{u}$ for $A_{d1}(D)$*

such that the components of $\mathbf{u}$ *corresponding to* R *and* R' *are* a *and* b *respectively.*

Proof. If D is a link projection, we arbitrarily add over/under information to crossings. Then we may assume that D is a link diagram. We fix a checkerboard coloring of D. Let c_R and $c_{R'}$ be the assigned integers to R and R' respectively by the fixed checkerboard coloring. By Lemma 6.4, there exists a kernel solution $\mathbf{w}$ for $A_{a1}(D)$ such that the components of $\mathbf{w}$ corresponding to R and R' are $(-1)^{c_R}a$ and $(-1)^{c_{R'}}b$ respectively. We apply Lemma 5.4 to $\mathbf{w}$. Then we obtain a kernel solution $\mathbf{u}$ for $A_{d1}(D)$ such that the components of $\mathbf{u}$ corresponding to R and R' are a and b respectively. □

The following lemma also has been proved by Ahara and Suzuki [1] for knot diagrams. They proved it as a corollary to Lemma 7.4. We give an alternative proof.

Lemma 7.5. *Let* D *be a link diagram or projection with* d *connected components and* n *crossings,* $n \geq 1$*. We suppose that there exists a reducible crossing* y *of* D*. Then there exist* $\mathbf{v}_y \in \mathbb{Z}^{n+d+1}$ *such that any components of* $A_{d1}(D)\mathbf{v}_y$ *are* 0 *but the component of* $A_{d1}(D)\mathbf{v}_y$ *to* y *is* 1.

Proof. If D is a link projection, we arbitrarily add over/under information to crossings. Then we may assume that D is a link diagram. By Lemma 6.5, there exist $\mathbf{w}_y \in \mathbb{Z}^{n+d+1}$ such that any components of $A_{a1}(D)\mathbf{w}_y$ are 0 but the component of $A_{a1}(D)\mathbf{w}_y$ to y is 1. For each region R, let w_R be the component of $\mathbf{w}_y$ to R. We denote the four corners touching x by C_1, C_2, C_3, C_4 clockwise, and the regions including C_j by R_j, $j = 1, 2, 3, 4$. We may assume that R_3 coincides with R_1. Then we have $w_{R_1} - w_{R_2} - w_{R_4} = \pm 1$. If $w_{R_1} - w_{R_2} - w_{R_4} = 1$, that is the crossing y is negative, we fix the checkerboard coloring such that the region R_1 is assigned 0 . Otherwise, we fix the checkerboard coloring such that the region R_1 is assigned 1 . For each region R, let c_R be the integer assigned by the fixed checkerboard coloring. Let $\mathbf{v}_y$ be the vector in $\mathbb{Z}^{n+d+1}$ such that the component to R is $(-1)^{c_R} w_R$. By similar argument to the proof of Lemma 5.4, it is shown that any components of $A_{d1}(D)\mathbf{v}_y$ are 0 but the component of $A_{d1}(D)\mathbf{v}_y$ to y is 1. □

Combining Lemma 7.5 with Theorem 3.1 (2), we obtain the proof of Theorem 3.1 (1), that is the existence of a solution of a definite integral region choice problem of the single counting rule for a knot diagram, by the same argument as that due to Ahara and Suzuki [1].

Proof of Theorem 3.1 (1). If the given knot diagram or projection D is irreducible, then we have $A_{d1}(D) = A_{d2}(D)$ and a solution of the double counting rule, which has been obtained, is also a solution of the single counting rule.

We suppose that the knot diagram or projection D has at least one reducible crossing. For a region R_j, let $\mathcal{X}_j$ be the set of reducible crossings touched by R_j twice, $j = 1, \cdots, n+2$. A set $\mathcal{X}_j$ might be empty. Applying Lemma 7.5 for each reducible crossing $y \in \mathcal{X}_j$, we obtain $\mathbf{v}_y \in \mathbb{Z}^{n+2}$ such that any components of $A_{d1}(D)\mathbf{v}_y$ are 0 but the component of $A_{d1}(D)\mathbf{v}_y$ to y is 1. We take $\mathbf{r}_j \in \mathbb{Z}^{n+2}$ such that any components of $\mathbf{r}_j$ are 0 but the j-th component is 1. Choosing R_j once corresponds with $\mathbf{r}_j$. By the definitions of the definite region choice matrices, we have

$$A_{d2}(D)\mathbf{r}_j - A_{d1}(D)\mathbf{r}_j = \sum_{y \in \mathcal{X}_j} A_{d1}(D)\mathbf{v}_y.$$

Applying Theorem 3.1 (2), which has been proved, we obtain a solution of double counting rule, $\mathbf{w} \in \mathbb{Z}^{n+2}$ with $A_{d2}(D)\mathbf{w}+\mathbf{c} = \mathbf{0}$. Let w_j be the j-th component of $\mathbf{w}$, $j = 1, \cdots, n+2$. We note $\mathbf{w} = \sum_j w_j \mathbf{r}_j$. We take $\mathbf{u} = \mathbf{w} + \sum_j w_j \sum_{y \in \mathcal{X}_j} \mathbf{v}_y$. Then we have

$$\begin{aligned} A_{d1}(D)\mathbf{u} &= A_{d1}(D)\mathbf{w} + \sum_j w_j \sum_{y \in \mathcal{X}_j} A_{d1}(D)\mathbf{v}_y \\ &= \sum_j w_j \left(A_{d1}(D)\mathbf{r}_j + \sum_{y \in \mathcal{X}_j} A_{d1}(D)\mathbf{v}_y \right) \\ &= \sum_j w_j A_{d2}(D)\mathbf{r}_j \\ &= A_{d2}(D)\mathbf{w} \\ &= -\mathbf{c}. \end{aligned}$$

Therefore we have $A_{d1}(D)\mathbf{u} + \mathbf{c} = \mathbf{0}$. □

REMARK 7.6. In Section 9, we prove that the first and second results of Theorem 3.1 are equivalent.

The proof of Lemma 7.5 for knot diagrams due to Ahara and Suzuki [1] implies the following fact. We give an alternative proof.

Lemma 7.7. *Let D be a link diagram or projection with d connected components and n crossings, $n \geq 1$. We suppose that there exists a reducible crossing y of D. We fix an arc γ in the link diagram D, and let R and R' be two regions which are the both sides of the arc γ. We take two arbitrary integers a and b. Then there exist $\mathbf{v}_y \in \mathbb{Z}^{n+d+1}$ such that any components of $A_{d1}(D)\mathbf{v}_y$ are 0 but the component of $A_{d1}(D)\mathbf{v}_y$ to y is 1, and such that the components of $\mathbf{v}_y$ corresponding to R and R' are a and b respectively.*

Proof. Applying Lemma 7.5, we obtain $\mathbf{v}'_y \in \mathbb{Z}^{n+d+1}$ such that any components of $A_{d1}(D)\mathbf{v}'_y$ are 0 but the component of $A_{d1}(D)\mathbf{v}'_y$ to y is 1. We denote the components of $\mathbf{v}'_y$ corresponding to R and R' by a' and b' respectively. Applying Lemma 7.4, there exists a kernel solution $\mathbf{u}$ for $A_{d1}(D)$ such that the components of $\mathbf{u}$ corresponding to R and R' are $a - a'$ and $b - b'$ respectively. Let $\mathbf{v}_y = \mathbf{u} + \mathbf{v}'_y$. Then we obtain the desired $\mathbf{v}_y$. □

We note that we use Lemma 7.5 but does not use Lemma 7.7 to prove Theorem 3.1 (1) in this article.

8. Region choice matrices of the double counting rule

From now on, we change link projections to link diagrams adding over/under information to crossings arbitrarily.

Applying the arguments in the original proofs of Theorem 3.1 and 4.1 in [1, 5] to link diagrams, the ranks of the definite and alternating region choice matrices are determined. In this section, we show that on the double counting rule.

Theorem 8.1. *Let D be a diagram of an l-component link. We assume that D has d connected components and n crossings, $n \geq 1$. Then each rank of the definite and alternating region choice matrices of the double counting rule, $A_{d2}(D)$ and $A_{a2}(D)$, is $n+d-l$, and each rank of the $\mathbb{Z}$-submodules $\{\mathbf{u} \in \mathbb{Z}^{n+d+1} \mid A_{d2}(D)\mathbf{u} = \mathbf{0}\}$ and $\{\mathbf{u} \in \mathbb{Z}^{n+d+1} \mid A_{a2}(D)\mathbf{u} = \mathbf{0}\}$ is $l+1$.*

If we transpose the incidence matrix induced by Cheng and Gao [4] and Hashizume [6], it is same as the definite region choice matrix of the single counting rule up to permutations of rows and columns. This transposed matrix also coincides with the alternating region choice matrix of the single counting rule modulo 2 up to permutations of rows and columns. For irreducible diagrams, Theorem 8.1 is an extension of their result on the rank of the incidence matrices.

It is well known that we can make any link diagram into a diagram of a trivial link after some crossing changes, and that some Reidemeister moves can transform any pair of non-trivial diagrams of a trivial link each other. We prove Theorem 8.1 using a similar argument to that in Appendix A of the article written by Ahara and Suzuki [1]. That means it is similar to the proof of the invariance for the Alexander polynomial in [2].

