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CURVES IN A SPACELIKE HYPERSURFACE IN MINKOWSKI SPACE-TIME

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Abstract

In mathematical physics, Minkowski space (or Minkowski space-time) is a combination of three-dimensional Euclidean space and time into a four-dimensional manifold.

The hyperbolic surface and de Sitter surface of a curve are defined in the spacelike hypersurface M in Minkowski 4-space and located, respectively, in hyperbolic 3-space and de Sitter 3-space. In this study, techniques from singularity theory were applied to obtain the generic shape of such surfaces and their singular value sets and the geometrical meanings of these singularities were investigated.

1. Introduction

Submanifolds in Lorentz-Minkowski space are investigated from various mathematical viewpoints and are of interest also in relativity theory. In recent years, the use of singularity theory has led to significant progress and many investigations have focused on the classification and characterisation of the singularity of submanifolds in both Euclidean spaces and semi-Euclidean spaces (see [1]-[8] and [10]).The results of the present study have complemented a whole study of the extrinsic geometry of curves in different ambient spaces, as mentioned above.

We considered a spacelike embedding $X : U \to \mathbb{R}^4_1$ from an open subset $U \subset \mathbb{R}^3$ and identified M and U through embedding X, where $\mathbb{R}^4_1$ is the Minkowski 4-space. For a curve $\gamma : I \to M$ with nowhere vanishing curvature, we defined a hyperbolic surface in hyperbolic space $H^3(-1)$ and a de Sitter surface in de Sitter space S^3_1 associated with curve γ. Singularity theory techniques, and in particular, the classical deformation theory, were applied for the study of the generic differential geometry of those surfaces and their singular sets.

This paper is organised as follows: Section 2 reviews some basic definitions of Minkowski 4-space, as well as the definition of A_k-singularities and discriminant sets, and reports the construction of a moving frame along γ together with Frenet-Serret type formulae; Sections 3 and 5 address the definition of two families of height functions on γ, namely timelike tangential height functions and spacelike tangential height functions, which measure the contact of curve t with special hyperplanes and whose differentiation yields invariants related to each surface. The hyperbolic surface of γ is described as the discriminant set of the family of timelike tangential height functions (Corollary 3.2) and de Sitter surface of γ is the

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discriminant set of the family of spacelike tangential height functions (Corollary 5.2). The theory of deformations provides a classification and a characterisation of the diffeomorphims type of such surfaces (Theorems 3.5 and 5.5). The sections also report on an investigation on the geometrical meaning of the invariants, and the results enable curve γ to be part of a slice surface (Propositions 3.6 and 5.6). When γ is not part of a slice surface, the contact of γ with a slice surface is characterised by the singularity types of both its hyperbolic surface (Proposition 3.7) and de Sitter surface (Proposition 5.7). Sections 4 and 6 provide examples of curves on spacelike hypersurface in $\mathbb{R}^4_1$ and the surfaces studied in [3].

2. Preliminaries

Minkowski space $\mathbb{R}^4_1$ is the vector space $\mathbb{R}^4$ endowed with the pseudo-scalar product $\langle x, y\rangle = -x_0y_0 + x_1y_1 + x_2y_2 + x_3y_3$, for any $x = (x_0, x_1, x_2, x_3)$ and $y = (y_0, y_1, y_2, y_3)$ in $\mathbb{R}^4_1$ (see, e.g., [9]). A non-zero vector $x \in \mathbb{R}^4_1$ is said to be *spacelike* if $\langle x, x\rangle > 0$, *lightlike* if $\langle x, x\rangle = 0$ and *timelike* if $\langle x, x\rangle < 0$, respectively. $\gamma : I \to \mathbb{R}^4_1$, with $I \subset \mathbb{R}$ open interval, is *spacelike* (resp. *timelike*) if tangent vector $\gamma'(t)$ is a *spacelike* (resp. *timelike*) vector for any $t \in I$.

