Laterveer, R.
Osaka J. Math.
60 (2023), 15–30

SOME NEW FANO VARIETIES WITH A MULTIPLICATIVE CHOW–KÜNNETH DECOMPOSITION

Robert LATERVEER

(Received January 21, 2021, revised September 7, 2021)

Abstract

Let Y be a smooth dimensionally transverse intersection of the Grassmannian $\mathrm{Gr}(2,n)$ with 3 Plücker hyperplanes. We show that Y admits a multiplicative Chow–Künneth decomposition, in the sense of Shen–Vial. As a consequence, a certain tautological subring of the Chow ring of powers of Y injects into cohomology.

1. Introduction

Given a smooth projective variety Y over $\mathbb{C}$, let $A^i(Y) := CH^i(Y)_{\mathbb{Q}}$ denote the Chow groups of Y, i.e. the groups of codimension i algebraic cycles on Y with $\mathbb{Q}$-coefficients, modulo rational equivalence. Let us write $A^*_{hom}(Y)$ and $A^*_{AJ}(Y)$ for the subgroups of homologically trivial (resp. Abel–Jacobi trivial) cycles. Intersection product defines a ring structure on $A^*(Y) = \bigoplus_i A^i(Y)$, the *Chow ring* of Y [16]. In the case of K3 surfaces, this ring structure has a peculiar property:

Theorem 1.1 (Beauville–Voisin [3])**.** *Let S be a K3 surface. The $\mathbb{Q}$-subalgebra*

$$R^*(S) := \langle A^1(S), c_j(S) \rangle \quad \subset \; A^*(S)$$

injects into cohomology under the cycle class map.

Inspired by the remarkable behaviour of K3 surfaces and of abelian varieties, Beauville [2] has famously conjectured that for certain special varieties, the Chow ring should admit a *multiplicative splitting*. To make concrete sense of Beauville's elusive "splitting property conjecture", Shen–Vial [42] have introduced the concept of *multiplicative Chow–Künneth decomposition*; let us abbreviate this to "MCK decomposition".

What can one say about the class of special varieties admitting an MCK decomposition? This class is not yet well-understood. Varieties with $A^*_{hom}() = 0$ (i.e. varieties with *trivial Chow groups*) admit an MCK decomposition, for trivial reasons. The question becomes interesting for varieties with $A^*_{AJ}() = 0$ (conjecturally, these are exactly the varieties with Hodge level at most 1, i.e. the Hodge numbers $h^{p,q}$ are zero for $|p-q| > 1$). It is known that hyperelliptic curves have an MCK decomposition [42, Example 8.16], but the very general curve of genus ≥ 3 does not have an MCK decomposition [13, Example 2.3] (for more details, cf. subsection 2.1 below). Also, there exist Fano threefolds that do not admit an MCK decomposition. On the positive side, here are some higher-dimensional varieties with

2020 Mathematics Subject Classification. 14C15, 14C25, 14C30.
Supported by ANR grant ANR-20-CE40-0023.

Hodge level 1 that are known to have an MCK decomposition:

- cubic threefolds and cubic fivefolds [7], [13];
- Fano threefolds of genus 8 [28];
- complete intersections of 2 quadrics [27];
- Gushel–Mukai fivefolds [26].

The goal of the present note is to add some new varieties with Hodge level 1 to this list:

Theorem (=Theorem 3.7)**.** *Let Y be a smooth dimensionally transverse intersection*

$$Y := \mathrm{Gr}(2,n)\cap H_1\cap H_2\cap H_3 \quad \subset \ \mathbb{P}^{\binom{n}{2}-1}\ ,$$

where $\mathrm{Gr}(2,n)$ *denotes the Grassmannian of 2-dimensional linear subspaces of a fixed n-dimensional vector space, and the H_j are Plücker hyperplanes. Then Y has an MCK decomposition.*

In case n is odd, a variety Y as in Theorem 3.7 has trivial Chow groups and so the statement is vacuously true. In case n is even, there is a curve C naturally associated to Y, and one has a relation of Chow motives

$$h(Y)\cong h(C)((1-\dim Y)/2)\oplus \bigoplus \mathbb{1}(*) \quad \text{in } \mathcal{M}_{\mathrm{rat}} \tag{1}$$

(cf. Theorem 3.2). The relation between Y and C has previously been studied on the level of Hodge theory in [8], and on the level of derived categories in [20], [21]. As a result of independent interest, we prove here (Theorem 3.2) that the relation (1) also holds on the level of Chow motives.

The existence of an MCK decomposition has profound intersection-theoretic consequences. This is exemplified by the following corollary, which is about a certain *tautological subring* of the Chow ring of powers of Y:

Corollary (=Corollary 4.1)**.** *Let Y be as in* Theorem 3.7*, and $m\in\mathbb{N}$. Let*

$$R^*(Y^m) := \Big\langle (p_i)^*\,\mathrm{Im}(A^*(\mathrm{Gr}(2,n))\to A^*(Y))\,,\ (p_{ij})^*(\Delta_Y)\Big\rangle\ \subset\ \ A^*(Y^m)$$

be the $\mathbb{Q}$-subalgebra generated by (pullbacks of) cycles coming from the Grassmannian and the diagonal $\Delta_Y\in A^*(Y\times Y)$. (Here p_i and p_{ij} denote the various projections from Y^m to Y resp. to $Y\times Y$). The cycle class map induces injections

$$R^*(Y^m)\ \hookrightarrow\ H^*(Y^m,\mathbb{Q})\quad \textit{for all } m\in\mathbb{N}\ .$$

Corollary 4.1 is somewhat surprising, because the corresponding statement for the associated curve C is *false*: in general it is *not* true that the $\mathbb{Q}$-subalgebra

$$\Big\langle (p_i)^*(h),\ (p_{ij})^*(\Delta_C)\Big\rangle\ \subset\ \ A^*(C^m)$$

injects into cohomology (cf. Proposition 4.3 for the precise statement). This means that the injection

$$A^*(C^m)\ \hookrightarrow\ A^*(Y^m)$$

induced by (1) does *not* send tautological cycles to tautological cycles !

Let us end this introduction with an open question. In view of Theorem 3.7, one might ask whether more generally smooth complete intersections of Grassmannians $\mathrm{Gr}(k,n)$ with an arbitrary number of Plücker hyperplanes have an MCK decomposition. This concerns in particular the Debarre–Voisin 20folds

$$\mathrm{Gr}(3,10)\cap H\ ,$$

which are Fano varieties of K3 type [6], and also the Fano eightfolds

$$\mathrm{Gr}(2,8)\cap H_1\cap\cdots\cap H_4\ ,$$

which are again of K3 type [41], [10]. Such varieties (being of Hodge level > 1) are out of scope of the argument of the present note.

