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A TABLE OF TWISTED KNOTS WITH CROSSING NUMBER 3

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Abstract

Virtual knot theory is a generalization of knot theory which is based on Gauss chord diagrams and link diagrams on closed orientable surfaces. Twisted knots are a generalization of virtual knots, which correspond to link diagrams in possibly non-orientable surfaces. In this paper, we construct a table of twisted knots with crossing numbers 3. We use the multivariable polynomial invariants, JKSS invariants of double covering of twisted knots, and the twisted knot quandles for classification of twisted knots. We determine invertibility, chirality and checkerboard colorability for some twisted knots in our table.

1. Introduction

Twisted knot theory was introduced by M. Bourgoin [1] as an extension of virtual knot theory. Twisted links correspond to stable equivalence classes of links in oriented 3-manifolds which are line bundles over closed surfaces [1], and virtual links correspond to those in oriented 3-manifolds which are line bundles over oriented closed surfaces [2].

A *virtual link diagram* is a generalization of a link diagram in $\mathbb{R}^2$ possibly with some crossing called *virtual crossings* which have no over/under information. A virtual crossing is depicted with an encircled double point. A *twisted link diagram* is a generalization of a virtual link diagram, possibly with *bars* which are small line segment intersecting the diagram transversely avoiding crossings. An example of a twisted link diagram is depicted on the left of Figure 1.

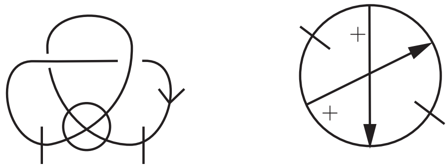

Fig. 1. An example of a twisted knot diagram and its Gauss chord diagram

Let D and D' be virtual (resp. twisted) link diagrams. We say that D is *equivalent* to D' as a virtual (resp. twisted) link if D is transformed to D', up to isotopies of $\mathbb{R}^2$, by a finite sequence of Reidemeister moves and virtual Reidemeister moves (resp. Reidemeister moves, virtual Reidemeister moves and twisted Reidemeister moves) depicted in Figures 2. A *virtual link* (resp. *twisted link*) is an equivalence class of a virtual (resp. twisted) link

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diagram.

A twisted link (diagram) of one component is called a twisted knot (diagram).

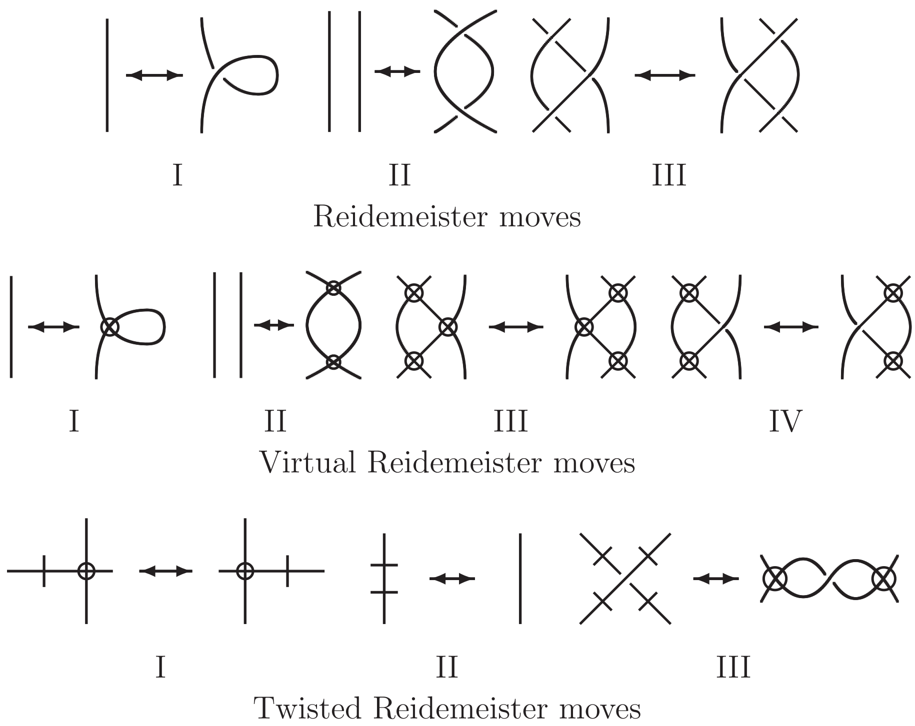


Fig. 2. Extended Reidemeister moves

For a twisted link diagram D, the *reverse* of D, denoted by $-D$, means a diagram obtained from D by reversing the orientation of D. The *mirror image* of D, denoted by D^*, means a diagram obtained from D by switching the over/under information, or equivalently the positive/negative information at every classical crossing of D. The *reflection image* of D, denoted by $D^\sharp$, is a diagram obtained from D by reflecting D along a line in $\mathbb{R}^2$.

Two twisted link diagrams D and D' are *weakly equivalent* if D' is equivalent to D, $-D$, D^* or $-D^*$ as a twisted link. (We note that for any twisted link diagram D, D^* is equivalent to $D^\sharp$ as a twisted link. In fact, it is shown in [12] that for any twisted link diagram D, $D^{*\sharp}$ is equivalent to D.) We also say that two twisted links $L = [D]$ and $L' = [D']$ are *weakly equivalent* if their representatives D and D' are weakly equivalent.

For a twisted link diagram D, we denote by $c(D)$ the number of classical crossings. The *crossing number* of a twisted link L, denoted by $c(L)$, is the minimum among $c(D)$ for all diagrams D representing L.

The first author classified twisted knots with crossing numbers up to 2 in [7]. A twisted knot L is said to be *pseudo prime* if there exists no diagram D representing L which is a connected sum of two diagrams D_1 and D_2 such that $c(D_1) > 0$, $c(D_2) > 0$ and $c(D_1) + c(D_2) = c(L)$. The purpose of this paper is to provide a table of weak equivalence classes of pseudo prime twisted knots with crossing numbers 3.

Theorem 1. *There exist* 82 *or* 81 *weak equivalence classes of pseudo prime twisted knots with crossing number* 3. *They are represented by the diagrams in* Table 3.

It is unknown to the authors if the twisted knot diagram 3_{82} in Table 3 is equivalent to a twisted knot diagram without crossings. If the diagram 3_{82} is (or is not) equivalent to one without crossings, then there exist 81 (or 82) weak equivalence classes of pseudo prime twisted knots with crossing number 3, which are represented by the diagrams 3_1 through 3_{81} (or 3_{82}).

The symbol 3_k ($k = 1, \ldots, 82$) will be used to denote the diagrams in Table 3 or the twisted knots represented by the diagrams.

We also investigate, for these twisted knots, invertibility, chirality, checkerboard colorability, and the twist numbers.

A twisted knot $L = [D]$ is *invertible* if D is equivalent to $-D$. Otherwise, it is *noninvertible*.

Theorem 2. (1) *The following twisted knots are invertible:*
3_1, 3_4, 3_5, 3_7, 3_8, 3_9, 3_{10}, 3_{11}, 3_{12}, 3_{13}, 3_{14}, 3_{15}, 3_{16}, 3_{17}, 3_{18}, 3_{19}, 3_{20}, 3_{31}, 3_{40}, 3_{42}, 3_{44}, 3_{46}, 3_{47}, 3_{50}, 3_{58}, 3_{60}, 3_{62}, 3_{65}, 3_{66}, 3_{67}, 3_{68}, 3_{69}, 3_{70}, 3_{72}, 3_{74}, 3_{78}, 3_{79}.

(2) *The following twisted knots are noninvertible:*
3_{24}, 3_{25}, 3_{26}, 3_{28}, 3_{29}, 3_{30}, 3_{33}, 3_{34}, 3_{35}, 3_{36}, 3_{37}, 3_{38}, 3_{43}, 3_{45}, 3_{48}, 3_{49}, 3_{51}, 3_{52}, 3_{53}, 3_{54}, 3_{56}, 3_{57}, 3_{59}, 3_{61}, 3_{63}, 3_{64}, 3_{73}, 3_{75}, 3_{76}, 3_{77}.

A twisted knot $L = [D]$ is (+)-*amphichiral* if D is equivalent to D^*. Otherwise, it is (+)-*chiral*. A twisted knot $L = [D]$ is (−)-*amphichiral* if D is equivalent to $-D^*$. Otherwise, it is (−)-*chiral*.

Theorem 3. *All twisted knots in* Table 3 *are* (+)-*chiral and* (−)-*chiral.*

For a twisted knot, the notion of *checkerboard colorability* was defined in [7]. The definition is given in Section 5.

Theorem 4. *The twisted knots* 3_{44}, 3_{58}, 3_{67}, 3_{70}, 3_{80}, 3_{81} *are checkerboard colorable and the remaining twisted knots in* Table 3 *are not.*

The *twisted number* of a twisted knot diagram D, denoted by $t(D)$, is the number of bars of D. The *twisted number* of a twisted knot L, denoted by $t(L)$, is the minimum among $t(D)$ for all diagrams representing L.

Theorem 5. (1) *The twisted number of the following twisted knots are* 1*:*
3_1, 3_2, 3_3, 3_4, 3_5, 3_6, 3_{24}, 3_{25}, 3_{26}, 3_{27}, 3_{28}, 3_{29}, 3_{30}, 3_{31}, 3_{32}, 3_{33}, 3_{34}, 3_{35}, 3_{36}, 3_{37}, 3_{38}, 3_{39}, 3_{40}, 3_{41}, 3_{42}.

(2) *The twisted number of the following twisted knots are* 2*:*
3_7, 3_8, 3_9, 3_{10}, 3_{11}, 3_{12}, 3_{13}, 3_{14}, 3_{15}, 3_{16}, $3_{21}, 3_{23}$, 3_{43}, 3_{45}, 3_{46}, 3_{47}, 3_{48}, 3_{49}, 3_{50}, 3_{51}, 3_{52}, 3_{53}, 3_{54}, 3_{55}, 3_{56}, 3_{57}, 3_{59}, 3_{60}, 3_{61}, 3_{62}, 3_{63}, 3_{64}, 3_{65}, 3_{66}, 3_{68}, 3_{69}, 3_{71}, 3_{72}.

(3) *The twisted number of the following twisted knots are* 3*:*
$3_{17}, 3_{18}, 3_{19}, 3_{20}, 3_{22}$.

(4) *The twisted number of the following twisted knots are* 1 *or* 3*:*
$3_{73}, 3_{74}, 3_{75}, 3_{76}, 3_{77}, 3_{78}, 3_{79}, 3_{82}$.

This paper is organized as follows: We recall Gauss chord diagrams in Section 2 and double covering diagrams in Section 3. In Section 4 we give the definitions and properties of some invariants of twisted knots which we use to obtain the results above. We introduce the notion of checkerboard colorability for twisted knots in Section 5. Section 6 is devoted to proofs of Theorems 2–5, and Section 7 to a proof of Theorem 1.

2. Gauss chord diagrams

In this section we recall Gauss chord diagrams for twisted knot diagrams.

Let C be the unit circle in $\mathbb{R}^2$, and let $P_1, \dots, P_{2n}$ be points on C appearing evenly in this order in the counterclockwise direction of C. A *Gauss chord diagram* with n chords (without bars) is the unit circle C equipped with n chords such that each chord is oriented and has a sign, positive/negative, $+/-$. We consider two Gauss chord diagrams G and G' are *equivalent* if G' is obtained by rotating G by $\frac{s\pi}{n}$ radian for some $s \in \{0, \dots, 2n-1\}$.

A *Gauss chord diagram* with n chords with m bars is a Gauss chord diagram with n chords equipped with m short line segments, called *bars*, intersecting C transversely avoiding the points $P_1, \dots, P_{2n}$ such that for each $k \in \{1, \dots, 2n\}$ the bars on the arc of C between P_{k-1} and P_k appear evenly on the arc.

Let D be a twisted knot diagram with n classical crossings and m bars. Consider an immersion $\eta : C \to \mathbb{R}^2$ which is an underlying immersion of the diagram D such that $\eta(P_k)$, $k = 1, \dots, 2n$, is the double point of the immersion corresponding to a classical crossing. We obtain a Gauss chord diagram with n chords such that each chord corresponds to a classical crossing such that the orientation of the chord is from the over crossing to the under crossing and the sign of the chord is the sign of the crossing. We put bars on the Gauss chord diagram corresponding to the bars of D. (If necessary, we change the immersion η so that the bars on the Gauss chord diagram appear evenly on each arc between P_{k-1} and P_k for each $k \in \{1, \dots, 2n\}$.) We call this the *Gauss chord diagram* associated with (or of) D. It is uniquely determined up to equivalence. We denote it by $G(D)$.

Conversely, for any Gauss chord diagram G with n chords with m (≥ 0) bars, there is a twisted knot diagram D such that G is the Gauss chord diagram of D. Such a diagram D is not determined uniquely. However, if D and D' are diagrams whose Gauss chord diagrams are the same, or equivalent, then D and D' are *strictly equivalent* as a twisted link, that is, D is transformed to D', up to isotopies of $\mathbb{R}^2$, by a finite sequence of virtual Reidemeister moves and twisted Reidemeister moves I depicted in Figures 2. Refer to [13] for virtual knots and [7] for twisted knots.