The norm of a vector $x \in \mathbb{R}^4_1$ is defined by $\|x\| = \sqrt{|\langle x, x\rangle|}$. For a non-zero vector $v \in \mathbb{R}^4_1$ and a real number c, *hyperplane* with *pseudo-normal* v is defined by

$$HP(v, c) = \left\{x \in \mathbb{R}^4_1 \;\middle|\; \langle x, v\rangle = c\right\}.$$

We call $HP(v, c)$ a *spacelike hyperplane*, a *timelike hyperplane* or a *lightlike hyperplane* if v is timelike, spacelike or lightlike, respectively. Let us now consider the pseudo-spheres in $\mathbb{R}^4_1$: The *hyperbolic 3-space* is defined by

$$H^3(-1) = \left\{x \in \mathbb{R}^4_1 \;\middle|\; \langle x, x\rangle = -1\right\},$$

and the de *Sitter 3-space* is denoted by

$$S^3_1 = \left\{x \in \mathbb{R}^4_1 \;\middle|\; \langle x, x\rangle = 1\right\}.$$

For any $x = (x_0, x_1, x_2, x_3)$, $y = (y_0, y_1, y_2, y_3)$, $z = (z_0, z_1, z_2, z_3) \in \mathbb{R}^4_1$, the pseudo vector product of x, y and z is defined as follows:

$$x \wedge y \wedge z = \begin{vmatrix} -e_0 & e_1 & e_2 & e_3 \\ x_0 & x_1 & x_2 & x_3 \\ y_0 & y_1 & y_2 & y_3 \\ z_0 & z_1 & z_2 & z_3 \end{vmatrix},$$

where $\{e_0, e_1, e_2, e_3\}$ is the canonical basis of $\mathbb{R}^4$.

Considering a spacelike embedding $X : U \to \mathbb{R}^4_1$ from an open subset $U \subset \mathbb{R}^3$, we write $M = X(U)$ and identify M and U through embedding X. X is said to be a *spacelike embedding* if the tangent space T_pM consists of spacelike vectors at any $p = X(u)$. Let $\bar{\gamma} : I \to U$ be a regular curve. Therefore, a curve $\gamma : I \to M \subset \mathbb{R}^4_1$ is defined by $\gamma(s) = X(\bar{\gamma}(s))$, and is a *curve in the spacelike hypersurface M*. Since γ is a spacelike curve, it can be reparametrized by the arc length s, which gives a unit tangent vector $t(s) = \gamma'(s)$. In this case, we call γ a *unit speed spacelike curve*. Since X is a spacelike embedding, a unit timelike normal vector field n along $M = X(U)$ is defined by

$$n(p) = \frac{X_{u_1}(u) \wedge X_{u_2}(u) \wedge X_{u_3}(u)}{\|X_{u_1}(u) \wedge X_{u_2}(u) \wedge X_{u_3}(u)\|}$$

for $p = X(u)$, where $X_{u_i} = \partial X/\partial u_i$, $i = 1, 2, 3$. n is *future directed* if $\langle n, e_0\rangle < 0$. We chose the orientation of M, such that n is future directed and we defined $n_\gamma(s) = n \circ \gamma(s)$, to obtain a unit timelike normal vector field n_γ along γ. Under the assumption that $\| \langle n_\gamma(s), t'(s)\rangle n_\gamma(s) + t'(s) \| \neq 0$, we defined

$$n_1(s) = \frac{\langle n_\gamma(s), t'(s)\rangle n_\gamma(s) + t'(s)}{\|\langle n_\gamma(s), t'(s)\rangle n_\gamma(s) + t'(s)\|}.$$

It follows that $\langle t, n_1\rangle = 0$ and $\langle n_\gamma, n_1\rangle = 0$. Therefore, a spacelike unit vector is defined by $n_2(s) = n_\gamma \wedge t(s) \wedge n_1(s)$, and a pseudo-orthonormal frame $\{n_\gamma, t(s), n_1(s), n_2(s)\}$ is called a *Lorentzian Darboux frame* along γ. By standard arguments, the Frenet-Serret type formulae for the above frame are given by

$$\begin{cases} n'_\gamma(s) = k_n(s)\, t(s) + \tau_1(s)\, n_1(s) + \tau_2(s)\, n_2(s), \\ \ t'(s) = k_n(s)\, n_\gamma(s) + k_g(s)\, n_1(s), \\ n'_1(s) = \tau_1(s)\, n_\gamma(s) - k_g(s)\, t(s) + \tau_g(s)\, n_2(s), \\ n'_2(s) = \tau_2(s)\, n_\gamma(s) - \tau_g(s)\, n_1(s), \end{cases}$$