Conventions. *In this note, the word variety will refer to a reduced irreducible scheme of finite type over $\mathbb{C}$. A subvariety is a (possibly reducible) reduced subscheme which is equidimensional.*

All Chow groups will be with rational coefficients: *we denote by $A_j(Y)$ the Chow group of j-dimensional cycles on Y with $\mathbb{Q}$-coefficients; for Y smooth of dimension n the notations $A_j(Y)$ and $A^{n-j}(Y)$ are used interchangeably. The notations $A^j_{\mathrm{hom}}(Y)$ and $A^j_{\mathrm{AJ}}(Y)$ will be used to indicate the subgroup of homologically trivial (resp. Abel–Jacobi trivial) cycles.*

The contravariant category of Chow motives (i.e., pure motives with respect to rational equivalence as in [39], [34]*) will be denoted $\mathcal{M}_{\mathrm{rat}}$.*

2. Preliminaries

2.1. MCK decomposition.

Definition 2.1 (Murre [33]). Let X be a smooth projective variety of dimension n. We say that X has a *CK decomposition* if there exists a decomposition of the diagonal

$$\Delta_X=\pi^0_X+\pi^1_X+\cdots+\pi^{2n}_X\quad\text{in } A^n(X\times X)\ ,$$

such that the π^i_X are mutually orthogonal idempotents and $(\pi^i_X)_*H^*(X,\mathbb{Q})=H^i(X,\mathbb{Q})$.

(NB: “CK decomposition” is shorthand for “Chow–Künneth decomposition”.)

Remark 2.2. The existence of a CK decomposition for any smooth projective variety is part of Murre’s conjectures [33], [17].

Definition 2.3 (Shen–Vial [42]). Let X be a smooth projective variety of dimension n. Let $\Delta^{\mathrm{sm}}_X\in A^{2n}(X\times X\times X)$ be the class of the small diagonal

$$\Delta^{\mathrm{sm}}_X:=\{(x,x,x)\mid x\in X\}\ \subset\ X\times X\times X\ .$$

An *MCK decomposition* is a CK decomposition $\{\pi^i_X\}$ of X that is *multiplicative*, i.e. it satisfies

$$\pi^k_X\circ\Delta^{\mathrm{sm}}_X\circ(\pi^i_X\times\pi^j_X)=0\quad\text{in } A^{2n}(X\times X\times X)\quad\text{for all } i+j\neq k\ .$$

(NB: “MCK decomposition” is shorthand for “multiplicative Chow–Künneth decomposition”.)

REMARK 2.4. The small diagonal (seen as a correspondence from $X\times X$ to X) induces the *multiplication morphism*

$$\Delta_X^{\mathrm{sm}}:\quad h(X)\otimes h(X)\ \to\ h(X)\quad \text{in } \mathcal{M}_{\mathrm{rat}}\ .$$

Let us assume X has a CK decomposition

$$h(X)=\bigoplus_{i=0}^{2n} h^i(X)\quad \text{in } \mathcal{M}_{\mathrm{rat}}\ .$$

By definition, this decomposition is multiplicative if for any i, j the composition

$$h^i(X)\otimes h^j(X)\ \to\ h(X)\otimes h(X)\ \xrightarrow{\Delta_X^{\mathrm{sm}}}\ h(X)\quad \text{in } \mathcal{M}_{\mathrm{rat}}$$

factors through $h^{i+j}(X)$.

If X has an MCK decomposition, then setting

$$A^i_{(j)}(X):=(\pi_X^{2i-j})_\ast A^i(X)\ ,$$

one obtains a bigraded ring structure on the Chow ring: that is, the intersection product sends $A^i_{(j)}(X)\otimes A^{i'}_{(j')}(X)$ to $A^{i+i'}_{(j+j')}(X)$.

It is expected that for any X with an MCK decomposition, one has

$$A^i_{(j)}(X)\stackrel{??}{=}0\quad \text{for } j<0\ ,\quad A^i_{(0)}(X)\cap A^i_{\mathrm{hom}}(X)\stackrel{??}{=}0\ ;$$

this is related to Murre's conjectures B and D, that have been formulated for any CK decomposition [33].

The property of having an MCK decomposition is restrictive, and is closely related to Beauville's "splitting property conjecture" [2]. To give an idea: hyperelliptic curves have an MCK decomposition [42, Example 8.16], but the very general curve of genus ≥ 3 does not have an MCK decomposition [13, Example 2.3]. As for surfaces: a smooth quartic in $\mathbb{P}^3$ has an MCK decomposition, but a very general surface of degree ≥ 7 in $\mathbb{P}^3$ should not have an MCK decomposition [13, Proposition 3.4]. There are examples of Fano threefolds that do not admit an MCK decomposition [13, Example 1.11].

For more detailed discussion, and examples of varieties with an MCK decomposition, we refer to [42, Section 8], as well as [48], [43], [14], [22], [32], [23], [24], [27], [30], [13].

2.2. The Franchetta property.

DEFINITION 2.5. Let $\mathcal{Y}\to B$ be a smooth projective morphism, where $\mathcal{Y}, B$ are smooth quasi-projective varieties. We say that $\mathcal{Y}\to B$ has the *Franchetta property in codimension* j if the following holds: for every $\Gamma\in A^j(\mathcal{Y})$ such that the restriction $\Gamma|_{Y_b}$ is homologically trivial for the very general $b\in B$, the restriction $\Gamma|_b$ is zero in $A^j(Y_b)$ for all $b\in B$.

We say that $\mathcal{Y}\to B$ has the *Franchetta property* if $\mathcal{Y}\to B$ has the Franchetta property in codimension j for all j.

This property is studied in [37], [4], [11], [12].

DEFINITION 2.6. Given a family $\mathcal{Y} \to B$ as above, with $Y := Y_b$ a fiber, we write

$$GDA^j_B(Y) := \mathrm{Im}\Big(A^j(\mathcal{Y}) \to A^j(Y)\Big)$$

for the subgroup of *generically defined cycles*. In a context where it is clear to which family we are referring, the index B will often be suppressed from the notation.

With this notation, the Franchetta property amounts to saying that $GDA^*_B(Y)$ injects into cohomology, under the cycle class map, for every fiber Y.

There is some flexibility with respect to the base B:

Lemma 2.7. *Let $\mathcal{Y} \to B$ be a smooth projective family, and $B_0 \subset B$ the intersection of a countable number of dense open subsets. Then $\mathcal{Y} \to B$ has the Franchetta property if and only if $\mathcal{Y} \to B_0$ has the Franchetta property.*

Proof. This follows from a well-known spread lemma [50, Lemma 3.2]. □

2.3. A Franchetta-type result.

Proposition 2.8. *Let M be a smooth projective variety with trivial Chow groups. Let $L_1, \ldots, L_r \to M$ be very ample line bundles, and let $\mathcal{Y} \to B$ be the universal family of smooth dimensionally transverse complete intersections of type*

$$Y = M \cap H_1 \cap \cdots \cap H_r\ , \quad H_j \in |L_j|\ .$$

Assume the fibers $Y = Y_b$ have $H^{\dim Y}_{tr}(Y, \mathbb{Q}) \neq 0$. There is an inclusion

$$\ker\Big(GDA^{\dim Y}_B(Y \times Y) \to H^{2\dim Y}(Y \times Y, \mathbb{Q})\Big) \ \subset\ \Big\langle (p_1)^* GDA^*_B(Y), (p_2)^* GDA^*_B(Y) \Big\rangle\ .$$

