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CONTACT GEOMETRY OF THIRD-ORDER PARTIAL DIFFERENTIAL EQUATIONS WITH TWO INDEPENDENT VARIABLES

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Abstract

We develop rigorously a geometric theory of third-order partial differential equations for a scalar function. Under our framework, we can define a notion of nilpotent graded Lie algebras as an invariant useful to study geometry of third-order equations. In terms of these graded Lie algebras, we provide a classification for some classes of third-order equations under a contact equivalence. By this classification, together with model equations, we also clarify several aspects for each subcategory of equations.

1. Introduction

Geometry of partial differential equations started from the classical theory by Monge, Lie, Darboux, Goursat, E. Cartan and others (cf. [2], [3], [5], [7]). In a series of research, they explored many techniques for the investigation into the geometric theory of differential equations. Notable topics include the reduction of the problem by some characteristic systems (e.g. Cauchy, Monge characteristics) and the establishment the notion of involutive systems. The former brought us the theory of quadrature via the reduction into lower dimensional spaces. On the other hand, the latter indicated a criterion for the existence of (local) solutions and the prolongation scheme by using the theory of (exterior) differential systems. By the improvement of these approaches, for partial differential equations up to second order, deep understanding have been obtained together with abundant examples. In particular, geometry of second-order partial differential equations has provided the various interesting subjects. Especially notable among them are applications to the study of surfaces with various curvature conditions (cf. [5], [9]) and relationship to parabolic geometry (cf. [17], [20]) modeled after the homogeneous space G/G', where G is a simple Lie group and G' is a parabolic subgroup of G. Nowadays, these geometric theory of second-order partial differential equations have been described by means of (exterior) differential systems (cf. [1], [6], [9], [19], [21]).

As we mentioned above, the geometry of partial differential equations up to second order gave substantial results rather than the formal theory. Succeeding to this tradition, in the present paper, we provide a rigorous development of the geometric theory of third-order partial differential equations for a scalar function. Roughly speaking, we formulate this geometry as the *theory of submanifolds R of the* 3*-jet space J^3 together with an appropri-*

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ate (*tangential*) *filtration*. This formulation enables us to clarify the notion of isomorphism (contact-equivalence) for all third-order equations. By utilizing this filtration, we can define a nilpotent graded Lie algebra called the *symbol algebra* of the third-order equations $R \subset J^3$ at each point as a fundamental invariant. In general, this notion of the symbol algebra clarify the analytic, algebraic and geometric aspects of differential equations, especially the existence of (local) solutions of equations and their degree of freedom, symmetries (infinitesimal automorphisms), the geometric behavior of the corresponding distribution, etc. Thus, we also study the structure of the symbol algebra in our situation.

Now let us proceed to describe the contents of each section. In section 2, we recall some notions and terminologies which are necessary for the geometric study of partial differential equations. Especially, we prepare basics of (linear) differential systems such as derived systems, weak-derived systems, Cauchy characteristic systems, (tangential) filtrations, and the symbol algebra $\mathfrak{s}(x)$ at each point x on R associated with a filtration. In section 3, we reconsider the 3-jet space J^3 as the manifold equipped with the filtration defined by the weak-derived systems $\partial^{(k)}C^3$ of the canonical contact system C^3. Then the symbol algebra $\mathfrak{m}$ at each point on the 3-jet space J^3 is isomorphic to the contact algebra $\mathfrak{c}^3$. Moreover, we also mention the graded Lie algebra automorphism group $Aut(\mathfrak{c}^3)$ of this contact algebra $\mathfrak{c}^3$. In section 4, we give the detailed formulation of a geometric theory of third-order partial differential equations. Here, our framework can be applied to any third-order equations for a scalar function with n independent variables. However, in this paper, we treat the case of two independent variables to obtain more concrete and essential results. Hence, the above discussion for the 3-jet space J^3 is also given in the case of two independent and one dependent variables. Then, we provide the rigorous formulation of geometry of third-order equations as the theory of submanifolds $(R; D^1, D^2, D^3)$ of the 3-jet space J^3, where R is a manifold together with a triplet of appropriate differential systems D^1, D^2 and D^3. Here, D^3, D^2 and D^1 are the restrictions to R of the canonical system C^3, the lift of the second-order canonical system C^2 and the lift of the first-order canonical system C^1 respectively. This geometric object $(R; D^1, D^2, D^3)$ satisfies an appropriate condition for the Lie bracket $[\ ,\ ]$ of vector fields, hence it defines the (*tangential*) *filtration* $TR \supset D^1 \supset D^2 \supset D^3$ on R (see, section 2.3). By utilizing this filtration, we can define the symbol algebra $\mathfrak{s}(v) = \bigoplus_{p=-1}^{-4} \mathfrak{s}_p(v)$ of $R \subset J^3$ at each $v \in R$. Then we provide the classification of these symbol algebras in the cases of submanifolds of codimension 1 (i.e. single equations) and submanifolds of codimension 3 (i.e. systems of three equations) as the main results (Theorem 4.7 and Theorem 4.11). From these results, we can show the *duality* between these two classes of equations. We do not discuss the cases of codimension 2 and 4 in this paper. Although the framework is slightly different, the case of codimension 4 is treated in [15]. We next reveal the analytic and algebraic aspects of each subcategory based on the above classification together with model equations. In particular, we establish involutivity and the uniqueness of the integral element (Corollary 4.9 and Corollary 4.10).

2. Differential systems and symbol algebras

2.1. Derived systems and weak derived systems. By a differential system (R, D), we mean a distribution D on a manifold R, that is, D is a subbundle of the tangent bundle TR of R. The sheaf of sections to D is denoted by $\mathcal{D} = \Gamma(D)$. The derived system ∂D of a differential system D is defined, in terms of sections, by $\partial\mathcal{D} := \mathcal{D} + [\mathcal{D},\mathcal{D}]$. In general, ∂D is obtained as a subsheaf of the tangent sheaf of R. Moreover, higher derived systems $\partial^k D$ are defined successively by $\partial^k\mathcal{D} := \partial(\partial^{k-1}\mathcal{D})$, where we set $\partial^0 D = D$ by convention. On the other hand, the k-th weak derived systems $\partial^{(k)}D$ of D are defined inductively by $\partial^{(k)}\mathcal{D} := \partial^{(k-1)}\mathcal{D}+[\mathcal{D},\partial^{(k-1)}\mathcal{D}]$, where $\partial^{(0)}D = D$. These derived systems are also interpreted by using annihilators as follows; Let $D = \{\varpi_1 = \cdots = \varpi_s = 0\}$ be a differential system on R. We denote by $D^\perp$ the annihilator subbundle of D in T^*R. Then the annihilator $(\partial D)^\perp$ of the first derived system $\partial D(=\partial^{(1)}D)$ of D is given by $(\partial D)^\perp = \{\varpi \in D^\perp \mid d\varpi \equiv 0 \mod D^\perp\}$. Moreover, for $k \geq 1$, the annihilator $(\partial^{k+1}D)^\perp$ of the $(k+1)$-th derived system of D is given by

$$(\partial^{k+1}D)^\perp = \{\varpi \in (\partial^k D)^\perp \mid d\varpi \equiv 0 \mod (\partial^k D)^\perp\}.$$

On the other hand, the annihilator $(\partial^{(k+1)}D)^\perp$ of the $(k+1)$-th weak derived system of D is also given by

$$(\partial^{(k+1)}D)^\perp = \{\varpi \in (\partial^{(k)}D)^\perp \mid d\varpi \equiv 0 \mod (\partial^{(k)}D)^\perp,\ (D)^\perp/(\partial^{(k)}D)^\perp \wedge (D)^\perp/(\partial^{(k)}D)^\perp\}.$$

We explain the meaning of the notation $\mod (D)^\perp/(\partial^{(k)}D)^\perp \wedge (D)^\perp/(\partial^{(k)}D)^\perp$. For 2-forms ω, η, we write $\omega \equiv \eta \mod (D)^\perp/(\partial^{(k)}D)^\perp \wedge (D)^\perp/(\partial^{(k)}D)^\perp$ if $\omega = \eta + \theta$ for some $\theta = \varpi_1 \wedge \varpi_2$, where 1-forms ϖ_1, ϖ_2 are any representatives of elements $[\varpi_1]$, $[\varpi_2]$ in $(D)^\perp/(\partial^{(k)}D)^\perp$. A differential system D is called *regular* (resp. *weakly regular*), if $\partial^k D$ (resp. $\partial^{(k)}D$) is a subbundle for each k. We set $D^{-1} := D$, $D^{-k} := \partial^{(k-1)}D$ $(k \geq 2)$, for a weakly regular differential system D. Then we have ([16, Proposition 1.1]);

(T1) There exists a unique positive integer μ such that

$$D^{-1} \subset D^{-2} \subset \cdots \subset D^{-k} \subset \cdots \subset D^{-(\mu-1)} \subset D^{-\mu} = D^{-(\mu+1)} = \cdots;$$

(T2) $[\mathcal{D}^p, \mathcal{D}^q] \subset \mathcal{D}^{p+q}$ for all $p,\ q < 0$.

Let D be a differential system on R defined by local 1-forms $\varpi_1, \ldots, \varpi_s$ such that $\varpi_1 \wedge \cdots \wedge \varpi_s \neq 0$ at each point, where s is the corank of D, namely, $D = \{\varpi_1 = \cdots = \varpi_s = 0\}$. Then the Cauchy characteristic system $Ch(D)$ is defined at each point $x \in R$ by

$$Ch(D)(x) := \{X \in D(x) \mid X \rfloor d\varpi_i \equiv 0 \pmod{\varpi_1, \ldots, \varpi_s} \quad \text{for } i = 1, \ldots, s\},$$

where $\rfloor$ denotes the interior product (i.e., $X\rfloor d\varpi(Y) = d\varpi(X,Y)$).