Lemma 8.2. *If we change a crossing of a link diagram admitting integral region choices, the changed diagram also admit it, the definite region choice matrix of double counting rule is preserved, and the row concerning this crossing is multiplied by -1 for the alternating region choice matrix of the double counting rule.*

Proof. It is cleared by the definitions of region choice matrices. □

REMARK 8.3. In [5], Harada proved that crossings changes preserve kernel solutions for alternating region choice matrices of the double counting rule on any knot diagrams. Lemma 8.2 implies that his claim holds on any link diagrams.

Lemma 8.4. *A Reidemeister move I between link diagrams admitting integral region choices preserves the rank of the $\mathbb{Z}$-submodules of kernel solutions for definite and alternating region choice matrices of the double counting rule.*

Fig. 13. Reidemeister moves I among the diagrams D_+, D, and D_-.

Proof (cf. [1]). Let the middle of Fig.13 be an arc of a link diagram D admitting an integral region choice, and we denote by D_+ and D_- the obtained diagrams from D by a Reidemeister move I at the arc as illustrated on the right side and the left side of Fig.13 respectively. We may order the regions of D such that the upper and lower regions on the middle of Fig.13 are ordered 1 and 2 respectively. We denote the definite and the alternating region choice matrices of the double counting rule for D by

$$A_{d2}(D) = \begin{pmatrix} \mathbf{a}_d & \mathbf{b}_d & P_d \end{pmatrix}, \quad A_{a2}(D) = \begin{pmatrix} \mathbf{a}_a & \mathbf{b}_a & P_a \end{pmatrix}.$$

Inserting the row and the column corresponding to the added crossing and the added region respectively, we obtain the matrices for D_+ and D_-,

$$A_{d2}(D_+) = A_{d2}(D_-) = \begin{pmatrix} 1 & 2 & 1 & \mathbf{0} \\ \mathbf{0} & \mathbf{a}_d & \mathbf{b}_d & P_d \end{pmatrix},$$

$$A_{a2}(D_+) = \begin{pmatrix} 1 & -2 & 1 & \mathbf{0} \\ \mathbf{0} & \mathbf{a}_a & \mathbf{b}_a & P_a \end{pmatrix}, \quad A_{a2}(D_-) = \begin{pmatrix} -1 & 2 & -1 & \mathbf{0} \\ \mathbf{0} & \mathbf{a}_a & \mathbf{b}_a & P_a \end{pmatrix}.$$

Each of the matrices for D_+ and D_- has one more column than D, and the equalities $\mathrm{rank}A_{d2}(D_\pm) = \mathrm{rank}A_{d2}(D) + 1$ and $\mathrm{rank}A_{a2}(D_\pm) = \mathrm{rank}A_{a2}(D) + 1$ hold. Then the $\mathbb{Z}$-submodules of the kernel solutions for these matrices have the same rank. □

Lemma 8.5. *A Reidemeister move II between link diagrams admitting integral region choices preserves the rank of the $\mathbb{Z}$-submodules of kernel solutions for definite and alternating region choice matrices of the double counting rule.*

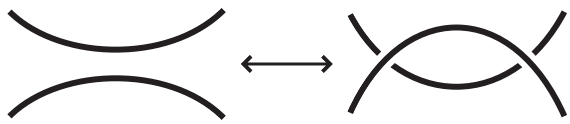


Fig. 14. A Reidemeister move II between the diagrams D and D'.

Proof (cf. [1]). Let the left side of Fig.14 be two arcs of a link diagram D admitting an integral region choice, and we denote by D' the obtained diagram from D by a Reidemeister move II around the arcs as illustrated on the right side of Fig.14.

If the Reidemeister move II does not change the number of the connected components of the diagram, and the regions appearing around the move are different each other, then we may order the regions of D such that the bottom, middle and top regions of the left side of Fig.14 are ordered 1, 2 and 3 respectively. We denote the definite and the alternating region choice matrices of the double counting rule for D by

$$A_{d2}(D) = \begin{pmatrix} \mathbf{a}_d & \mathbf{b}_d & \mathbf{c}_d & P_d \end{pmatrix}, \quad A_{a2}(D) = \begin{pmatrix} \mathbf{a}_a & \mathbf{b}_a & \mathbf{c}_a & P_a \end{pmatrix}.$$

Inserting the rows and the column corresponding to the added crossings and the added region respectively, we obtain the matrices for D',

$$A_{d2}(D') = \begin{pmatrix} 1 & 1 & 1 & 0 & 1 & O \\ 1 & 1 & 0 & 1 & 1 & \\ \mathbf{0} & \mathbf{a}_d & \mathbf{b}'_d & \mathbf{b}''_d & \mathbf{c}_d & P_d \end{pmatrix},$$

where $\mathbf{b}'_d + \mathbf{b}''_d = \mathbf{b}_d$, and

$$A_{a2}(D') = \begin{pmatrix} 1 & -1 & 1 & 0 & -1 & O \\ -1 & 1 & 0 & -1 & 1 & \\ \mathbf{0} & \mathbf{a}_a & \mathbf{b}'_a & \mathbf{b}''_a & \mathbf{c}_a & P_a \end{pmatrix},$$

where $\mathbf{b}'_a + \mathbf{b}''_a = \mathbf{b}_a$. Each of the matrices for D' has two more columns than D. Adding the fourth column to the third column, and taking the first column off the second, third and fifth columns on $A_{d2}(D')$, we obtain the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & O \\ 1 & 0 & 0 & 1 & 0 & \\ \mathbf{0} & \mathbf{a}_d & \mathbf{b}_d & \mathbf{b}''_d & \mathbf{c}_d & P_d \end{pmatrix},$$

and the equality $\mathrm{rank} A_{d2}(D') = \mathrm{rank} A_{d2}(D) + 2$. Adding the fourth column to the third column, and the first column to the second and fifth columns, and taking the first column off the third column on $A_{a2}(D')$, we obtain the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & O \\ -1 & 0 & 0 & -1 & 0 & \\ \mathbf{0} & \mathbf{a}_a & \mathbf{b}_a & \mathbf{b}''_a & \mathbf{c}_a & P_a \end{pmatrix},$$

and the equality $\mathrm{rank} A_{a2}(D') = \mathrm{rank} A_{a2}(D) + 2$. Then the $\mathbb{Z}$-submodules of the kernel solutions for these matrices have the same rank.

If the Reidemeister move II does not change the number of the connected components of the diagram, and the top and bottom regions coincide, then we may order the regions of D such that the upper and middle regions of the left side of Fig.14 are ordered 1, and 2 respectively. We denote the definite and the alternating region choice matrices of the double counting rule for D by

$$A_{d2}(D) = \begin{pmatrix} \mathbf{a}_d & \mathbf{b}_d & P_d \end{pmatrix}, \quad A_{a2}(D) = \begin{pmatrix} \mathbf{a}_a & \mathbf{b}_a & P_a \end{pmatrix}.$$

Inserting the rows and the column corresponding to the added crossings and the added region respectively, we obtain the matrices for D',

$$A_{d2}(D') = \begin{pmatrix} 1 & 2 & 1 & 0 & O \\ 1 & 2 & 0 & 1 & \\ \mathbf{0} & \mathbf{a}_d & \mathbf{b}'_d & \mathbf{b}''_d & P_d \end{pmatrix},$$

where $\mathbf{b}'_d + \mathbf{b}''_d = \mathbf{b}_d$, and

$$A_{a2}(D') = \begin{pmatrix} 1 & -2 & 1 & 0 & O \\ -1 & 2 & 0 & -1 & \\ \mathbf{0} & \mathbf{a}_a & \mathbf{b}'_a & \mathbf{b}''_a & P_a \end{pmatrix},$$

where $\mathbf{b}'_a + \mathbf{b}''_a = \mathbf{b}_a$. Each of the matrices for D' has two more columns than D. Adding the fourth column to the third column, taking the first column off the third column, and taking the first column multiplied by 2 off the second column on $A_{d2}(D')$, we obtain the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 & O \\ 1 & 0 & 0 & 1 & \\ \mathbf{0} & \mathbf{a}_d & \mathbf{b}_d & \mathbf{b}''_d & P_d \end{pmatrix},$$

and the equality $\mathrm{rank} A_{d2}(D') = \mathrm{rank} A_{d2}(D) + 2$. Adding the fourth column to the third column, the first column multiplied by 2 to the second column, and taking the first column off the third column on $A_{a2}(D')$, we obtain the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 & O \\ -1 & 0 & 0 & -1 & \\ \mathbf{0} & \mathbf{a}_a & \mathbf{b}_a & \mathbf{b}''_a & P_a \end{pmatrix}$$

and the equality $\mathrm{rank}A_{a2}(D') = \mathrm{rank}A_{a2}(D) + 2$. Then the $\mathbb{Z}$-submodules of the kernel solutions for these matrices have the same rank.