Proof. This is essentially Voisin's "spread" result [49, Proposition 1.6] (cf. also [31, Proposition 5.1] for a reformulation of Voisin's result). We give a proof which is somewhat different from [49]. Let $\bar{B} := \mathbb{P}H^0(M, L_1 \oplus \cdots \oplus L_r)$ (so $B \subset \bar{B}$ is a Zariski open), and let us consider the projection

$$\pi\colon\ \ \mathcal{Y} \times_{\bar{B}} \mathcal{Y}\ \to\ M \times M\ .$$

Using the very ampleness assumption, one finds that π is a $\mathbb{P}^s$-bundle over $(M \times M) \setminus \Delta_M$, and a $\mathbb{P}^t$-bundle over Δ_M. That is, π is what is termed a *stratified projective bundle* in [11]. As such, [11, Proposition 5.2] implies the equality

$$GDA^*_B(Y \times Y) = \mathrm{Im}\Big(A^*(M \times M) \to A^*(Y \times Y)\Big) + \Delta_* GDA^*_B(Y)\ , \tag{2}$$

where $\Delta\colon Y \to Y \times Y$ is the inclusion along the diagonal. As M has trivial Chow groups, $A^*(M \times M)$ is generated by $A^*(M) \otimes A^*(M)$. Base-point freeness of the L_j implies that

$$GDA^*_B(Y) = \mathrm{Im}(A^*(M) \to A^*(Y))\ .$$

The equality (2) thus reduces to

$$GDA^*_B(Y \times Y) = \Big\langle (p_1)^* GDA^*_B(Y), (p_2)^* GDA^*_B(Y), \Delta_Y \Big\rangle$$

(where p_1, p_2 denote the projection from $S \times S$ to first resp. second factor). The assumption that Y has non-zero transcendental cohomology implies that the class of Δ_Y is not decomposable in cohomology. It follows that

$$\mathrm{Im}\Big(GDA^{\dim Y}_B(Y\times Y)\to H^{2\dim Y}(Y\times Y,\mathbb{Q})\Big) = \mathrm{Im}\Big(\mathrm{Dec}^{\dim Y}(Y\times Y)\to H^{2\dim Y}(Y\times Y,\mathbb{Q})\Big)\oplus \mathbb{Q}[\Delta_Y]\ ,$$

where we use the shorthand

$$\mathrm{Dec}^j(Y\times Y) := \Big\langle (p_1)^\ast GDA^\ast_B(Y), (p_2)^\ast GDA^\ast_B(Y)\Big\rangle \cap A^j(Y\times Y)$$

for the *decomposable cycles*. We now see that if $\Gamma\in GDA^{\dim Y}(Y\times Y)$ is homologically trivial, then Γ does not involve the diagonal and so $\Gamma\in \mathrm{Dec}^{\dim Y}(Y\times Y)$. This proves the proposition. □

Corollary 2.9. *Let $\mathcal{Y}\to B$ be as in* Proposition 2.8. *Assume that $\mathcal{Y}\to B$ has the Franchetta property. Then for any fiber Y the cycle class map induces an injection*

$$GDA^{\dim Y}(Y\times Y)\ \hookrightarrow\ H^{2\dim Y}(Y\times Y,\mathbb{Q})\ .$$

Proof. This is immediate from Proposition 2.8: the Franchetta property for $\mathcal{Y}\to B$, combined with the Künneth decomposition in cohomology, implies that the right-hand side of Proposition 2.8 injects into cohomology. □

2.4. A CK decomposition.

Lemma 2.10. *Let M be a smooth projective variety with trivial Chow groups. Let $Y\subset M$ be a smooth complete intersection of dimension* $\dim Y = d$ *defined by ample line bundles. The variety Y has a self-dual CK decomposition $\{\pi^j_Y\}$ with the property that*

$$h^j(Y) := (Y,\pi^j_Y,0) = \oplus\mathbb{1}(\ast)\quad in\ \mathcal{M}_{\mathrm{rat}}\quad \forall\ j\neq d\ .$$

Moreover, this CK decomposition is generically defined*: writing $\mathcal{Y}\to B$ for the universal family (of complete intersections of the type of Y), there exist relative projectors $\pi^j_{\mathcal{Y}}\in A^d(\mathcal{Y}\times_B\mathcal{Y})$ such that $\pi^j_Y=\pi^j_{\mathcal{Y}}|_b$ (where $Y=Y_b$ for $b\in B$).*

Proof. This is a standard construction, one can look for instance at [38] (in case d is odd, which will be the case in this note, the "variable motive" $h(Y)^{\mathrm{var}}$ of [38, Theorem 4.4] coincides with $h^d(Y)$). □

3. Main results

3.1. An isomorphism of motives.

Definition 3.1. Let V be a vector space of dimension n, and let

$$\mathrm{Gr}(2,n) := \mathrm{Gr}(2,V)\ \subset\ \mathbb{P}(\wedge^2 V)$$

be the Grassmannian (parametrizing 2-dimensional subspaces of V) in its Plücker embedding. Assuming n is even, let

$$\mathrm{Pf}\ \subset\ \mathbb{P}(\wedge^2 V^\vee)$$

denote the projective dual of $\mathrm{Gr}(2,n)\subset\mathbb{P}(\wedge^2 V)$, called the *Pfaffian*. (The Pfaffian Pf is a hypersurface of degree $n/2$ and singular locus of codimension 7.)

Assume n is even. Given a linear subspace $U \subset \wedge^2 V$ of codimension 3, one can define varieties by intersecting on the Grassmannian side and on the Pfaffian side:

$$Y = Y_U := \mathrm{Gr}(2,V) \cap \mathbb{P}(U) \quad \subset \mathbb{P}(\wedge^2 V)\,,$$
$$C = C_U := \mathrm{Pf} \cap \mathbb{P}(U^\perp) \quad \subset \mathbb{P}(\wedge^2 V^\vee)\,.$$

We say that Y and C are *dual*. For U generic, the intersections Y and C are smooth and dimensionally transverse, of dimension $2(n-2)-3$ resp. 1.

Theorem 3.2. *Let Y be a smooth dimensionally transverse intersection*

$$Y := \mathrm{Gr}(2,n) \cap H_1 \cap H_2 \cap H_3\,,$$

where the H_j are Plücker hyperplanes.
(i) *Assume n is odd. Then $A^*_{hom}(Y) = 0$.*
(ii) *Assume that n is even, and that Y has a smooth dual curve C. There is an isomorphism*

$$h^d(Y) \cong h^1(C)((1-d)/2) \quad \textit{in } \mathcal{M}_{\mathrm{rat}}\,,$$

where $d := \dim Y$ and $h^d(Y)$ is as in Lemma (2.10).

Proof. This is a special case of [29, Theorem 3.17]. Since this is crucial to the present note, let us include a (sketch of) proof.

With notation as in Definition 3.1, let us consider

$$Q := \Big\{(T,\mathbb{C}\omega) \in \mathrm{Gr}(2,V)\times\mathbb{P}(U^\perp) \,\Big|\, \omega|_T = 0\Big\} \quad \subset \mathrm{Gr}(2,V)\times\mathbb{P}(U^\perp)\,,$$

the so-called *Cayley hypersurface*. There is a diagram

$$\begin{array}{ccccccccc} & & Q_Y & & \hookrightarrow & Q & \hookleftarrow & Q_C & \\ (3) & \swarrow & & {}^{p}\swarrow & & & \searrow^{q} & & \searrow \\ Y & \hookrightarrow & \mathrm{Gr}(2,V) & & & & \mathbb{P}(U^\perp) & \hookleftarrow & C \end{array}$$

Here, C is defined to be the empty set for n is odd, and the dual curve $C \subset \mathrm{Pf}$ in case n is even. The morphisms p and q are induced by the natural projections, and the closed subvarieties $Q_Y, Q_C \subset Q$ are defined as $p^{-1}(Y)$ resp. $q^{-1}(C)$.