2.2. Symbol algebra of regular differential system. Let (R, D) be a weakly regular differential system such that $TR = D^{-\mu} \supset D^{-(\mu-1)} \supset \cdots \supset D^{-1} =: D$. For all $x \in R$, we set $\mathfrak{g}_{-1}(x) := D^{-1}(x) = D(x)$, $\mathfrak{g}_p(x) := D^p(x)/D^{p+1}(x)$ $(p = -2, -3, \ldots, -\mu)$, and $\mathfrak{m}(x) := \bigoplus_{p=-1}^{-\mu} \mathfrak{g}_p(x)$. Then, $\dim \mathfrak{m}(x) = \dim R$ holds. We set $\mathfrak{g}_p(x) = \{0\}$ when $p \leq -\mu - 1$. For $X \in \mathfrak{g}_p(x)$, $Y \in \mathfrak{g}_q(x)$, the Lie bracket $[X, Y] \in \mathfrak{g}_{p+q}(x)$ is defined as follows; Let ϖ_p be the projection of $D^p(x)$ onto $\mathfrak{g}_p(x)$ and $\tilde{X} \in \mathcal{D}^p$, $\tilde{Y} \in \mathcal{D}^q$ be any extensions such that $\varpi_p(\tilde{X}_x) = X$ and $\varpi_q(\tilde{Y}_x) = Y$. Then $[\tilde{X}, \tilde{Y}] \in \mathcal{D}^{p+q}$, and we define $[X, Y] := \varpi_{p+q}([\tilde{X}, \tilde{Y}]_x) \in \mathfrak{g}_{p+q}(x)$. It does not depend on the choice of the extensions. Hence, $\mathfrak{m}(x)$ is a nilpotent graded Lie algebra. We call $(\mathfrak{m}(x),\ [\ ,\ ])$ the *symbol algebra* of (R, D) at x. Note that the symbol algebra $(\mathfrak{m}(x),\ [\ ,\ ])$ satisfies the generating conditions $[\mathfrak{g}_p, \mathfrak{g}_{-1}] = \mathfrak{g}_{p-1}$ $(p < 0)$. For two differential systems (R, D) and (R', D'), we define (local) contact transformations ϕ from R to R' by (local) diffeomorphisms $\phi : R \to R'$ satisfying $\phi_* D = D'$. The symbol algebra is an invariant of differential systems under contact transformations. Namely, if there exists a (local) contact transformation $\phi : R \to R'$, then we obtain the graded Lie algebra isomorphism $\mathfrak{m}(x) \cong \mathfrak{m}(\phi(x))$ at each point x ([16]).

2.3. Filtered manifolds and symbol algebras. Morimoto introduced the notion of a filtered manifold as a generalization of weakly regular differential systems ([12]). We define a filtered manifold (R, F) by a pair of a manifold R and a tangential filtration F. Here, a tangential filtration F on R is a sequence $\{F^p\}_{p<0}$ of subbundles of the tangent bundle TR and the following conditions are satisfied;

(M1) $F^p \supset F^{p+1},\ F^0 = \{0\},\ \bigcup_{p\leq 0} F^p = TR,$

(M2) $[\mathcal{F}^p, \mathcal{F}^q] \subset \mathcal{F}^{p+q}$ for all $p,\ q < 0$,

where $\mathcal{F}^p = \Gamma(F^p)$ is the space of sections of F^p. Let (R, F) be a filtered manifold. For $x \in R$, we set $\mathfrak{f}_p(x) := F^p(x)/F^{p+1}(x)$ and $\mathfrak{f}(x) := \bigoplus_{p<0} \mathfrak{f}_p(x)$. For $X \in \mathfrak{f}_p(x),\ Y \in \mathfrak{f}_q(x)$, the Lie bracket $[X, Y] \in \mathfrak{f}_{p+q}(x)$ is defined in the same way as before. The Lie algebra $\mathfrak{f}(x)$ is also a nilpotent graded Lie algebra. We call $(\mathfrak{f}(x),\ [\ ,\ ])$ the *symbol algebra* of (R, F) at x. In general it does not satisfy the generating conditions. Suppose (R, F) and (R', F') are filtered manifolds. Then, (local) contact transformations between (R, F) and (R', F') are defined by (local) diffeomorphisms $\phi : R \to R'$ such that $\phi_* F^p = F^{p\prime}$. This symbol algebra is also an invariant of filtered manifolds under contact transformations ([12]).

3. Third jet space with a tangential filtration

In this section, we reconsider the 3-jet space. The 3-jet space $J^3(\mathbb{R}^2, \mathbb{R})$ with two independent variables and one dependent variable is expressed as;

$$J^3(\mathbb{R}^2, \mathbb{R}) := \{(x_1, x_2, z, p_1, p_2, p_{11}, p_{12}, p_{22}, p_{111}, p_{112}, p_{122}, p_{222})\}. \tag{1}$$

The contact system $C^3 := \{\varpi_0 = \varpi_1 = \varpi_2 = \varpi_{11} = \varpi_{12} = \varpi_{22} = 0\}$ on this jet space J^3 is given by the 1-forms;

$$\begin{aligned} \varpi_0 &:= dz - p_1 dx_1 - p_2 dx_2, & \varpi_1 &:= dp_1 - p_{11} dx_1 - p_{12} dx_2, \\ \varpi_2 &:= dp_2 - p_{12} dx_1 - p_{22} dx_2, & \varpi_{11} &:= dp_{11} - p_{111} dx_1 - p_{112} dx_2, \\ \varpi_{12} &:= dp_{12} - p_{112} dx_1 - p_{122} dx_2, & \varpi_{22} &:= dp_{22} - p_{122} dx_1 - p_{222} dx_2. \end{aligned}$$

Then we can show easily the following facts. The canonical contact system C^3 is weakly-regular and the weak derived systems define the filtration $T(J^3(\mathbb{R}^2, \mathbb{R})) = \partial^{(3)} C^3 \supset \partial^{(2)} C^3 \supset \partial C^3 \supset C^3$ on this 3-jet space. Here, the first derived system ∂C^3 and the second derived system $\partial^{(2)} C^3$ are the pull-back of the second-order canonical system C^2 on $J^2(\mathbb{R}^2, \mathbb{R})$ and the first-order canonical system C^1 on $J^1(\mathbb{R}^2, \mathbb{R})$ respectively. Namely, we have;

$$\partial C^3 = \{\varpi_0 = \varpi_1 = \varpi_2 = 0\} = (\pi^3_2)^{-1}_* C^2, \quad \partial^{(2)} C^3 = \{\varpi_0 = 0\} = (\pi^3_1)^{-1}_* C^1, \tag{2}$$

where $\pi^3_2 : J^3(\mathbb{R}^2, \mathbb{R}) \to J^2(\mathbb{R}^2, \mathbb{R})$ and $\pi^3_1 : J^3(\mathbb{R}^2, \mathbb{R}) \to J^1(\mathbb{R}^2, \mathbb{R})$ denote the projections of the fibrations as the jet bundles. Hence, according to the manner of section 2.2, we can define the symbol algebra $\mathfrak{m} = \mathfrak{m}_{-4} \oplus \mathfrak{m}_{-3} \oplus \mathfrak{m}_{-2} \oplus \mathfrak{m}_{-1}$ at each point on this 3-jet space. Now, let us take the coframe $\left\{\varpi_0, \varpi_i, \varpi_{ij}, \omega_i := dx_i, \pi_{ijk} := dp_{ijk}\right\}$ $(1 \leq i \leq j \leq k \leq 2)$ on the 3-jet space. We also take the dual frame $\left\{\frac{\partial}{\partial z}, \frac{\partial}{\partial p_i}, \frac{\partial}{\partial p_{ij}}, \frac{d}{dx_i}, \frac{\partial}{\partial p_{ijk}}\right\}$, where

$$\frac{d}{dx_1} = \frac{\partial}{\partial x_1} + p_1\frac{\partial}{\partial z} + p_{11}\frac{\partial}{\partial p_1} + p_{12}\frac{\partial}{\partial p_2} + p_{111}\frac{\partial}{\partial p_{11}} + p_{112}\frac{\partial}{\partial p_{12}} + p_{122}\frac{\partial}{\partial p_{22}},$$
$$\frac{d}{dx_2} = \frac{\partial}{\partial x_2} + p_2\frac{\partial}{\partial z} + p_{12}\frac{\partial}{\partial p_1} + p_{22}\frac{\partial}{\partial p_2} + p_{112}\frac{\partial}{\partial p_{11}} + p_{122}\frac{\partial}{\partial p_{12}} + p_{222}\frac{\partial}{\partial p_{22}}.$$

Then, we have the explicit description of the bracket product of $\mathfrak{m}$ as follows.

$$\left[\frac{\partial}{\partial p_{111}}, \frac{d}{dx_1}\right] = \left[\frac{\partial}{\partial p_{112}}, \frac{d}{dx_2}\right] = \frac{\partial}{\partial p_{11}}, \quad \left[\frac{\partial}{\partial p_{112}}, \frac{d}{dx_1}\right] = \left[\frac{\partial}{\partial p_{122}}, \frac{d}{dx_2}\right] = \frac{\partial}{\partial p_{12}},$$
$$\left[\frac{\partial}{\partial p_{122}}, \frac{d}{dx_1}\right] = \left[\frac{\partial}{\partial p_{222}}, \frac{d}{dx_2}\right] = \frac{\partial}{\partial p_{22}}, \quad \left[\frac{\partial}{\partial p_{11}}, \frac{d}{dx_1}\right] = \left[\frac{\partial}{\partial p_{12}}, \frac{d}{dx_2}\right] = \frac{\partial}{\partial p_{1}},$$
$$\left[\frac{\partial}{\partial p_{12}}, \frac{d}{dx_1}\right] = \left[\frac{\partial}{\partial p_{22}}, \frac{d}{dx_2}\right] = \frac{\partial}{\partial p_{2}}, \quad \left[\frac{\partial}{\partial p_{1}}, \frac{d}{dx_1}\right] = \left[\frac{\partial}{\partial p_{2}}, \frac{d}{dx_2}\right] = \frac{\partial}{\partial z},$$

and the other brackets are trivial. Each component of $\mathfrak{m}$ is given by $\mathfrak{m}_{-1} = \left\{\frac{d}{dx_i}, \frac{\partial}{\partial p_{ijk}}\right\}$, $\mathfrak{m}_{-2} = \left\{\frac{\partial}{\partial p_{ij}}\right\}$, $\mathfrak{m}_{-3} = \left\{\frac{\partial}{\partial p_{i}}\right\}$ and $\mathfrak{m}_{-4} = \left\{\frac{\partial}{\partial z}\right\}$. On the other hand, this bracket product can be also calculated in terms of the following structure equations (differential invariant equations) in the sense of E. Cartan;

$$d\varpi_0 \equiv \omega_1 \wedge \varpi_1 + \omega_2 \wedge \varpi_2 \qquad \text{mod } \varpi_0, \varpi_1 \wedge \varpi_2, \varpi_i \wedge \varpi_{jk}, \varpi_{ij} \wedge \varpi_{kl}, \tag{3}$$

$$\begin{cases} d\varpi_1 \equiv \omega_1 \wedge \varpi_{11} + \omega_2 \wedge \varpi_{12} & \text{mod } \varpi_0, \varpi_1, \varpi_2, \varpi_{ij} \wedge \varpi_{kl}, \\ d\varpi_2 \equiv \omega_1 \wedge \varpi_{12} + \omega_2 \wedge \varpi_{22} & \text{mod } \varpi_0, \varpi_1, \varpi_2, \varpi_{ij} \wedge \varpi_{kl}. \end{cases}$$

$$\begin{cases} d\varpi_{11} \equiv \omega_1 \wedge \pi_{111} + \omega_2 \wedge \pi_{112} & \text{mod } \varpi_0, \varpi_i, \varpi_{ij}, \\ d\varpi_{12} \equiv \omega_1 \wedge \pi_{112} + \omega_2 \wedge \pi_{122} & \text{mod } \varpi_0, \varpi_i, \varpi_{ij}, \\ d\varpi_{22} \equiv \omega_1 \wedge \pi_{122} + \omega_2 \wedge \pi_{222} & \text{mod } \varpi_0, \varpi_i, \varpi_{ij}. \end{cases}$$