If the Reidemeister move II changes the number of the connected components of the diagram, then we may order the regions of D such that the bottom, middle and top regions of the left side of Fig.14 are ordered 1, 2 and 3 respectively. We denote the definite and the alternating region choice matrices of the double counting rule for D by

$$A_{d2}(D) = \begin{pmatrix}\mathbf{a}_d & \mathbf{b}_d & \mathbf{c}_d & P_d\end{pmatrix}, \quad A_{a2}(D) = \begin{pmatrix}\mathbf{a}_a & \mathbf{b}_a & \mathbf{c}_a & P_a\end{pmatrix}.$$

Inserting the rows and the column corresponding to the added crossings and the added region respectively, we obtain the matrices for D',

$$A_{d2}(D') = \begin{pmatrix} 1 & 1 & 1 & 1 & O \\ 1 & 1 & 1 & 1 & \\ \mathbf{0} & \mathbf{a}_d & \mathbf{b}_d & \mathbf{c}_d & P_d \end{pmatrix},$$

$$A_{a2}(D') = \begin{pmatrix} 1 & -1 & 1 & -1 & O \\ -1 & 1 & -1 & 1 & \\ \mathbf{0} & \mathbf{a}_a & \mathbf{b}_a & \mathbf{c}_a & P_a \end{pmatrix}.$$

Each of the matrices for D' has one more column than D, and the equalities $\mathrm{rank}A_{d2}(D') = \mathrm{rank}A_{d2}(D) + 1$ and $\mathrm{rank}A_{a2}(D') = \mathrm{rank}A_{a2}(D) + 1$ hold. Then the $\mathbb{Z}$-submodules of the kernel solutions for these matrices have the same rank. □

Lemma 8.6. *A Reidemeister moves III between link diagrams admitting integral region choices preserves the ranks of the $\mathbb{Z}$-submodules of kernel solutions for definite and alternating region choice matrices of the double counting rule.*

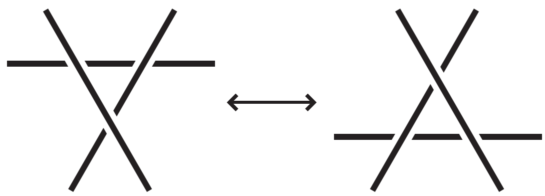


Fig. 15. A Reidemeister move III between the diagrams D_∇ and D_Δ.

Proof (cf. [1]). Let the left side of Fig.15 be a neighborhood of a triangle region on a link diagram D_∇ admitting an integral region choice, and we denote by D_Δ the obtained diagram from D_∇ by a Reidemeister move III around the triangle region as illustrated on the right side of Fig.15.

If regions appearing around the move are different each other, then we may order the regions of D_∇ such that the triangle region on Fig.15 is ordered 1 and that other six regions around are ordered 2, 3, 4, 5, 6, 7 clockwise from top left. The definite and the alternating region choice matrices of the double counting rule for D_∇ are

$$A_{d2}(D_\nabla) = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 1 & \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 & O \\ 1 & 0 & 0 & 0 & 1 & 1 & 1 & \\ \mathbf{0} & \mathbf{a}_d & \mathbf{b}_d & \mathbf{c}_d & \mathbf{d}_d & \mathbf{e}_d & \mathbf{f}_d & P_d \end{pmatrix},$$

$$A_{a2}(D_\nabla) = \begin{pmatrix} -1 & -1 & 1 & 0 & 0 & 0 & 1 & \\ 1 & 0 & -1 & 1 & -1 & 0 & 0 & O \\ 1 & 0 & 0 & 0 & -1 & 1 & -1 & \\ \mathbf{0} & \mathbf{a}_a & \mathbf{b}_a & \mathbf{c}_a & \mathbf{d}_a & \mathbf{e}_a & \mathbf{f}_a & P_a \end{pmatrix},$$

where the upper left crossing, the upper right crossing, and the lower crossing of the triangle of D_∇ are ordered 1,2, and 3. We order the crossings of D_Δ such that the lower right crossing, the lower left crossing, and the upper crossing are ordered 1,2, and 3, and that the others are ordered as D_∇. Then we obtain the matrices for D_Δ,

$$A_{d2}(D_\Delta) = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 1 & 0 & \\ 1 & 1 & 0 & 0 & 0 & 1 & 1 & O \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & \\ \mathbf{0} & \mathbf{a}_d & \mathbf{b}_d & \mathbf{c}_d & \mathbf{d}_d & \mathbf{e}_d & \mathbf{f}_d & P_d \end{pmatrix},$$

$$A_{a2}(D_\Delta) = \begin{pmatrix} -1 & 0 & 0 & 1 & -1 & 1 & 0 & \\ 1 & -1 & 0 & 0 & 0 & -1 & 1 & O \\ 1 & -1 & 1 & -1 & 0 & 0 & 0 & \\ \mathbf{0} & \mathbf{a}_a & \mathbf{b}_a & \mathbf{c}_a & \mathbf{d}_a & \mathbf{e}_a & \mathbf{f}_a & P_a \end{pmatrix}.$$

After multiplying the top three rows on $A_{d_2}(D_\nabla)$ by -1 and multiplying the first column by -1, if we add the first column to the j-th columns, $j = 2, 3, \cdots, 7$, then we obtain $A_{d_2}(D_\Delta)$. The matrices $A_{d2}(D_\nabla)$ and $A_{d2}(D_\Delta)$ have the same size and the same rank, then the $\mathbb{Z}$-submodules of the kernel solutions for the matrices have the same rank. Adding the first column multiplied by $(-1)^{j+1}$ to the j-th columns, $j = 2, 3, \cdots, 7$, we obtain $A_{a2}(D_\Delta)$ from $A_{a2}(D_\nabla)$. The matrices $A_{a2}(D_\nabla)$ and $A_{a2}(D_\Delta)$ have the same size and the same rank, then the $\mathbb{Z}$-submodules of the kernel solutions for these matrices have the same rank.

If the top left and top right regions of the left side of Fig.15 coincide on D_∇, and if this and the other regions appearing around the move are different each other, then we may order the regions of D_∇ such that the triangle region on Fig.15 is ordered 1 and that other five regions around are ordered 2, 3, 4, 5, 6 clockwise from top left. The definite and the alternating region choice matrices of the double counting rule for D_∇ are

$$A_{d2}(D_\nabla) = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 1 & \\ 1 & 1 & 1 & 1 & 0 & 0 & O \\ 1 & 0 & 0 & 1 & 1 & 1 & \\ \mathbf{0} & \mathbf{a}_d & \mathbf{b}_d & \mathbf{d}_d & \mathbf{e}_d & \mathbf{f}_d & P_d \end{pmatrix},$$

$$A_{a2}(D_\nabla) = \begin{pmatrix} -1 & -1 & 1 & 0 & 0 & 1 & \\ 1 & 1 & -1 & -1 & 0 & 0 & O \\ 1 & 0 & 0 & -1 & 1 & -1 & \\ \mathbf{0} & \mathbf{a}_a & \mathbf{b}_a & \mathbf{d}_a & \mathbf{e}_a & \mathbf{f}_a & P_a \end{pmatrix}.$$

Then we obtain the matrices for D_Δ,

$$A_{d2}(D_\Delta) = \begin{pmatrix} 1 & 1 & 0 & 1 & 1 & 0 & \\ 1 & 1 & 0 & 0 & 1 & 1 & O \\ 1 & 2 & 1 & 0 & 0 & 0 & \\ \mathbf{0} & \mathbf{a}_d & \mathbf{b}_d & \mathbf{d}_d & \mathbf{e}_d & \mathbf{f}_d & P_d \end{pmatrix},$$

$$A_{a2}(D_\Delta) = \begin{pmatrix} -1 & 1 & 0 & -1 & 1 & 0 & \\ 1 & -1 & 0 & 0 & -1 & 1 & O \\ 1 & -2 & 1 & 0 & 0 & 0 & \\ \mathbf{0} & \mathbf{a}_a & \mathbf{b}_a & \mathbf{d}_a & \mathbf{e}_a & \mathbf{f}_a & P_a \end{pmatrix}.$$

After multiplying the top three rows on $A_{d_2}(D_\nabla)$ by -1 and multiplying the first column by -1, if we add the first column to the j-th columns, $j = 3, \cdots, 6$ and the first column multiplied by 2 to the second column, then we obtain $A_{d_2}(D_\Delta)$. The matrices $A_{d2}(D_\nabla)$ and $A_{d2}(D_\Delta)$ have the same size and the same rank, then the $\mathbb{Z}$-submodules of the kernel solutions for the matrices have the same rank. Adding the first column to the third, fourth, and sixth columns, and taking the first column off the fifth column and the first column multiplied by 2 off the second column, we obtain $A_{a2}(D_\Delta)$ from $A_{a2}(D_\nabla)$. The matrices $A_{a2}(D_\nabla)$ and $A_{a2}(D_\Delta)$ have the same size and the same rank, then the $\mathbb{Z}$-submodules of the kernel solutions for these matrices have the same rank.