The restriction of p to $Q \setminus Q_Y$ is trivial with fibre $Q_u \cong \mathbb{P}^1$, while the restriction of p to Q_Y is Zariski locally trivial with fibre $Q_{Y,y} \cong \mathbb{P}^2$. This allows us to relate the motives of Q and Y: an application of the "motivic Cayley trick" [18, Corollary 3.2] gives an isomorphism

$$(4) \qquad \begin{aligned} h(Q) &\cong h(Y)(-2) \oplus h(\mathrm{Gr}(2,n)) \oplus h(\mathrm{Gr}(2,n))(-1) \\ &\cong h(Y)(-2) \oplus \bigoplus \mathbb{1}(*) \quad \text{in } \mathcal{M}_{\mathrm{rat}}\,. \end{aligned}$$

The restriction of q to Q_C is *piecewise trivial* (in the sense of [40, Section 4.2]) with constant fiber F_1, while the restriction of q to $Q \setminus Q_C$ is piecewise trivial with constant fiber F_2. The fibers F_1 and F_2 are explicitly known; they have only algebraic cohomology [29, Lemma 3.5]. This allows to relate Q and C on the level of the Grothendieck ring of varieties, and hence also on the level of cohomology:

$$(5)\qquad h(Q)\cong h(C)(2-n)\oplus\bigoplus \mathbb{1}(*)\quad\text{in } \mathcal{M}_{\rm hom}\ .$$

(Here the convention is that $h(C)=0$ in case n is odd.)

Combining (4) and (5), we find a split injection of homological motives

$$(6)\qquad h^d(Y)\ \hookrightarrow\ h^1(C)((1-d)/2)\quad\text{in } \mathcal{M}_{\rm hom}\ .$$

Let us now consider things family-wise. Writing $B_0\subset \bar{B}:=\mathbb{P}H^0(\mathbb{P}(\wedge^2 V),\mathcal{O}(1)^{\oplus 3})$ for the dense open parametrizing sections such that both $Y_b:=\mathrm{Gr}(2,n)\cap H_1^b\cap H_2^b\cap H_3^b$ and the dual curve $C_b\subset \mathrm{Pf}$ are smooth and dimensionally transverse (and in addition C_b is contained in the non-singular locus $\mathrm{Pf}^\circ\subset\mathrm{Pf}$), we have universal families

$$\mathcal{Y}\to B_0\ ,\ \ \mathcal{C}\to B_0\ .$$

The above construction can be performed for every fiber $Y=Y_b$ of the family $\mathcal{Y}\to B_0$. A Hilbert schemes argument [29, Proposition 2.11] then allows to find *generically defined* correspondences (with respect to B_0) inducing the split injection (6). Then, the Franchetta-type result (Proposition 2.8) allows to lift the split injection (6) to an injection of Chow groups:

$$(7)\qquad A^\ast_{hom}(Y)=A^\ast(h^d(Y))\ \hookrightarrow\ A^\ast_{hom}(h^1(C)((1-d)/2))=A^1_{hom}(C)\ .$$

We conclude from (7) that $A^\ast_{AJ}(Y)=0$ and so Y is Kimura finite-dimensional (i.e. $h(Y)$ is finite-dimensional in the sense of [19]). Combining (4) and (5), we find a numerical equality $\dim H^d(Y,\mathbb{Q})=\dim H^1(C,\mathbb{Q})$ and so the injection (6) is actually an isomorphism of homological motives. Using Kimura finite-dimensionality of both sides, it follows that (6) is also an isomorphism of Chow motives:

$$h^d(Y)\ \xrightarrow{\cong}\ h^1(C)((1-d)/2)\quad\text{in } \mathcal{M}_{\rm rat}\ .$$

This proves the theorem. □

3.2. Some instances of the Franchetta property.

NOTATION 3.3. Let $\bar{B}$ and B_0 be as in the proof of Theorem 3.2, and let $B\supset B_0$ be the set parametrizing smooth dimensionally transverse intersections $Y_b=\mathrm{Gr}(2,n)\cap H_1\cap H_2\cap H_3$; there is a universal family

$$\mathcal{Y}\ \to\ B\ .$$

Assuming n is even, let us write

$$\mathcal{C}\ \to\ B_0$$

for the universal family of smooth dual curves $C_b\subset\mathrm{Pf}^\circ$, as in the proof of Theorem 3.2.

Proposition 3.4. *The following families have the Franchetta property:*
(i) *the family* $\mathcal{Y}\to B$*;*
(ii) *the family* $\mathcal{C}\to B_0$.

Proof. For (i), let us note that the statement is vacuously true in case n is odd, because then each fiber Y_b has trivial Chow groups (Theorem 3.2(i)). Let us now assume that n is even, say $n = 2m$. We observe that the projection

$$\bar{\mathcal{Y}} \ \to \ \mathrm{Gr}(2,n)$$

is a projective bundle, and so (reasoning with the projective bundle formula, or directly applying [11, Proposition 5.2]) one finds that for any fiber $Y := Y_b$ there is equality

$$GDA^j(Y) = \mathrm{Im}\Big(A^j(\mathrm{Gr}(2,n)) \to A^j(Y)\Big) .$$

We know from Theorem 3.2(ii) that the only non-trivial Chow group of Y is

$$A^{(d+1)/2}(Y) = A^{(2(n-2)-3+1)/2}(Y) = A^{2m-3}(Y) ,$$

and so we only need to prove that $GDA^{2m-3}(Y)$ injects into cohomology. The Chow ring of the Grassmannian is

$$A^*(\mathrm{Gr}(2,n)) = \langle h,\ c\rangle ,$$

where $c := c_2(Q) \in A^2(\mathrm{Gr}(2,n))$ is the second Chern class of the tautological quotient bundle [9], and so

$$A^{2m-4}(\mathrm{Gr}(2,n)) \ \xrightarrow{\cdot h}\ A^{2m-3}(\mathrm{Gr}(2,n))$$

is surjective (and hence an isomorphism, by hard Lefschetz). Let $\tau\colon Y \to \mathrm{Gr}(2,n)$ denote the inclusion morphism. The normal bundle formula tells us that the composition

$$A^{2m-4}(\mathrm{Gr}(2,n)) \ \xrightarrow{\cdot h}\ A^{2m-3}(\mathrm{Gr}(2,n)) \ \xrightarrow{\tau^*}\ A^{2m-3}(Y) \ \xrightarrow{\tau_*}\ A^{2m}(\mathrm{Gr}(2,n))$$

is a non-zero multiple of

$$A^{2m-4}(\mathrm{Gr}(2,n)) \ \xrightarrow{\cdot h^4}\ A^{2m}(\mathrm{Gr}(2,n)) .$$

This last map is the same as

$$H^{4m-8}(\mathrm{Gr}(2,n),\mathbb{Q}) \ \xrightarrow{\cdot h^4}\ H^{4m}(\mathrm{Gr}(2,n),\mathbb{Q}) ,$$

which is an isomorphism thanks to hard Lefschetz for the $(4m-4)$-dimensional variety $\mathrm{Gr}(2,n)$. This proves the required injectivity of $GDA^{2m-3}(Y)$ into cohomology.