In fact, the bracket product of this symbol algebra $\mathfrak{m}^3$ can be also described in terms of the following tensor space decomposition (cf. [21]).

$$\mathfrak{c}^3 = \mathfrak{c}_{-4} \oplus \mathfrak{c}_{-3} \oplus \mathfrak{c}_{-2} \oplus \mathfrak{c}_{-1}, \tag{4}$$

where $\mathfrak{c}_{-4} = \mathbb{R}$, $\mathfrak{c}_{-3} = V^*$, $\mathfrak{c}_{-2} = S^2(V^*)$ and $\mathfrak{c}_{-1} = V \oplus S^3(V^*)$. Here V is a 2-dimensional vector space and the bracket relation of this algebra $\mathfrak{c}^3$ is defined through the dual pairing between V and V^* such that V and $S^3(V^*)$ are both abelian subspaces of $\mathfrak{c}_{-1}$. From the correspondence between the direction generated by $\frac{d}{dx_i}$ (resp. $\frac{\partial}{\partial p_{ijk}}$) and the component V (resp. $S^3(V^*)$), we have the graded Lie algebra isomorphism $\mathfrak{m} \cong \mathfrak{c}^3$. Here, the directions generated by $\frac{\partial}{\partial p_{ij}}, \frac{\partial}{\partial p_i}$ and $\frac{\partial}{\partial z}$ correspond to the components $\mathfrak{c}_{-2} = S^2(V^*)$, $\mathfrak{c}_{-3} = V^*$ and $\mathfrak{c}_{-4} = \mathbb{R}$ respectively (cf. [16]).

In the rest of this section, we recall the structure of graded Lie algebra automorphism group $Aut(\mathfrak{c}^3)$ of the above symbol algebra $\mathfrak{c}^3 = \mathfrak{c}_{-4} \oplus \mathfrak{c}_{-3} \oplus \mathfrak{c}_{-2} \oplus \mathfrak{c}_{-1}$, where $\mathfrak{c}_{-4} = \mathbb{R}$, $\mathfrak{c}_{-3} = V^*$, $\mathfrak{c}_{-2} = S^2(V^*)$ and $\mathfrak{c}_{-1} = V \oplus S^3(V^*)$. In the following discussion, we refer to [19]. Let

$\kappa : \mathfrak{c}_{-1} \to \mathfrak{c}_{-1}/S^3(V^*)$ be the projection. Then $\kappa_0 := \kappa|_V$ is a linear isomorphism of V onto $\mathfrak{c}_{-1}/S^3(V^*)$. From the bracket product of $\mathfrak{c}^3$, we have $S^3(V^*) = \{X \in \mathfrak{c}_{-1} \mid [X, \mathfrak{c}_{-2}] = 0\}$, hence it follows that $\phi(S^3(V^*)) = S^3(V^*)$ for $\phi \in Aut(\mathfrak{c}^3)$. Thus this automorphism ϕ derives a unique linear isomorphism $\hat{\phi} : \mathfrak{c}_{-1}/S^3(V^*) \to \mathfrak{c}_{-1}/S^3(V^*)$ such that $\hat{\phi} \cdot \kappa = \kappa \cdot \phi$. We define the closed normal subgroup $N(\mathfrak{c}^3)$ of $Aut(\mathfrak{c}^3)$ by setting

$$N(\mathfrak{c}^3) := \left\{\phi \in Aut(\mathfrak{c}^3) \mid \phi|_{\mathfrak{c}_{-4}} = id_{\mathfrak{c}_{-4}},\ \hat{\phi} := id_{\mathfrak{c}_{-1}/S^3(V^*)}\right\}. \tag{5}$$

Furthermore, we define the homomorphism $\chi : GL(V) \times GL(\mathbb{R}) \to Aut(\mathfrak{c}^3)$ for $a \in GL(V)$ and $b \in GL(\mathbb{R})$ by putting

$$\chi(a,b)|_V = a, \quad \chi(a,b)|_{\mathbb{R}} = b \cdot id_{\mathfrak{c}_{-4}}, \quad \chi(a,b)|_{V^*} = b \cdot (a^*)^{-1},$$
$$\chi(a,b)|_{S^2(V^*)} = b \cdot \otimes^2 (a^*)^{-1}, \quad \chi(a,b)|_{S^3(V^*)} = b \cdot \otimes^3 (a^*)^{-1},$$

where a^* is the adjoint linear map. We set $G_0(\mathfrak{c}^3) = \chi(GL(V) \times GL(\mathbb{R}))$ and let $S(\mathfrak{c}^3)$ be the set of abelian subalgebras $\hat{V}$ of $\mathfrak{c}^3$ such that $\mathfrak{c}_{-1} = \hat{V} \oplus S^3(V^*)$. Then we have (Proposition 3.7 in [19]);

(J-1) $N(\mathfrak{c}^3)$ is canonically isomorphic to the vector group $S^4(V^*)$.

(J-2) $G_0(\mathfrak{c}^3) = \left\{\phi \in Aut(\mathfrak{c}^3) \mid \phi(V) = V\right\}$ and $Aut(\mathfrak{c}^3) = G_0(\mathfrak{c}^3) \cdot N(\mathfrak{c}^3)$ (semi-direct product).

By the relation; $S^4(V^*) \cong \left\{\rho : V \to S^3(V^*) \text{ (linear)} \mid v_1 \rfloor \rho(v_2) = v_2 \rfloor \rho(v_1),\ v_1, v_2 \in V\right\}$, we describe the action of $N(\mathfrak{c}^3)$ on $\mathfrak{c}^3$. For $\rho \in S^4(V^*)$, we define the element $A_\rho \in N(\mathfrak{c}^3)$ by $A_\rho|_{\mathfrak{c}_{-4}} = id_{\mathfrak{c}_{-4}}$, $A_\rho|_{S^r(V^*)} = id_{S^r(V^*)}$ (for $r = 1, 2, 3$) and $A_\rho|_V = id_V + \rho$. Then the correspondence $S^4(V^*) \ni \rho \mapsto A_\rho \in N(\mathfrak{c}^3)$ gives a group isomorphism in (J-1).

4. Contact geometry of third-order partial differential equations

In this section, we develop contact geometry of partial differential equations of third order as the theory of submanifolds of the 3-jet space J^3 equipped with the filtration $T(J^3(\mathbb{R}^2, \mathbb{R})) = \partial^{(3)}C^3 \supset \partial^{(2)}C^3 \supset \partial C^3 \supset C^3$.

DEFINITION 4.1. Let R be a (regular) submanifold of $J^3(\mathbb{R}^2, \mathbb{R})$ and let ι be the inclusion of R into $J^3(\mathbb{R}^2, \mathbb{R})$. We set the two projections $p^3_i = \pi^3_i \cdot \iota : R \to J^i(\mathbb{R}^2, \mathbb{R})$ ($i = 1, 2$). Now, we impose the following conditions on R.

(c.1) The projection $p^3_2 : R \to J^2(\mathbb{R}^2, \mathbb{R})$ is a submersion.

(c.2) The first prolongation $p^{(1)} : R^{(1)} \to R$ of R is onto. Here, $R^{(1)} = \bigcup_{x \in R} R^{(1)}_x$ is defined by
$R^{(1)}_x = \left\{v \subset T_xR \mid v \text{ is a 2-dim. integral element of } D^3(x) \text{ transversal to } ker(p^3_2)_*\right\}$,
where $D^3 := \iota_*^{-1}C^3$ is a differential system on R obtained by the pull-back by ι and an integral element of D^3 at $x \in R$ is a subspace v of T_xR satisfying $\iota^*\varpi_0|_v = \iota^*\varpi_i|_v = \iota^*\varpi_{ij}|_v = d\iota^*\varpi_0|_v = d\iota^*\varpi_i|_v = d\iota^*\varpi_{ij}|_v = 0$ for the annihilators of D^3.

Then, for the canonical contact systems C^i ($i = 1, 2, 3$) on each jet space $J^i(\mathbb{R}^2, \mathbb{R})$, all of the pullbacks $D^k := (p^3_k)_*^{-1}C^k (k = 1, 2)$ and $D^3 := \iota_*^{-1}C^3$ are differential systems on R (i.e. D^i has constant rank) from the condition (c.1). We call the triplet (D^1, D^2, D^3) *the third-order contact system* on R. We also call the quadruple $(R; D^1, D^2, D^3)$ a *geometric third-order partial differential equation* for a scalar function of two independent variables.

Remark 4.2. From now on, we sometimes omit the notation of the pull-back for the annihilators of differential systems on R.

We explain the meaning of these conditions more precisely. The first condition (c.1) implies that the system of equations R contains no equations up to second order. Namely, R is an essential third-order equation. On the other hand, the second condition (c.2) means that there exists a 2-dimensional integral element v of (R, D^3) at each $x \in R$. In this connection, we state that these integral elements are the candidates for the tangent spaces at $x \in R$ of the 2-dimensional integral manifolds (i.e. local solutions) of D^3.

According to this formulation, we define the isomorphism (contact-equivalence) of third-order equations as follows.

Definition 4.3. Let $(R; D^1, D^2, D^3)$ and $(\hat{R}; \hat{D}^1, \hat{D}^2, \hat{D}^3)$ be the third-order contact systems as above. Then we call a local diffeomorphism $\phi : R \to \hat{R}$ such that $\phi_* D^i = \hat{D}^i$ $(i = 1, 2, 3)$ a contact transformation between R and $\hat{R}$.

We emphasize that this notion of isomorphisms can be applicable to a wide range of equations. In particular, the above isomorphism is different from a local diffeomorphism $\psi : R \to \hat{R}$ such that $\psi_* D^3 = \hat{D}^3$, that is, a transformation preserving the canonical differential system D^3 of third order. If D^3 is weakly-regular, the above isomorphism ψ becomes the isomorphism preserving the filtration. In fact, for single equations (i.e. submanifolds of codimension one) R, D^3 is weakly-regular. However, for systems of equations (i.e. submanifolds of higher codimension), we can not have the weak-regularity of D^3. Thus, we need to define the notion of isomorphisms as above to treat the case of each codimension uniformly.

Remark 4.4. We have given the above geometric formulation along the spirit of *contact geometry of second order* by K. Yamaguchi (cf. [19], [21]).