where $k_n(s) = -\langle n_\gamma(s), t'(s)\rangle$, $\tau_1(s) = \langle n_1(s), n'_\gamma(s)\rangle$, $\tau_2(s) = \langle n_2(s), n'_\gamma(s)\rangle$, $k_g(s) = \|\langle n_\gamma(s), t'(s)\rangle n_\gamma(s) + t'(s)\| = \|-k_n(s) n_\gamma(s) + t'(s)\|$ and $\tau_g(s) = \langle -n'_2(s), n_1(s)\rangle$. The invariant k_n is called a normal curvature, τ_1 is a first normal torsion, τ_2 is a second normal torsion, k_g is a geodesic curvature, and τ_g is a geodesic torsion.

By assumption, $k_g(s) = \|\langle n_\gamma(s), t'(s)\rangle n_\gamma(s) + t'(s)\| \neq 0$, so that $k_g(s) > 0$.

Definition 2.1. Let $F : \mathbb{R}^4_1 \to \mathbb{R}$ be a submersion and $\gamma : I \to M$ a regular curve. γ and $F^{-1}(0)$ have contact of order k at s_0 if function $g(s) = F \circ \gamma(s)$ satisfies $g(s_0) = g'(s_0) = \cdots = g^{(k)}(s_0) = 0$ and $g^{(k+1)}(s_0) \neq 0$, i.e., g has an A_k-singularity at s_0.

Let $G : \mathbb{R} \times \mathbb{R}^r, (s_0, x_0) \to \mathbb{R}$ be a family of germs of functions. We call G an r-parameter deformation of f if $f(s) = G_{x_0}(s)$. Supposing f has an A_k-singularity ($k \geq 1$) at s_0, we write

$$j^{(k-1)}\left(\frac{\partial G}{\partial x_i}(s, x_0)\right)(s_0) = \sum_{j=0}^{k-1} \alpha_{ji}(s - s_0)^j,$$

for $i = 1, \ldots, r$. Then, G is a *versal deformation* if the $k \times r$ matrix of coefficients (α_{ji}) has rank k ($k \leq r$) (see [1]).

The *discriminant set* of G is

$$\mathcal{D}_G = \left\{ x \in (\mathbb{R}^r, x_0) \;\middle|\; G = \frac{\partial G}{\partial s} = 0 \ at\ (s, x)\ for\ some\ s \in (\mathbb{R}, s_0) \right\}$$

and the *bifurcation set* of G is

$$\mathcal{B}_G = \left\{ x \in (\mathbb{R}^r, x_0) \;\middle|\; \frac{\partial G}{\partial s} = \frac{\partial^2 G}{\partial s^2} = 0 \ at\ (s, x)\ for\ some\ s \in (\mathbb{R}, s_0) \right\}.$$

The next result is from reference [1].

Theorem 2.2. *Let $G : \mathbb{R} \times \mathbb{R}^r, (s_0, x_0) \to \mathbb{R}$ be an r-parameter deformation of f, such that f has an A_k-singularity at s_0. Supposing G is a versal deformation, $\mathcal{D}_G$ is locally diffeomorphic to*

(1) $C \times \mathbb{R}^{r-2}$ *if* $k = 2$,

(2) $SW \times \mathbb{R}^{r-3}$ *if* $k = 3$,

where $C = \{(x_1, x_2) \mid x_1^2 = x_2^3\}$ *is the ordinary cusp and* $SW = \{(x_1, x_2, x_3) \mid x_1 = 3u^4 + u^2v, x_2 = 4u^3 + 2uv, x_3 = v\}$ *is the swallowtail surface.*

In Sections 3 and 5, special families of functions on curves in M were used for the study of the hyperbolic surface and de Sitter surface. In fact, such surfaces are the discriminant sets of those families.

3. Timelike tangential height functions

This section introduces the family of timelike tangential height functions on a curve in a spacelike hypersurface M, and addresses the definition and study of the hyperbolic surface given by the discriminant set of this family.