As for (ii), one can either prove this directly, or can reduce to (i) via the generically defined isomorphism

$$A^1_{hom}(C) \ \xrightarrow{\cong}\ A^{2m-3}_{hom}(Y)$$

given by Theorem 3.2(ii). □

Proposition 3.5. *The following families have the Franchetta property:*
(i) *the family* $\mathcal{C} \times_{B_0} \mathcal{C} \to B_0$;
(ii) *the family* $\mathcal{Y} \times_B \mathcal{Y} \to B$.

Proof. (i) Let $\bar{\mathcal{C}} \subset \mathrm{Pf} \times \bar{B}$ denote the projective closure of $\mathcal{C}$, and let us consider the projection

$$\pi\colon \bar{\mathcal{C}} \times_{\bar{B}} \bar{\mathcal{C}} \ \to\ \mathrm{Pf} \times \mathrm{Pf} \ .$$

This is a *stratified projective bundle* (in the sense of [11]). As such, [11, Proposition 5.2] implies the equality

$$GDA^*_{B_0}(C\times C) = \mathrm{Im}\Big(A^*(\mathrm{Pf}^\circ\times\mathrm{Pf}^\circ)\to A^*(C\times C)\Big) + \Delta_* GDA^*_{B_0}(C)\,, \tag{8}$$

where $\Delta\colon C \to C\times C$ is the inclusion along the diagonal, and $\mathrm{Pf}^\circ\subset\mathrm{Pf}$ denotes the non-singular locus of the Pfaffian. As Pf° has the Chow–Künneth property [29, Example 2.7], $A^*(\mathrm{Pf}^\circ\times\mathrm{Pf}^\circ)$ is generated by $A^*(\mathrm{Pf}^\circ)\otimes A^*(\mathrm{Pf}^\circ)$. The equality (8) thus simplifies to

$$GDA^*_{B_0}(C\times C) = \Big\langle (p_j)^*\,\mathrm{Im}(A^j(\mathrm{Pf}^\circ)\to A^*(C))\,,\,\Delta_C\Big\rangle\,. \tag{9}$$

We now proceed to check that $GDA^j_{B_0}(C\times C)$ injects into cohomology:

In case $j=1$, we know that Δ_C is linearly independent from the decomposable classes

$$\Big\langle (p_j)^*\,\mathrm{Im}(A^j(\mathrm{Pf}^\circ)\to A^*(C))\Big\rangle$$

in cohomology (indeed, we may assume that C has genus >0, for otherwise the statement is vacuously true). The required injectivity then reduces to Proposition 3.4(ii).

In case $j=2$, we know that $A^1(\mathrm{Pf}^\circ)$ is 1-dimensional, generated by a hyperplane class H (cf. Lemma 3.6 below). Since $C\subset\mathbb{P}^2$ is a plane curve, clearly we have an equality

$$\Delta_C\cdot(p_i)^*(H) = \sum_{r=0}^{2}\frac{1}{\deg C}\,(p_1)^*(H^r)\cdot(p_2)^*(H^{2-r})\quad\text{in } A^2(C\times C)\,,$$

and so

$$\begin{aligned} GDA^2_{B_0}(C\times C) &= \Big\langle (p_j)^*\,\mathrm{Im}(A^j(\mathrm{Pf}^\circ)\to A^*(C))\,,\,\Delta_C\Big\rangle\cap A^2(C\times C)\\ &= \Big\langle (p_j)^*\,\mathrm{Im}(A^j(\mathrm{Pf}^\circ)\to A^*(C))\Big\rangle\cap A^2(C\times C)\,. \end{aligned}$$

The required injectivity then reduces to Proposition 3.4(ii).

In the above, we have used the following lemma:

Lemma 3.6. *Let* $\mathrm{Pf}^\circ\subset\mathrm{Pf}$ *denote (as above) the non-singular locus of the Pfaffian. We have*

$$A^1(\mathrm{Pf}^\circ)\cong\mathbb{Q}[H]\,.$$

Proof. (of the lemma.) We consider

$$\widetilde{\mathrm{Pf}} := \Big\{(\omega,K)\in\mathrm{Pf}\times\mathrm{Gr}(2,n)\,\Big|\,K\subset\ker\omega\Big\}\quad\subset\ \mathrm{Pf}\times\mathrm{Gr}(2,n)\,.$$

The projection $\widetilde{\mathrm{Pf}}\to\mathrm{Gr}(2,n)$ is a projective bundle (and so $\widetilde{\mathrm{Pf}}$ is smooth), and the projection $\widetilde{\mathrm{Pf}}\to\mathrm{Pf}$ is an isomorphism over the non-singular locus (and so $\widetilde{\mathrm{Pf}}\to\mathrm{Pf}$ is a resolution of singularities).

Being a projective bundle over a Grassmannian, $\widetilde{\mathrm{Pf}}$ has Picard number 2:

$$A^1(\widetilde{\mathrm{Pf}})=\mathbb{Q}^2\,.$$

The complement of (the isomorphic pre-image of) Pf° inside $\widetilde{\mathrm{Pf}}$ is an irreducible divisor D (it is a partial flag variety). The localization sequence

$$A_*(D)\ \to\ A^1(\widetilde{\mathrm{Pf}})\ \to\ A^1(\mathrm{Pf}^\circ)\ \to\ 0$$

then gives the result. □

(ii) Again, we may assume that n is even (for otherwise the statement is vacuously fulfilled). In view of Lemma 2.7, it will suffice to prove the Franchetta property for $\mathcal{Y}\times_{B_0}\mathcal{Y}\to B_0$. Thanks to Theorem 3.2(ii), for any fiber $Y=Y_b$ with $b\in B_0$ we have split injections

$$A^j(Y\times Y)\ \hookrightarrow\ A^{j+1-d}(C\times C)\oplus\bigoplus A^*(C)\oplus\mathbb{Q}^s\ .$$

The isomorphism of Theorem 3.2 being generically defined, there are also split injections

$$GDA^j(Y\times Y)\ \hookrightarrow\ GDA^{j+1-d}(C\times C)\oplus\bigoplus GDA^*(C)\oplus\mathbb{Q}^s\ .$$

The required injectivity now follows from (i) and Proposition 3.4(ii). □

3.3. MCK.

Theorem 3.7. *Let Y be a smooth dimensionally transverse intersection*

$$Y:=\mathrm{Gr}(2,n)\cap H_1\cap H_2\cap H_3\ ,$$

where the H_j are Plücker hyperplanes. Then Y has an MCK decomposition.

Proof. In case n is odd, Y has trivial Chow groups (Theorem 3.2(i)) and so the statement is vacuously true. In case $n=4$, Y is a rational curve and again the statement is vacuously true. We may thus suppose that n is even and ≥ 6. We have the following general result:

Proposition 3.8. *Let $\mathcal{Y}\to B$ be a family of smooth projective varieties, verifying*
(a1) *the fibers Y_b are of odd dimension $d\geq 5$ and*

$$A^j_{hom}(Y_b)=0\quad\forall j>(d+1)/2\ \ \forall\, b\in B\ ;$$

(a2) *the fibers Y_b have a generically defined Künneth decomposition, i.e. there exist $\{\pi^j_{\mathcal{Y}}\}\in A^d(\mathcal{Y}\times_B\mathcal{Y})$ such that the fiberwise restriction $\pi^j_{Y_b}:=\pi^j_{\mathcal{Y}}|_b\in A^d(Y_b\times Y_b)$ is a Künneth decomposition for all $b\in B$;*
(a3) *the family $\mathcal{Y}\times_B\mathcal{Y}\to B$ has the Franchetta property.*
Then $\{\pi^j_{Y_b}\}$ is an MCK decomposition for any $b\in B$.