Therefore there is a natural bijection between the equivalence classes of Gauss chord diagrams with n chords and m bars and the strict equivalence classes of twisted link diagrams with n classical crossings and m bars. We assume that the orientation of a circle of a Gauss chord diagram is counterclockwise.

Figures 3, 4 and 1 are examples of (classical, virtual and twisted) knot diagrams and their Gauss chord diagrams.

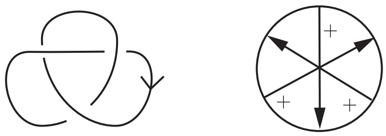


Fig. 3. An example of a knot diagram and its Gauss chord diagram

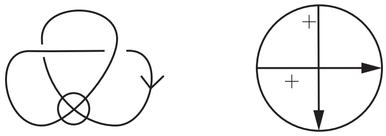


Fig. 4. An example of a virtual knot diagram and its Gauss chord diagram

When we apply Reidemeister moves I, II, III and twisted Reidemeister moves II and III to twisted knot diagrams, then the Gauss chord diagrams change as in Figure 5.

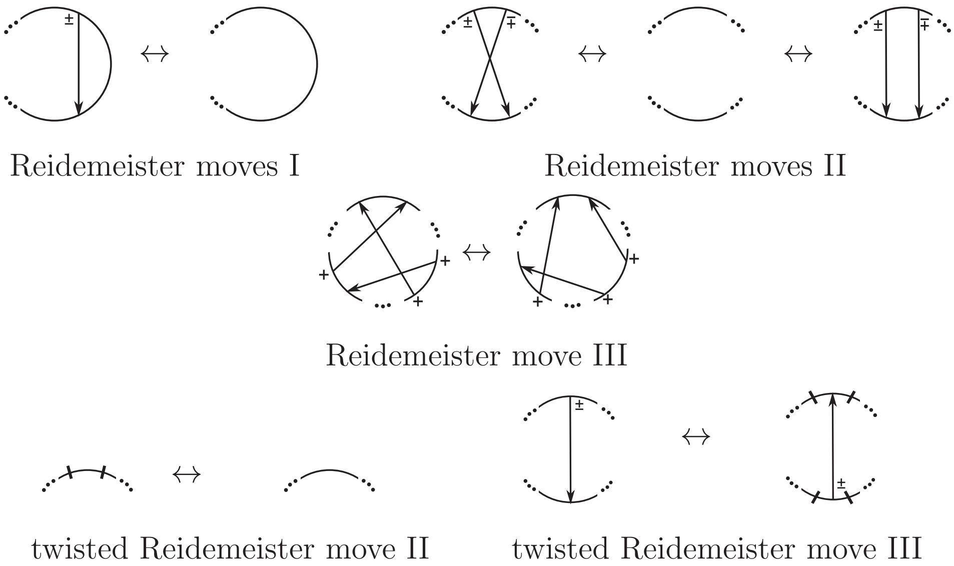


Fig. 5. Deformation of a Gauss chord diagram

The following lemma is directly seen from the definition.

Lemma 6. *Let D be a twisted link diagram and let $G(D)$ be the Gauss chord diagram associated with D.*

(1) *The Gauss chord diagram $G(-D)$ of the reverse $-D$ is obtained from $G(D)$ by reflecting $G(D)$ along a line in $\mathbb{R}^2$ and reversing the orientation of the circle.*
(2) *The Gauss chord diagram $G(D^*)$ of the mirror image D^* of D is obtained from $G(D)$ by reversing the orientation and switching the sign of every chord.*
(3) *The Gauss chord diagram $G(D^\sharp)$ of the reflection image $D^\sharp$ of D is obtained from $G(D)$ by switching the sign of every chord.*

3. Double covering diagrams

A method of constructing a virtual link diagram, called a *double covering diagram*, from a twisted link diagram was introduced in [11]. We recall this construction. Let D be a twisted link diagram with bars $b_1, \dots, b_m$. Assume that D is on the left of the y-axis and all bars are parallel to the x-axis with disjoint y-coordinates. Let $s(D)$ be a twisted link diagram which is obtained from D by reflection along the y-axis and switching over/under information of every classical crossing of D. Thus $s(D) = D^{\sharp *}$. See Figure 6. We denote a bar of $s(D)$

corresponding b_i by $s(b_i)$.

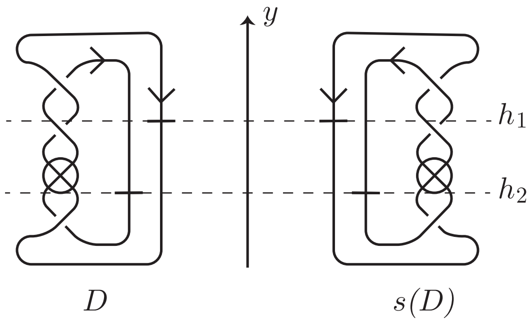


Fig. 6. A twisted link diagram D and its counterpart $s(D)$

For horizontal lines $h_1, \ldots, h_m$ such that h_i contains b_i and $s(b_i)$, we replace each part of $D \amalg s(D)$ in a neighborhood of h_i, $N(h_i)$ as in Figure 7 to obtain a virtual link diagram. We call this diagram the *double covering diagram* of D and denote it by $\widetilde{D}$.

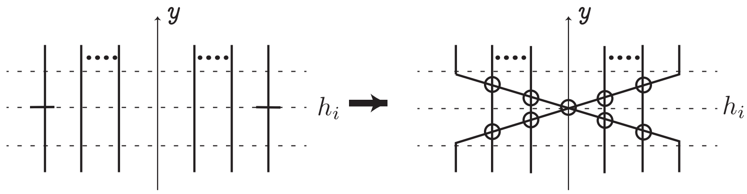


Fig. 7. Replacement in $N(h_i)$

Figure 8 shows the double covering diagram of the twisted link diagram D in Figure 6.

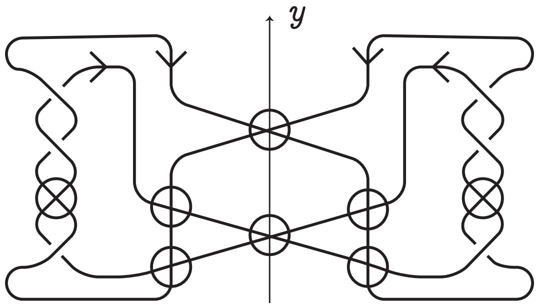


Fig. 8. The double covering diagram of D

Theorem 7 ([11])**.** *Let D and D' be twisted link diagrams, and $\widetilde{D}$ and $\widetilde{D'}$ their double covering diagrams, respectively. If D and D' are equivalent as twisted links, then $\widetilde{D}$ and $\widetilde{D'}$ are equivalent as virtual links.*

This theorem implies that for any invariant f of virtual links, if D and D' are equivalent as twisted links then $f(\widetilde{D}) = f(\widetilde{D'})$. Thus, for any invariant f of virtual links, we obtain an invariant $\widetilde{f}$ of twisted links by defining $\widetilde{f}(D)$ to be $f(\widetilde{D})$.

4. Invariants of twisted knots

In this section we introduce some invariants of twisted links: the X-polynomial defined in [6], the twisted JKSS invariant $\widetilde{Z}$ defined in [8], the twisted quandle $\widetilde{\mathbf{Q}}$ and the twisted n-coloring number $\widetilde{\mathrm{col}}_n$ defined in [7].

First we introduce the X-polynomial defined in [6].

A local replacement at a classical crossing of a twisted link diagram as in Figure 9 indicated A or B is called an A-*splice* or a B-*splice*, respectively. When we apply an A-splice or a B-splice at a classical crossing, put *a pole* at the place of edge whose orientation is different as in Figure 9. A *state* of a twisted link diagram D is a diagram with possibly some poles which is obtained from D by applying an A-splice or a B-splice at each classical crossing of D. Note that a state has no classical crossings anymore.

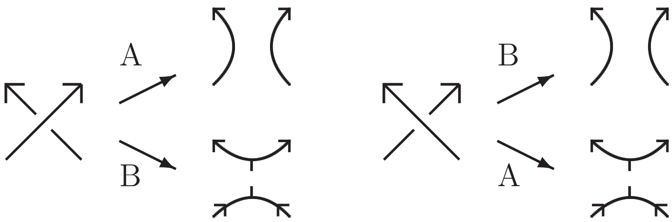
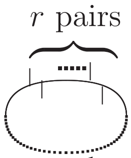
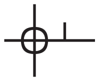
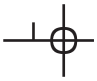


Fig. 9. Splice

For a loop ℓ of a state of D, the number of poles on ℓ is even, since the orientations of the arcs of ℓ divided by the poles change alternately at each pole. An index of ℓ, $\iota(\ell) \in \mathbb{Z}$, is defined as follows, where we ignore orientations of arcs of the state.

(i) $\iota($  $) = r$, where $2r$ poles appear on both sides alternately, and the dotted line may have some virtual crossings and some bars.

(ii) $\iota($  $) = \iota($  $)$,

(iii) $\iota($  $) = \iota($ $)$ and

(iv) $\iota($  $) = \iota($  $)$,

where in (ii) a pole passes through a virtual crossing, in (iii) a pair of poles on the same side is cancelled, and in (iv) a pole passes through a bar.

Note that $\iota(\ell) = 0$ if the number of bars on ℓ is odd by (ii), (iii) and (iv).

For a twisted link diagram D, let S be a state of D and $\omega(D)$ be the writhe of D, which is the number of the positive crossings minus that of negative ones. We denote by $\natural S$ the number of A-splices minus that of B-splices applied on D to obtain the state S. The number of loops of S is denoted by $\sharp S$. The number of loops of S which have odd numbers of bars on them is denoted by $\sharp_o S$. The number of loops of S whose indices are i, is denoted by $\tau_i(S)$.

Definition 8. For a state S, we define $\langle\!\langle D|S\rangle\!\rangle$ by

$$\langle\!\langle D|S\rangle\!\rangle = A^{\natural S}(-A^2 - A^{-2})^{\sharp S} M^{\sharp_o S} d_1^{\tau_1(S)} d_2^{\tau_2(S)} \cdots \qquad \in \mathbb{Z}[A, A^{-1}, M, d_1, d_2, \dots],$$

and we define $\langle\!\langle D\rangle\!\rangle$ by $\langle\!\langle D\rangle\!\rangle = \sum_S \langle\!\langle D|S\rangle\!\rangle$, where S runs over all states of D. Let $X(D) = (-A^3)^{-\omega(D)}\langle\!\langle D\rangle\!\rangle$. We call this *$X$-polynomial* of D.

Theorem 9 ([6]). *The polynomial $X(D)$ is an invariant of a twisted link.*

Lemma 10. *Let D be a twisted link diagram.*

(1) $X(-D) = X(D)$.
(2) $X(D^*) = \overline{X(D)}$, *where $\overline{X(D)}$ is a polynomial obtained from $X(D)$ by replacing A with A^{-1}.*

Proof. (1) Each state S of D is also regarded as a state, say S', of $-D$. Since $S = S'$ and each classical crossing of S has the same sign as the corresponding crossing of S', we have $\langle\!\langle D|S\rangle\!\rangle = \langle\!\langle -D|S'\rangle\!\rangle$. Since $w(D) = w(-D)$, we have $X(D) = X(-D)$.

(2) There is a bijection between states S of D and states S' of D^* such that $S = S'$ but A-splices and B-splices to obtain S from D are switched from those to obtain S' from D^*. Thus $\sharp_o S' = \sharp_o S$, $\tau_k(S') = \tau_k(S)$ for $k = 1, \dots$ and $\natural S' = -\natural S$. Therefore $\langle\!\langle D^*|S'\rangle\!\rangle = \overline{\langle\!\langle D|S\rangle\!\rangle}$. Since $w(D^*) = -w(D)$, we have $X(D^*) = \overline{X(D)}$. □

REMARK 11. Dye and Kauffman [3] and Miyazawa [16] introduced independently a multivariable polynomial invariant of virtual links, which we denote by R and call it the Dye-Kauffman-Miyazawa multivariable polynomial invariant or the R-polynomial. For a virtual link diagram D, the invariant $R(D)$ is defined by

$$(-A^3)^{-\omega(D)}\sum_S A^{\natural S}(-A^2 - A^{-2})^{\sharp S'} d_1^{\tau_1(S)} d_2^{\tau_2(S)} \cdots \qquad \in \mathbb{Z}[A, A^{-1}, d_1, d_2, \dots],$$

where $\sharp S' = \sharp S - 1 - \sum_{n=1}^{\infty} \tau_n(S)$ and if $\tau_n(S) = 0$, $d_n^{\tau_n(S)} = 1$.

Our invariant X for twisted links is, in a sense, an extension of the invariant R for virtual links. Namely for any virtual link diagram D, $R(D)$ and $X(D)$ are the same up to multiplication of powers of $-A^2 - A^{-2}$.

We introduce the twisted JKSS invariant $\widetilde{Z}$ of twisted links defined in [8]. This invariant can be defined using double covering in Section 3 from the JKSS invariant Z for virtual links defined in [17], namely, for a twisted link diagram D, $\widetilde{Z}(D)$ is $Z(\widetilde{D})$ where $\widetilde{D}$ is the double covering diagram of D. Here we explain how to define or compute $\widetilde{Z}(D)$ directly from D without taking double covering introduced in [8].