A family of functions on a curve $\gamma : I \to M \subset \mathbb{R}^4_1$ is defined as

$$H_t^T : I \times H^3(-1) \to \mathbb{R}; \quad (s, v) \mapsto \langle t(s), v \rangle.$$

We call H_t^T a *family of timelike tangential height functions of* γ, and $(h_t^T)_v(s) = H_t^T(s, v)$ is denoted for any fixed $v \in H^3(-1)$. The family H_t^T measures the contact of the curve t with spacelike hyperplanes in $\mathbb{R}^4_1$, which is, generically, of order k, $k = 1, 2, 3$.

The conditions that characterise the A_k-singularity, $k = 1, 2, 3$ can be obtained in Proposition 3.1.

The proof of (2) in the following proposition leads to $k_g^2(s) > k_n^2(s)$, therefore, we can assume that there exists an interval I, such that $k_g^2(s) > k_n^2(s)$ for $s \in I$. Towards avoiding complicated situations, we have assumed $(k_n\tau_2 + k_g\tau_g)(s) \neq 0$ for any $s \in I$.

Proposition 3.1. *Let* $\gamma : I \to M$ *be a unit speed curve with* $k_g(s) \neq 0$ *and* $(k_n\tau_2 + k_g\tau_g)(s) \neq 0$. *Therefore,*

(1) $(h_t^T)_v(s) = 0$ *if and only if there exist* $\mu, \lambda, \eta \in \mathbb{R}$, *such that* $-\mu^2 + \lambda^2 + \eta^2 = -1$ *and* $v = \mu n_\gamma(s) + \lambda n_1(s) + \eta n_2(s)$.

(2) $(h_t^T)_v(s) = (h_t^T)'_v(s) = 0$ *if and only if there exists* $\theta \in \mathbb{R}$, *such that*

$$v = \frac{\cosh\theta}{\sqrt{k_g^2(s) - k_n^2(s)}}\Big(k_g(s)n_\gamma(s) + k_n(s)n_1(s)\Big) + \sinh\theta n_2(s).$$

(3) $(h_t^T)_v(s) = (h_t^T)'_v(s) = (h_t^T)''_v(s) = 0$ *if and only if*

$$v = \frac{\cosh\theta}{\sqrt{k_g^2(s) - k_n^2(s)}}\Big(k_g(s)n_\gamma(s) + k_n(s)n_1(s)\Big) + \sinh\theta n_2(s),$$

$$\tanh\theta = \frac{k_g k_n' + k_g^2\tau_1 - k_n^2\tau_1 - k_n k_g'}{(k_n\tau_2 + k_g\tau_g)\sqrt{k_g^2 - k_n^2}}(s).$$

(4) $(h_t^T)_v(s) = (h_t^T)'_v(s) = (h_t^T)''_v(s) = (h_t^T)'''_v(s) = 0$ *if and only if*

$$v = \frac{\cosh\theta}{\sqrt{k_g^2(s)-k_n^2(s)}}\Big(k_g(s)n_\gamma(s)+k_n(s)n_1(s)\Big)+\sinh\theta n_2(s),$$

$$\tanh\theta = \frac{k_gk_n'+k_g^2\tau_1-k_n^2\tau_1-k_nk_g'}{(k_n\tau_2+k_g\tau_g)\sqrt{k_g^2-k_n^2}}(s) \textit{ and } \rho(s)=0, \textit{ where}$$

$$\rho(s) = \Big((-k_gk_n''-k_gk_n\tau_2^2-2k_gk_g'\tau_1-k_g^2\tau_1'-k_g^2\tau_g\tau_2+2k_nk_n'\tau_1+k_n^2\tau_1'-k_n^2k_g\tau_2+k_g''k_n-k_gk_n\tau_g^2)(k_n\tau_2+k_g\tau_g)+(k_gk_n'+k_g^2\tau_1-k_n^2\tau_1-k_nk_g')(2k_n'\tau_2+k_n\tau_1\tau_g+k_n\tau_2'+2k_g'\tau_g+k_g\tau_1\tau_2+k_g\tau_g')\Big)(s).$$