If the top left, top right, and the bottom middle regions of the left side of Fig.15 coincide on D_∇, then this and the other regions appearing around the move are different each other. We may order the regions of D_∇ such that the triangle region on Fig.15 is ordered 1 and that other four regions around are ordered 2, 3, 4, 5 clockwise from top left. The definite and the alternating region choice matrices of the double counting rule for D_∇ are

$$A_{d2}(D_\nabla) = \begin{pmatrix} 1 & 1 & 1 & 0 & 1 & \\ 1 & 1 & 1 & 1 & 0 & O \\ 1 & 1 & 0 & 1 & 1 & \\ \mathbf{0} & \mathbf{a}_d & \mathbf{b}_d & \mathbf{d}_d & \mathbf{f}_d & P_d \end{pmatrix},$$

$$A_{a2}(D_\nabla) = \begin{pmatrix} -1 & -1 & 1 & 0 & 1 & \\ 1 & 1 & -1 & -1 & 0 & O \\ 1 & 1 & 0 & -1 & -1 & \\ \mathbf{0} & \mathbf{a}_a & \mathbf{b}_a & \mathbf{d}_a & \mathbf{f}_a & P_a \end{pmatrix}.$$

Then we obtain the matrices for D_Δ,

$$A_{d2}(D_\Delta) = \begin{pmatrix} 1 & 2 & 0 & 1 & 0 & \\ 1 & 2 & 0 & 0 & 1 & O \\ 1 & 2 & 1 & 0 & 0 & \\ \mathbf{0} & \mathbf{a}_d & \mathbf{b}_d & \mathbf{d}_d & \mathbf{f}_d & P_d \end{pmatrix},$$

$$A_{a2}(D_\Delta) = \begin{pmatrix} -1 & 2 & 0 & -1 & 0 & \\ 1 & -2 & 0 & 0 & 1 & O \\ 1 & -2 & 1 & 0 & 0 & \\ \mathbf{0} & \mathbf{a}_a & \mathbf{b}_a & \mathbf{d}_a & \mathbf{f}_a & P_a \end{pmatrix}.$$

After multiplying the top three rows on $A_{d_2}(D_\nabla)$ by -1 and multiplying the first column by -1, if we add the first column to the third, fourth and fifth columns, and the first column multiplied by 3 to the second column, then we obtain $A_{d_2}(D_\Delta)$. The matrices $A_{d2}(D_\nabla)$ and $A_{d2}(D_\Delta)$ have the same size and the same rank, then the $\mathbb{Z}$-submodules of the kernel solutions for the matrices have the same rank. Adding the first column to the third, fourth, and fifth columns, and taking the first column multiplied by 3 off the second columns, we obtain $A_{a2}(D_\Delta)$ from $A_{a2}(D_\nabla)$. The matrices $A_{a2}(D_\nabla)$ and $A_{a2}(D_\Delta)$ have the same size and the same rank, then the $\mathbb{Z}$-submodules of the kernel solutions for these matrices have the same rank.

If the top left and top right regions of the left side of Fig.15 coincide on D_∇, and if the bottom left and bottom right regions also coincide on D_∇, we may order the regions of D_∇ such that the triangle region on Fig.15 is ordered 1 and that other four regions around are ordered 2, 3, 4, 5 clockwise from top left. The definite and the alternating region choice matrices of the double counting rule for D_∇ are

$$A_{d2}(D_\nabla) = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & \\ 1 & 1 & 1 & 1 & 0 & O \\ 1 & 0 & 0 & 2 & 1 & \\ \mathbf{0} & \mathbf{a}_d & \mathbf{b}_d & \mathbf{d}_d & \mathbf{e}_d & P_d \end{pmatrix},$$

$$A_{a2}(D_\nabla) = \begin{pmatrix} -1 & -1 & 1 & 1 & 0 & \\ 1 & 1 & -1 & -1 & 0 & O \\ 1 & 0 & 0 & -2 & 1 & \\ \mathbf{0} & \mathbf{a}_a & \mathbf{b}_a & \mathbf{d}_a & \mathbf{e}_a & P_a \end{pmatrix}.$$

Then we obtain the matrices for D_Δ,

$$A_{d2}(D_\Delta) = \begin{pmatrix} 1 & 1 & 0 & 1 & 1 & \\ 1 & 1 & 0 & 1 & 1 & O \\ 1 & 2 & 1 & 0 & 0 & \\ \mathbf{0} & \mathbf{a}_d & \mathbf{b}_d & \mathbf{d}_d & \mathbf{e}_d & P_d \end{pmatrix},$$

$$A_{a2}(D_\Delta) = \begin{pmatrix} -1 & 1 & 0 & -1 & 1 & \\ 1 & -1 & 0 & 1 & -1 & O \\ 1 & -2 & 1 & 0 & 0 & \\ \mathbf{0} & \mathbf{a}_a & \mathbf{b}_a & \mathbf{d}_a & \mathbf{e}_a & P_a \end{pmatrix}.$$

After multiplying the top three rows on $A_{d_2}(D_\nabla)$ by -1 and multiplying the first column by -1, if we add the first column to the third and fifth columns, and the first column multiplied by 2 to the second and fourth columns, then we obtain $A_{d_2}(D_\Delta)$. The matrices $A_{d2}(D_\nabla)$ and $A_{d2}(D_\Delta)$ have the same size and the same rank, then the $\mathbb{Z}$-submodules of the kernel solutions for the matrices have the same rank. Adding the first column multiplied by -2, 1, 2 and -1 to the second, third, fourth, and fifth columns respectively, we obtain $A_{a2}(D_\Delta)$

from $A_{a2}(D_\nabla)$. The matrices $A_{a2}(D_\nabla)$ and $A_{a2}(D_\Delta)$ have the same size and the same rank, then the $\mathbb{Z}$-submodules of the kernel solutions for these matrices have the same rank.

The proof for the other cases are given by the similar arguments as above, or reduced to the one of the above cases applying Lemma 8.2. □

Proof of Theorem 8.1. It is well known that we can make any link diagram into a diagram of a trivial link after some crossing changes, and that some Reidemeister moves can transform the non-trivial diagram of a trivial link to the split sum of the knot diagram with only one crossing and the $l-1$ copies of the trivial knot diagram, which is given in Example 4.6. On this obtained diagram D_0 with certain orders of crossings and regions, we obtain

$$A_{d2}(D_0) = \begin{pmatrix} 2 & 1 & 1 & 0 & 0 & 0 & \dots & 0 \end{pmatrix},$$

and

$$A_{a2}(D_0) = \begin{pmatrix} -2\varepsilon & \varepsilon & \varepsilon & 0 & 0 & 0 & \dots & 0 \end{pmatrix},$$

where $\varepsilon = 1$ if the crossing is positive, otherwise $\varepsilon = -1$, and the number of 0 appearing on each matrix is $l-1$. The both matrices have $l+2$ columns. Their ranks are equal to $1 = 1 + l - l$. Then the desired claim holds for D_0. By Lemma 8.2, 8.4, 8.5, and 8.6, crossing changes and Reidemeister moves preserve this claim. Then this theorem is proved. □

REMARK 8.7. As commented in Remark 6.2 and 7.2, Ahara and Suzuki [1] and Harada [5] showed that Reidemeister moves and crossing changes preserve the existence of the kernel solution, and that the knot diagram with only one crossing has a kernel solution, to prove Lemma 6.1 and 7.1 for a knot diagram. Their arguments are extended to that for the proof of Theorem 8.1, though Harada did not describe how alternating region choice matrices are affected.

9. Region choice matrices of the single counting rule

Before determining the rank of region choice matrices of the single counting rule, we observe the images of the homomorphisms $\Phi_{d1}(D)$, $\Phi_{d2}(D)$, $\Phi_{a1}(D)$, $\Phi_{a2}(D)$ defined in Section 5. The arguments in the proofs of Theorem 3.1 (1) and 4.1 (1) given in Sections 7 and 6 imply the following result.

Lemma 9.1. *Let D be a diagram of an l-component link. We assume that D has d connected components and n crossings, $n \geq 1$. We take $\mathbf{c} \in \mathbb{Z}^n$.*

(1) *If there exists a solution $\mathbf{w} \in \mathbb{Z}^{n+d+1}$ such that $A_{d2}(D)\mathbf{w} + \mathbf{c} = \mathbf{0}$, then there exists a solution $\mathbf{u} \in \mathbb{Z}^{n+d+1}$ such that $A_{d1}(D)\mathbf{u} + \mathbf{c} = \mathbf{0}$.*

(2) *If there exists a solution $\mathbf{w} \in \mathbb{Z}^{n+d+1}$ such that $A_{a2}(D)\mathbf{w} + \mathbf{c} = \mathbf{0}$, then there exists a solution $\mathbf{u} \in \mathbb{Z}^{n+d+1}$ such that $A_{a1}(D)\mathbf{u} + \mathbf{c} = \mathbf{0}$.*

By a similar argument, the converse of Lemma 9.1 is shown as follows.

Lemma 9.2. *Let D be a diagram of an l-component link. We assume that D has d connected components and n crossings $x_1, \cdots, x_n$, $n \geq 1$. Let $R_1, \cdots, R_{n+d+1}$ be the regions of D. We take $\mathbf{c} \in \mathbb{Z}^n$.*

(1) *If there exists a solution* $\mathbf{u} \in \mathbb{Z}^{n+d+1}$ *such that* $A_{d1}(D)\mathbf{u} + \mathbf{c} = \mathbf{0}$*, then there exists a solution* $\mathbf{w} \in \mathbb{Z}^{n+d+1}$ *such that* $A_{d2}(D)\mathbf{w} + \mathbf{c} = \mathbf{0}$.
(2) *If there exists a solution* $\mathbf{u} \in \mathbb{Z}^{n+d+1}$ *such that* $A_{a1}(D)\mathbf{u} + \mathbf{c} = \mathbf{0}$*, then there exists a solution* $\mathbf{w} \in \mathbb{Z}^{n+d+1}$ *such that* $A_{a2}(D)\mathbf{w} + \mathbf{c} = \mathbf{0}$.

Proof. If the given diagram D is irreducible, then we have $A_{d1}(D) = A_{d2}(D)$ and $A_{a1}(D) = A_{a2}(D)$.

We suppose that the diagram D has at least one reducible crossing. For a region R_j, let $\mathcal{X}_j$ be the set of reducible crossings touched by R_j twice, $j = 1, \cdots, n + d + 1$. A set $\mathcal{X}_j$ might be empty. We take $\mathbf{r}_j \in \mathbb{Z}^{n+d+1}$ such that any components of $\mathbf{r}_j$ are 0 but the j-th component is 1. Choosing R_j once corresponds with $\mathbf{r}_j$.