Let D be a twisted link diagram with n classical crossings $c_1, \dots, c_n$. We define a $4n \times 4n$ matrix $\widetilde{M}$, by $\widetilde{M} = \mathrm{diag}(\widetilde{M}_1, \dots, \widetilde{M}_n)$, where $\widetilde{M}_i = \widetilde{M}_+$ (or $\widetilde{M}_-$) if the crossing c_i is positive (or negative). Here $\widetilde{M}_+$ and $\widetilde{M}_-$ are 4×4 matrices as follows,

$$\widetilde{M}_+ = \begin{pmatrix} 1-x & -y & 0 & 0 \\ -xy^{-1} & 0 & 0 & 0 \\ 0 & 0 & 1-x & -y \\ 0 & 0 & -xy^{-1} & 0 \end{pmatrix} \text{ and } \widetilde{M}_- = \begin{pmatrix} 0 & -x^{-1}y & 0 & 0 \\ -y^{-1} & 1-x^{-1} & 0 & 0 \\ 0 & 0 & 0 & -x^{-1}y \\ 0 & 0 & -y^{-1} & 1-x^{-1} \end{pmatrix}.$$

For a twisted link diagram D, the graph $|D|$ is obtained from D by replacing all classical crossings of D with vertices. We denote by the same symbols $c_1, \dots, c_n$ the vertices of $|D|$. The graph $|D|$ is immersed in $\mathbb{R}^2$ and the multiple points of $|D|$ are virtual crossings of D. For each vertex c_i of $|D|$, consider an open regular neighborhood $N(c_i, |D|)$ of c_i in $|D|$. Then $N(c_i, |D|) - \{c_i\}$ is the union of four open arcs, which we call the *short edges around* c_i. According to the position, we denote by $i_0^-, i_1^-, i_0^+, i_1^+$ the short edges as in Figure 10.

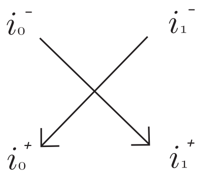


Fig. 10. Labels of four edges

Each edge of $|D|$ may have bars on it. For each vertex c_i of $|D|$, we denote by i_0^-, i_1^-, i_0^+, and i_1^+ the short edges around c_i as before. We denote $i_\epsilon \overset{e}{\leftarrow} j_\lambda$ (or $i_\epsilon \overset{o}{\leftarrow} j_\lambda$) for $\epsilon, \lambda \in \{0, 1\}$, if two short edges i_ϵ^- and j_λ^+ are on the same edge of $|D|$ and there are an even (or odd) number of bars on the edge.

We define a $4n \times 4n$ matrix, $\widetilde{P} = (\tilde{p}_{kl})$ as follows. For each $i, j \in \{1, \dots, n\}$,

$$\tilde{p}_{(4i-3+a)(4j-3+b)} = \begin{cases} 1 & \left(i_a \overset{e}{\leftarrow} j_b,\ i_{3-a} \overset{e}{\leftarrow} j_{3-b},\ i_a \overset{o}{\leftarrow} j_{3-b} \text{ or } i_{3-a} \overset{o}{\leftarrow} j_b\right) \\ 0 & (\text{otherwise}) \end{cases},$$

where $a, b \in \{0, 1, 2, 3\}$. Note that i_k^- and j_k^- are not defined for $k \in \{2, 3\}$. We assume that $i_k \overset{e}{\leftarrow} j_l$ and $i_k \overset{o}{\leftarrow} j_l$ are false when $k \in \{2, 3\}$ or $l \in \{2, 3\}$.

Theorem 12 ([8]). *For a twisted link diagram D, $\widetilde{Z}(D) := det(\widetilde{M} - \widetilde{P})$ is an invariant of the twisted link up to multiplication by powers of $x^{\pm 1}$, i.e., for any twisted link diagram D' representing the same twisted link with D, we have $\widetilde{Z}(D') = x^m \widetilde{Z}(D)$ for some $m \in \mathbb{Z}$.*

Let L be a twisted link. For a twisted link diagram D of L, we call $\widetilde{Z}(D)$ the *twisted JKSS invariant* of D (or L), which is also denoted by $\widetilde{Z}(L)$.

REMARK 13. To avoid the ambiguity of multiplication by powers of $x^{\pm 1}$, we will take a normalization as follows: For a twisted link diagram D, let N be the minimal degree of x in $\widetilde{Z}(D)$, then we normalize $\widetilde{Z}(D)$ by $x^{-N}\widetilde{Z}(D)$. After this normalization, $\widetilde{Z}(D)$ is well-defined as a twisted link invariant and it belongs to $\mathbb{Z}[x, y, y^{-1}]$.

A *quandle* is a set Y with a binary operator $*$ satisfying the following conditions.

(i) For any $x \in Y$, $x * x = x$.
(ii) For any $y \in Y$, the map $S_y : Y \longrightarrow Y$ defined by $x \mapsto x * y$ is a bijection.
(iii) For any $x, y, z \in Y$, $(x * y) * z = (x * z) * (y * z)$.

The knot quandle of a classical link was defined in [4, 15]. Kauffman [13] defined the knot quandle $\mathbf{Q}(D)$ of a virtual link diagram D as a virtual link invariant. The *twisted knot quandle* of a twisted link diagram D, denoted by $\widetilde{\mathbf{Q}}(D)$, was defined in [6] by a similar way with the twisted knot group defined in [1]. We explain the definition of $\widetilde{\mathbf{Q}}(D)$ below. However, it can be also defined by using double cover in Section 3 such that $\widetilde{\mathbf{Q}}(D) = \mathbf{Q}(\widetilde{D})$.

Let D be a twisted link diagram. Let $e_1, \dots, e_p$ be the edges of D. The generating set is $\{x_1, y_1, x_2, y_2, \dots, x_p, y_p\}$ where x_i and y_i $(i = 1, \dots, p)$ are symbols associated to each edge e_i. For a positive crossing, a negative crossing, a virtual crossing, and a bar, we associate four or two relations given in Table 1 (i), (ii), (iii) and (iv), respectively, where x_i, y_i, x_{i+1}, y_{i+1}, x_j, y_j and x_{j+1}, y_{j+1} are symbols as in Figure 11 (i), (ii), (iii) and (iv), respectively.

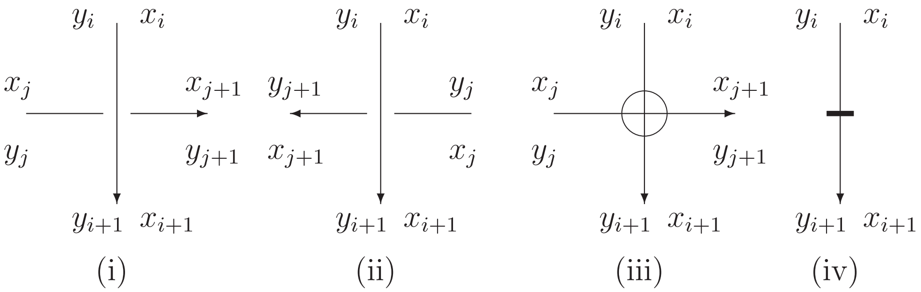


Fig. 11. Generators of edges

Table 1. Relation of generators around crossings

(i)	(ii)	(iii)	(iv)
$x_{i+1} = x_i$ $x_{j+1} = x_j * x_i$ $y_{i+1} = y_i * y_j$ $y_{j+1} = y_j$	$x_{i+1} = x_i$ $x_j = x_{j+1} * x_i$ $y_i = y_{i+1} * y_j$ $y_{j+1} = y_j$	$x_{i+1} = x_i$ $x_{j+1} = x_j$ $y_{i+1} = y_i$ $y_{j+1} = y_j$	$x_{i+1} = y_i$ $y_{i+1} = x_i$

Theorem 14 ([7]). *The twisted knot quandle of a twited link is an invariant of an twisted link.*

For $n \in \mathbb{N}$, let $\mathcal{D}_n$ be a dihedral quandle, which is $\mathbb{Z}/n\mathbb{Z}$ with $x * y = 2y - x$. For a twisted link diagram D, let $e_1, \dots, e_p$ be the edges of D and $\{x_1, y_1, x_2, y_2, \dots, x_p, y_p\}$ the generating set of the quandle $\widetilde{\mathbf{Q}}(D)$ as before. A *coloring* of D by $\mathcal{D}_n$ is a homomorphism from $\widetilde{\mathbf{Q}}(D)$ to $\mathcal{D}_n$, which is a map sending the generators to elements of $\mathcal{D}_n$ (denoted by the same symbols) such that they satisfy the equations in Table 1 (i), (ii), (iii) and (iv). The number of colorings of D is denoted by $\widetilde{\mathrm{col}}_n(D)$, which we call the *twisted n-coloring number* of D.

Proposition 15 ([7]). *$\widetilde{\mathrm{col}}_n(D)$ is an invariant of a twisted link D.*

Proposition 16 ([7], cf. [10]). *Let D be a twisted link diagram. If $\widetilde{\mathrm{col}}_n(D) < n^2$, then D is not equivalent to a virtual link diagram.*

5. Checkerboard colorability

In this section, we discuss checkerboard colorability.

An *abstract link diagram* is a pair (Σ, D_Σ) of a compact, orientable or non-orientable surface Σ and a link diagram D_Σ in Σ such that $|D_\Sigma|$ is a deformation retract of Σ, where $|D_\Sigma|$ is the graph of Σ obtained from D_Σ by replacing each crossing with a 4-valent vertex. An example of an abstract link is depicted on the right of Figures 12. For details of abstract link

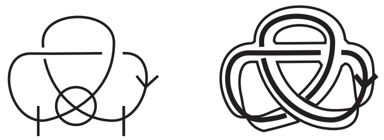


Fig. 12. A twisted link diagram and an abstract link diagram

diagrams, refer to [1, 9, 11].

The map from the set of virtual link diagrams to that of abstract link diagrams is defined in the figure depicted as in Figure 13. We call the abstract link diagram obtained from a twisted link diagram this way the *abstract link diagram associated with D*. The abstract link diagram on the right of Figure 12 is the one associated with the twisted link diagram on the left of Figure 12.

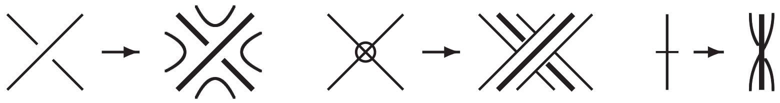


Fig. 13. Twisted link diagram and abstract link diagram

Let D be a twisted link diagram and (Σ, D_Σ) the abstract link diagram associated with D. The diagram D is said to be *checkerboard colorable* if the abstract link diagram (Σ, D_Σ) admits checkerboard coloring, namely each region of $\Sigma - |D_\Sigma|$ can be painted black and white such that colors of two adjacent regions are different. In Figure 14, we show an example of a checkerboard colorable twisted link diagram and an abstract link diagram associated with it.

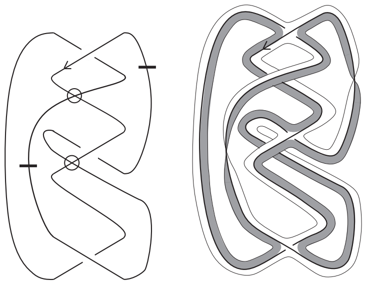


Fig. 14. An example of a checkerboard colorable twisted link diagram and the abstract link diagram associated with it

A twisted link is said to be *checkerboard colorable* if there exists a checkerboard colorable diagram representing the twisted link.

Theorem 17 ([7]). *Let D be a checkerboard colorable twisted link diagram.*

(1) *For any state S of D,*

$$\sharp_o S = 0 \quad \textit{and} \quad \sum_{n=1}^{\infty} n\tau_n(S) \in 2\mathbb{Z}.$$

(2) *Each term of the invariant $X(D)$ is written in a form*

$$f(A)d_1^{k_1}d_2^{k_2}\cdots$$

for some $f(A) \in \mathbb{Z}[A, A^{-1}]$ and non-negative integers $k_1, k_2, \ldots$ with $\sum_{n=1}^{\infty} nk_n \in 2\mathbb{Z}$.

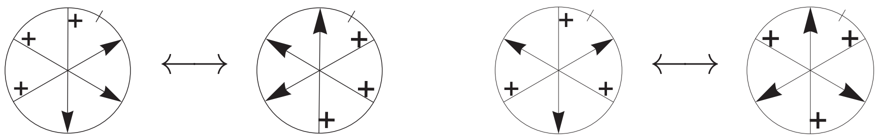
6. Proofs of Theorems 2, 3, and 4

We discuss our results.

Proof of Theorem 2. (1) Let D be a diagram in the list of (1) in Theorem 2 and let $-D$ be the inverse of D. Let $G(D)$ be the Gauss chord diagram of D and $G(-D)$ is the Gauss chord diagram of $-D$. Recall that $G(-D)$ is obtained from $G(D)$ by reflecting along a line in $\mathbb{R}^2$ and reversing an orientation of a circle of $G(D)$.

If D is 3_9 or 3_{16}, then the Gauss chord diagram $G(D)$ is the equivalent to $G(-D)$. If D is one of 3_7, 3_{11}, 3_{13}, 3_{15}, 3_{17}, 3_{18}, 3_{19}, 3_{20}, 3_{46}, 3_{47}, 3_{50}, 3_{60}, 3_{62}, 3_{65}, 3_{74}, 3_{78} and 3_{79}, then $G(-D)$ is obtained, up to equivalence, from $G(D)$ by a twisted Reidemeister move III. If D is one of 3_8, 3_{14}, 3_{44}, 3_{66}, 3_{67}, 3_{68}, 3_{69}, 3_{70}, and 3_{72} then $G(-D)$ is obtained, up to equivalence, from $G(D)$ by applying two twisted Reidemeister moves III.