(5) $(h_t^T)_v(s)=(h_t^T)_v'(s)=(h_t^T)_v''(s)=(h_t^T)_v'''(s)=(h_t^T)_v^{(4)}(s)=0$ *if and only if*

$$v = \frac{\cosh\theta}{\sqrt{k_g^2(s)-k_n^2(s)}}\Big(k_g(s)n_\gamma(s)+k_n(s)n_1(s)\Big)+\sinh\theta n_2(s),$$

$$\tanh\theta = \frac{k_gk_n'+k_g^2\tau_1-k_n^2\tau_1-k_nk_g'}{(k_n\tau_2+k_g\tau_g)\sqrt{k_g^2-k_n^2}}(s) \textit{ and } \rho(s)=\rho'(s)=0.$$

Proof. By definition, $(h_t^T)_v(s)=0$ if and only if $\langle t(s),v\rangle=0$. This is equivalent to $v=\mu n_\gamma(s)+\lambda n_1(s)+\eta n_2(s)$, where $\mu,\lambda,\eta\in\mathbb{R}$ and $-\mu^2+\lambda^2+\eta^2=-1$ so that (1) follows. For (2), $(h_t^T)_v(s)=(h_t^T)_v'(s)=0$ if and only if $v=\mu n_\gamma(s)+\lambda n_1(s)+\eta n_2(s)$ with $-\mu^2+\lambda^2+\eta^2=-1$ and $\langle t'(s),v\rangle=-\mu k_n+\lambda k_g=0$. This is equivalent to

$$v = \frac{\cosh\theta}{\sqrt{k_g^2(s)-k_n^2(s)}}\Big(k_g(s)n_\gamma(s)+k_n(s)n_1(s)\Big)+\sinh\theta n_2(s).$$

For (3), $(h_t^T)_v(s)=(h_t^T)_v'(s)=(h_t^T)_v''(s)=0$ if and only if

$$v = \frac{\cosh\theta}{\sqrt{k_g^2(s)-k_n^2(s)}}\Big(k_g(s)n_\gamma(s)+k_n(s)n_1(s)\Big)+\sinh\theta n_2(s) \text{ and } \langle t''(s),v\rangle=0.$$

Since $t''(s)=(k_n^2(s)-k_g^2(s))t(s)+(k_n'(s)+k_g(s)\tau_1(s))n_\gamma(s)+(k_n(s)\tau_1(s)+k_g'(s))n_1(s)+(k_n(s)\tau_2(s)+k_g(s)\tau_g(s))n_2(s)$, the previous assertion is equivalent to

$$v = \frac{\cosh\theta}{\sqrt{k_g^2(s)-k_n^2(s)}}\Big(k_g(s)n_\gamma(s)+k_n(s)n_1(s)\Big)+\sinh\theta n_2(s)$$

and $\tanh\theta = \dfrac{k_gk_n'+k_g^2\tau_1-k_n^2\tau_1-k_nk_g'}{(k_n\tau_2+k_g\tau_g)\sqrt{k_g^2-k_n^2}}(s).$

Items (4) and (5) were calculated by Frenet-Serret type formulae of γ. Since such calculations are laborious and long, details have been omitted. □

Following Proposition 3.1, we defined the invariant

$$\rho(s) = \Big((-k_gk_n''-k_gk_n\tau_2^2-2k_gk_g'\tau_1-k_g^2\tau_1'-k_g^2\tau_g\tau_2+2k_nk_n'\tau_1+k_n^2\tau_1'-k_n^2k_g\tau_2+k_g''k_n-k_gk_n\tau_g^2)(k_n\tau_2+k_g\tau_g)+(k_gk_n'+k_g^2\tau_1-k_n^2\tau_1-k_nk_g')(2k_n'\tau_2+k_n\tau_1\tau_g+k_n\tau_2'+2k_g'\tau_g+k_g\tau_1\tau_2+k_g\tau_g')\Big)(s)$$

of the curve γ. The geometrical meaning of this invariant will be studied.