Now, we study various geometric properties of $(R; D^1, D^2, D^3)$. We prove the following fundamental properties of D^i.

Lemma 4.5. *Let R be a submanifold of $J^3(\mathbb{R}^2, \mathbb{R})$ and (D^1, D^2, D^3) be the third-order contact system on R. Then we have*

(i) $\partial D^3 \subset D^2$ and $\partial D^i = D^{i-1}$ for $i = 1, 2$, *where* $D^0 := TR$.

(ii) $Ch(D^i) = ker(p^3_i)_*$ for $i = 1, 2$.

Proof. For the inclusion $\iota : R \hookrightarrow J^3(\mathbb{R}^2, \mathbb{R})$, we set $x^*_i = x_i \cdot \iota$, $z^* = z \cdot \iota$, $p^*_i = p_i \cdot \iota$, $p^*_{ij} = p_{ij} \cdot \iota$ and $p^*_{ijk} = p_{ijk} \cdot \iota$, where $(x_i, z, p_i, p_{ij}, p_{ijk})$ is the canonical coordinate on the 3-jet space. Then, from the condition (c.1), D^k $(k = 1, 2, 3)$ is defined by the (linearly independent) 1-forms as follows;

$$\varpi^*_0 = dz^* - \sum_{i=1}^{2} p^*_i dx^*_i, \quad \varpi^*_i = dp^*_i - \sum_{j=1}^{2} p^*_{ij} dx^*_j, \quad \varpi^*_{ij} = dp^*_{ij} - \sum_{k=1}^{2} p^*_{ijk} dx^*_k,$$

$$D^3 = \left\{\varpi^*_0 = \varpi^*_i = \varpi^*_{ij} = 0\right\}, \; D^2 = \left\{\varpi^*_0 = \varpi^*_i = 0\right\}, \; D^1 = \left\{\varpi^*_0 = 0\right\},$$

where the indices of p^*_{ij} and p^*_{ijk} are symmetric. Now we recall the structure equations (3) of the canonical systems C^3 or the weak-derived systems $\partial^{(k)} C^3$ on the 3-jet space. Then we first have $d\varpi_0^* \equiv d\varpi_1^* \equiv d\varpi_2^* \equiv 0 \bmod \varpi_0^*, \varpi_i^*, \varpi_{ij}^*$ as the structure equation of D^3. This means that $\partial\mathcal{D}^3 \subset \mathcal{D}^2$ holds. In general, we can not have $\partial\mathcal{D}^3 = \mathcal{D}^2$ (see section 4.1). We next have $d\varpi_0^* \equiv 0,\ d\varpi_i^* \not\equiv 0,\ \bmod\ \varpi_0^*, \varpi_i^*$ as the structure equation of D^2. Hence, we have $\partial\mathcal{D}^2 = \mathcal{D}^1$. Finally, for the corank 1 differential system D^1, we have $\partial D^1 = TR$ by the same argument. Thus we have the statement of (i).

For the statement (ii), from the structure equation of D^k $(k = 1, 2)$ and the description of each fiber $ker(p_k^3)_*$ at each $x \in R$; $ker(p_1^3)_*(x) = \left\{X \in T_xR \mid dx_i^* = dz^* = dp_i^* = 0\right\}$, $ker(p_2^3)_*(x) = \left\{X \in T_xR \mid dx_i^* = dz^* = dp_i^* = dp_{ij}^* = 0\right\}$. Then we can show the agreement $Ch(D^i) = ker(p_i^3)_*$ for $i = 1, 2$. □

Lemma 4.6. *Let R be a submanifold of $J^3(\mathbb{R}^2,\mathbb{R})$ and (D^1, D^2, D^3) be the third-order contact system on R. Then the sequence $TR \supset D^1 \supset D^2 \supset D^3$ becomes a filtration in the sense of* section 2.3. *Namely, the quadruple $(R; D^1, D^2, D^3)$ is a filtered manifold.*

Proof. To establish the statement, we reset the subbundles of the tangent bundle TR; $F^k := TR$ $(k \le -4)$, $F^{-3} := D^1$, $F^{-2} := D^2$ and $F^{-1} := D^3$. Then, from the condition (i) of Lemma 4.5, we have $[\mathcal{F}^{-1}, \mathcal{F}^{-1}] \subset \mathcal{F}^{-2}$, $[\mathcal{F}^{-1}, \mathcal{F}^{-2}] \subset [\mathcal{F}^{-2}, \mathcal{F}^{-2}] \subset \mathcal{F}^{-3}$. Moreover, it is clear that $[\mathcal{F}^{-1}, \mathcal{F}^{-3}]$, $[\mathcal{F}^{-2}, \mathcal{F}^{-2}]$, $[\mathcal{F}^{-2}, \mathcal{F}^{-3}]$ and $[\mathcal{F}^{-3}, \mathcal{F}^{-3}]$ are subsheafs of $\mathcal{F}^{-4} = \Gamma(TR)$. Thus, we obtain the condition $[\mathcal{F}^p, \mathcal{F}^q] \subset \mathcal{F}^{p+q}$ for $p, q < 0$. □

From Lemma 4.6, we can define the symbol algebra $\mathfrak{s}(x)$ as a fundamental invariant of $(R; D^1, D^2, D^3)$ following the manner in section 2.3. We set $\mathfrak{s}_{-4}(x) = T_xR/D^1(x)$, $\mathfrak{s}_{-3}(x) = D^1(x)/D^2(x)$, $\mathfrak{s}_{-2}(x) = D^2(x)/D^3(x)$, $\mathfrak{s}_{-1}(x) = D^3(x)$ at each $x \in R$. Then the *symbol algebra* $\mathfrak{s}(x)$ of R at $x \in R$ is written as

$$\mathfrak{s}(x) = \mathfrak{s}_{-4}(x) \oplus \mathfrak{s}_{-3}(x) \oplus \mathfrak{s}_{-2}(x) \oplus \mathfrak{s}_{-1}(x). \tag{6}$$

In fact, in this case, we can discuss the detailed method of the calculation of the bracket product by utilizing the annihilators of $D^1 = \{\varpi_0 = 0\}$, $D^2 = \{\varpi_0 = \varpi_i = 0\}$ and $D^3 = \left\{\varpi_0 = \varpi_i = \varpi_{ij} = 0\right\}$. The defining 1-forms $\varpi_0, \varpi_i, \varpi_{ij}$ of D^1, D^2 and D^3 defines a basis $\{A\}$ of $\mathfrak{s}_{-4}(x)$, $\{B_1, B_2\}$ of $\mathfrak{s}_{-3}(x)$ and $\{C_{11}, C_{12}, C_{22}\}$ of $\mathfrak{s}_{-2}(x)$ so that

$$\varpi_0(\tilde{A}) = 1,\ \pi_{-4}(\tilde{A}) = A,\ \varpi_i(\tilde{B}_j) = \delta^i_j,\ \pi_{-3}(\tilde{B}_i) = B_i\ (\tilde{B}_i \in D^1(x)),$$
$$\varpi_{ij}(\tilde{C}_{kl}) = \delta^{ij}_{kl}\ (1 \le i \le j \le 2, 1 \le k \le l \le 2),\ \pi_{-2}(\tilde{C}_{ij}) = C_{ij}\ (\tilde{C}_{ij} \in D^2(x)),$$

where $\pi_{-4} : T_xR \to \mathfrak{s}_{-4}(x)$, $\pi_{-3} : D^1(x) \to \mathfrak{s}_{-3}(x)$ and $\pi_{-2} : D^2(x) \to \mathfrak{s}_{-2}(x)$ are the projections. We also use the notation δ^{ij}_{kl} defined by $\delta^{ij}_{kl} := \begin{cases} 1 & \text{if } i = k \text{ and } j = l \\ 0 & \text{if otherwise.} \end{cases}$

Then we calculate brackets in symbol algebra as follows;

(1) $[\mathfrak{s}_{-1}(x), \mathfrak{s}_{-1}(x)]$. For $X_1, X_2 \in \mathfrak{s}_{-1}(x) = D^3(x)$, we calculate

$$d\varpi_{ij}(X_1, X_2) = \tilde{X}_1\varpi_{ij}(\tilde{X}_2) - \tilde{X}_2\varpi_{ij}(\tilde{X}_1) - \varpi_{ij}([\tilde{X}_1, \tilde{X}_2]) = -\varpi_{ij}([\tilde{X}_1, \tilde{X}_2]).$$

Hence, by setting $\gamma_{ij} = -d\varpi_{ij}(X_1, X_2)$, we have $[X_1, X_2] = \gamma_{11}C_{11} + \gamma_{12}C_{12} + \gamma_{22}C_{22} \in \mathfrak{s}_{-2}(x)$.

(2) $[\mathfrak{s}_{-1}(x), \mathfrak{s}_{-2}(x)]$. For $X \in \mathfrak{s}_{-1}(x) = D^3(x)$ and $Y \in \mathfrak{s}_{-2}(x)$, we calculate

$$d\varpi_i(X, \tilde{Y}_x) = \tilde{X}_x(\varpi_i(\tilde{Y})) - \tilde{Y}_x(\varpi_i(\tilde{X})) - \varpi_i([\tilde{X}, \tilde{Y}]_x) = -\varpi_i([\tilde{X}, \tilde{Y}]_x).$$

Similarly, we have $d\varpi_i(X_1, X_2) = 0$ for $X_1, X_2 \in \mathfrak{s}_{-1}(x)$. Hence $d\varpi_i(X, \tilde{Y}_x)$ depends only on $X \in \mathfrak{s}_{-1}(x), Y \in \mathfrak{s}_{-2}(x)$. Thus, by putting, $\beta_i = -d\varpi_i(X, \tilde{Y}_x)$, we have $[X, Y] = \beta_1 B_1 + \beta_2 B_2 \in \mathfrak{s}_{-3}(x)$. Moreover, it follows that, for $X \in \mathfrak{s}_{-1}(x)$,

(7) $$X \rfloor d\varpi_i(Y) = 0 \text{ for all } i \text{ and } \forall Y \in D^2(x) \text{ if and only if } [X, \mathfrak{s}_{-2}(x)] = 0.$$

(3) $[\mathfrak{s}_{-1}(x), \mathfrak{s}_{-3}(x)]$. For $X \in \mathfrak{s}_{-1}(x) = D^3(x)$ and $Z \in \mathfrak{s}_{-3}(x)$, we calculate

$$d\varpi_0(X, \tilde{Z}_x) = \tilde{X}_x(\varpi_0(\tilde{Z})) - \tilde{Z}_x(\varpi_0(\tilde{X})) - \varpi_0([\tilde{X}, \tilde{Z}]_x) = -\varpi_0([\tilde{X}, \tilde{Z}]_x).$$