If D is one of the remaining diagrams in the list of (1) in Theorem 2, then $G(-D)$ is obtained, up to equivalence, from $G(D)$ by applying three twisted Reidemeister moves III. For example, the cases of 3_1 and 3_4 are shown in Figure 15.



3_1 3_4

Fig. 15. Transform a diagram to its inverse

(2) Let D be a diagram listed in (2) of Theorem 2 and let $-D$ be the inverse of D. The twisted JKSS invariants of D and $-D$ are distinct, and hence D is not equivalent to $-D$. □

In Table 4 we show the multivariable polynomial invariants of the twisted knots in Table 3.

Proof of Theorem 3. Let D be a diagram in Table 3. The multivariable polynomial invariant of D^* (or $-D^*$) is obtained from the invariant of D by replacing A with A^{-1} (Lemma 10). Since the invariants are distinct, we see that D is not equivalent to D^* (or $-D^*$). □

Proof of Theorem 4. The twisted knot diagrams 3_{44}, 3_{58}, 3_{67}, 3_{70}, 3_{80}, 3_{81} in Table 3 are checkerboard colorable twisted knot diagrams. In Figure 16. checkerboard colorings of twisted link diagrams of 3_{44}, 3_{58}, 3_{70}, 3_{80}, and 3_{81} in Table 3 are depicted. A checkerboard colorable twisted link diagram of 3_{67} is shown in Figure 14.

The multivariable polynomial invariants of the remaining diagrams in Table 3 have a term of M. Thus they are not checkerboaed colorable from Theorem 17. □

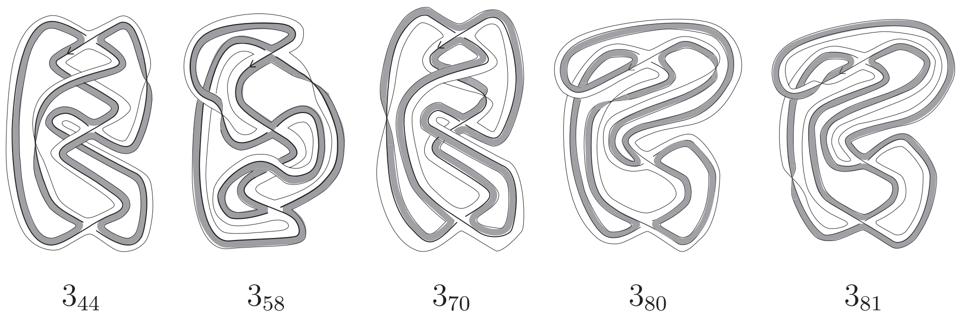


Fig. 16. Checkerboard colorable twisted link diagrams of 3_{44}, 3_{58}, 3_{70}, 3_{80}, and 3_{81}

Proposition 18. *Let D and D' be equivalent twisted link diagrams. If the number of bars of D is even (or odd), then that of D' is even (or odd).*

Proof. The number of bars are increased or decreased by even number under the Reidemeister moves, the virtual Reidemeister moves, and the twisted Reidemeister moves. Thus we have the result. □

Proposition 19. *Let L be a twisted link. Then the minimum number of bars of L is equal or greater than the degree of M of $X(L)$. For a twisted link L, the degree of M of any term with M of $X(L)$ is even (or odd), if and only if the number of bars of a diagram of L is even (or odd).*

Proof. Let D be a twisted link diagram of a twisted link L. The degree of M of $X(L)$ is a number of loops of a state of L with odd number of bars. Then the number of bars of D is equal or greater than the degree of M of $X(L)$. Suppose that the number of bars in D is even (or odd). Then the number of bars in a state S of D is even (or odd) from the definition of the multivariable polynomial invariant. Thus the degree of M is even (or odd) in $\langle\!\langle D|S \rangle\!\rangle$ since the number of loops with odd number of bars of S is even (or odd). Noting that $X(D) \neq 0$, we have the result. If the number of loops with odd number of bars in a state S of D is even (or odd), then the number of bars in S is even (or odd). Thus the number of bars in D is even (or odd) from the definition of the multivariable polynomial invariant. □

Proof of Theorem 5.

(1) The diagrams in the statement have a bar. From Proposition 19 we have the conclusion.

(2) (or (3)) The diagrams in the statement have two bars (or three bars).The degree of M of their X-polynomial invariant is 2 (or 3). Thus we conclude the minimum numbers of bars of them are 2 (or 3) from Proposition 19.

(4) The degrees of M of the X-polynomial invariants of the diagrams in the statement are 1. Thus we conclude the minimum number of bars of them are equal or grater than 1 from Proposition 19. □

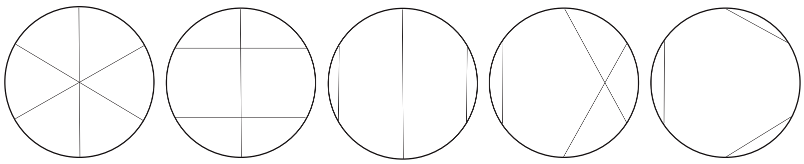
7. Proof of Theorem 1

First we consider possible underlying Gauss chord diagrams with 3 chords. Underlying Gauss chord diagrams mean Gauss chord diagrams without orientations, signs of chords and all bars. There are 5 underlying Gauss chord diagrams with 3 chords as depicted in Figure 17. Recall that we consider (underlying) Gauss chord diagrams up to equivalence, i.e., up to rotation.

We call the underlying Gauss chord diagram of (1) in the figure to be of *wheel type* and the one of (2) to be of *turtle type*.

If a twisted knot L is represented by a diagram D whose underlying Gauss chord diagram is (3), (4) or (5) in the figure, then L is not pseudo prime. Thus we consider twisted knots which can be represented by diagrams whose underlying Gauss chord diagrams are of wheel type or of turtle type.

We list all Gauss chord diagrams G, without bars, by considering all possible orientations and signs on the chords for the underlying Gauss chord diagram of wheel type and that of turtle type, and then remove duplications by applying rotations and also taking $-G$, G^* and $G^\sharp$, where $-G$ is a Gauss chord diagram obtained from G by reflecting G along a line in $\mathbb{R}^2$ and reversing an orientation of a circle of G. Now we consider all possible cases of bars for each Gauss chord diagram. By twisted Reidemeister move II, we may assume that each arc of C between P_{k-1} and P_k has at most one bar. We remove Gauss chord diagrams which are duplications or whose underlying Gauss chord diagrams are (3), (4) or (5) in Figure 17 after applying rotations, taking $-G$, G^*, $G^\sharp$ and applying moves in Figure 5 if possible. Then we obtain 23 Gauss chord diagrams of wheel type and 59 Gauss chord diagrams of turtle type. Table 3 shows twisted knot diagrams whose Gauss chord diagrams are these 82. The task is now to show that these 82 twisted knot diagrams are not weakly equivalent.



(1) (2) (3) (4) (5)

Fig. 17. Underlying Gauss chord diagrams

We denote by 3_k, for $k = 1, \ldots, 82$, the diagrams in Table 3 or the twisted knots represented by the diagrams. At this moment, as twisted knots, there might be some duplications.

Lemma 20. *Twisted knots* 3_1, 3_2, 3_3, 3_4, 3_5, 3_6, 3_7, 3_8, 3_9, 3_{10}, 3_{11}, 3_{12}, 3_{13}, 3_{14}, 3_{15}, 3_{16}, 3_{17}, 3_{18}, 3_{19}, 3_{20}, 3_{21}, 3_{22}, 3_{23}, 3_{25}, 3_{27}, 3_{29}, 3_{31}, 3_{43}, 3_{44}, 3_{45}, 3_{46}, 3_{49}, 3_{51}, 3_{52}, 3_{53}, 3_{55}, 3_{56}, 3_{57}, 3_{58}, 3_{59}, 3_{61}, 3_{64}, 3_{65}, 3_{68}, 3_{71} *are not weakly equivalent each other and they are not weakly equivalent to the rest of twisted knots in* Table 3.

Proof. We compute the X-polynomials for all diagrams in Table 3 and the result is given in Table 4. Let L be a twisted knot represented by a diagram D listed in the lemma. By Lemma 10, the X-polynomial $X(L) = X(-L) = X(D)$ is found in the table and the X-polynomial $X(L^*) = X(-L^*) = \overline{X(D)}$ is obtained from $X(D)$ by replacing A with A^{-1}. Comparing the two polynomials $X(D)$ and $\overline{X(D)}$ with the all polynomials in Table 4, we see that L is not weakly equivalent to other twisted knots in Table 3. □

Lemma 21. *The twisted knots in* Table 3 *which are not listed in* Lemma 20 *are divided into the following* 14 *groups such that two twisted knots belonging to distinct groups are not weakly equivalent:*
$\{ 3_{24}, 3_{26}, 3_{73} \}$, $\{ 3_{28}, 3_{30}, 3_{77} \}$, $\{ 3_{32}, 3_{74} \}$, $\{ 3_{33}, 3_{35}, 3_{36}, 3_{38}, 3_{75}, 3_{76} \}$,
$\{ 3_{34}, 3_{37} \}$, $\{ 3_{39}, 3_{41}, 3_{78}, 3_{79} \}$, $\{ 3_{40}, 3_{42} \}$, $\{ 3_{47}, 3_{48} \}$, $\{ 3_{50}, 3_{66} \}$,
$\{ 3_{54}, 3_{69} \}$, $\{ 3_{60}, 3_{72} \}$, $\{ 3_{62}, 3_{63} \}$, $\{ 3_{67}, 3_{70} \}$, $\{ 3_{80}, 3_{81} \}$.

Proof. Let D and D' be twisted knot diagrams belonging to distinct groups in the lemma. Comparing $X(D)$ and $\overline{X(D)}$ with $X(D')$ using Table 3, we see that D is not weakly equivalent to D'. □

In Table 5 we show the twisted JKSS invariant of the twisted knots in Table 3.

Lemma 22. (i) *The following twisted knots are not weakly equivalent each other:* $3_{33}, 3_{34}, 3_{35}, 3_{36}, 3_{37}, 3_{38}, 3_{40}, 3_{41}, 3_{42}, 3_{47}, 3_{48}, 3_{50}, 3_{54}, 3_{60}, 3_{62}, 3_{63}, 3_{66}, 3_{67}, 3_{69}, 3_{70}, 3_{72}, 3_{73}, 3_{75}, 3_{76}, 3_{77}$ *and* 3_{79}.

(ii) *The twisted knot diagram* 3_{24} (*or* 3_{26}) *is not weakly equivalent to* 3_{73}.
(iii) *The twisted knot diagram* 3_{28} (*or* 3_{30}) *is not weakly equivalent to* 3_{77}.
(iv) *The twisted knot diagram* 3_{39} (*or* 3_{78}) *is not weakly equivalent to* 3_{41}.
(v) *The twisted knot diagram* 3_{39} (*or* 3_{78}) *is not weakly equivalent to* 3_{79}.

Proof. (i) For the twisted knot diagrams, $3_{33}, 3_{34}, 3_{35}, 3_{36}, 3_{37}, 3_{38}, 3_{41}, 3_{47}, 3_{48}, 3_{50}, 3_{54}, 3_{67}, 3_{70}, 3_{72}, 3_{73}, 3_{75}, 3_{76}, 3_{77}$ and 3_{79}, we have the result from twisted JKSS invariants. For the twisted knot diagrams, $3_{40}, 3_{42}, 3_{60}, 3_{62}, 3_{63}, 3_{66}$ and 3_{69}, we have the result from twisted JKSS invariants and X-polynomial invariants.

(ii), (iii), (iv), (v) We have the result from twisted JKSS invariant. □

In Table 2, we show the multivariable polynomial invariants of the double covering diagrams of the twisted link diagrams $3_{24}, 3_{26}, 3_{28}, 3_{30}, 3_{32}, 3_{39}, 3_{74}, 3_{78}, 3_{80}$, and 3_{81} in Table 3.

From the multivariable polynomial invariants of double covering diagrams and the multivariable polynomial invariants, we have the following.

Lemma 23. *The twisted knot diagrams* $3_{24}, 3_{26}, 3_{74}, 3_{32}, 3_{28}, 3_{30}, 3_{78}$ *and* 3_{39} *in* Table 3 *are not equivalent each other.*

The twisted 3-coloring number of 3_{80} (or 3^*_{80}, -3_{80}, -3^*_{80}), $\widetilde{\mathrm{col}}_3(3_{80})$ (or $\widetilde{\mathrm{col}}_3(3^*_{80})$, $\widetilde{\mathrm{col}}_3(-3_{80})$, $\widetilde{\mathrm{col}}_3(-3^*_{80})$) is 3 and that of 3_{81} (or 3^*_{81}, -3_{81}, -3^*_{81}), $\widetilde{\mathrm{col}}_3(3_{81})$ (or $\widetilde{\mathrm{col}}_3(3^*_{81})$, $\widetilde{\mathrm{col}}_3(-3_{81})$, $\widetilde{\mathrm{col}}_3(-3^*_{81})$) is 9. Then we have the following.