Motivated by the calculations of this proposition, we defined a surface and its singular locus. Let $\gamma : I \to M$ be a unit speed curve with $k_g(s) \neq 0$ and $(k_n\tau_2 + k_g\tau_g)(s) \neq 0$. A surface $S_\gamma : I \times \mathbb{R} \to H^3(-1)$ is defined by

$$S_\gamma(s,\theta) = \frac{\cosh\theta}{\sqrt{k_g^2(s) - k_n^2(s)}}\Big(k_g(s)n_\gamma(s) + k_n(s)n_1(s)\Big) + \sinh\theta n_2(s).$$

We call S_γ a *hyperbolic surface* of γ. Since we have assumed $k_g^2(s) > k_n^2(s)$ for any $s \in I$, the hyperbolic surface exists. We now define $CH_\gamma = S_\gamma(s,\theta(s))$, where $\tanh\theta(s) = \dfrac{k_gk_n' + k_g^2\tau_1 - k_n^2\tau_1 - k_nk_g'}{(k_n\tau_2 + k_g\tau_g)\sqrt{k_g^2 - k_n^2}}(s)$, which is generically a curve. We call CH_γ a *hyperbolic curve* of γ. By Theorem 3.5 (1), this curve is the locus of the singular points of the hyperbolic surface of γ.

Corollary 3.2. *The hyperbolic surface of γ is the discriminant set $\mathcal{D}_{H_t^T}$ of the family of timelike tangential height functions H_t^T.*

Proof. The proof follows from the definition of the discriminant set given in Section 2 and Proposition 3.1 (2). □

In the following proposition, we show the family of timelike tangential height functions on a curve in M is a versal deformation of an A_k-singularity, $k = 2, 3$, of its members. Furthermore, we will study the geometric meaning of the invariant ρ. We write $\lambda_0(s) = \left(k_gk_n' + k_g^2\tau_1 - k_n^2\tau_1 - k_nk_g'\right)(s)$.

Proposition 3.3. *Let $\gamma : I \to M$ be a unit speed curve with $k_g(s) \neq 0$ and $(k_n\tau_2 + k_g\tau_g)(s) \neq 0$.*

(a) *If $(h_t^T)_{v_0}$ has an A_2-singularity at s_0, then H_t^T is a versal deformation of $(h_t^T)_{v_0}$.*
(b) *If $(h_t^T)_{v_0}$ has an A_3-singularity at s_0 and $\lambda_0(s_0) \neq 0$ (which is a generic condition), then H_t^T is a versal deformation of $(h_t^T)_{v_0}$.*

Proof. The family of timelike tangential height functions is given by

$$H_t^T(s,v) = -v_0x_0'(s) + v_1x_1'(s) + v_2x_2'(s) + v_3x_3'(s),$$

where $v = (v_0, v_1, v_2, v_3)$, $t(s) = (x_0'(s), x_1'(s), x_2'(s), x_3'(s))$ and $v_0 = \sqrt{1 + v_1^2 + v_2^2 + v_3^2}$.

Thus

$$\frac{\partial H_t^T}{\partial v_i}(s,v) = x_i'(s) - \frac{v_i}{v_0}x_0'(s),$$

for $i = 1, 2, 3$. Therefore, the 1-jet of $\dfrac{\partial H_t^T}{\partial v_i}(s,v)$ at s_0 is given by

$$x_i'(s_0) - \frac{v_i}{v_0}x_0'(s_0) + \left(x_i''(s_0) - \frac{v_i}{v_0}x_0''(s_0)\right)(s - s_0)$$

and the 2-jet of $\dfrac{\partial H_t^T}{\partial v_i}(s,v)$ at s_0 is given by

$$x_i'(s_0) - \frac{v_i}{v_0}x_0'(s_0) + \left(x_i''(s_0) - \frac{v_i}{v_0}x_0''(s_0)\right)(s - s_0) + \frac{1}{2}\left(x_i'''(s_0) - \frac{v_i}{v_0}x_0'''(s_0)\right)(s - s_0)^2.$$

First, we assumed that $(h_t^T)_v$ has an A_2-singularity at $s = s_0$, and show that the rank of the matrix

$$B = \begin{pmatrix} x_1'(s_0) - \frac{v_1}{v_0}x_0'(s_0) & x_2'(s_0) - \frac{v_2}{v_0}x_0'(s_0) & x_3'(s_0) - \frac{v_3}{v_0}x_0'(s_0) \\ x_1''(s_0) - \frac{v_1}{v_0}x_0''(s_0) & x_2''(s_0) - \frac{v_2}{v_0}x_0''(s_0) & x_3''(s_0) - \frac{v_3}{v_0}x_0''(s_0) \end{pmatrix}$$

is two.