Similarly, we have $d\varpi_0(X_1, X_2) = d\varpi_0(X_1, \tilde{Y}_x) = 0$ for $X_1, X_2 \in \mathfrak{s}_{-1}(x),\ Y \in \mathfrak{s}_{-2}(x)$. Hence $d\varpi_0(X, \tilde{Z}_x)$ depends only on $X \in \mathfrak{s}_{-1}(x), Z \in \mathfrak{s}_{-3}(x)$. Thus, by setting, $\alpha = -d\varpi_0(X, \tilde{Z}_x)$, we have $[X, Z] = \alpha A \in \mathfrak{s}_{-4}(x)$. Moreover, it follows that, for $X \in \mathfrak{s}_{-1}(x)$,

(8) $$X \rfloor d\varpi_0(Y) = 0 \text{ for } \forall Y \in D^1(x) \text{ if and only if } [X, \mathfrak{s}_{-3}(x)] = 0.$$

From the above description of $\mathfrak{s}(x)$, we obtain that ϖ_0, ϖ_i and ϖ_{ij}, all together, define bases of $\mathfrak{m}_{-4}(x)$, $\mathfrak{m}_{-3}(x)$ and $\mathfrak{m}_{-2}(x)$ of the symbol algebra $\mathfrak{m}(x) = \mathfrak{m}_{-4}(x) \oplus \mathfrak{m}_{-3}(x) \oplus \mathfrak{m}_{-2}(x) \oplus \mathfrak{m}_{-1}(x)$ ($\cong \mathfrak{c}^3$) of the 3-jet space $J^3(\mathbb{R}^2, \mathbb{R})$ at each $x \in J^3(\mathbb{R}^2, \mathbb{R})$. Then $\mathfrak{s}(x)$ is a graded Lie subalgebra of $\mathfrak{m}(x)$ satisfying $\mathfrak{s}_{-4}(x) = \mathfrak{m}_{-4}(x)$, $\mathfrak{s}_{-3}(x) = \mathfrak{m}_{-3}(x)$ and $\mathfrak{s}_{-2}(x) = \mathfrak{m}_{-2}(x)$. We also recall the compatibility condition (c.2) in Definition 4.1. If we put $\mathfrak{f}(x) := (p^3_2)^{-1}_*(x) = Ch(D^2)(x) = T_xR \cap Ch(C^2)(x)$, then this condition means that there exists a 2-dimensional integral element V of D^3 at each $x \in R$ such that $\mathfrak{s}_{-1}(x) = V \oplus \mathfrak{f}(x)$, where V is an abelian subalgebra in $\mathfrak{s}(x)$. In summary, the symbol algebra $\mathfrak{s}(x) = \mathfrak{s}_{-4}(x) \oplus \mathfrak{s}_{-3}(x) \oplus \mathfrak{s}_{-2}(x) \oplus \mathfrak{s}_{-1}(x)$ at $x \in R$ can be treated as the graded subalgebra of the symbol algebra $\mathfrak{m}(x) \cong \mathfrak{c}^3$ and hence we obtain;

(9) $$\begin{aligned} &\mathfrak{s}_{-4}(x) \cong \mathbb{R}, \quad \mathfrak{s}_{-3}(x) \cong V^*, \quad \mathfrak{s}_{-2}(x) \cong S^2(V^*), \\ &\mathfrak{s}_{-1}(x) = V \oplus \mathfrak{f}(x), \quad \mathfrak{f}(x) \subset S^3(V^*). \end{aligned}$$

Thus $\mathfrak{f}(x) \subset S^3(V^*)$ is also an invariant of R under contact transformations in Definition 4.3. The classification of the symbol algebras $\mathfrak{s}(x)$ can be reduced to the classification of the subspaces $\mathfrak{f}(x)$. To discuss the classification of $\mathfrak{f}$, we mention the graded Lie algebra automorphism group of $\mathfrak{s}$ in the following. We discuss the graded Lie algebra automorphism group $Aut(\mathfrak{s})$ of the above symbol algebra $\mathfrak{s}$. In this context, we can apply Corollary 5.8 in [19] to our situation. Namely, we have ([19, Corollary 5.8]);

(R-1) Each $\phi \in Aut(\mathfrak{s})$ has a unique extension $\hat{\phi} \in Aut(\mathfrak{c}^3)$, that is, $\hat{\phi}|_{\mathfrak{s}} = \phi$. Hence $Aut(\mathfrak{s})$ can be treated as a closed subgroup of $Aut(\mathfrak{c}^3)$ and is given by $Aut(\mathfrak{s}) = \left\{\phi \in Aut(\mathfrak{c}^3) \mid \phi(\mathfrak{s}) = \mathfrak{s}\right\} = \left\{\phi \in Aut(\mathfrak{c}^3) \mid \phi(\mathfrak{s}_{-1}) = \mathfrak{s}_{-1} \subset \mathfrak{c}_{-1}\right\}$.

(R-2) Let $N(\mathfrak{c}^3)$ be the group in (5), then $N(\mathfrak{s}) = Aut(\mathfrak{s}) \cap N(\mathfrak{c}^3)$ is a closed normal subgroup of $Aut(\mathfrak{s})$ and is isomorphic to the vector group $\mathfrak{f}^{(1)}$, where $\mathfrak{f}^{(1)}$ is defined by $\mathfrak{f}^{(1)} = \left\{\rho : V \to \mathfrak{f} \subset S^3(V^*) : \text{linear} \mid X \rfloor \rho(Y) = Y \rfloor \rho(X),\ \text{for } X, Y \in V\right\}$. Moreover, $N(\mathfrak{s})$ acts simply transitively on $S(\mathfrak{s})$, where $S(\mathfrak{s})$ is the set of abelian subalgebras $\hat{V}$ such that $\mathfrak{s}_{-1} = \hat{V} \oplus \mathfrak{f}$.

(R-3) $Aut(\mathfrak{s}) = G_0(\mathfrak{s}) \cdot N(\mathfrak{s})$ (semi-direct product), where $G_0(\mathfrak{s}) = Aut(\mathfrak{s}) \cap G_0(\mathfrak{c}^3) = \{\sigma \in Aut(\mathfrak{s}) \mid \sigma(V) = V\}$.

(R-4) $N(\mathfrak{s}) = \left\{\phi \in Aut(\mathfrak{s}) \mid \phi|_{\mathfrak{s}_p} = id_{\mathfrak{s}_p}\ p < -1\right\}$.

Here we pay attention to the action of $Aut(\mathfrak{c}^3)$ in the previous section and the structure of the symbol algebras $\mathfrak{s}$ of $(R; D^1, D^2, D^3)$. It is sufficient to consider the orbit decomposition of the subspaces $\mathfrak{f} \subset S^3(V^*)$ for the adjoint action of $GL(V)$ on $S^3(V^*)$ to classify the symbol algebra $\mathfrak{s}$ under the graded Lie algebra isomorphisms. From now on, we reveal the several aspects for each category of equations from the viewpoint of this symbol algebra.

4.1. The case of codimension three. In this subsection, we investigate the system of three third-order equations geometrically;

$$F_i(x_1, x_2, z, p_1, p_2, p_{11}, p_{12}, p_{22}, p_{111}, p_{112}, p_{122}, p_{222}) = 0, \quad (i = 1, 2, 3). \tag{10}$$

Of course, we impose the conditions in Definition 4.1 on the third-order equation. Hence we have a third-order equation $(R; D^1, D^2, D^3)$. Here R is a 9-dimensional submanifold and D^3 is a rank 3 differential system. In this subsection, we call $(R; D^1, D^2, D^3)$ the third-order equation of codimension three. In this case, we have $\dim \mathfrak{f}(x) = 1$ for the symbol algebra $\mathfrak{s}(x) = \mathfrak{s}_{-4}(x) \oplus \mathfrak{s}_{-3}(x) \oplus \mathfrak{s}_{-2}(x) \oplus \mathfrak{s}_{-1}(x)$ at each point $x \in R$.

Theorem 4.7. *Let $(R; D^1, D^2, D^3)$ be a third-order equation of codimension three. We choose a basis $\{e_1, e_2\}$ of V and the dual basis $\left\{e_1^*, e_2^*\right\}$ of V^*. We also take a basis $\left\{e_1^* \circledcirc e_1^* \circledcirc e_1^*, 3e_1^* \circledcirc e_1^* \circledcirc e_2^*, 3e_1^* \circledcirc e_2^* \circledcirc e_2^*, e_2^* \circledcirc e_2^* \circledcirc e_2^*\right\}$ of $S^3(V^*)$, where $e_i \circledcirc e_j \circledcirc e_k = \frac{1}{3!}\sum_{\sigma \in S_3} e_{\sigma(i)} \otimes e_{\sigma(j)} \otimes e_{\sigma(k)}$. Then the symbol algebra $\mathfrak{s}$ at each point is classified into the following four types in terms of lines $\mathfrak{f} \subset S^3(V^*)$.*

(I) $\mathfrak{f} = \langle e_1^* \circledcirc e_1^* \circledcirc e_1^* \rangle$,
(II) $\mathfrak{f} = \langle e_1^* \circledcirc e_1^* \circledcirc e_1^* + e_2^* \circledcirc e_2^* \circledcirc e_2^* \rangle$,
(III) $\mathfrak{f} = \langle 3e_1^* \circledcirc e_2^* \circledcirc e_2^* \rangle$,
(IV) $\mathfrak{f} = \langle e_1^* \circledcirc e_1^* \circledcirc e_1^* - 3e_1^* \circledcirc e_2^* \circledcirc e_2^* \rangle$.