Proof. (of Proposition 3.8.) Condition (a1) implies (via the Bloch–Srinivas argument, cf. [5]) that for every fiber Y_b there exists a curve C_b and a split injection of motives

$$h(Y_b)\ \hookrightarrow\ h(C_b)((1-d)/2)\oplus\bigoplus\mathbb{1}(*)\quad\text{in }\mathcal{M}_{\mathrm{rat}}\ . \tag{10}$$

Condition (a3) implies that the Künneth decomposition $\{\pi^j_{Y_b}\}$ of (a2) is a self-dual CK decomposition. Let $h(Y_b)=\oplus_j h^j(Y_b)$ denote the corresponding decomposition of the motive of X. Using the injection (10), one finds that $h^j(Y_b)=\oplus\mathbb{1}(*)$ for all $j\neq d$, while for $j=d$ one finds a split injection

$$h^d(Y_b)\ \hookrightarrow\ h^1(C_b)((1-d)/2)\quad\text{in }\mathcal{M}_{\mathrm{rat}}\ . \tag{11}$$

Let us now establish that the CK decomposition $\{\pi^j_{Y_b}\}$ is MCK. By definition, what we need to check is that the cycle

$$\Gamma_{ijk}:=\pi^k_{Y_b}\circ\Delta^{\mathrm{sm}}_{Y_b}\circ(\pi^i_{Y_b}\times\pi^j_{Y_b})\quad\in\ A^{2d}(Y_b\times Y_b\times Y_b)$$

is zero for all $i+j\neq k$.

Let us assume at least one of the integers i, j, k is different from d. In this case, there is an injection

$$\Gamma_{ijk} \ \in\ (\pi^{2d-i}_{Y_b}\times\pi^{2d-j}_{Y_b}\times\pi^{k}_{Y_b})_\ast A^{2d}(Y_b\times Y_b\times Y_b)\ \hookrightarrow\ \bigoplus A^\ast(Y_b\times Y_b)\,,$$

and this injection sends generically defined cycles to generically defined cycles. But Γ_{ijk} is generically defined and homologically trivial, and so the Franchetta property for $\mathcal{Y}\times_B\mathcal{Y}\to B$ gives the required vanishing $\Gamma_{ijk}=0$.

Next, let us assume $i=j=k=d$. In this case, the injection of motives (11) induces an injection of Chow groups

$$\Gamma_{ijk} \ \in\ (\pi^{d}_{Y_b}\times\pi^{d}_{Y_b}\times\pi^{d}_{Y_b})_\ast A^{2d}(Y_b\times Y_b\times Y_b)\ \hookrightarrow\ A^{(d+3)/2}(C_b\times C_b\times C_b)\,.$$

But the right-hand side vanishes for dimension reasons for any $d\geq 5$, and so $\Gamma_{ijk}=0$. □

Let us now consider the family $\mathcal{Y}\to B$ of all smooth complete intersections $\mathrm{Gr}(2,n)\cap H_1\cap H_2\cap H_3$, where $n\geq 6$ is even. Each fiber Y_b has a generically defined CK decomposition $\{\pi^j_{Y_b}\}$ (Lemma 2.10). To check that $\{\pi^j_{Y_b}\}$ is MCK, it suffices to do this over a dense open of B; for instance we may take $B_0\subset B$ the locus as before where Y_b has a smooth dual curve C_b contained in Pf°. Let us check that $\mathcal{Y}\to B_0$ verifies the conditions of Proposition 3.8. Condition (a1) is immediate from Theorem 3.2(ii). Condition (a2) is fulfilled by the $\{\pi^j_{Y_b}\}$. As for condition (a3), this is Proposition 3.5(ii). This ends the proof. □

4. The tautological ring

4.1. A positive result.

Corollary 4.1. *Let Y be as in* Theorem 3.7*, and $m\in\mathbb{N}$. Let*

$$R^\ast(Y^m):=\Big\langle (p_i)^\ast\,\mathrm{Im}(A^\ast(\mathrm{Gr}(2,n))\to A^\ast(Y)),\ (p_{ij})^\ast(\Delta_Y)\Big\rangle\ \subset\ \ A^\ast(Y^m)$$

be the $\mathbb{Q}$-subalgebra generated by (pullbacks of) cycles coming from $\mathrm{Gr}(2,n)$ *and (pullbacks of) the diagonal $\Delta_Y\in A^d(Y\times Y)$. (Here p_i and p_{ij} denote the various projections from Y^m to Y resp. to $Y\times Y$). The cycle class map induces injections*

$$R^\ast(Y^m)\ \hookrightarrow\ H^\ast(Y^m,\mathbb{Q})\ \ \textit{for all } m\in\mathbb{N}\,.$$

Proof. This is inspired by the analogous result for cubic hypersurfaces [12, Section 2.3], which in turn is inspired by analogous results for hyperelliptic curves [44], [45] (cf. Remark 4.2 below) and for K3 surfaces [51].

The Chow ring $A^\ast(\mathrm{Gr}(2,n))$ is generated by the Plücker polarization $h\in A^1(\mathrm{Gr}(2,n))$ and the Chern class $c_2(Q)\in A^2(\mathrm{Gr}(2,n))$, where $Q\to\mathrm{Gr}(2,n)$ is the universal quotient bundle [9]. As in [12, Section 2.3], let us write

$$o:=\frac{1}{\deg(h^d)}\,h^d\ \in\ A^d(Y)\,,\quad c:=c_2(Q)|_Y\ \in\ A^2(Y)\,,$$

and

$$\tau := \pi_Y^d = \Delta_Y - \sum_{j\neq d} \pi_Y^j \ \in \ A^d(Y\times Y)\,,$$

where the π_Y^j are as above, and $d := \dim Y$.