We calculated the Gram-Schmidt matrix of $\widetilde{B} = v_0 B$, and denoted the lines of $\widetilde{B}$ by

$$F = (x_1'(s_0)v_0 - x_0'(s_0)v_1, x_2'(s_0)v_0 - x_0'(s_0)v_2, x_3'(s_0)v_0 - x_0'(s_0)v_3),$$

$$G = (x_1''(s_0)v_0 - x_0''(s_0)v_1, x_2''(s_0)v_0 - x_0''(s_0)v_2, x_3''(s_0)v_0 - x_0''(s_0)v_3).$$

Since $\langle v, v\rangle = -1$, $\langle t(s), t(s)\rangle = 1$, $\langle t(s), v\rangle = 0$, $\langle t'(s), v\rangle = 0$ and $\langle t'(s), t'(s)\rangle = k_g^2(s) - k_n^2(s)$, we have the following Euclidean inner product

$$F.F = v_0^2 - (x_0')^2, \quad F.G = -x_0'x_0'' \quad \text{and} \quad G.G = v_0^2(k_g^2(s) - k_n^2(s)) - (x_0'')^2.$$

Therefore, the Gram-Schmidt matrix of $\widetilde{B}$ is

$$G_{\widetilde{B}} = \begin{pmatrix} v_0^2 - (x_0')^2 & -x_0'x_0'' \\ -x_0'x_0'' & v_0^2(k_g^2(s) - k_n^2(s)) - (x_0'')^2 \end{pmatrix}.$$

Through a Lorentzian motion of the curve, we can assume $n_\gamma(s_0) = (1, 0, 0, 0)$. In this case, $x_0'(s_0) = 0$, $x_0''(s_0) = k_n(s_0)$ and $v_0 = \dfrac{k_g(s_0)\cosh\theta_0}{\sqrt{k_g^2(s) - k_n^2(s)}}$. Therefore, the determinant of $G_{\widetilde{B}}$ is

$$v_0^2\left(k_g^2(s_0) - k_n^2(s_0)\right)\left(v_0^2 - (x_0')^2\right) - {v_0}^2(x_0'')^2 = \frac{k_g^2(s_0)\cosh^2\theta_0}{k_g^2(s_0) - k_n^2(s_0)}\left(k_g^2(s_0)\cosh^2\theta_0 - k_n^2(s_0)\right),$$

which is different from zero, since $k_g^2(s_0) > k_n^2(s_0)$. Consequently, the rank of the matrix B is two, and assertion (a) follows.

We now assume $(h_t^T)_v$ has an A_3-singularity at $s = s_0$. In this case, the determinant of the 3×3 matrix

$$A = \begin{pmatrix} x_1'(s_0) - \frac{v_1}{v_0}x_0'(s_0) & x_2'(s_0) - \frac{v_2}{v_0}x_0'(s_0) & x_3'(s_0) - \frac{v_3}{v_0}x_0'(s_0) \\ x_1''(s_0) - \frac{v_1}{v_0}x_0''(s_0) & x_2''(s_0) - \frac{v_2}{v_0}x_0''(s_0) & x_3''(s_0) - \frac{v_3}{v_0}x_0''(s_0) \\ x_1'''(s_0) - \frac{v_1}{v_0}x_0'''(s_0) & x_2'''(s_0) - \frac{v_2}{v_0}x_0'''(s_0) & x_3'''(s_0) - \frac{v_3}{v_0}x_0'''(s_0) \end{pmatrix}$$

is nonzero. Denoting

$$a = \begin{pmatrix} x_0'(s_0) \\ x_0''(s_0) \\ x_0'''(s_0) \end{pmatrix}, b_i = \begin{pmatrix} x_i'(s_0) \\ x_i''(s_0) \\ x_i'''(s_0) \end{pmatrix},$$

for $i = 1, 2, 3$, then

$$\det A = \frac{v_0}{v_0}\det(b_1\ b_2\ b_3) - \frac{v_1}{v_0}\det(a\ b_2\ b_3) - \frac{v_2}{v_0}\det(b_1\ a\ b_3) - \frac{v_3}{v_0}\det(b_1\ b_2\ a).$$