Proof. Since $\mathfrak{f} \subset S^3(V^*)$ is a 1-dimensional subspace (i.e. line), the orbit decomposition of $\mathfrak{f}$ under the adjoint action of $GL(V)$ on $S^3(V^*)$ corresponds to the classification of the cubic (binary) form $ax^3 + 3bx^2y + 3cxy^2 + dx^3$ under the congruence, i.e., the equivalence between cubic forms. In this context, the normal form of this cubic form is well-known traditionally (e.g., p. 274–278 in [18] or p. 263–267 in [8]);

$$\text{(I) } x^3, \quad \text{(II) } x^3 + y^3, \quad \text{(III) } 3xy^2, \quad \text{(IV) } x^3 - 3xy^2.$$

Thus, we obtain the classification of the statement by rewriting the above normal forms in terms of the basis of $S^3(V^*)$. □

In connection with the above-mentioned classification of the symbol algebras, we explain here the process of this classification from the point of view of the structure equation. For defining 1-forms of the differential systems $D^1 = \{\varpi_0 = 0\}$, $D^2 = \{\varpi_0 = \varpi_i = 0\}$ and $D^3 = \left\{\varpi_0 = \varpi_i = \varpi_{ij} = 0\right\}$, the isomorphism of the symbol algebra $\mathfrak{s}$ corresponds to the transformation of the structure equations associated with the change of the 1-forms;

$$\begin{pmatrix} \hat{\omega}_1 \\ \hat{\omega}_2 \end{pmatrix} = P^{-1} \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}, \qquad \begin{pmatrix} \hat{\varpi}_1 \\ \hat{\varpi}_2 \end{pmatrix} = {}^tP \begin{pmatrix} \varpi_1 \\ \varpi_2 \end{pmatrix}, \qquad \begin{pmatrix} \hat{\varpi}_{11} & \hat{\varpi}_{12} \\ \hat{\varpi}_{12} & \hat{\varpi}_{22} \end{pmatrix} = {}^tP \begin{pmatrix} \varpi_{11} & \varpi_{12} \\ \varpi_{12} & \varpi_{22} \end{pmatrix} P, \tag{11}$$

where $P \in GL(2, \mathbb{R})$. Under this change of these 1-forms, the structure equations up to second-order are preserved, that is,

(12) $$d\varpi_0 \equiv \omega_1 \wedge \varpi_1 + \omega_2 \wedge \varpi_2 \qquad \text{mod } \varpi_0, \varpi_1 \wedge \varpi_2, \varpi_i \wedge \varpi_{jk}, \varpi_{ij} \wedge \varpi_{kl},$$

$$\begin{cases} d\varpi_1 \equiv \omega_1 \wedge \varpi_{11} + \omega_2 \wedge \varpi_{12} & \text{mod } \varpi_0, \varpi_1, \varpi_2, \varpi_{ij} \wedge \varpi_{kl}, \\ d\varpi_2 \equiv \omega_1 \wedge \varpi_{12} + \omega_2 \wedge \varpi_{22} & \text{mod } \varpi_0, \varpi_1, \varpi_2, \varpi_{ij} \wedge \varpi_{kl} \end{cases}$$

are invariant. Namely, the structure equation of the third-order system D^3 is essential for the classification. Here, we remark that ω_i, ϖ_i, ϖ_{ij} and π_{ijk} correspond to the components of the symbol algebra, V, V^*, $S^2(V^*)$ and $\mathfrak{f}(\subset S^3(V^*))$ respectively. Under this correspondence between 1-forms and the components of the symbol algebra $\mathfrak{s}$, the transformations (11) of 1-forms are equivalent to the representations of $GL(V)$ on V (change of basis), V^* (dual representation) and $S^2(V^*)$ (adjoint representation) respectively. Then the change of 1-forms π_{ijk} appeared in the (third-order) structure equation of D^3 corresponds to the congruence of cubic forms. Thus, we can show that the normal form of cubic forms also gives the normal form of the structure equation equivalent to the normal form of the symbol algebra.

REMARK 4.8. In order to derive a series of these invariant properties, we adopted the formulation as in Definition 4.1. In the above structure equation, the 2-forms $\varpi_i \wedge \varpi_j$, $\varpi_i \wedge \varpi_{jk}$ and $\varpi_{ij} \wedge \varpi_{kl}$ appearing at the behind of mod are very closely related to our contact-equivalence.

Now, we describe here the structure equation of D^3 representing the symbol algebras each of the above four types.

(I) $\mathfrak{f} = \langle e_1^* \circledcirc e_1^* \circledcirc e_1^* \rangle$.

$$\begin{cases} d\varpi_{11} \equiv \omega_1 \wedge \pi & \text{mod } \varpi_0, \varpi_i, \varpi_{ij}, \\ d\varpi_{12} \equiv 0 & \text{mod } \varpi_0, \varpi_i, \varpi_{ij}, \\ d\varpi_{22} \equiv 0 & \text{mod } \varpi_0, \varpi_i, \varpi_{ij}. \end{cases}$$

(II) $\mathfrak{f} = \langle e_1^* \circledcirc e_1^* \circledcirc e_1^* + e_2^* \circledcirc e_2^* \circledcirc e_2^* \rangle$.

$$\begin{cases} d\varpi_{11} \equiv \omega_1 \wedge \pi & \text{mod } \varpi_0, \varpi_i, \varpi_{ij}, \\ d\varpi_{12} \equiv 0 & \text{mod } \varpi_0, \varpi_i, \varpi_{ij}, \\ d\varpi_{22} \equiv \omega_2 \wedge \pi & \text{mod } \varpi_0, \varpi_i, \varpi_{ij}. \end{cases}$$

(III) $\mathfrak{f} = \langle 3e_1^* \circledcirc e_2^* \circledcirc e_2^* \rangle$.

$$\begin{cases} d\varpi_{11} \equiv 0 & \text{mod } \varpi_0, \varpi_i, \varpi_{ij}, \\ d\varpi_{12} \equiv \omega_2 \wedge \pi & \text{mod } \varpi_0, \varpi_i, \varpi_{ij}, \\ d\varpi_{22} \equiv \omega_1 \wedge \pi & \text{mod } \varpi_0, \varpi_i, \varpi_{ij}. \end{cases}$$

(IV) $\mathfrak{f} = \langle e_1^* \circledcirc e_1^* \circledcirc e_1^* - 3e_1^* \circledcirc e_2^* \circledcirc e_2^* \rangle$.

$$\begin{cases} d\varpi_{11} \equiv \omega_1 \wedge \pi & \text{mod } \varpi_0, \varpi_i, \varpi_{ij}, \\ d\varpi_{12} \equiv -\,\omega_2 \wedge \pi & \text{mod } \varpi_0, \varpi_i, \varpi_{ij}, \\ d\varpi_{22} \equiv -\,\omega_1 \wedge \pi & \text{mod } \varpi_0, \varpi_i, \varpi_{ij}. \end{cases}$$

We remark here that the structure equations in (12) up to second order are omitted, because the difference does not appear for each case. However, we emphasize that the precise bracket product of each symbol algebra can be calculated in terms of both of structure equations. Now, we obtain the following corollary through process of analyzing these symbol algebras.

Corollary 4.9. *Let* $(R; D^1, D^2, D^3)$ *be a third-order equation of codimension three. Among of the symbol algebras of the four types, only the type* (I) *is involutive. In contrast to the type* (I)*, for other three types* (*i.e.,* (II), (III) *and* (IV)*-type*)*, the integral element* V *is unique, i.e.,* $R^{(1)} \cong R$ *and* $\mathfrak{f}^{(1)} = \{0\}$. *Namely, these are finite types.*

Proof. To establish the statement, we first clarify the first prolongation $R^{(1)}$ for each case. For this purpose, we introduce the Grassmann bundle $J(D^3, 2) := \bigcup_{x \in R} J_x$ over R, where J_x is defined by $J_x := \left\{ w \subset T_x R \mid w \text{ is a 2-dimensional subspace of } D^3(x) \right\}$. Let $\Pi : J(D^3, 2) \to R$ be the projection and U be a small open neighborhood of a point in R. Then $\Pi^{-1}(U)$ is covered by 3 open sets in $J(D^3, 2)$; $\Pi^{-1}(U) = U_{\omega_1\omega_2} \cup U_{\omega_1\pi} \cup U_{\omega_2\pi}$, where each open set is $U_{\omega_1\omega_2} := \left\{ v \in \Pi^{-1}(U) \mid \omega_1|_v \wedge \omega_2|_v \neq 0 \right\}$, $U_{\omega_1\pi} := \left\{ v \in \Pi^{-1}(U) \mid \omega_1|_v \wedge \pi|_v \neq 0 \right\}$ and $U_{\omega_2\pi} := \left\{ v \in \Pi^{-1}(U) \mid \omega_2|_v \wedge \pi|_v \neq 0 \right\}$. To clarify the integral elements with the transversality condition, we explicitly describe the defining equation of $R^{(1)}$ in terms of the inhomogeneous Grassmann coordinate of fibers in $U_{\omega_1\omega_2}$. For $\hat{w} \in U_{\omega_1\omega_2}$, $\hat{w}$ is a 2-dimensional subspace of $D^3(w)$, where $p^{(1)}(\hat{w}) = w$. Hence, by restricting π to $\hat{w}$, we can introduce the inhomogeneous coordinate p_i^1 $(i = 1, 2)$ of fibers of $J(D^3, 2)$ around $\hat{w}$ with $\pi|_{\hat{w}} = p_1^1(\hat{w})\omega_1|_{\hat{w}} + p_2^1(\hat{w})\omega_2|_{\hat{w}}$. Moreover, $\hat{w}$ satisfies the condition $d\varpi_{ij}|_{\hat{w}} \equiv 0$ for the structure equation of each type. We describe this condition for each type by using the inhomogeneous coordinate. In the case of (I), we have $d\varpi_{11}|_{\hat{w}} \equiv \omega_1|_{\hat{w}} \wedge \pi|_{\hat{w}} \equiv p_2^1(\hat{w})\omega_1|_{\hat{w}} \wedge \omega_2|_{\hat{w}}$. Hence we obtain the defining equations $f = 0$ of $R^{(1)}$ in $U_{\omega_1\omega_2}$ of $J(D, 2)$, where $f = p_1^2$, that is, $\{f = 0\} \subset U_{\omega_1\omega_2}$. Then df does not vanish on $\{f = 0\}$. Thus $p^{(1)} : R^{(1)} \to R$ is a $\mathbb{R}$-bundle. In other three cases (i.e., (II), (III) and (IV)-type), we can see that $p^{(1)} : R^{(1)} \to R$ is a section of the Grassmann bundle $J(D^3, 2) := \bigcup_{x \in R} J_x$ by the same argument.

We next check involutivity for the above four types by applying the Cartan–Kähler theorem for linear Pfaffian systems (e.g., section 4 and 5 in [9]). We set the tableau matrix A consisting of the 1-form π appeared in each structure equation as follows;

$$\text{(I)} \begin{pmatrix} \pi & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \text{(II)} \begin{pmatrix} \pi & 0 \\ 0 & 0 \\ 0 & \pi \end{pmatrix}, \text{(III)} \begin{pmatrix} 0 & 0 \\ 0 & \pi \\ \pi & 0 \end{pmatrix}, \text{(IV)} \begin{pmatrix} \pi & 0 \\ 0 & -\pi \\ -\pi & 0 \end{pmatrix}. \tag{13}$$

In all cases, we show that the reduced characters of A are $s_1 = 1$, $s_2 = 0$. Among of the above four types, only the type (I) satisfies the equality of the Cartan test (i.e., dim $R_x = s_1 + 2s_2 = 1$) and the other three types do not satisfy this equality, because dim $R_x = 0$, $s_1 = 1$. □

From Corollary 4.9 and the structure equations of D^3, we obtain the following characterization of involutive systems.

Corollary 4.10. *Let* $(R; D^1, D^2, D^3)$ *be a third-order equation of codimension three. Then* R *is involutive if and only if this admits a one-dimensional Cauchy characteristic system.*

This result is analogous to the characterization of second-order regular overdetermined involutive systems by E. Cartan ([2]). In the following paragraph, we clarify some properties for the third-order equations which have the regular symbol of each type. Namely, we assume that the symbol algebra $\mathfrak{s}(x)$ is isomorphic to a fixed algebra at each point $x \in R$.