Lemma 24. *The twisted knot diagrams* 3_{80} *and* 3_{81} *in* Table 3 *are not weakly equivalent.*

Proof of Theorem 1. Form Lemmas 20, 22, 23 and 24 we have the conclusion. □

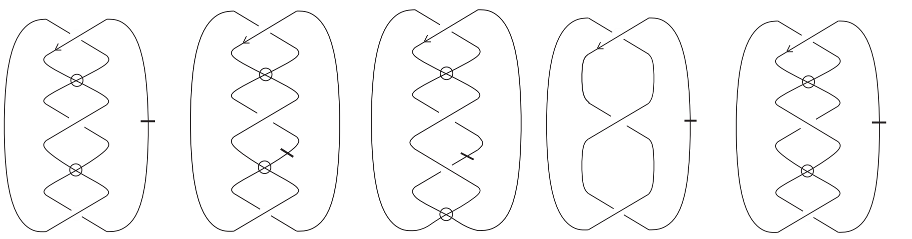
Table 2. The X-polynomial invariants of double covering diagrams

3_{24}	$-(A^4+1)A^{-26}[-2A^2(A^4-1)d_3-2A^4(A^4-1)d_2+2A^2(A^4-1)(A^4+1)d_1d_2$ $+(A^4-1)(A^4+1)(2A^4+1)d_1^2+2A^6(A^4-1)^2d_1+A^{16}-A^{12}-A^8+A^4+1]$
3_{26}	$-(A^4+1)A^{-26}[-2A^4(A^4-1)d_2+(A^4-1)(A^4+1)(2A^4+1)d_1^2$ $+2A^{10}(A^4-1)d_1+A^{16}-A^{12}-A^8+A^4+1]$
3_{74}	$(A^4+1)[-2(A^4-1)d_1+(A^{12}-A^8+2A^4-1)A^2]A^{-24}$
3_{32}	$-(A^4+1)[2(A^4-1)A^6d_1+A^{12}+A^8-2A^4+1]A^{-22}$
3_{28}	$-(A^4+1)A^{-6}[2A^8(A^4-1)d_2+A^8(A^4-1)(A^4+1)(A^4-2)d_1^2$ $-2A^2(A^4-1)(A^8-A^4+1)d_1+2A^{12}-A^8-A^4+1]$
3_{30}	$-(A^4+1)A^{-6}[2A^6(A^4-1)d_3+2A^8(A^4-1)d_2-2A^6(A^4-1)(A^4+1)d_1d_2$ $+A^8(A^4-1)(A^4+1)(A^4-2)d_1^2+2A^2(A^4-1)^2d_1+2A^{12}-A^8-A^4+1]$
3_{78}	$-(A^4+1)[-2A^6(A^4-1)(2A^4-1)d_1+A^{20}-2A^{16}+A^{12}+3A^8-3A^4+1]A^{-6}$
3_{39}	$-(A^4+1)[-2A^2(A^4-1)(A^8-A^4+1)d_1+A^{20}-2A^{16}+3A^{12}-A^8-A^4+1]A^{-6}$
3_{80}	$(A^4+1)(A^{24}-A^{20}+3A^{16}-2A^{12}+2A^8-2A^4+1)$
3_{81}	$(A^4+1)(A^{24}-A^{20}+3A^{16}-2A^{12}+2A^8-2A^4+1)$

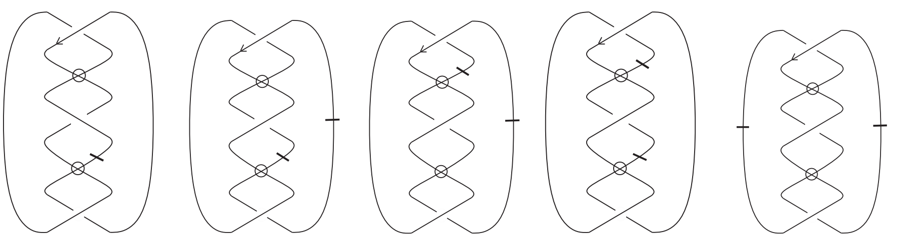
Question Is the twisted knot diagram 3_{82} equivalent to the trivial non-orientable curve (a trivial loop with a bar) ?

For twisted knot diagram 3_{82}, its X-polynomial invariant, twisted JKSS invariant, twisted 3-coloring number, twisted biquandle coloring, and X-polynomial of a double covering diagram of 3_{82} are equal to those of the trivial non-orientable curve.

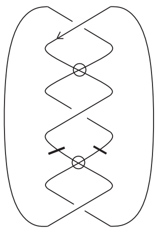
Table 3. Table of pseudo prime twisted knots



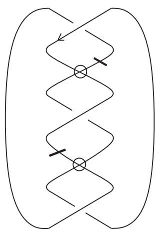
3_1 3_2 3_3 3_4 3_5



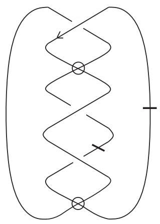
3_6 3_7 3_8 3_9 3_{10}



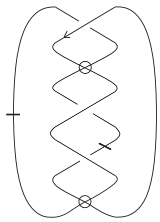
3_{11}



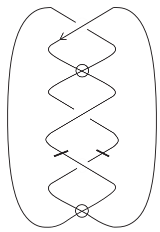
3_{12}



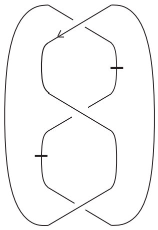
3_{13}



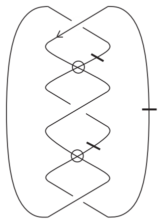
3_{14}



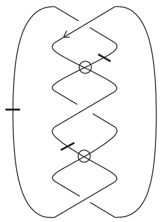
3_{15}



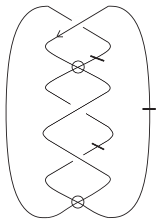
3_{16}



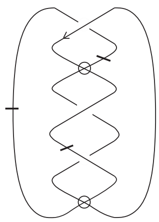
3_{17}



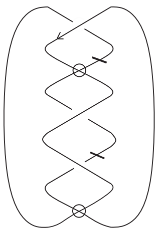
3_{18}



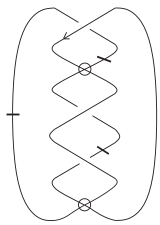
3_{19}



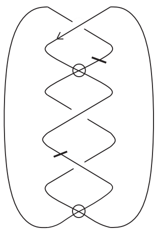
3_{20}



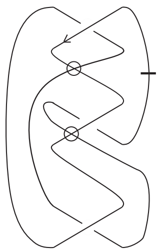
3_{21}



3_{22}



3_{23}



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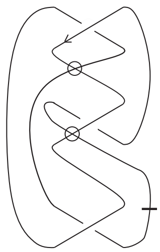

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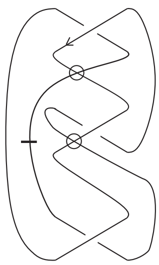

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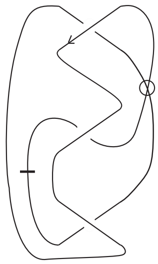

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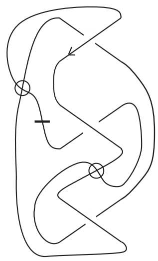

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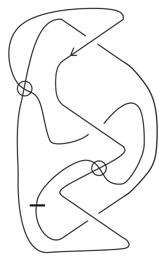

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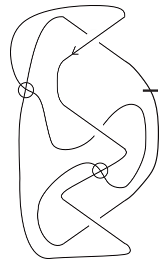

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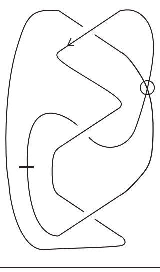

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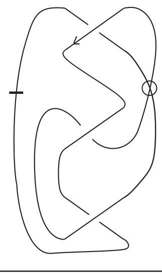

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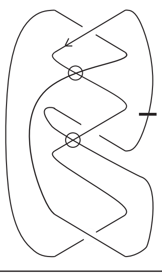

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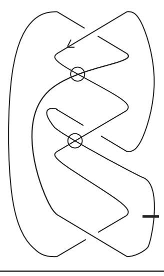

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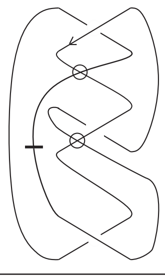

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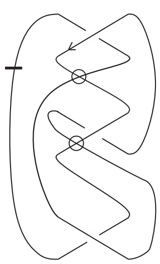
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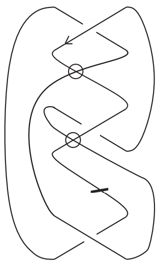
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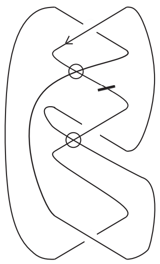
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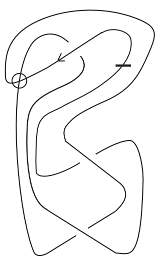
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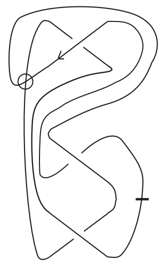
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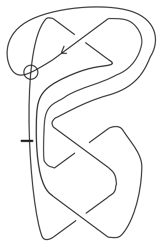
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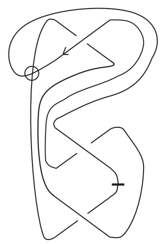
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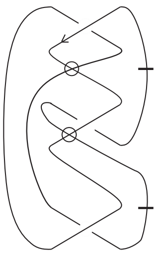
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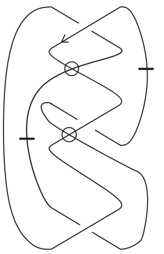
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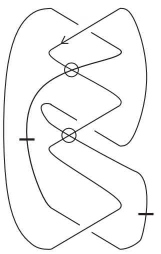
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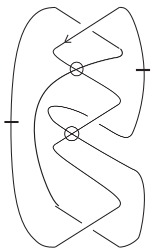
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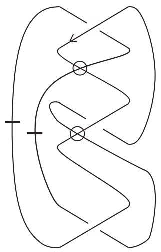
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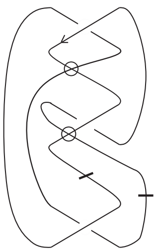
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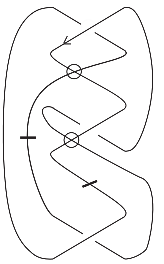
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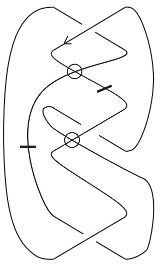
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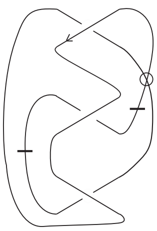
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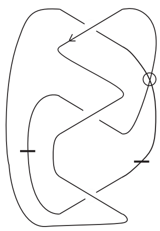
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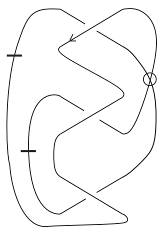
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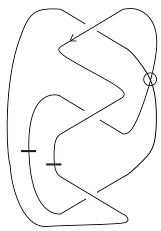
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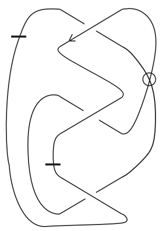
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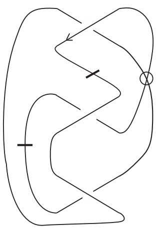
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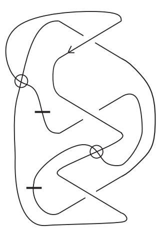
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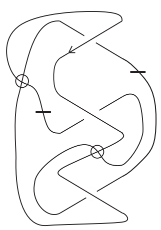
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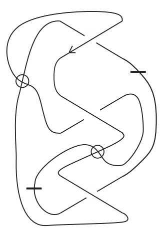
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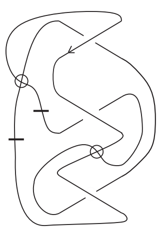
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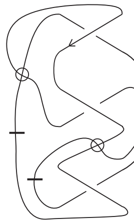

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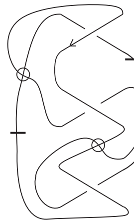

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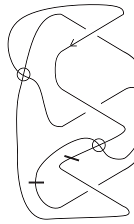

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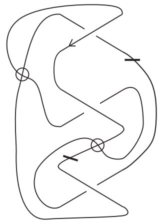

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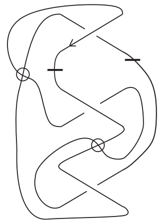

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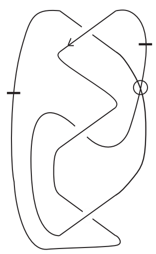

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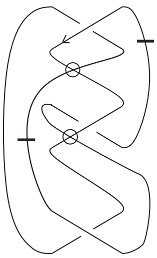

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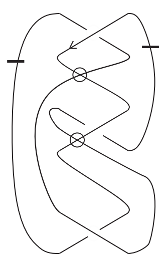

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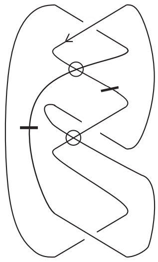