(1) We suppose that $A_{d1}(D)\mathbf{u} + \mathbf{c} = \mathbf{0}$. Let u_j be the j-th component of $\mathbf{u}$, $j = 1, \cdots, n + d + 1$. Applying Theorem 7.3 for each reducible crossing $y \in \mathcal{X}_j$, we obtain $\mathbf{v}'_y \in \mathbb{Z}^{n+d+1}$ such that any components of $A_{d2}(D)\mathbf{v}'_y$ are 0 but the component of $A_{d2}(D)\mathbf{v}'_y$ to y is 1. By the definitions of the definite region choice matrices, we have

$$A_{d2}(D)\mathbf{r}_j - A_{d1}(D)\mathbf{r}_j = \sum_{y \in \mathcal{X}_j} A_{d2}(D)\mathbf{v}'_y.$$

We note $\mathbf{u} = \sum_j u_j \mathbf{r}_j$. We take $\mathbf{w} = \mathbf{u} - \sum_j u_j \sum_{y \in \mathcal{X}_j} \mathbf{v}'_y$. Then we have

$$\begin{aligned} A_{d2}(D)\mathbf{w} &= A_{d2}(D)\mathbf{u} - \sum_j u_j \sum_{y \in \mathcal{X}_j} A_{d2}(D)\mathbf{v}'_y \\ &= \sum_j u_j \left(A_{d2}(D)\mathbf{r}_j - \sum_{y \in \mathcal{X}_j} A_{d2}(D)\mathbf{v}'_y \right) \\ &= \sum_j u_j A_{d1}(D)\mathbf{r}_j \\ &= A_{d1}(D)\mathbf{u} \\ &= -\mathbf{c}. \end{aligned}$$

Therefore we have $A_{d2}(D)\mathbf{w} + \mathbf{c} = \mathbf{0}$.

(2) We suppose that $A_{a1}(D)\mathbf{u} + \mathbf{c} = \mathbf{0}$. Let u_j be the j-th component of $\mathbf{u}$, $j = 1, \cdots, n + d + 1$. Applying Theorem 6.3 for each reducible crossing $y \in \mathcal{X}_j$, we obtain $\mathbf{v}'_y \in \mathbb{Z}^{n+d+1}$ such that any components of $A_{a2}(D)\mathbf{v}'_y$ are 0 but the component of $A_{a2}(D)\mathbf{v}'_y$ to y is 1. By the definitions of the definite region choice matrices, we have

$$A_{a2}(D)\mathbf{r}_j - A_{a1}(D)\mathbf{r}_j = \sum_{y \in \mathcal{X}_j} A_{a2}(D)\mathbf{v}'_y.$$

We note $\mathbf{u} = \sum_j u_j \mathbf{r}_j$. We take $\mathbf{w} = \mathbf{u} - \sum_j u_j \sum_{y \in \mathcal{X}_j} \mathbf{v}'_y$. Then we have

$$\begin{aligned} A_{a2}(D)\mathbf{w} &= A_{a2}(D)\mathbf{u} - \sum_j u_j \sum_{y \in \mathcal{X}_j} A_{a2}(D)\mathbf{v}'_y \\ &= \sum_j u_j \left(A_{a2}(D)\mathbf{r}_j - \sum_{y \in \mathcal{X}_j} A_{a2}(D)\mathbf{v}'_y \right) \end{aligned}$$

$$\begin{aligned} &= \sum_j u_j A_{a1}(D)\mathbf{r}_j \\ &= A_{a1}(D)\mathbf{u} \\ &= -\mathbf{c}. \end{aligned}$$

Therefore we have $A_{a2}(D)\mathbf{w} + \mathbf{c} = \mathbf{0}$.

□

By Lemmas 9.1 and 9.2, we obtain the following result.

Theorem 9.3. *Let D be a link diagram. We assume that D has at least one crossing.*

(1) *The image of the homomorphism $\Phi_{d1}(D)$ coincides with the image of the homomorphism $\Phi_{d2}(D)$.*
(2) *The image of the homomorphism $\Phi_{a1}(D)$ coincides with the image of the homomorphism $\Phi_{a2}(D)$.*

The above theorem implies that the existence of a solution of the single counting rule coincides with that of the double counting rule for each of the both integral region choice problem. Particularly, the first and second results of Theorem 3.1 are equivalent, and the first and second results of Theorem 4.1 are equivalent.

By Theorems 8.1, 9.3, and the homomorphism theorem, we obtain the following theorem.

Theorem 9.4. *Let D be a diagram of an l-component link. We assume that D has d connected components and n crossings, $n \geq 1$. Then each rank of the definite and alternating region choice matrices of the single counting rule, $A_{d1}(D)$ and $A_{a1}(D)$, is $n + d - l$, and each rank of the $\mathbb{Z}$-submodules $\{\mathbf{u} \in \mathbb{Z}^{n+d+1} \mid A_{d1}(D)\mathbf{u} = \mathbf{0}\}$ and $\{\mathbf{u} \in \mathbb{Z}^{n+d+1} \mid A_{a1}(D)\mathbf{u} = \mathbf{0}\}$ is $l + 1$.*

In [4], Chen and Gao determined the $\mathbb{Z}_2$-rank of the incidence matrix for connected link diagrams. In [6], Hashizume generalized their result to disconnected diagrams and connected diagrams, and determined the rank, which she called the $\mathbb{Z}_2$-dimension, of the $\mathbb{Z}_2$-submodule of kernel solutions for the homomorphism induced from region crossing changes on diagrams. The ranks obtained in Theorems 8.1 and 9.4 are the same values as their ranks. If we transpose their incidence matrix, it coincides with the definite region choice matrix of the single counting rule up to permutations of rows and columns. This transposed matrix also coincides with the alternating region choice matrix of the single counting rule modulo 2 up to permutations of rows and columns. Hence Theorems 8.1 and 9.4 are integral extensions of their results.

10. Standard kernel solutions of the double counting rule

In [6, 7], Hashizume studied structures of the $\mathbb{Z}_2$-homomorphism induced by region crossing changes on link diagrams. Particularly, she gave a basis of the kernel of the homomorphism on an irreducible link diagram. In this section, we observe the kernels of the $\mathbb{Z}$-homomorphisms Φ_{a2} and Φ_{d2} given in Section 5.

Lemma 10.1. *On any link diagram with at least one crossing, assigning a same integer to all regions gives a kernel solution for an alternating region choice matrix of the double counting rule.*

Proof. Any crossing is touched by four corners of regions, and we have $p - p + p - p = 0$ for any integer p. □

We denote by $\mathbf{u}_\infty$ the kernel solution assigning 1 to all regions, as illustrated in Fig.16.

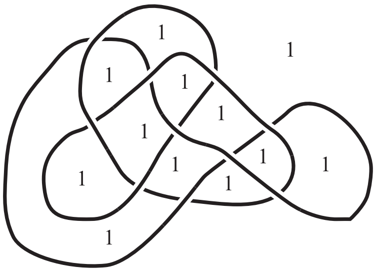


Fig. 16. The kernel solution $\mathbf{u}_\infty$.

Let D be an oriented link diagram with ordered link components, and D_i be a sub-diagram of D representing i-th link component, $i = 1, \cdots, l$. If the oriented link diagram D is a diagram on the plane $\mathbb{R}^2$, we may denote the unbounded region by $R_\infty(D)$. If D is a diagram on the sphere $S^2 = \mathbb{R}^2 \cup \{\infty\}$, we may denote the region including the infinite point by $R_\infty(D)$. For each sub-diagram D_i, we denote by $\mathbf{u}_i$ the kernel solution obtained by Lemma 5.2 from a componentwise Alexander numbering associated with D_i such that the region $R_\infty(D)$ is assigned 0. We shall call $\mathbf{u}_i$ the *standard kernel solution associated with* D_i. Fig.17, same as Fig.5, gives examples of standard kernel solutions on the same link diagram as Fig.16 and Fig.6. The kernel solution given by an Alexander numbering for D is equal to $r\mathbf{u}_\infty + \sum_{i=1}^{l} \mathbf{u}_i$ where the region $R_\infty(D)$ is assigned the integer r in the Alexander numbering.

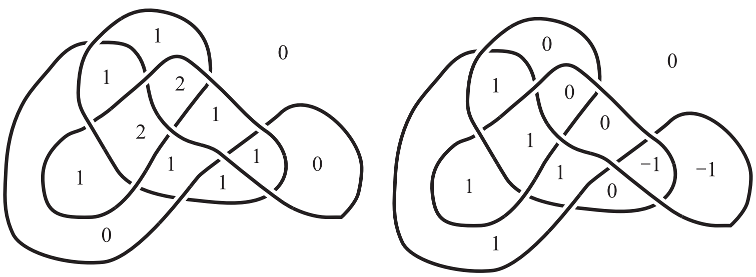


Fig. 17. The standard kernel solutions $\mathbf{u}_1$ and $\mathbf{u}_2$.