First we investigate the case of (I) $\mathfrak{f} = \langle e_1^* \circledcirc e_1^* \circledcirc e_1^* \rangle$. For the bracket product, we show that $\mathfrak{s}_{-1}$ generates the 1-dimensional subspace of $\mathfrak{s}_{-2}$. Moreover, this involutive subcategory satisfies the condition $\dim Ch(D^3)(x) = 1$ at each point $x \in R$. From the proof of Corollary 4.9, the Cartan character is $s_1 = 1$ and the Cartan integer is also 1. By the Cartan-Kähler theorem, (local) solutions depends on a scalar function of one variable, and these solutions can be constructed by using the reduction of (R, D^3) into the leaf (moduli) space $(R/Ch(D^3), D_{R/Ch(D^3)})$ coming from the complete integrability of $Ch(D^3)$. More precisely, locally, $R/Ch(D^3)$ is an 8-dimensional manifold and $D_{R/Ch(D^3)}$ is a rank 2 differential system. Then, by utilizing the fibration $\Psi : R \to R/Ch(D^3)$, we can construct the (local) solution of D^3 as the lifts of the integral curves of $D_{R/Ch(D^3)}$. Now, we give a model equation belonging to this class; namely

$$\frac{\partial^3 z}{\partial x_1^2 \partial x_2} = \frac{\partial^3 z}{\partial x_1 \partial x_2^2} = \frac{\partial^3 z}{\partial x_2^3} = 0,$$

for a scalar function $z = z(x_1, x_2)$. Next, we study the case of (II), (III), and (IV), that is, $\mathfrak{f}$ is one of the three types $\langle e_1^* \circledcirc e_1^* \circledcirc e_1^* + e_2^* \circledcirc e_2^* \circledcirc e_2^* \rangle$, $\langle 3e_1^* \circledcirc e_2^* \circledcirc e_2^* \rangle$ or $\langle e_1^* \circledcirc e_1^* \circledcirc e_1^* - 3e_1^* \circledcirc e_2^* \circledcirc e_2^* \rangle$. These three subcategories have common properties that is in contrast to the involutive-type (I). As we mentioned in Corollary 4.9, these equations have a unique integral element at each point. For each category consisting of these equations, the analysis of infinitesimal automorphisms (symmetry algebras) is a very important problem. However, these symbol algebras do not satisfy the generating condition $[\mathfrak{s}_k, \mathfrak{s}_{-1}] = \mathfrak{s}_{k-1}$. Indeed, for the bracket product, we show that $\mathfrak{s}_{-1}$ generates the 2-dimensional subspace of $\mathfrak{s}_{-2}$. Hence, we mention that Tanaka's inequality $\dim Aut \leq \dim \mathfrak{g}(\mathfrak{s})$ ([16], Corollary of Theorem 8.4) can not be applied to the analysis of the symmetry for these equations. We give here the model equations for each class.

$$\frac{\partial^3 z}{\partial x_1^3} - \frac{\partial^3 z}{\partial x_2^3} = \frac{\partial^3 z}{\partial x_1^2 \partial x_2} = \frac{\partial^3 z}{\partial x_1 \partial x_2^2} = 0, \quad \text{in case of (II)},$$

$$\frac{\partial^3 z}{\partial x_1^3} = \frac{\partial^3 z}{\partial x_1^2 \partial x_2} = \frac{\partial^3 z}{\partial x_2^3} = 0, \quad \text{in case of (III)},$$

$$\frac{\partial^3 z}{\partial x_1^3} + \frac{\partial^3 z}{\partial x_1 \partial x_2^2} = \frac{\partial^3 z}{\partial x_1^2 \partial x_2} = \frac{\partial^3 z}{\partial x_2^3} = 0, \quad \text{in case of (IV)}.$$

Finally, let us discuss the type-changing phenomenon in the case of codimension 3. We already showed the degeneration of the bracket product for type (I) compared with (II), (III) and (IV) types, that is, the degeneration of the generating direction by $\mathfrak{s}_{-1}$. Accordingly, we note that the notion of type-changing equations can be also defined for these systems of three equations by perturbations of equations which have the regular symbol of type (I). Here, we recall that second-order type-changing equations are defined by perturbations of parabolic equations (see [14]). Thus, we have formulated an example of the third-order version of second-order type-changing equations.

4.2. The case of codimension one (i.e., hypersurfaces). In this subsection, we treat the following single equation;

$$F(x_1, x_2, z, p_1, p_2, p_{11}, p_{12}, p_{22}, p_{111}, p_{112}, p_{122}, p_{222}) = 0. \tag{14}$$

We also assume for these equations the conditions in Definition 4.1. Hence, $R = \{F = 0\}$ is a (regular) hypersurface in $J^3(\mathbb{R}^2, \mathbb{R})$. Moreover, a quadruple $(R; D^1, D^2, D^3)$ is a third-order contact system associated with a equation (14). Here R is a 11-dimensional manifold and the third-order canonical system D^3 is a rank 5 differential system. In this subsection, we call $(R; D^1, D^2, D^3)$ the third-order equation of codimension one. In this case, we have dim $\mathfrak{f}(x) = 3$ for the symbol algebra $\mathfrak{s}(x) = \mathfrak{s}_{-4}(x) \oplus \mathfrak{s}_{-3}(x) \oplus \mathfrak{s}_{-2}(x) \oplus \mathfrak{s}_{-1}(x)$ at each point $x \in R$. Then our statement is the following.

Theorem 4.11. *Let $(R; D^1, D^2, D^3)$ be a third-order equation of codimension one. Then the symbol algebra $\mathfrak{s}$ at each point is classified into the following four types.*

(I) $\mathfrak{f} = \langle 3e_1^* \circledcirc e_1^* \circledcirc e_2^*,\ 3e_1^* \circledcirc e_2^* \circledcirc e_2^*,\ e_2^* \circledcirc e_2^* \circledcirc e_2^* \rangle$,
(II) $\mathfrak{f} = \langle e_1^* \circledcirc e_1^* \circledcirc e_1^* - e_2^* \circledcirc e_2^* \circledcirc e_2^*,\ 3e_1^* \circledcirc e_1^* \circledcirc e_2^*,\ 3e_1^* \circledcirc e_2^* \circledcirc e_2^* \rangle$,
(III) $\mathfrak{f} = \langle e_1^* \circledcirc e_1^* \circledcirc e_1^*,\ 3e_1^* \circledcirc e_1^* \circledcirc e_2^*,\ e_2^* \circledcirc e_2^* \circledcirc e_2^* \rangle$,
(IV) $\mathfrak{f} = \langle e_1^* \circledcirc e_1^* \circledcirc e_1^* + 3e_1^* \circledcirc e_2^* \circledcirc e_2^*,\ 3e_1^* \circledcirc e_1^* \circledcirc e_2^*,\ e_2^* \circledcirc e_2^* \circledcirc e_2^* \rangle$.

Proof. To prove our assertion, we make full use of the duality $V \cong V^*$ to avoid the direct classification of subspaces $\mathfrak{f} \subset S^3(V^*)$ of codimension 1 (i.e. dimension 3). Under this duality, let us describe the symbol algebra $\mathfrak{c} = \mathfrak{c}_{-4} \oplus \mathfrak{c}_{-3} \oplus \mathfrak{c}_{-2} \oplus \mathfrak{c}_{-1}$ of the 3-jet space by $\mathfrak{c}_{-4} = \mathbb{R}$, $\mathfrak{c}_{-3} = V$, $\mathfrak{c}_{-2} = S^2(V)$, $\mathfrak{c}_{-1} = V^* \oplus S^3(V)$ and consider the classification of the 1-dimensional subspaces $\mathfrak{f} \subset S^3(V)$ under the adjoint action of $GL(V^*)$ on $S^3(V)$. It is clear that this classification is essentially equal to the classification in the previous subsection, hence the list of the classification of the 1-dimensional subspaces is given by (I) $\mathfrak{f} = \langle e_1 \circledcirc e_1 \circledcirc e_1 \rangle$, (II) $\langle e_1 \circledcirc e_1 \circledcirc e_1 + e_2 \circledcirc e_2 \circledcirc e_2 \rangle$, (III) $\langle e_1 \circledcirc e_2 \circledcirc e_2 \rangle$ and (IV) $\langle e_1 \circledcirc e_1 \circledcirc e_1 - e_1 \circledcirc e_2 \circledcirc e_2 \rangle$. Now we recall again the duality $V \cong V^*$. We have the one to one correspondence (i.e. Grassmann duality) between $\mathfrak{f} \subset S^3(V)$ and $\mathfrak{f}^\perp \subset S^3(V^*)$, where $\mathfrak{f}^\perp$ denotes the annihilator of $\mathfrak{f}$. Hence the classification of subspaces $\mathfrak{f}^\perp \subset S^3(V^*)$ of codimension 1 can be reduced to the classification of 1-dimensional subspaces $\mathfrak{f} \subset S^3(V)$. We obtain the required list by rewriting the above list in term of the annihilator $\mathfrak{f}^\perp$. □

In connection with this classification, by using the famous covariants (e.g., discriminant) of cubic forms, we can assign the list of the classification to given single equations at pointwise level. For detailed commentary of the covariants for cubic forms, refer to [4], [8] and [11]. Here we describe the structure equations corresponding to each symbol algebra.