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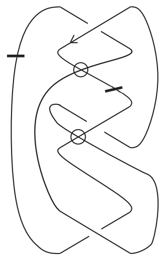

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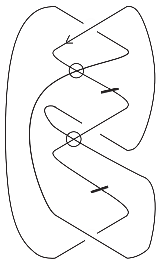

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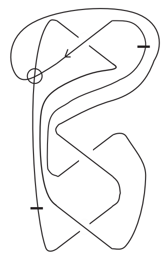

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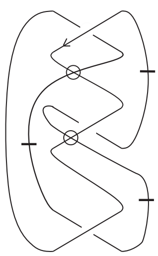

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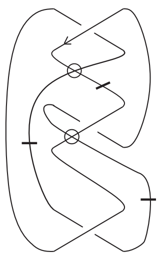

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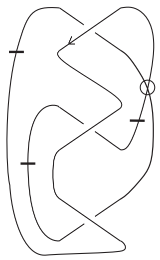

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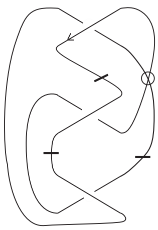

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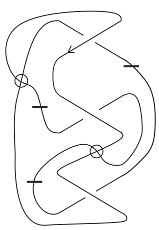

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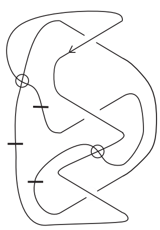

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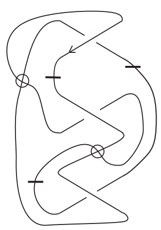

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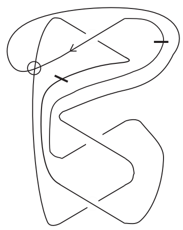

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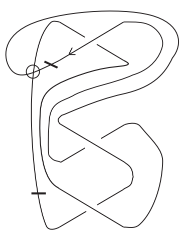

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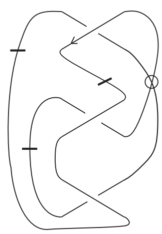

3_{82}

Table 4. The X-polynomials of twisted knots in Table 3

	The X-polynomial
3_1	$(A^4+1)\left[(A^4+1)^2d_1^2-2A^4(A^4+1)d_1-A^4(A^8-A^4+1)\right]A^{-18}M$
3_2	$-(A^4+1)\left[(A^8-1)d_1+A^{12}-A^8+A^4\right]A^{-18}M$
3_3	$(A^4+1)\left[\left(A^4+1\right)^2d_1^2-\left(A^4+1\right)^2d_1-A^8\right]A^{-10}M$
3_4	$-(A^4+1)\left(A^{12}+A^4-1\right)A^{-18}M$
3_5	$\left(A^4+1\right)\left[\left(A^4+1\right)^2d_1^2-2A^4\left(A^4+1\right)d_1-1\right]A^{-10}M$
3_6	$-\left(A^4+1\right)\left[\left(A^8-1\right)d_1+1\right]A^{-10}M$
3_7	$(A^4+1)\left[(A^4+1)^2M^2d_1+A^4(2A^4-1)d_1-A^4(A^8+3A^4+2)M^2\right]A^{-18}$
3_8	$-(A^4+1)\left[A^4(A^4+1)d_1^2-(A^4+1)^2M^2d_1+2(A^8+A^4)M^2+A^{12}-2A^8\right]A^{-18}$
3_9	$-(A^4+1)\left[(A^4-2A^8)d_1+(A^{12}+2A^8-1)M^2\right]A^{-18}$
3_{10}	$(A^4+1)\left[(A^4+1)^2d_1^3-3A^4d_1-A^8(A^4+1)M^2\right]A^{-18}$
3_{11}	$(A^4+1)\left[(A^8-A^4+1)d_1-A^8(A^4+1)^2M^2\right]A^{-18}$
3_{12}	$-\left(A^4+1\right)\left[(A^8-1)M^2+A^{12}-A^8+A^4\right]A^{-18}$
3_{13}	$(A^4+1)\left[(A^4+1)^2M^2d_1+(A^8+A^4-1)d_1-3A^4(A^4+1)M^2\right]A^{-10}$
3_{14}	$(A^4+1)\left[-(A^4+1)d_1^2+(A^4+1)^2M^2d_1-2A^4(A^4+1)M^2+A^4\right]A^{-10}$
3_{15}	$-(A^4+1)\left[-(A^4+1)^2d_1^3+(A^8+A^4+1)d_1+A^4(A^4+1)M^2\right]A^{-10}$
3_{16}	$(A^4+1)\left[(A^8+A^4-1)d_1+(-2A^8-A^4+1)A^{-10}M^2\right]$
3_{17}	$-(A^4+1)\left[2(A^8+A^4)d_1-(A^4+1)^2M^2+A^{12}-A^8+A^4\right]A^{-18}M$
3_{18}	$(A^4+1)\left[(A^4+1)^2M^3-A^4(A^8+A^4+3)MA^{-18}\right]$
3_{19}	$(A^4+1)\left[(A^4+1)^2M^2-2A^8-2A^4-1\right]A^{-10}M$
3_{20}	$(A^4+1)\left[-2(A^8+A^4)d_1+(A^4+1)^2M^2-1\right]A^{-10}M$
3_{21}	$(A^4+1)\left[A^4d_1+(A^4+1)^2M^2d_1-(2A^8+3A^4+1)M^2\right]A^{-10}$
3_{22}	$(A^4+1)\left[-(A^4+1)^2d_1+(A^4+1)^2M^2-A^8\right]A^{-10}M$
3_{23}	$(A^4+1)\left[-A^4(A^4+1)d_1^2+(A^4+1)^2M^2d_1-(A^4+1)^2M^2+A^4\right]A^{-10}$
3_{24}	$-(A^4+1)\left[(A^4+1)d_1+A^2(A^2-1)(A^2+1)^2-1\right]A^{-14}M$
3_{25}	$-(A^4+1)\left[(A^2+1)(A^4+1)d_1+A^8-A^4-2A^2-1\right]A^{-14}M$
3_{26}	$-(A^4+1)\left[(A^4+1)d_1+A^2(A^2-1)(A^2+1)^2-1\right]A^{-14}M$
3_{27}	$-(A^4+1)\left[(A^4+1)d_1+A^6-A^4-1\right]A^{-8}M$
3_{28}	$-(A^4+1)\left[A^6(A^4+1)d_1-2A^6-A^4+A^2+1\right]A^{-4}M$
3_{29}	$-(A^4+1)\left[(A^6+1)(A^4+1)d_1-2A^6-2A^4+A^2\right]A^{-4}M$
3_{30}	$-(A^4+1)\left[A^6(A^4+1)d_1-2A^6-A^4+A^2+1\right]A^{-4}M$
3_{31}	$-(A^4+1)\left[(A^4+1)d_1+A^6-2\right]A^{-12}M$
3_{32}	$-(A^4+1)(A^6+A^4-1)A^{-12}M$

3_{33}	$-(A^4+1)\left[(A^4+1)d_1-1\right]A^{-6}M$
3_{34}	$(A^4+1)\left[-(A^2+1)(A^4+1)d_1+A^4+A^2+1\right]A^{-8}M$
3_{35}	$-(A^4+1)\left[(A^4+1)d_1-1\right]A^{-6}M$
3_{36}	$-(A^4+1)\left[(A^4+1)d_1-1\right]A^{-6}M$
3_{37}	$(A^4+1)\left[-(A^2+1)(A^4+1)d_1+A^4+A^2+1\right]A^{-8}M$
3_{38}	$-(A^4+1)\left[(A^4+1)d_1-1\right]A^{-6}M$
3_{39}	$-(A^4+1)(A^{10}-A^6-A^4+A^2+1)A^{-4}M$
3_{40}	$-(A^4+1)\left[(A^4+1)d_1+A^{10}-A^6-2A^4+A^2\right]A^{-4}M$
3_{41}	$-(A^4+1)(A^{10}-A^6-A^4+A^2+1)A^{-4}M$
3_{42}	$-(A^4+1)\left[(A^4+1)d_1+A^{10}-A^6-2A^4+A^2\right]A^{-4}M$
3_{43}	$-(A^4+1)\left[-(A^2+1)d_2-(2A^4+A^2)d_1+(A^4+1)(A^4+A^2+1)M^2\right]A^{-14}$
3_{44}	$[-A^{12}-A^{10}-\left(A^4+1\right)^2d_1^2+2A^4+A^2+1]\times A^{-14}$
3_{45}	$-(A^4+1)\left[-(2A^4+A^2)d_1+(A^8+A^6+2A^4+A^2+1)M^2-A^2-1\right]A^{-14}$
3_{46}	$-(A^4+1)\left[-d_3+(A^4+1)M^2+(A^2-1)(A^3+A)^2\right]A^{-14}$
3_{47}	$-(A^4+1)\left[(A^6-A^2-1)d_1+(A^4+1)M^2+A^8-A^4\right]A^{-14}$
3_{48}	$-(A^4+1)\left[(A^6-A^2-1)d_1+(A^4+1)M^2+A^8-A^4\right]A^{-14}$
3_{49}	$-(A^4+1)\left[-(A^2+1)d_1+(A^2+1)(A^4+1)M^2-1\right]A^{-12}$
3_{50}	$-(A^4+1)\left[-d_1+(A^4+1)M^2+A^2(A^2-1)(A^2+1)^2\right]A^{-14}$
3_{51}	$-(A^4+1)\left[-(A^2+1)d_2-A^2(A^4+A^2+1)d_1+(2A^2+1)(A^4+1)M^2\right]A^{-8}$
3_{52}	$-(A^4+1)\left(-A^2(A^2+1)d_2-(A^6+A^2+1)d_1+(2A^2+1)(A^4+1)M^2\right]A^{-8}$
3_{53}	$-(A^4+1)\left[-(A^2+1)d_1+(A^2+1)(A^4+1)M^2-A^4\right]A^{-8}$
3_{54}	$(A^4+1)\left[d_1-(A^4+1)M^2\right]A^{-6}$
3_{55}	$-(A^4+1)\left(-(A^6+A^2+1)d_1+(2A^2+1)(A^4+1)M^2-A^2(A^2+1)\right]A^{-8}$
3_{56}	$-(A^4+1)\left[-A^2(A^2+1)d_1+(A^2+1)(A^4+1)M^2-1\right]A^{-8}$
3_{57}	$-(A^4+1)\left[-A^4(2A^2+1)d_1+(A^4+1)(A^6+A^2+1)M^2-A^4(A^2+1)\right]A^{-4}$
3_{58}	$-(A^4+1)\left[A^6(A^4+1)d_1^2-2A^6-A^4+A^2+1\right]A^{-4}$
3_{59}	$-(A^4+1)\left[A^4(A^2+1)d_2-A^4(2A^2+1)d_1+(A^4+1)(A^6+A^2+1)M^2\right]A^{-4}$
3_{60}	$-(A^4+1)\left[-A^6d_1+A^6(A^4+1)M^2+(A^2+1)(-A^4+1)\right]A^{-4}$
3_{61}	$-(A^4+1)\left[A^4(A^6-2A^2-1)d_1+(A^2+1)(A^4+1)M^2-A^4\right]A^{-4}$
3_{62}	$-(A^4+1)\left[-(A^6+A^4-1)d_1+A^6(A^4+1)M^2-A^6+A^2\right]A^{-4}$
3_{63}	$-(A^4+1)\left[-(A^6+A^4-1)d_1+A^6(A^4+1)M^2-A^6+A^2\right]A^{-4}$
3_{64}	$-(A^4+1)\left[-A^4d_2+A^4(A^6-2A^2-1)d_1+(A^2+1)(A^4+1)M^2\right]A^{-4}$
3_{65}	$-(A^4+1)\left[-A^6d_3+A^6(A^4+1)M^2-A^6-A^4+A^2+1\right]A^{-4}$
3_{66}	$-(A^4+1)\left[-d_1+(A^4+1)M^2+A^2(A^2-1)(A^2+1)^2\right]A^{-14}$
3_{67}	$-\left(A^4+1\right)^2[d_1^2+A^4+1]A^{-6}$

3_{68}	$(A^4+1)\left[d_3-(A^4+1)M^2\right]A^{-6}$
3_{69}	$(A^4+1)\left[d_1-(A^4+1)M^2\right]A^{-6}$
3_{70}	$-(A^4+1)^2[d_1^2+A^4+1]A^{-6}$
3_{71}	$-(A^4+1)\left[-A^2(A^4+A^2+1)d_1+(2A^2+1)(A^4+1)M^2-A^2-1\right]A^{-8}$
3_{72}	$-(A^4+1)\left[-A^6d_1+A^6(A^4+1)M^2-A^6-A^4+A^2+1\right]A^{-4}$
3_{73}	$-(A^4+1)\left[(A^4+1)d_1+A^2(A^2-1)(A^2+1)^2-1\right]A^{-14}M$
3_{74}	$-(A^4+1)(A^6+A^4-1)A^{-12}M$
3_{75}	$-(A^4+1)\left[(A^4+1)d_1-1\right]A^{-6}M$
3_{76}	$-(A^4+1)\left[(A^4+1)d_1-1\right]A^{-6}M$
3_{77}	$-(A^4+1)\left[A^6(A^4+1)d_1-2A^6-A^4+A^2+1\right]A^{-4}M$
3_{78}	$-(A^4+1)(A^{10}-A^6-A^4+A^2+1)A^{-4}M$
3_{79}	$-(A^4+1)(A^{10}-A^6-A^4+A^2+1)A^{-4}M$
3_{80}	$-(A^4+1)(A^{10}-A^6-A^4+A^2+1)A^4$
3_{81}	$-(A^4+1)(A^{10}-A^6-A^4+A^2+1)A^4$
3_{82}	$-(A^4+1)A^{-2}M$