Theorem 10.2. *Let D be an oriented link diagram with l ordered link components and at least one crossing, and $R_\infty = R_\infty(D)$ be the above region. The set of the above kernel solutions $\mathbf{u}_1, \cdots, \mathbf{u}_l$, and $\mathbf{u}_\infty$ is a basis of the kernel of the homomorphism induced by the alternating integral region choice problem of double counting rule.*

Proof. For the linear independence of $\mathbf{u}_1, \cdots, \mathbf{u}_l, \mathbf{u}_\infty$, it is sufficient to prove that the standard kernel solutions are linearly independent, since the region R_∞ is assigned 1 by $\mathbf{u}_\infty$ and 0 by $\mathbf{u}_i$, $i = 1, \cdots, l$. We use an induction on l. If D is a knot diagram, the standard kernel solution associated with D has at least one component equal to 1 or -1. Then it is linearly independent. We assume that $l \geq 2$ and that the standard kernel solutions are linearly independent on any oriented link diagram with less components than l. Let D be an oriented link diagram with l components. We take a point p on $\partial R_\infty(D)$ except crossings. We may assume that p lies on l-th link component diagram D_l. We suppose $\sum_{i=1}^{l} n_i\mathbf{u}_i = \mathbf{0}$, $n_i \in \mathbb{Z}$. Let R_p be a region of D with $p \in \partial R_p$ and $R_p \neq R_\infty$. The region R_p is assigned 1 or -1 by $\mathbf{u}_l$ and 0 by each of the other standard kernel solutions. Then we have $n_l = 0$ and $\sum_{i=1}^{l-1} n_i\mathbf{u}_i = \mathbf{0}$. On the diagram obtained from D ignoring D_l, the standard kernel solutions are linearly independent by the assumption of the induction. Then $n_1, \cdots, n_{l-1}$ should be 0. Hence the standard kernel solutions on D are linearly independent.

Let $\mathbf{x}$ be a kernel solution of the homomorphism $\Phi_{a2}(D)$. By Theorem 8.1, the rank of the kernel of the homomorphism $\Phi_{a2}(D)$ is $l+1$. Then $\mathbf{x}, \mathbf{u}_1, \cdots, \mathbf{u}_l, \mathbf{u}_\infty$ are linearly dependent, since $\mathbf{u}_1, \cdots, \mathbf{u}_l, \mathbf{u}_\infty$ are linearly independent. Hence $\mathbf{x} = \sum_{i=1}^{l} q_i\mathbf{u}_i + q_\infty\mathbf{u}_\infty$ holds for certain rational numbers $q_1, \cdots, q_l, q_\infty$. We show that $q_1, \cdots, q_l, q_\infty$ are integers. The region R_∞ is assigned 1 by $\mathbf{u}_\infty$ and 0 by $\mathbf{u}_i$, $i = 1, \cdots, l$. Hence q_∞ is an integer since the components of $\mathbf{x}$ are integers. Then it is sufficient to show that $q_1, \cdots, q_l$ are integers if the all components of $\sum_{i=1}^{l} q_i\mathbf{u}_i$ are integers. We use an induction on l. If D is a knot diagram, the standard kernel solution associated with D has at least one component equal to 1 or -1. Then $q_1 \in \mathbb{Z}$ holds. We assume that $l \geq 2$ and that the desired claim holds for any oriented link diagram with less components than l. Let D be an oriented link diagram with l components. The above region R_p is assigned 1 or -1 by $\mathbf{u}_l$ and 0 by each of the other standard kernel solutions. Hence $q_l \in \mathbb{Z}$ holds and all components of $\sum_{i=1}^{l-1} q_i\mathbf{u}_i$ are integers. We apply the assumption of the induction to the diagram obtained from D ignoring D_l. Then we have $q_1, \cdots, q_{l-1} \in \mathbb{Z}$.

Therefore $\mathbf{x}$ is a linear combination of $\mathbf{u}_1, \cdots, \mathbf{u}_l, \mathbf{u}_\infty$ over $\mathbb{Z}$. Then the set of the kernel solutions $\mathbf{u}_1, \cdots, \mathbf{u}_l$, and $\mathbf{u}_\infty$ is a basis of the kernel of the homomorphism $\Phi_{a2}(D)$. □

For the above link diagram D, we fix a checkerboard coloring. We apply Lemma 5.3 to the above basis $\mathbf{u}_1, \cdots, \mathbf{u}_l, \mathbf{u}_\infty$. Then we obtain kernel solutions of the definite integral region choice problem of double counting rule. We denote them by $\bar{\mathbf{u}}_1, \cdots, \bar{\mathbf{u}}_l, \bar{\mathbf{u}}_\infty$.

Theorem 10.3. *Let D be an oriented link diagram with l ordered link components and at least one crossing, and $R_\infty = R_\infty(D)$ be the above region. The set of the above kernel solutions $\bar{\mathbf{u}}_1, \cdots, \bar{\mathbf{u}}_l$, and $\bar{\mathbf{u}}_\infty$ is a basis of the kernel of the homomorphism induced by the definite integral region choice problem of double counting rule.*

Proof. The linear independence of $\bar{\mathbf{u}}_1, \cdots, \bar{\mathbf{u}}_l, \bar{\mathbf{u}}_\infty$ is shown by the similar argument to that for $\mathbf{u}_1, \cdots, \mathbf{u}_l, \mathbf{u}_\infty$ in the proof of Theorem 10.2. Let $\mathbf{y}$ be a kernel solution of the homomorphism $\Phi_{d2}(D)$. By Theorem 8.1, the rank of the kernel of the homomorphism $\Phi_{d2}(D)$ is $l+1$. Then $\mathbf{y}, \bar{\mathbf{u}}_1, \cdots, \bar{\mathbf{u}}_l, \bar{\mathbf{u}}_\infty$ are linearly dependent, since $\bar{\mathbf{u}}_1, \cdots, \bar{\mathbf{u}}_l, \bar{\mathbf{u}}_\infty$ are linearly independent. By the similar argument to that for a kernel solution $\mathbf{x}$ of $\Phi_{a2}(D)$ in the proof of Theorem

10.2, $\mathbf{y}$ is a linear combination of $\bar{\mathbf{u}}_1, \cdots, \bar{\mathbf{u}}_l, \bar{\mathbf{u}}_\infty$ over $\mathbb{Z}$. Therefore the set of the kernel solutions $\bar{\mathbf{u}}_1, \cdots, \bar{\mathbf{u}}_l$, and $\bar{\mathbf{u}}_\infty$ is a basis of the kernel of the homomorphism $\Phi_{d2}(D)$. □

Fig.18 gives an example of the basis obtained by Theorem 10.3.

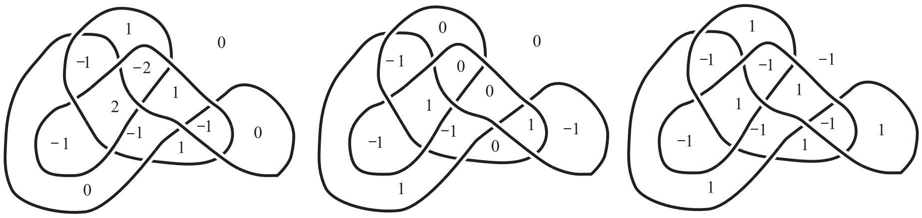


Fig. 18. The basis $\bar{\mathbf{u}}_1$, $\bar{\mathbf{u}}_2$, $\bar{\mathbf{u}}_\infty$.

Theorems 10.2 and 10.3 are extensions of the result due to Hashizume [6] on a region crossing change. Her basis of the kernel of $\mathbb{Z}_2$-homomorphism induced by region crossing changes is the same as the basis given in Theorem 10.2 and the basis given in Theorem 10.3 modulo 2.

11. Images of homomorphisms induced by integral region choice problems

Let D be a link diagram with at least one crossing. For each $i = 1, 2$, the system of linear equations $A_{di}(D)\mathbf{u} + \mathbf{c} = \mathbf{0}$ (resp. $A_{ai}(D)\mathbf{u} + \mathbf{c} = \mathbf{0}$) is solvable if and only if $\mathbf{c}$ lies in the image of the homomorphism $\Phi_{di}(D)$ (resp. $\Phi_{ai}(D)$). In this section, we discuss about the images of the homomorphisms induced by integral region choice problems.

By Theorems 3.1 and 4.1, the homomorphisms $\Phi_{d1}(D)$, $\Phi_{d2}(D)$, $\Phi_{a1}(D)$, $\Phi_{a2}(D)$ defined in Section 5 are surjective if D is a knot diagram. Otherwise they are not surjective in general by Theorems 8.1 and 9.4.

In [4], Cheng and Gao proved that a region crossing change on a 2-component link diagram is an unknotting operation if and only if the linking number is even, showing that changing two crossings of different components on a 2-component link diagram is represented by certain region crossing changes.