Moreover, for any $1 \le i < j \le m$ let us write

$$\begin{aligned} o_i &:= (p_i)^*(o) \ \in \ A^d(Y^m)\,,\\ h_i &:= (p_i)^*(h) \ \in \ A^1(Y^m)\,,\\ c_i &:= (p_i)^*(c) \ \in \ A^2(Y^m)\,,\\ \tau_{ij} &:= (p_{ij})^*(\tau) \ \in \ A^d(Y^m)\,. \end{aligned}$$

Note that (by definition) we have

$$R^*(Y^m) = \left\langle o_i, h_i, c_i, \tau_{ij}\right\rangle \ \subset \ A^*(Y^m)\,.$$

Let us now define the $\mathbb{Q}$-subalgebra

$$\bar{R}^*(Y^m) := \left\langle o_i, h_i, c_i, \tau_{ij}\right\rangle \ \subset \ H^*(Y^m, \mathbb{Q})$$

(where i ranges over $1 \le i \le m$, and $1 \le i < j \le m$); this is the image of $R^*(Y^m)$ in cohomology. One can prove (just as [12, Lemma 2.11] and [51, Lemma 2.3]) that the $\mathbb{Q}$-algebra $\bar{R}^*(Y^m)$ is isomorphic to the free graded $\mathbb{Q}$-algebra generated by o_i, h_i, c_i, τ_{ij}, modulo the following relations:

(12) $$h_i\cdot o_i = c_i\cdot o_i = 0,\quad c_i^{(d+1)/2} = 0,\quad c_i^{(d-1)/2} = \lambda h_i^{d-1},\quad \ldots,\quad h_i^d = \deg(h^d)\, o_i\,;$$

(13) $$\tau_{ij}\cdot o_i = \tau_{ij}\cdot h_i = \tau_{ij}\cdot c_i = 0,\quad \tau_{ij}\cdot\tau_{ij} = -b_d\, o_i\cdot o_j\,;$$

(14) $$\tau_{ij}\cdot\tau_{ik} = \tau_{jk}\cdot o_i\,;$$

(15) $$\sum_{\sigma\in\mathfrak{S}_{b_d+2}} \prod_{i=1}^{b_d/2+1} \tau_{\sigma(2i-1),\sigma(2i)} = 0\,.$$

Here $\lambda \in \mathbb{Q}$, and the dots "…" in (12) indicate certain relations of type $c_i^{m_j} h_i^{n_j} = \lambda_j h_i^{2m_j+n_j}$. By definition, $b_d := \dim H^d(Y, \mathbb{Q})$ and $\mathfrak{S}_r$ denotes the symmetric group on r elements.

To prove Corollary 4.1, it suffices to check that all these relations are verified modulo rational equivalence. The relations (12) take place in $R^*(Y)$ and so they follow from the Franchetta property for Y (Proposition 3.4). The relations (13) take place in $R^*(Y^2)$. The last relation is trivially verified, because (Y being Fano) $A^{2d}(Y^2) = \mathbb{Q}$. As for the other relations of (13), these follow from the Franchetta property for $Y\times Y$ (Proposition 3.5).

Relation (14) takes place in $R^*(Y^3)$ and follows from the MCK decomposition. Indeed, we have

$$\Delta_Y^{\rm sm}\circ(\pi_Y^d\times\pi_Y^d) = \pi_Y^{2d}\circ\Delta_Y^{\rm sm}\circ(\pi_Y^d\times\pi_Y^d) \ \text{ in } A^{2d}(Y^3)\,,$$

which (using Lieberman's lemma) translates into

$$(\pi_Y^d\times\pi_Y^d\times\Delta_Y)_*\Delta_Y^{\rm sm} = (\pi_Y^d\times\pi_Y^d\times\pi_Y^{2d})_*\Delta_Y^{\rm sm} \ \text{ in } A^{2d}(Y^3)\,,$$

which means that

$$\tau_{13}\cdot\tau_{23}=\tau_{12}\cdot o_3 \quad \text{in } A^{2d}(Y^3)\ .$$

Finally, relation (15), which takes place in $R^*(Y^{b_d+2})$, is related to the Kimura finite-dimensionality relation [19]: relation (15) expresses the vanishing

$$\mathrm{Sym}^{b_d+2} H^d(Y,\mathbb{Q})=0\ ,$$

where $H^d(Y,\mathbb{Q})$ is seen as a super vector space. This relation is also verified modulo rational equivalence, (i.e., relation (15) is also true in $A^{d(b_d+2)}(Y^{b_d+2})$): relation (15) involves a cycle in

$$A^*(\mathrm{Sym}^{b_d+2} h^d(Y))\ ,$$

and $\mathrm{Sym}^{b_d+2} h^d(Y)$ is 0 because Y has Kimura finite-dimensional motive (Theorem 3.2).

This ends the proof. □

Remark 4.2. Given any curve C and an integer $m\in\mathbb{N}$, one can define the *tautological ring*

$$R^*(C^m):=\langle (p_i)^*(K_C),(p_{ij})^*(\Delta_C)\rangle \quad \subset\ A^*(C^m)$$

(where p_i, p_{ij} denote the various projections from C^m to C resp. $C\times C$). Tavakol has proven [45, Corollary 6.4] that if C is a hyperelliptic curve, the cycle class map induces injections

$$R^*(C^m)\ \hookrightarrow\ H^*(C^m,\mathbb{Q}) \quad \text{for all } m\in\mathbb{N}\ .$$

On the other hand, there exist curves for which the tautological ring $R^*(C^3)$ does *not* inject into cohomology, cf. Proposition 4.3 below.

4.2. A negative result.

Proposition 4.3. *Let*

$$Y:=\mathrm{Gr}(2,n)\cap H_1\cap H_2\cap H_3$$

be a very general intersection of the Grassmannian with 3 Plücker hyperplanes, where n is even and $8\le n\le 2000$. Let $C\subset$ Pf *be the curve dual to Y* (Definition 3.1). *The $\mathbb{Q}$-subalgebra*

$$R^*(C^m):=\langle (p_i)^*(K_C),\ (p_{ij})^*(\Delta_C)\rangle \ \subset\ A^*(C^m)$$

does not *inject into cohomology for $m=3$.*

Proof. The point is that C is a plane curve of degree $n/2$, and that the general plane curve of degree $n/2$ arises in this way [1]. Using the spread lemma (Lemma 2.7), it follows that the assumption that $R^*(C^m)$ injects into cohomology for the very general C as in Proposition 4.3 would imply that $R^*(C^m)$ injects into cohomology for every plane curve of degree $n/2$. Taking $m=3$, this would mean that every plane curve of degree $n/2$ has a self-dual MCK decomposition. As explained in [15, Proposition 7.1] and [13, Remark 2.4], this would imply that for every plane curve C of degree $n/2$ the Ceresa cycle

$$C - [-1]_*(C) \in A_1(\mathrm{Jac}(C))$$

is algebraically trivial. But this is known to be false for the Fermat curve of degree between 4 and 1000, cf. [36]. □

Acknowledgements. Thanks to Lie Fu and Charles Vial for lots of inspiring exchanges around MCK.

References

[1] A. Beauville: *Determinantal hypersurfaces*, Michigan Math. J. **48** (2000), 39–64.

[2] A. Beauville: *On the splitting of the Bloch–Beilinson filtration*; in Algebraic Cycles and Motives, London Math. Soc. Lecture Notes **344**, Cambridge University Press, Cambridge, 2007.

[3] A. Beauville and C. Voisin: *On the Chow ring of a K3 surface*, J. Algebraic. Geom. **13** (2004), 417–426.

[4] N. Bergeron and Z. Li: *Tautological classes on moduli space of hyperKähler manifolds*, Duke Math. J. **168** (2019), 1179–1230.

[5] S. Bloch and V. Srinivas: *Remarks on correspondences and algebraic cycles*, Amer. J. Math. **105** (1983), 1235–1253.

[6] O. Debarre and C. Voisin: *Hyper-Kähler fourfolds and Grassmann geometry*, J. Reine Angew. Math. **649** (2010), 63–87.

[7] H. Diaz: *The Chow ring of a cubic hypersurface*, to appear in Int. Math. Res. Not. IMRN.

[8] R. Donagi: *On the geometry of Grassmannians*, Duke Math. J. **44** (1977), 795–837.