Table 5. The twisted JKSS invariants of twisted knots in Table 3

Knot	Twisted JKSS invariant
$3_1(\sim -3_1)$	$(x-1)(x+1)\left(x^2-x+1\right)(y-1)(y+1)(x-y)(x+y)y^{-2}$
$3_1{}^*(\sim -3_1{}^*)$	$-(x-1)(x+1)\left(x^2-x+1\right)(y-1)(y+1)(x-y)(x+y)y^{-2}$
3_2, -3_2	$(x-1)(x+1)\left(x^2-x+1\right)(y-1)(y+1)(x-y)(x+y)y^{-2}$
$3_2{}^*$, $-3_2{}^*$	$-(x-1)(x+1)\left(x^2-x+1\right)(y-1)(y+1)(x-y)(x+y)y^{-2}$
3_3, -3_3, $3_3{}^*$, $-3_3{}^*$	0
$3_4(\sim -3_4)$, $3_4{}^*(\sim -3_4{}^*)$	0
$3_5(\sim -3_5)$	$(x-1)(x+1)(y-1)(y+1)(x-y)(x+y)y^{-2}$
$3_5{}^*(\sim -3_5{}^*)$	$-(x-1)(x+1)(y-1)(y+1)(x-y)(x+y)y^{-2}$
3_6, -3_6	$(x-1)(x+1)(y-1)(y+1)(x-y)(x+y)y^{-2}$
$3_6{}^*$, $-3_6{}^*$	$-(x-1)(x+1)(y-1)(y+1)(x-y)(x+y)y^{-2}$
$3_7(\sim -3_7)$, $3_7{}^*(\sim -3_7{}^*)$	$\left(x^2-x+1\right)^2(y-1)(y+1)(x-y)(x+y)y^{-2}$
$3_8(\sim -3_8)$, $3_8{}^*(\sim -3_8{}^*)$	$(y-1)(y+1)(x-y)(x+y)(x^3+y^2)(xy^2+1)y^{-4}$
$3_9(\sim -3_9)$, $3_9{}^*(\sim -3_9{}^*)$	$\left(x^2-x+1\right)^2(y-1)(y+1)(x-y)(x+y)y^{-2}$
$3_{10}(\sim -3_{10})$, $3_{10}{}^*(\sim -3_{10}{}^*)$	$(y-1)(y+1)(x-y)(x+y)\left(y^2-y+1\right)\left(y^2+y+1\right)\left(x^2-xy+y^2\right)$ $\times\left(x^2+xy+y^2\right)y^{-6}$
$3_{11}(\sim -3_{11})$, $3_{11}{}^*(\sim -3_{11}{}^*)$	$\left(x^2-x+1\right)^2(y-1)(y+1)(x-y)(x+y)y^{-2}$
$3_{12}(\sim -3_{12})$, $3_{12}{}^*(\sim -3_{12}{}^*)$	0
$3_{13}(\sim -3_{13})$, $3_{13}{}^*(\sim -3_{13}{}^*)$	$(y-1)(y+1)(x-y)(x+y)y^{-2}$
$3_{14}(\sim -3_{14})$, $3_{14}{}^*(\sim -3_{14}{}^*)$	$(y-1)(y+1)(x-y)(x+y)\left(x+y^2\right)^2y^{-4}$

Knot	Twisted JKSS invariant
$3_{15}, -3_{15}, 3_{15}{}^{*}, -3_{15}{}^{*}$	$-(y-1)(y+1)(x-y)(x+y)(xy-x-y^2)(x+y^2-y)$ $\times(x+y^2+y)(xy+x+y^2)y^{-6}$
$3_{16}(\sim -3_{16}), 3_{16}{}^{*}(\sim -3_{16}{}^{*})$	$(y-1)(y+1)(x-y)(x+y)y^{-2}$
$3_{17} \sim -3_{17}$	$(x-1)(x+1)\left(x^2-x+1\right)(y-1)(y+1)(x-y)(x+y)y^{-2}$
$3_{17}{}^{*} \sim -3_{17}{}^{*}$	$-(x-1)(x+1)\left(x^2-x+1\right)(y-1)(y+1)(x-y)(x+y)y^{-2}$
$3_{18}(\sim -3_{18}), 3_{18}{}^{*}(\sim -3_{18}{}^{*})$	0
$3_{19}(\sim -3_{19}), 3_{19}{}^{*}(\sim -3_{19}{}^{*})$	0
$3_{20}(\sim -3_{20})$	$(x-1)(x+1)(y-1)(y+1)(x-y)(x+y)y^{-2}$
$3_{20}{}^{*}(\sim -3_{20}{}^{*})$	$-(x-1)(x+1)(y-1)(y+1)(x-y)(x+y)y^{-2}$
$3_{21}, -3_{21}, 3_{21}{}^{*}, -3_{21}{}^{*}$	$(y-1)(y+1)(x-y)(x+y)y^{-2}$
$3_{22}, -3_{22}, 3_{22}{}^{*}, -3_{22}{}^{*}$	0
$3_{23}, -3_{23}, 3_{23}{}^{*}, -3_{23}{}^{*}$	$(y-1)(y+1)(x-y)(x+y)\left(x+y^2\right)^2 y^{-4}$
3_{24}	$(x-1)(x+1)\left(x^2-x+1\right)(y+1)(x+y)(xy-x+y)y^{-2}$
-3_{24}	$(x-1)(x+1)\left(x^2-x+1\right)(y+1)(x-y+1)(x+y)y^{-1}$
$3_{24}{}^{*}$	$-(x-1)(x+1)\left(x^2-x+1\right)(y+1)(x-y+1)(x+y)y^{-1}$
$-3_{24}{}^{*}$	$-(x-1)(x+1)\left(x^2-x+1\right)(y+1)(x+y)(xy-x+y)y^{-2}$
3_{25}	$(x-1)(y+1)(x+y)(xy-x+y)\left(x^3+x^2y-2xy+y+1\right)y^{-2}$
-3_{25}	$(x-1)(y+1)(x-y+1)(x+y)\left(x^3y+x^3-2x^2+x+y\right)y^{-2}$
$3_{25}{}^{*}$	$-(x-1)(y+1)(x-y+1)(x+y)\left(x^3y+x^3-2x^2+x+y\right)y^{-2}$
$-3_{25}{}^{*}$	$-(x-1)(y+1)(x+y)(xy-x+y)\left(x^3+x^2y-2xy+y+1\right)y^{-2}$
3_{26}	$(x-1)(x+1)\left(x^2-x+1\right)(y+1)(x+y)(xy-x+y)y^{-2}$
-3_{26}	$(x-1)(x+1)\left(x^2-x+1\right)(y+1)(x-y+1)(x+y)y^{-1}$
$3_{26}{}^{*}$	$-(x-1)(x+1)\left(x^2-x+1\right)(y+1)(x-y+1)(x+y)y^{-1}$
$-3_{26}{}^{*}$	$-(x-1)(x+1)\left(x^2-x+1\right)(y+1)(x+y)(xy-x+y)y^{-2}$
$3_{27}, -3_{27}, 3_{27}{}^{*}, -3_{27}{}^{*}$	0
3_{28}	$(1-x)(x+1)(y+1)(x+y)$
-3_{28}	$-(x-1)(x+1)(y+1)(x+y)y^{-2}$
$3_{28}{}^{*}$	$(x-1)(x+1)(y+1)(x+y)y^{-2}$
$-3_{28}{}^{*}$	$(x-1)(x+1)(y+1)(x+y)$
3_{29}	$(1-x)(y+1)(x+y)\left(x^2y+x^2-2xy+x+y\right)$
-3_{29}	$-(x-1)(y+1)(x+y)\left(x^2+xy-2x+y+1\right)y^{-3}$
$3_{29}{}^{*}$	$(x-1)(y+1)(x+y)\left(x^2+xy-2x+y+1\right)y^{-3}$
$-3_{29}{}^{*}$	$(x-1)(y+1)(x+y)\left(x^2y+x^2-2xy+x+y\right)$
3_{30}	$(1-x)(x+1)(y+1)(x+y)$
-3_{30}	$-(x-1)(x+1)(y+1)(x+y)y^{-2}$
$3_{30}{}^{*}$	$(x-1)(x+1)(y+1)(x+y)y^{-2}$
$-3_{30}{}^{*}$	$(x-1)(x+1)(y+1)(x+y)$
$3_{31}(\sim -3_{31})$	$(x-1)\left(x^2+1\right)\left(x^2-x+1\right)(y+1)(x+y)y^{-1}$
$3_{31}{}^{*}(\sim -3_{31}{}^{*})$	$-(x-1)\left(x^2+1\right)\left(x^2-x+1\right)(y+1)(x+y)y^{-1}$

Knot	Twisted JKSS invariant
$3_{32}, -3_{32}$	$2(x-1)\left(x^2-x+1\right)(y+1)(x+y)y^{-1}$
$3_{32}{}^*, -3_{32}{}^*$	$-2(x-1)\left(x^2-x+1\right)(y+1)(x+y)y^{-1}$
3_{33}	$-(x-1)(y+1)(x+y)\left(x^3y-x^3-x^2y^2+2x^2-x-y^2+y\right)y^{-3}$
-3_{33}	$-(x-1)(y+1)(x+y)\left(x^3y-x^3-x^2y^2+2xy^2-x-y^2+y\right)y^{-1}$
$3_{33}{}^*$	$(x-1)(y+1)(x+y)\left(x^3y-x^3-x^2y^2+2xy^2-x-y^2+y\right)y^{-1}$
$-3_{33}{}^*$	$(x-1)(y+1)(x+y)\left(x^3y-x^3-x^2y^2+2x^2-x-y^2+y\right)y^{-3}$
3_{34}	$(x-1)(y+1)(x+y)\left(x^2-xy-2x+2y^2-y+1\right)y^{-3}$
-3_{34}	$(x-1)(y+1)(x+y)\left(x^2y^2-x^2y+2x^2-2xy^2-xy+y^2\right)y^{-1}$
$3_{34}{}^*$	$-(x-1)(y+1)(x+y)\left(x^2y^2-x^2y+2x^2-2xy^2-xy+y^2\right)y^{-1}$
$-3_{34}{}^*$	$-(x-1)(y+1)(x+y)\left(x^2-xy-2x+2y^2-y+1\right)y^{-3}$
3_{35}	$(x-1)(y+1)(x+y)\left(2x^2y-x^2-2xy-x+2y\right)y^{-2}$
-3_{35}	$(x-1)(y+1)(x+y)\left(2x^2-xy-2x-y+2\right)y^{-1}$
$3_{35}{}^*$	$-(x-1)(y+1)(x+y)\left(2x^2-xy-2x-y+2\right)y^{-1}$
$-3_{35}{}^*$	$-(x-1)(y+1)(x+y)\left(2x^2y-x^2-2xy-x+2y\right)y^{-2}$
3_{36}	$-(x-1)(y+1)(x+y)$ $\times\left(x^3-2x^2y^2+x^2y-2x^2+2xy^2+xy+x-2y^2\right)y^{-3}$
-3_{36}	$(x-1)(y+1)(x+y)$ $\times\left(2x^3-x^2y^2-x^2y-2x^2+2xy^2-xy+2x-y^2\right)y^{-1}$
$3_{36}{}^*$	$-(x-1)(y+1)(x+y)$ $\times\left(2x^3-x^2y^2-x^2y-2x^2+2xy^2-xy+2x-y^2\right)y^{-1}$
$-3_{36}{}^*$	$(x-1)(y+1)(x+y)$ $\times\left(x^3-2x^2y^2+x^2y-2x^2+2xy^2+xy+x-2y^2\right)y^{-3}$
3_{37}	$-(x-1)(y+1)(x-2y+1)(x+y)y^{-2}$
-3_{37}	$-(x-1)(y+1)(x+y)(xy-2x+y)y^{-1}$
$3_{37}{}^*$	$(x-1)(y+1)(x+y)(xy-2x+y)y^{-1}$
$-3_{37}{}^*$	$(x-1)(y+1)(x-2y+1)(x+y)y^{-2}$
3_{38}	$-(x-1)(y+1)(x+y)\left(x^3-x^2y-y+1\right)y^{-2}$
-3_{38}	$-(x-1)(y+1)(x+y)\left(x^3y-x^3-x+y\right)y^{-1}$
$3_{38}{}^*$	$(x-1)(y+1)(x+y)\left(x^3y-x^3-x+y\right)y^{-1}$
$-3_{38}{}^*$	$(x-1)(y+1)(x+y)\left(x^3-x^2y-y+1\right)y^{-2}$
$3_{39}, -3_{39}$	$-2(x-1)(y+1)(x+y)y^{-1}$
$3_{39}{}^*, -3_{39}{}^*$	$2(x-1)(y+1)(x+y)y^{-1}$
$3_{40}(\sim -3_{40})$	$-2(x-1)\left(x^2-x+1\right)(y+1)(x+y)y^{-1}$
$3_{40}{}^*(\sim -3_{40}{}^*)$	$2(x-1)\left(x^2-x+1\right)(y+1)(x+y)y^{-1}$
$3_{41}, -3_{41}$	$(x-1)\left(x^2-4x+1\right)(y+1)(x+y)y^{-1}$
$3_{41}{}^*, -3_{41}{}^*$	$-(x-1)\left(x^2-4x+1\right)(y+1)(x+y)y^{-1}$
$3_{42}(\sim -3_{42})$	$-(x-1)\left(x^2+1\right)(y+1)(x+y)y^{-1}$
$3_{42}{}^*(\sim -3_{42}{}^*)$	$(x-1)\left(x^2+1\right)(y+1)(x+y)y^{-1}$
$3_{43}, -3_{43}{}^*$	$(y+1)(x+y)\left(x^3-x^2y-x^2+xy^2+xy+x-y^2-y\right)$