For example, the canonical diagram of (2, 4)-torus link changes to a diagram of the trivial link by one region choice. Otherwise there exists an equipment of integers to the crossings on this diagram such that the definite and alternating integral region choice problem does not have any solution. On the canonical diagram of (2, 4)-torus link with certain orders of crossings and regions, the definite region choice matrix is

$$A_d = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{pmatrix}$$

and the alternating region choice matrix is

$$A_a = \begin{pmatrix} -1 & 1 & 1 & -1 & 0 & 0 \\ 0 & 1 & 1 & -1 & -1 & 0 \\ 0 & 1 & 1 & 0 & -1 & -1 \\ -1 & 1 & 1 & 0 & 0 & -1 \end{pmatrix}.$$

The system of linear equations $A_d\mathbf{u} + \mathbf{c} = \mathbf{0}$ has a solution $\mathbf{u} \in \mathbb{Z}^6$ if and only if $c_1 - c_2 + c_3 - c_4 = 0$ holds where c_i is the i-th component of $\mathbf{c} \in \mathbb{Z}^4$. The system of linear equations $A_a\mathbf{u} + \mathbf{c} = \mathbf{0}$ is solvable if and only if $c_1 - c_2 + c_3 - c_4 = 0$ holds. Then in the case $(c_1, c_2, c_3, c_4) = (1, 0, 0, -1)$, any $\mathbf{u} \in \mathbb{Z}^6$ does not hold $A_d\mathbf{u} + \mathbf{c} = \mathbf{0}$ or $A_a\mathbf{u} + \mathbf{c} = \mathbf{0}$, though we have

$$A_d \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} = - \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix} \in \mathbb{Z}_2{}^4,$$

$$A_a \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 0 \\ -1 \end{pmatrix} = - \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix} \in \mathbb{Z}_2{}^4.$$

In [7], Hashizume gave a generating system of the image of the $\mathbb{Z}_2$-homomorphism induced by region crossing changes on a link diagram. Their results include the following results.

Lemma 11.1 ([4, 7])**.** *Let D be a connected diagram of two-component link with n crossings. We take two distinct crossings x, y of D of arcs in different link components. There exist $\mathbf{v}_{xy} \in \mathbb{Z}_2^{n+2}$ such that any components of $A_{d1}(D)\mathbf{v}_{xy} \in \mathbb{Z}_2{}^n$ are 0 but the components of $A_{d1}(D)\mathbf{v}_{xy} \in \mathbb{Z}_2{}^n$ to x and y are 1.*

Theorem 11.2 ([7])**.** *Let D be a connected diagram of two-component link with n crossings. The image of the $\mathbb{Z}_2$-homomorphism induced by region crossing changes is generated by the elements in $\mathbb{Z}_2{}^n$ of the following two types:*

(1) *any components are 0 but the only one component corresponding to a crossing in same link component is 1;*
(2) *any components are 0 but the only two components corresponding to two distinct crossings of distinct link components are 1.*

We note that a connected diagram of two-component link has at least two crossings because of the Jordan curve theorem.

We extend Lemma 11.1 to the alternating integral region choice problem as follows, where we define $\varepsilon_x = 1$ for a positive crossing x and $\varepsilon_x = -1$ for a negative crossing x.

Lemma 11.3. *Let $D = D_1 \cup D_2$ be a connected diagram of two-component oriented link with n crossings, where D_1 and D_2 are sub-diagram of D representing the first and the second components respectively. We take two distinct crossings x, y of D_1 and D_2. We suppose that D_2 crosses D_1 from the right to the left at x.*

(1) *If D_2 crosses D_1 from the left to the right at y, then there exist $\mathbf{v}_{xy} \in \mathbb{Z}^{n+2}$ such that any components of $A_{a2}(D)\mathbf{v}_{xy} \in \mathbb{Z}^n$ are 0 but the components of $A_{a2}(D)\mathbf{v}_{xy} \in \mathbb{Z}^n$ to x and y are ε_x and ε_y.*

(2) *If D_2 crosses D_1 from the right to the left at y, then there exist $\mathbf{v}_{xy} \in \mathbb{Z}^{n+2}$ such that any components of $A_{a2}(D)\mathbf{v}_{xy} \in \mathbb{Z}^n$ are 0 but the components of $A_{a2}(D)\mathbf{v}_{xy} \in \mathbb{Z}^n$ to x and y are ε_x and $-\varepsilon_y$.*

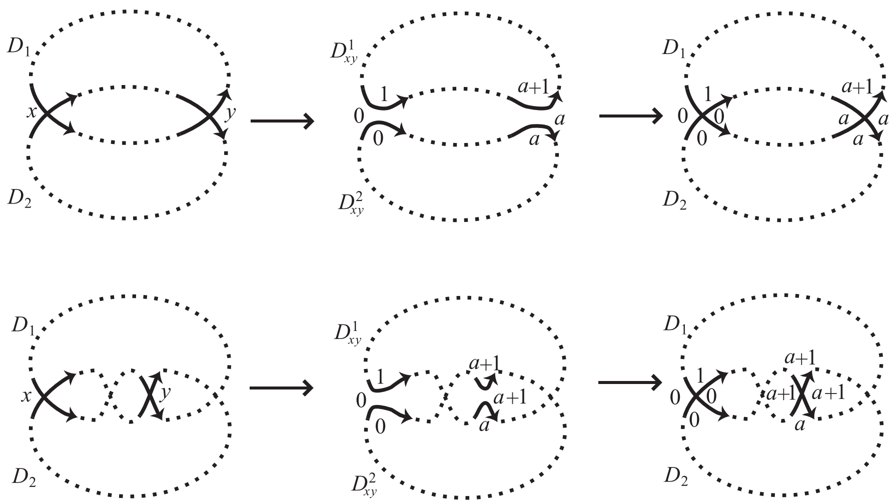


Fig. 19. Obtaining $\mathbf{v}_{xy}$ in the two cases.

Proof. We splice at x. Let γ_1 and γ_2 be oriented arcs appearing after the splice at x on the obtained diagram. We suppose that γ_1 lies on the left of γ_2. We splice at y. We obtain a new diagram of a two-component link as illustrated on the middle of Fig.19, where the cases (1) and (2) are described in the upper and lower rows respectively. We denote the sub-diagram of the link component including the arc γ_i by D^i_{xy}, $i = 1, 2$. For the diagram $D^1_{xy} \cup D^2_{xy}$, we take the componentwise Alexander numbering associated with D^1_{xy} such that the right and left regions of γ_1 are assigned 0 and 1 respectively. We denote by a the integer assigned to the right region of two oriented arcs which appear after the splice at y. In the case (1), D^1_{xy} includes the left of these two arcs, then the region between the arcs is assigned a, and the left region is assigned $a + 1$. In the case (2), D^1_{xy} includes the right of these two arcs, then the both regions adjacent to the left arc are assigned $a + 1$. By Lemma 5.2, this numbering gives a kernel solution for $A_{a2}(D^1_{xy} \cup D^2_{xy})$ if $D^1_{xy} \cup D^2_{xy}$ has at least one crossing. We unsplice $D^1_{xy} \cup D^2_{xy}$ at x and y. We assign the same integers to all regions of D as $D^1_{xy} \cup D^2_{xy}$, where the two regions splitting at x are assigned 0, and where the two regions splitting at y are assigned a and $a + 1$ in the case (1) and (2) respectively, as illustrated on the right of Fig.19. Then the obtained assignment of integers to regions is the desired $\mathbf{v}_{xy} \in \mathbb{Z}^{n+2}$ in the both cases. □

By Lemmas 9.1 (2) and 11.3, we obtain the following result.

Corollary 11.4. *Let $D = D_1 \cup D_2$ be a connected diagram of two-component oriented link with n crossings, where D_1 and D_2 are sub-diagram of D representing the first and the second components respectively. We take two distinct crossings x, y of D_1 and D_2. We suppose that D_2 crosses D_1 from the right to the left at x.*

(1) *If D_2 crosses D_1 from the left to the right at y, then there exist $\mathbf{v}_{xy} \in \mathbb{Z}^{n+2}$ such that any components of $A_{a1}(D)\mathbf{v}_{xy} \in \mathbb{Z}^n$ are 0 but the components of $A_{a1}(D)\mathbf{v}_{xy} \in \mathbb{Z}^n$ to x and y are ε_x and ε_y.*

(2) *If D_2 crosses D_1 from the right to the left at y, then there exist $\mathbf{v}_{xy} \in \mathbb{Z}^{n+2}$ such that any components of $A_{a1}(D)\mathbf{v}_{xy} \in \mathbb{Z}^n$ are 0 but the components of $A_{a1}(D)\mathbf{v}_{xy} \in \mathbb{Z}^n$ to x and y are ε_x and $-\varepsilon_y$.*

We note that Lemma 11.1 is a modulo 2 reduction of Corollary 11.4.

To construct a generating system of the image of $\Phi_{a2}(D)$ for a connected diagram of a 2-component link $D = D_1 \cup D_2$ extending Theorem 11.2, we do not need all pair of the distinct crossings of D_1 and D_2, since we obtain the following result by easy calculation.

Corollary 11.5. *Let $D = D_1 \cup D_2$ be a connected diagram of two-component oriented link with n crossings, where D_1 and D_2 are sub-diagram of D representing the first and the second components respectively. We suppose that there exist three distinct crossings x, y, z of D_1 and D_2, and that D_2 crosses D_1 from the right to the left at x. Let $\mathbf{v}_{xy}, \mathbf{v}_{xz} \in \mathbb{Z}^n$ be obtained by Lemma 11.3.*

(1) *If D_2 crosses D_1 from the left to the right at y and z, then any components of $A_{a2}(D)(\mathbf{v}_{xy} - \mathbf{v}_{xz})$ are 0 but the components of $A_{a2}(D)(\mathbf{v}_{xy} - \mathbf{v}_{xz})$ to y and z are ε_y and $-\varepsilon_z$ respectively.*

(2) *If D_2 crosses D_1 from the left to the right at y and from the right to the left at z, then any components of $A_{a2}(D)(\mathbf{v}_{xy} - \mathbf{v}_{xz})$ are 0 but the components of $A_{a2}(D)(\mathbf{v}_{xy} - \mathbf{v}_{xz})$ to y and z are ε_y and ε_z respectively.*

(3) *If D_2 crosses D_1 from the right to the left at y and z, then any components of $A_{a2}(D)(\mathbf{v}_{xz} - \mathbf{v}_{xy})$ are 0 but the components of $A_{a2}(D)(\mathbf{v}_{xz} - \mathbf{v}_{xy})$ to y and z are ε_y and $-\varepsilon_z$ respectively.*

We note that the above $\mathbf{v}_{xy} - \mathbf{v}_{xz}$ or $\mathbf{v}_{xz} - \mathbf{v}_{xy}$ are not equal to $\mathbf{v}_{yz}$ obtained in the proof of Lemma 11.3 generally.