[9] D. Eisenbud and J. Harris: 3264 and All That: A Second Course in Algebraic Geometry, Cambridge University Press, Cambridge, 2016.

[10] E. Fatighenti and G. Mongardi: *Fano varieties of K3 type and IHS manifolds*, arXiv:1904.05679.

[11] L. Fu, R. Laterveer and Ch. Vial: *The generalized Franchetta conjecture for some hyper-Kähler varieties*, with an appendix joint with M. Shen, J. Math. Pures Appl. (9) **130** (2019), 1–35.

[12] L. Fu, R. Laterveer and Ch. Vial: *The generalized Franchetta conjecture for some hyper-Kähler varieties, II*, J. Éc. Polytec. Math. **8** (2021), 1065–1097.

[13] L. Fu, R. Laterveer and Ch. Vial: *Multiplicative Chow–Künneth decompositions and varieties of cohomological K3 type*, Ann. Mat. Pura Appl. (4) **200** (2021), 2085–2126.

[14] L. Fu, Z. Tian and Ch. Vial: *Motivic hyper-Kähler resolution conjecture, I: generalized Kummer varieties*, Geom. Topol. **23** (2019), 427–492.

[15] L. Fu and Ch. Vial: *Distinguished cycles on varieties with motive of abelian type and the section property*, J. Algebraic Geom. **29** (2020), 53–107.

[16] W. Fulton: Intersection Theory, Springer–Verlag, Ergebnisse der Mathematik, Berlin, 1984.

[17] U. Jannsen: *On finite-dimensional motives and Murre's conjecture*; in Algebraic Cycles and Motives, Cambridge University Press, Cambridge, 2007.

[18] Q. Jiang: *On the Chow theory of projectivization*, arXiv:1910.06730v1.

[19] S.-I. Kimura: *Chow groups are finite dimensional, in some sense*, Math. Ann. **331** (2005), 173–201.

[20] A. Kuznetsov: *Hyperplane sections and derived categories*, Izv. Ross. Akad. Nouk. Ser. Mat. **70** (2006) 23–128 (in Russian); translation in Izv. Math. **70** (2006), 447–547.

[21] A. Kuznetsov: *Homological projective duality for Grassmannians of lines*, math.AG/0610957.

[22] R. Laterveer: *A remark on the chow ring of Küchle fourfolds of type* $d3$, Bull. Aust. Math. Soc. **100** (2019), 410–418.

[23] R. Laterveer: *Algebraic cycles and Verra fourfolds*, Tohoku Math. J. (2) **72** (2020), 451–485.

[24] R. Laterveer: *On the Chow ring of certain Fano fourfolds*, Ann. Univ. Paedagog. Crac. Stud. Math. **19** (2020), 39–52.

[25] R. Laterveer: *On the Chow ring of Fano varieties of type* $S2$, Abh. Math. Semin. Univ. Hambg. **90** (2020), 17–28.

[26] R. Laterveer: *Algebraic cycles and Gushel–Mukai fivefolds*, J. Pure Appl. Algebra **225** (2021), Paper No. 106582, 18pp.
[27] R. Laterveer: *Algebraic cycles and intersections of 2 quadrics*, Mediterr. J. Math. **18** (2021), Paper No. 146, 22pp.
[28] R. Laterveer: *Algebraic cycles and Fano threefolds of genus 8*, preprint.
[29] R. Laterveer: *Motives and the Pfaffian–Grassmannian equivalence*, J. Lond. Math. Soc. **104** (2021) 1738–1764.
[30] R. Laterveer: *Algebraic cycles and intersections of three quadrics*, Mathematical Proceedings of the Cambridge Philosophical Society (2021), 1–19.
[31] R. Laterveer, J. Nagel and C. Peters: *On complete intersections in varieties with finite-dimensional motive*, Q. J. Math. **70** (2019), 71–104.
[32] R. Laterveer and Ch. Vial: *On the Chow ring of Cynk–Hulek Calabi–Yau varieties and Schreieder varieties*, Canad. J. Math. **72** (2020), 505–536.
[33] J. Murre: *On a conjectural filtration on the Chow groups of an algebraic variety, parts I and II*, Indag. Math. (N.S.) **4** (1993), 177–201.
[34] J. Murre, J. Nagel and C. Peters: Lectures on the Theory of Pure Motives, University Lecture Series **61**, American Mathematical Society, Providence, RI, 2013.
[35] A. Negut, G. Oberdieck and Q. Yin: *Motivic decompositions for the Hilbert scheme of points of a K3 surface*, arXiv:1912.09320v1.
[36] N. Otsubo: *On the Abel–Jacobi maps of Fermat Jacobians*, Math. Z. **270** (2012), 423–444.
[37] N. Pavic, J. Shen and Q. Yin: *On O'Grady's generalized Franchetta conjecture*, Int. Math. Res. Not. **2017** (2017), 4971–4983.
[38] C. Peters: *On a motivic interpretation of primitive, variable and fixed cohomology*, Math. Nachr. **292** (2019), 402–408.
[39] T. Scholl: *Classical motives*; in Motives, Proc. Sympos. Pure Math. **55**, Amer. Math. Soc., Proviedence, RI 1994.
[40] J. Sebag: *Intégration motivique sur les schémas formels*, Bull. Soc. Math. France **132** (2004), 1–54.
[41] E. Segal and R. Thomas: *Quintic threefolds and Fano elevenfolds*, J. Reine Angew. Math. **743** (2018), 245–259.
[42] M. Shen and Ch. Vial: The Fourier Transform for Certain HyperKähler Fourfolds, Mem. Amer. Math. Soc. **240** (2016), no. 1139.
[43] M. Shen and Ch. Vial: *The motive of the Hilbert cube* $X^{[3]}$, Forum Math. Sigma **4** (2016), Paper No. e30, 55pp.
[44] M. Tavakol: *The tautological ring of the moduli space* M_2^{rt}, Int. Math. Res. Not. **2014** (2014), 6661–6683.
[45] M. Tavakol: *Tautological classes on the moduli space of hyperelliptic curves with rational tails*, J. Pure Appli. Algebra **222** (2018), 2040–2062.
[46] Ch. Vial: *Projectors on the intermediate algebraic Jacobians*, New York J. Math. **19** (2013), 793–822.
[47] Ch. Vial: *Niveau and coniveau filtrations on cohomology groups and Chow groups*, Proc. Lond. Math. Soc. **106** (2013), 410–444.
[48] Ch. Vial: *On the motive of some hyperkähler varieties*, J. Reine Angew. Math. **725** (2017), 235–247.
[49] C. Voisin: *The generalized Hodge and Bloch conjectures are equivalent for general complete intersections, II*, J. Math. Sci. Univ. Tokyo **22** (2015), 491–517.
[50] C. Voisin: Chow Rings, Decomposition of the Diagonal, and the Topology of Families, Princeton University Press, Princeton and Oxford, 2014.
[51] Q. Yin: *Finite-dimensionality and cycles on powers of K3 surfaces*, Comment. Math. Helv. **90** (2015), 503–511.

Institut de Recherche Mathématique Avancée
CNRS – Université de Strasbourg
7 Rue René Descartes, 67084 Strasbourg CEDEX
FRANCE

e-mail: robert.laterveer@math.unistra.fr