Knot	Twisted JKSS invariant
	$\times\left(x^2y-x^2+xy^2-xy+x-y^2+y-1\right)y^{-2}$
-3_{43}, $3_{43}{}^*$	$(y+1)(x+y)\left(x^3y+x^3-x^2y^2-x^2y-x^2+xy^2+xy-y^2\right)$ $\times\left(x^2y^2-x^2y+x^2-xy^2+xy-x+y^2-y\right)y^{-4}$
$3_{44}(\sim-3_{44})$, $3_{44}{}^*(\sim-3_{44}{}^*)$	$(y-1)(y+1)(x-y)(x+y)\left(x^3+y^2\right)\left(xy^2+1\right)y^{-4}$
3_{45}, $-3_{45}{}^*$	$(y+1)\left(x^3-x^2+x-y\right)(x+y)\left(x^2y-x^2+x-1\right)y^{-2}$
-3_{45}, $3_{45}{}^*$	$(y+1)(x+y)\left(x^3-x^2y+xy-y\right)\left(x^2y-xy+y-1\right)y^{-2}$
$3_{46}(\sim-3_{46})$, $3_{46}{}^*(\sim-3_{46}{}^*)$	$(y-1)(y+1)(x-y)(x+y)\left(y^2-y+1\right)$ $\times\left(y^2+y+1\right)\left(x^2-xy+y^2\right)\left(x^2+xy+y^2\right)y^{-6}$
$3_{47}(\sim-3_{47})$, $3_{47}{}^*(\sim-3_{47}{}^*)$	$(y+1)\left(x^3-y\right)(x+y)\left(x^2y-1\right)y^{-2}$
3_{48}, $-3_{48}{}^*$	$(y+1)(x+y)\left(x^2+xy-2y\right)(2xy-y-1)y^{-2}$
-3_{48}, $3_{48}{}^*$	$(y+1)\left(2x^2-x-y\right)(x+y)(xy+x-2)y^{-2}$
3_{49}, $-3_{49}{}^*$	$(x-1)^2(y+1)(x+y)$ $\times\left(x^3y-x^3+x^2y+2x^2+xy^2+xy-x+y\right)y^{-2}$
-3_{49}, $3_{49}{}^*$	$(x-1)^2(y+1)(x+y)$ $\times\left(x^3y-x^2y^2+x^2y+x^2+2xy^2+xy-y^2+y\right)y^{-2}$
$3_{50}(\sim-3_{50})$, $3_{50}{}^*(\sim-3_{50}{}^*)$	$(y+1)(x+y)\left(2x^3y-x^2y-x^2-xy^2-xy+2y\right)y^{-2}$
3_{51}, $-3_{51}{}^*$	$-(y+1)(x+y)\left(x^2-xy^2-xy+y^2\right)\left(xy^2-xy+x-y^2\right)y^{-2}$
-3_{51}, $3_{51}{}^*$	$(y+1)(x+y)\left(x-y^2+y-1\right)\left(x^2-xy-x+y^2\right)y^{-4}$
3_{52}, $-3_{52}{}^*$	$-(y+1)(x+y)\left(x^2-xy^2-xy+y^2\right)\left(xy^2-xy+x-y^2\right)y^{-2}$
-3_{52}, $3_{52}{}^*$	$(y+1)(x+y)\left(x-y^2+y-1\right)\left(x^2-xy-x+y^2\right)y^{-4}$
3_{53}, $-3_{53}{}^*$	$(x-1)^2(y+1)(x+y)$
-3_{53}, $3_{53}{}^*$	$(x-1)^2(y+1)(x+y)y^{-2}$
3_{54}, $-3_{54}{}^*$	$-(y+1)(x+y)\left(x^2y-xy-x+y\right)\left(x^2y-x^2-xy+y\right)y^{-2}$
-3_{54}, $3_{54}{}^*$	$-(y+1)\left(x^2-x-y+1\right)(x+y)\left(x^2-xy-x+1\right)y^{-2}$
3_{55}, -3_{55}, $3_{55}{}^*$, $-3_{55}{}^*$	$(y-1)(y+1)(x-y)(x+y)y^{-2}$
3_{56}, $-3_{56}{}^*$	$(x-1)^2(y+1)(x+y)$
-3_{56}, $3_{56}{}^*$	$(x-1)^2(y+1)(x+y)y^{-2}$
3_{57}, $-3_{57}{}^*$	$(y+1)(x+y)(xy+x-2y)(2xy-x-y)y^{-2}$
-3_{57}, $3_{57}{}^*$	$(y+1)(2x-y-1)(x+y-2)(x+y)y^{-2}$
$3_{58}(\sim-3_{58})$, $3_{58}{}^*(\sim-3_{58}{}^*)$	$(y-1)(y+1)(x-y)(x+y)\left(x+y^2\right)^2y^{-4}$
3_{59}, $-3_{59}{}^*$	$(y+1)(x+y)\left(x^2+xy^2-xy-y^2\right)\left(xy^2+xy-x-y^2\right)y^{-2}$
-3_{59}, $3_{59}{}^*$	$(y+1)(x+y)\left(x^2+xy-x-y^2\right)\left(x+y^2-y-1\right)y^{-4}$
$3_{60}(\sim-3_{60})$, $3_{60}{}^*(\sim-3_{60}{}^*)$	$(y-1)(y+1)(x-y)(x+y)y^{-2}$
3_{61}, $-3_{61}{}^*$	$(x-1)^2(y+1)(x+y)$
-3_{61}, $3_{61}{}^*$	$(x-1)^2(y+1)(x+y)y^{-2}$
$3_{62}(\sim-3_{62})$, $3_{62}{}^*(\sim-3_{62}{}^*)$	$(y+1)\left(x^2-y\right)(x+y)(xy-1)y^{-2}$
3_{63}, $-3_{63}{}^*$	$(y+1)(x+y)(xy+x-2y)(2xy-x-y)y^{-2}$
-3_{63}, $3_{63}{}^*$	$(y+1)(2x-y-1)(x+y-2)(x+y)y^{-2}$
3_{64}, $-3_{64}{}^*$	$-y^{-4}(y+1)(x+y)$

Knot	Twisted JKSS invariant
	$\times[xy^6-(2x^2+2x-1)y^5-(3x^2-9x+3)xy^4$ $+(x^3-3x^2-3x+1)xy^3+3x^3y^2-(x+1)x^3y+x^4]$
-3_{64}, $3_{64}{}^*$	$-y^{-4}(y+1)(x+y)$ $\times[y^6-(xy^5+1)y^5+3xy^4+(x^2-3x-3)xy^3$ $+y^3-(3x^2-9x+3)xy^2+(x^3-2x^2-2x+1)xy+x^3]$
$3_{65}(\sim-3_{65})$, $3_{65}{}^*(\sim-3_{65}{}^*)$	$-(y+1)(x+y)\left(y^2-y+1\right)$ $\times\left(x^2-xy+y^2\right)\left(x^3y^3+x^3-2x^2y^3-2xy^3+y^6+y^3\right)y^{-6}$
$3_{66}(\sim-3_{66})$, $3_{66}{}^*(\sim-3_{66}{}^*)$	$(y+1)\left(x^2-y\right)(x+y)(xy-1)y^{-2}$
$3_{67}(\sim-3_{67})$, $3_{67}{}^*(\sim-3_{67}{}^*)$	$(y-1)(y+1)(x-y)(x+y)\left(2x^2y^2+x^2-2xy^2+y^4+2y^2\right)y^{-4}$
$3_{68}(\sim-3_{68})$, $3_{68}{}^*(\sim-3_{68}{}^*)$	$-(y+1)\left(y^2-y+1\right)(x+y)\left(x^2-xy+y^2\right)\left(x-y^3\right)\left(x^2-y^3\right)y^{-6}$
$3_{69}(\sim-3_{69})$, $3_{69}{}^*(\sim-3_{69}{}^*)$	$(y+1)\left(x^2-y\right)(x+y)(xy-1)y^{-2}$
$3_{70}(\sim-3_{70})$, $3_{70}{}^*(\sim-3_{70}{}^*)$	$(y-1)(y+1)(x-y)(x+y)$ $\times\left(x^4y^2-2x^3y^2+x^3+4x^2y^2+xy^4-2xy^2+y^2\right)y^{-4}$
3_{71}, -3_{71}, $3_{71}{}^*$, $-3_{71}{}^*$	$(y-1)(y+1)(x-y)(x+y)y^{-2}$
$3_{72}(\sim-3_{72})$, $3_{72}{}^*(\sim-3_{72}{}^*)$	$-(y+1)(x+y)\left(x^3y-2x^2y+x^2+xy^2-2xy+y\right)y^{-2}$
3_{73}	$(x-1)(y+1)(x+y)y^{-2}(x^4y-x^4-x^3y^2+2x^3y+x^2y^3$ $-x^2y^2-xy^2+2xy-x+y^3-y^2+y)$
-3_{73}	$(x-1)(y+1)(x+y)y^{-3}(x^4y^2-x^4y+x^4-x^3y^3+2x^3y^2$ $-x^3y-x^2y+x^2+2xy^2-xy-y^3+y^2)$
$3_{73}{}^*$	$-(x-1)(y+1)(x+y)y^{-3}(x^4y^2-x^4y+x^4-x^3y^3+2x^3y^2$ $-x^3y-x^2y+x^2+2xy^2-xy-y^3+y^2)$
$-3_{73}{}^*$	$-(x-1)(y+1)(x+y)y^{-2}(x^4y-x^4-x^3y^2+2x^3y+x^2y^3$ $-x^2y^2-xy^2+2xy-x+y^3-y^2+y)$
$3_{74}(\sim-3_{74})$	$2(x-1)\left(x^2-x+1\right)(y+1)(x+y)y^{-1}$
$3_{74}{}^*(\sim-3_{74}{}^*)$	$-2(x-1)\left(x^2-x+1\right)(y+1)(x+y)y^{-1}$
3_{75}	$(x-1)(y+1)(x-2y+1)(x+y)$
-3_{75}	$(x-1)(y+1)(x+y)(xy-2x+y)y^{-3}$
$3_{75}{}^*$	$-(x-1)(y+1)(x+y)(xy-2x+y)y^{-3}$
$-3_{75}{}^*$	$(1-x)(y+1)(x-2y+1)(x+y)$
3_{76}	$(x-1)(y+1)(x+y)(x^3-2x^2y-xy^3+4xy^2-2xy+x-y^3)y^{-3}$
-3_{76}	$-(x-1)(y+1)(x+y)(x^3-x^2y^3+2x^2y^2-4x^2y+x^2+2xy^2-y^3)y^{-2}$
$3_{76}{}^*$	$(x-1)(y+1)(x+y)(x^3-x^2y^3+2x^2y^2-4x^2y+x^2+2xy^2-y^3)y^{-2}$
$-3_{76}{}^*$	$-(x-1)(y+1)(x+y)(x^3-2x^2y-xy^3+4xy^2-2xy+x-y^3)y^{-3}$
3_{77}	$(x-1)(y+1)(x+y)\left(xy-2x-2y^2+y\right)y^{-1}$
-3_{77}	$(x-1)(y+1)(x+y)\left(xy-2x-2y^2+y\right)y^{-3}$
$3_{77}{}^*$	$-(x-1)(y+1)(x+y)\left(xy-2x-2y^2+y\right)y^{-3}$
$-3_{77}{}^*$	$-(x-1)(y+1)(x+y)\left(xy-2x-2y^2+y\right)y^{-1}$
$3_{78}(\sim-3_{78})$	$-2(x-1)(y+1)(x+y)y^{-1}$
$3_{78}{}^*(\sim-3_{78}{}^*)$	$2(x-1)(y+1)(x+y)y^{-1}$
$3_{79}(\sim-3_{79})$	$(x-1)(x^2-4x+1)(y+1)(y^2-y+1)(x+y)(x^2-xy+y^2)y^{-3}$
$3_{79}{}^*(\sim-3_{79}{}^*)$	$-(x-1)(x^2-4x+1)(y+1)(y^2-y+1)(x+y)(x^2-xy+y^2)y^{-3}$

Knot	Twisted JKSS invariant
$3_{80}, -3_{80}, 3_{80}{}^{*}, -3_{80}{}^{*}$	0
$3_{81}, -3_{81}, 3_{81}{}^{*}, -3_{81}{}^{*}$	0
$3_{82}, -3_{82}, 3_{82}{}^{*}, -3_{82}{}^{*}$	0

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