We obtain a basis of the image of the homomorphism of the alternating integral region choice problem as belows.

Theorem 11.6. *Let $D = D_1 \cup D_2$ be a connected diagram of two-component oriented link with n crossings $x_1, \cdots, x_n$, where D_1 and D_2 are sub-diagram of D representing the first and the second components respectively. We suppose that each of $x_1, \cdots, x_k$ $(k < n)$ is a crossing in D_1 or a crossing in D_2, $x_{k+1}, \cdots, x_n$ are crossings of D_1 and D_2, and D_2 crosses D_1 from the right to the left at x_n. We take $\mathbf{e}_1, \cdots, \mathbf{e}_{n-1} \in \mathbb{Z}^n$ as belows:*

(1) *for $i = 1, \cdots, k$, let $\mathbf{e}_i$ be the element of $\mathbb{Z}^n$ such that the i-th component is 1 and the others are 0;*

(2) *for $i = k+1, \cdots, n-1$, let $\mathbf{e}_i$ be the element of $\mathbb{Z}^n$ such that the n-th component is ε_{x_n} and the i-th component is ε_{x_i} (resp. $-\varepsilon_{x_i}$) if D_2 crosses D_1 at x_i from the left to*

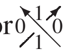
the right (resp. from the right to the left), and that the others are 0.

Then the set of $\mathbf{e}_1, \cdots, \mathbf{e}_{n-1}$ *is a basis of the image of the homomorphism induced by the alternating integral region choice problem. Therefore the systems of linear equations* $A_{a1}(D_1 \cup D_2)\mathbf{u} + \mathbf{c} = \mathbf{0}$ *and* $A_{a2}(D_1 \cup D_2)\mathbf{w} + \mathbf{c} = \mathbf{0}$ *have solutions* $\mathbf{u}, \mathbf{w} \in \mathbb{Z}^{n+2}$ *if and only if* $\mathbf{c} \in \mathbb{Z}^n$ *is a linear combination of* $\mathbf{e}_1, \cdots, \mathbf{e}_{n-1}$.

Proof. By Theorem 6.3 and Lemma 11.3, $\mathbf{e}_1, \cdots, \mathbf{e}_{n-1}$ are elements in the image of the homomorphism $\Phi_{a2}(D_1 \cup D_2)$. They are linearly independent by the construction. Let $\mathbf{c}$ be an element of the image of the homomorphism $\Phi_{a2}(D_1 \cup D_2)$. By Theorem 8.1, the rank of the image of the homomorphism $\Phi_{a2}(D_1 \cup D_2)$ is $n + 1 - 2 = n - 1$. Hence $\mathbf{c}, \mathbf{e}_1, \cdots, \mathbf{e}_{n-1}$ are linearly dependent since $\mathbf{e}_1, \cdots, \mathbf{e}_{n-1}$ are linearly independent. By the construction of $\mathbf{e}_1, \cdots, \mathbf{e}_{n-1}$, it is shown that $\mathbf{c}$ is a linear combination of $\mathbf{e}_1, \cdots, \mathbf{e}_{n-1}$ over $\mathbb{Z}$. Then the set of $\mathbf{e}_1, \cdots, \mathbf{e}_{n-1}$ is a basis of the image of the homomorphism $\Phi_{a2}(D_1 \cup D_2)$.

By Theorem 9.3 (2), the image of the homomorphism $\Phi_{a1}(D_1 \cup D_2)$ coincides with the image of the homomorphism $\Phi_{a2}(D_1 \cup D_2)$. □

We note that the modulo 2 reduction of Theorem 11.6 implies Theorem 11.2.

From the above basis, we obtain a basis of the image of the homomorphism of the definite integral region choice problem as belows. Let $D = D_1 \cup D_2$ be a connected diagram of two-component oriented link with n crossings $x_1, \cdots, x_n$. Let $R_1, \cdots, R_{n+2}$ be the regions of D. We suppose that each of $x_1, \cdots, x_k$ $(k < n)$ is a crossing in D_1 or a crossing in D_2, $x_{k+1}, \cdots, x_n$ are crossings of D_1 and D_2, and D_2 crosses D_1 from the right to the left at x_n. For each of $\mathbf{e}_1, \cdots, \mathbf{e}_{n-1}$ obtained by Theorem 11.6, there exists a solution $\mathbf{v}_i \in \mathbb{Z}^{n+2}$ of $A_{a2}(D)\mathbf{v}_i = \mathbf{e}_i$. We take the checkerboard coloring such that the left and right regions of both oriented arcs crossing at x_n are assigned 0 as  or . For each $i = 1, \cdots, k$, there exists a solution $\bar{\mathbf{v}}_i \in \mathbb{Z}^{n+2}$ of $A_{d2}(D)\bar{\mathbf{v}}_i = \mathbf{e}_i$ by Theorem 7.3. For each $i = k + 1, \cdots, n - 1$, we multiply the j-th component of $\mathbf{v}_i$ by -1 if the region R_j is assigned the checkerboard index 1. We denote by $\bar{\mathbf{v}}_i$ the obtained element of $\mathbb{Z}^{n+2}$ from $\mathbf{v}_i$. Let $\bar{\mathbf{e}}_i = A_{d2}(D)\bar{\mathbf{v}}_i$. Then the n-th component of $\bar{\mathbf{e}}_i$ becomes 1, the i-th component becomes 1 or -1, and the others are 0.

Theorem 11.7. *The set of the above* $\mathbf{e}_1, \cdots, \mathbf{e}_k, \bar{\mathbf{e}}_{k+1}, \cdots, \bar{\mathbf{e}}_{n-1}$ *is a a basis of the image of the homomorphism induced by the definite integral region choice problem on the connected diagram of two-component link* $D = D_1 \cup D_2$. *Therefore the systems of linear equations* $A_{d1}(D_1 \cup D_2)\mathbf{u} + \mathbf{c} = \mathbf{0}$ *and* $A_{d2}(D_1 \cup D_2)\mathbf{w} + \mathbf{c} = \mathbf{0}$ *have solutions* $\mathbf{u}, \mathbf{w} \in \mathbb{Z}^{n+2}$ *if and only if* $\mathbf{c} \in \mathbb{Z}^n$ *is a linear combination of* $\mathbf{e}_1, \cdots, \mathbf{e}_k, \bar{\mathbf{e}}_{k+1}, \cdots, \bar{\mathbf{e}}_{n-1}$.

Proof. By the construction, $\mathbf{e}_1, \cdots, \mathbf{e}_k, \bar{\mathbf{e}}_{k+1}, \cdots, \bar{\mathbf{e}}_{n-1}$ are elements in the image of the homomorphism $\Phi_{d2}(D_1 \cup D_2)$ and linearly independent. Let $\mathbf{c}$ be an element of the image of the homomorphism $\Phi_{d2}(D_1 \cup D_2)$. By Theorem 8.1, the rank of the image of the homomorphism $\Phi_{d2}(D_1 \cup D_2)$ is $n + 1 - 2 = n - 1$. Hence $\mathbf{c}, \mathbf{e}_1, \cdots, \mathbf{e}_k, \bar{\mathbf{e}}_{k+1}, \cdots, \bar{\mathbf{e}}_{n-1}$ are linearly dependent since $\mathbf{e}_1, \cdots, \mathbf{e}_k, \bar{\mathbf{e}}_{k+1}, \cdots, \bar{\mathbf{e}}_{n-1}$ are linearly independent. By the construction of $\mathbf{e}_1, \cdots, \mathbf{e}_k, \bar{\mathbf{e}}_{k+1}, \cdots, \bar{\mathbf{e}}_{n-1}$, it is shown that $\mathbf{c}$ is a linear combination of $\mathbf{e}_1, \cdots, \mathbf{e}_k, \bar{\mathbf{e}}_{k+1}, \cdots, \bar{\mathbf{e}}_{n-1}$ over $\mathbb{Z}$. Then the set of $\mathbf{e}_1, \cdots, \mathbf{e}_k, \bar{\mathbf{e}}_{k+1}, \cdots, \bar{\mathbf{e}}_{n-1}$ is a basis of the image of the homomorphism $\Phi_{d2}(D_1 \cup D_2)$.

By Theorem 9.3 (1), the image of the homomorphism $\Phi_{d1}(D_1 \cup D_2)$ coincides with the image of the homomorphism $\Phi_{d2}(D_1 \cup D_2)$. □

REMARK 11.8. In [7], Hashizume gave a generating system of the image of the $\mathbb{Z}_2$-homomorphism induced by region crossing changes for a link diagram with arbitrary number of link components. Her generating system includes that in Theorem 11.2. For each of the $\mathbb{Z}$-homomorphisms induced by integral region choice problems on a link diagram with arbitrary number of link components, we are finding a basis of the image.

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