(I) $\mathfrak{f} = \langle 3e_1^* \circledcirc e_1^* \circledcirc e_2^*,\ 3e_1^* \circledcirc e_2^* \circledcirc e_2^*,\ e_2^* \circledcirc e_2^* \circledcirc e_2^* \rangle$.

$$\begin{cases} d\varpi_{11} \equiv \omega_2 \wedge \pi_{112} & \mod \varpi_0, \varpi_i, \varpi_{ij}, \\ d\varpi_{12} \equiv \omega_1 \wedge \pi_{112} + \omega_2 \wedge \pi_{122} & \mod \varpi_0, \varpi_i, \varpi_{ij}, \\ d\varpi_{22} \equiv \omega_1 \wedge \pi_{122} + \omega_2 \wedge \pi_{222} & \mod \varpi_0, \varpi_i, \varpi_{ij}. \end{cases}$$

(II) $\mathfrak{f} = \langle e_1^* \circledcirc e_1^* \circledcirc e_1^* - e_2^* \circledcirc e_2^* \circledcirc e_2^*,\ 3e_1^* \circledcirc e_1^* \circledcirc e_2^*,\ 3e_1^* \circledcirc e_2^* \circledcirc e_2^* \rangle$.

$$\begin{cases} d\varpi_{11} \equiv \omega_1 \wedge \pi_{111} + \omega_2 \wedge \pi_{112} & \mod \varpi_0, \varpi_i, \varpi_{ij}, \\ d\varpi_{12} \equiv \omega_1 \wedge \pi_{112} + \omega_2 \wedge \pi_{122} & \mod \varpi_0, \varpi_i, \varpi_{ij}, \\ d\varpi_{22} \equiv \omega_1 \wedge \pi_{122} - \omega_2 \wedge \pi_{111} & \mod \varpi_0, \varpi_i, \varpi_{ij}. \end{cases}$$

(III) $\mathfrak{f} = \langle e_1^* \circledcirc e_1^* \circledcirc e_1^*,\ 3e_1^* \circledcirc e_1^* \circledcirc e_2^*,\ e_2^* \circledcirc e_2^* \circledcirc e_2^* \rangle$.

$$\begin{cases} d\varpi_{11} \equiv \omega_1 \wedge \pi_{111} + \omega_2 \wedge \pi_{112} & \mod \varpi_0, \varpi_i, \varpi_{ij}, \\ d\varpi_{12} \equiv \omega_1 \wedge \pi_{112} & \mod \varpi_0, \varpi_i, \varpi_{ij}, \\ d\varpi_{22} \equiv \qquad\qquad\quad \omega_2 \wedge \pi_{222} & \mod \varpi_0, \varpi_i, \varpi_{ij}. \end{cases}$$

(IV) $\mathfrak{f} = \langle e_1^* \circledcirc e_1^* \circledcirc e_1^* + 3e_1^* \circledcirc e_2^* \circledcirc e_2^*,\ 3e_1^* \circledcirc e_1^* \circledcirc e_2^*,\ e_2^* \circledcirc e_2^* \circledcirc e_2^* \rangle$.

$$\begin{cases} d\varpi_{11} \equiv \omega_1 \wedge \pi_{111} + \omega_2 \wedge \pi_{112} & \mod \varpi_0, \varpi_i, \varpi_{ij}, \\ d\varpi_{12} \equiv \omega_1 \wedge \pi_{112} + \omega_2 \wedge \pi_{111} & \mod \varpi_0, \varpi_i, \varpi_{ij}, \\ d\varpi_{22} \equiv \omega_1 \wedge \pi_{111} + \omega_2 \wedge \pi_{222} & \mod \varpi_0, \varpi_i, \varpi_{ij}. \end{cases}$$

Now, let us investigate the various aspects of single equations. In all cases, we have dim $R_x^{(1)} = 3$ for each point $x \in R$ by the same argument as in the proof of Corollary 4.9. In contrast to the case of codimension three, we can show that the above four types are all involutive by the following discussion. We set the tableau matrix A consisting of the 1-forms π_{ijk} appeared in each structure equation;

$$\text{(15)} \qquad \text{(I)} \begin{pmatrix} 0 & \pi_{112} \\ \pi_{112} & \pi_{122} \\ \pi_{122} & \pi_{222} \end{pmatrix}, \ \text{(II)} \begin{pmatrix} \pi_{111} & \pi_{112} \\ \pi_{112} & \pi_{122} \\ \pi_{122} & -\pi_{111} \end{pmatrix}, \ \text{(III)} \begin{pmatrix} \pi_{111} & \pi_{112} \\ \pi_{112} & 0 \\ 0 & \pi_{222} \end{pmatrix}, \ \text{(IV)} \begin{pmatrix} \pi_{111} & \pi_{112} \\ \pi_{112} & \pi_{111} \\ \pi_{111} & \pi_{222} \end{pmatrix}.$$

In this expression of A, the characters of A are given by $s_1 = 2,\ s_2 = 1$ for (I), (III), (IV) and $s_1 = 3,\ s_2 = 0$ for (II). We show that the type (II) is involutive because the above A of type (II) satisfies the equality dim $R_x = 3 = s_1 + 2s_2 = 3$. On the other hand, for other three cases, we fail to pass the Cartan test by the inequality dim $R_x = 3 < s_1 + 2s_2 = 2 + 2 = 4$. However, it is possible to break through this situation by the appropriate transformation of each structure equation. In the following, we calculate only the case of type (III). By using the matrix $P := \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$, we transform the structure equation of type (III) associated with the change of the 1-forms in (11). Then we have the (reduced) tableau matrix $\hat{A} := \begin{pmatrix} \pi_{111} + 3\pi_{112} + \pi_{222} & -\pi_{111} - \pi_{112} + \pi_{222} \\ -\pi_{111} - \pi_{112} + \pi_{222} & \pi_{111} - \pi_{112} + \pi_{222} \\ \pi_{111} - \pi_{112} + \pi_{222} & -\pi_{111} + 3\pi_{112} + \pi_{222} \end{pmatrix}$. In this description, 1-forms $\pi_{111} + 3\pi_{112} + \pi_{222},\ -\pi_{111} - \pi_{112} + \pi_{222},\ \pi_{111} - \pi_{112} + \pi_{222}$ in the first column are linearly independent. Hence we have the reduced characters $s_1 = 3,\ s_2 = 0$ of $\hat{A}$ and the equality dim $R_x^{(1)} = s_1 + 2s_2 = 3$ in the Cartan test. Consequently, we show that the type (III) is involutive. The type (I) and (IV) are also involutive by the same argument. By the Cartan–Kähler theorem, local solutions depend on three scalar functions of one variable. We also mention that the generating condition $[\mathfrak{s}_{-1}, \mathfrak{s}_{-k}] = \mathfrak{s}_{-k-1}$ ($k = 1, 2, 3$) is satisfied in all cases.

In this manner, the symbol algebras of all types have some common properties. On the other hand, we can provide the individual property that some cases have. As an example, we discuss the covariant systems of $(R; D^1, D^2, D^3)$. In general, covariant systems mean the invariant subsheafs of D^3 under contact transformations. For third-order single equations, we note that the covariant systems can be defined for types (I) and (III) from the decomposability of 2-forms in the structure equations.

From now on, we assume the regularity of the symbol algebras for equations. We first discuss the case of (I) $\mathfrak{f} = \langle 3e_1^* \circledcirc e_1^* \circledcirc e_2^*,\ 3e_1^* \circledcirc e_2^* \circledcirc e_2^*,\ e_2^* \circledcirc e_2^* \circledcirc e_2^* \rangle$. For this case, we can introduce the covariant system $E = \left\{\varpi_0 = \varpi_i = \varpi_{ij} = \omega_2 = \pi_{112} = 0\right\}$. This covariant system can be regarded as the third-order version of Monge characteristic systems for second-order parabolic equations. However, this covariant system has a decisive difference compared with the second-order parabolic case in the following sense.

Proposition 4.12. *Let* $(R; D^1, D^2, D^3)$ *be a third-order equation belonging to type* (I). *Then, the covariant system* E *is not completely integrable.*

Proof. For the defining 1-forms $\varpi_0, \varpi_i, \varpi_{ij}, \omega_2$ and π_{112} of E, we have

$$d\varpi_{22} \equiv \omega_1 \wedge \pi_{122} \not\equiv 0 \quad \mathrm{mod}\ \varpi_0,\ \varpi_i,\ \varpi_{ij},\ \omega_2,\ \pi_{112}. \qquad \square$$

Hence, at least by using the method similar to the second-order parabolic case, we can not derive the subcategory consisting of equations like Goursat equations ([6]). Another approach is needed to discover such a subcategory.

We next investigate the case of (III) $\mathfrak{f} = \langle e_1^* \circledcirc e_1^* \circledcirc e_1^*,\ 3e_1^* \circledcirc e_1^* \circledcirc e_2^*,\ e_2^* \circledcirc e_2^* \circledcirc e_2^* \rangle$. In this case, we can provide a pair of differential systems; $E_1 = \left\{\varpi_0 = \varpi_i = \varpi_{ij} = \omega_1 = \pi_{112} = 0\right\}$ and $E_2 = \left\{\varpi_0 = \varpi_i = \varpi_{ij} = \omega_2 = \pi_{222} = 0\right\}$ as a covariant system. This covariant system can be regarded as the third-order version of Monge characteristic systems in the case of second-order hyperbolic equations. As an application of this covariant system E_i, the quadrature initiated by Darboux is expected. Now, we give the simplest model equations belonging to each class.

$$\frac{\partial^3 z}{\partial x_1^3} = 0 \quad \text{in case of (I)}, \qquad \frac{\partial^3 z}{\partial x_1^3} + \frac{\partial^3 z}{\partial x_2^3} = 0 \quad \text{in case of (II)},$$
$$\frac{\partial^3 z}{\partial x_1 \partial x_2^2} = 0 \quad \text{in case of (III)}, \qquad \frac{\partial^3 z}{\partial x_1^3} - \frac{\partial^3 z}{\partial x_1 \partial x_2^2} = 0 \quad \text{in case of (IV)}.$$

These model equations can be regarded as the dual equations of the model equations of codimension 3 given in the previous subsection. We also note that the notion of type-changing equations can be defined for these single equations by perturbations of equations which have the regular symbol of type (I) ([14]).

In the rest of this subsection, we mention the KdV equation which is an important soliton equation;

$$\frac{\partial z}{\partial x_2} + \frac{\partial^3 z}{\partial x_1^3} + 6z\frac{\partial z}{\partial x_1} = 0. \tag{16}$$

We explain how this equation can be understood in our framework. For this purpose, we rewrite the KdV equation (16) as the equation $p_2 + p_{111} + 6zp_1 = 0$. Then we have the description of the defining 1-forms of D^3, especially $\varpi_{11} := dp_{11} + (p_2 + 6zp_1)dx_1 - p_{112}dx_2, \varpi_{12} :=$

$dp_{12} - p_{112}dx_1 - p_{122}dx_2$ and $\varpi_{22} := dp_{22} - p_{122}dx_1 - p_{222}dx_2$. The exterior derivative of ϖ_{11} satisfies

$$\begin{aligned} d\varpi_{11} &= (dp_2 + 6zdp_1 + 6p_1dz) \wedge dx_1 - dp_{112} \wedge dx_2, \\ &\equiv dx_2 \wedge \{dp_{112} + (p_{22} + 6zp_{12} + 6p_1p_2)dx_1\}, \quad \bmod \varpi_0,\ \varpi_i,\ \varpi_{ij}. \end{aligned}$$

Thus, if we take the coframe $\left\{\varpi_0, \varpi_i, \varpi_{ij}, \omega_i, \pi_{ijk}\right\}$ on the equation manifold R by putting $\omega_1 := dx_1,\ \omega_2 := dx_2,\ \pi_{112} := dp_{112} + (p_{22} + 6zp_{12} + 6p_1p_2)dx_1$ and $\pi_{122} := dp_{122},\ \varpi_{222} := dp_{222}$, then we have the structure equation of type (I). Namely, we have shown that the KdV equation (16) belongs to type (I).

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