On the other hand,

$$(\gamma' \wedge \gamma'' \wedge \gamma''')(s_0) = (-\det(b_1\ b_2\ b_3), -\det(a\ b_2\ b_3), -\det(b_1\ a\ b_3), -\det(b_1\ b_2\ a)).$$

Therefore,

$$\det A = \left\langle \left(\frac{v_0}{v_0}, \frac{v_1}{v_0}, \frac{v_2}{v_0}, \frac{v_3}{v_0}\right), (\gamma' \wedge \gamma'' \wedge \gamma''')(s_0) \right\rangle = \frac{\cosh\theta_0 \left(k_g k_n' + k_g^2\tau_1 - k_n^2\tau_1 - k_n k_g'\right)^2}{v_0\sqrt{k_g^2 - k_n^2}(k_n\tau_2 + k_g\tau_g)}(s_0).$$

If $(h_t^T)_{v_0}$ has an A_3-singularity at s_0 and $\lambda_0(s_0) \neq 0$, then $\det A \neq 0$ and H_t^T is a versal deformation of $(h_t^T)_{v_0}$, which completes the proof. □

According to Proposition 3.3, if $(h_t^T)_{v_0}$ has an A_3-singularity at s_0 and $\lambda_0(s_0) \neq 0$, then H_t^T is a versal deformation of $(h_t^T)_{v_0}$. Let us now investigate what occurs if $\lambda_0(s_0) = 0$.

First, we must define a new deformation of $(h_t^T)_{v_0}$ and prove it is a versal deformation. Then, the Recognition Lemma is applied for cuspidal beaks, or cuspidal lips given in [6].

Using Proposition 3.1 with $\lambda_0(s_0) = 0$, $(h_t^T)_{v_0}$ has an A_3-singularity at s_0 if and only if $\theta = 0$, $v(s_0) = \dfrac{1}{\sqrt{k_g^2(s_0) - k_n^2(s_0)}}\left(k_g n_\gamma + k_n n_1\right)(s_0)$, $\rho(s_0) = 0$ and $\rho'(s_0) \neq 0$, where

$$\rho'(s_0) = \frac{-1}{\sqrt{k_g^2(s_0) - k_n^2(s_0)}}(-3\lambda_0'(s_0)\lambda_1(s_0) + \lambda_2(s_0)) \neq 0,$$

$$\lambda_1(s) = (k_n'\tau_2 + k_n\tau_2' + k_g'\tau_g + k_g\tau_g')(s),$$

$$\begin{aligned}\lambda_2(s) = (&k_g k_n''' + 3k_g'' k_g\tau_1 + 3k_g'\tau_1' k_g + k_g^2\tau_1'' + k_g^2\tau_g\tau_2' - k_g^2\tau_g^2\tau_1 - k_n\tau_1\tau_2 k_g^2 + k_n\tau_1\tau_2 k_g\tau_g \\ &- 3k_n k_n''\tau_1 - 3k_n k_n'\tau_1' - k_n^2\tau_1'' + k_n k_g''' + k_n^2\tau_1\tau_g^2 + k_g^2\tau_1\tau_2^2 - k_n^2\tau_1\tau_2^2 + k_n^2\tau_2\tau_g' - k_n^2\tau_2'\tau_g \\ &- k_g^2\tau_g'\tau_2 + 2\tau_1^2 k_n' k_g - 2\tau_1^2 k_g' k_n)(s).\end{aligned}$$

We now define a deformation $\widetilde{H} : I \times H^3(-1) \times \mathbb{R} \to \mathbb{R}$ by $\widetilde{H}(s, v, u) = H_t^T(s, v) + u(s - s_0)^2 = \langle t(s), v\rangle + u(s - s_0)^2$. The germ at $(s_0, v_0, 0)$ represented by $\widetilde{H}$ is considered.

Proposition 3.4. *If $(h_t^T)_{v_0}$ has an A_3-singularity at s_0 and $\lambda_0(s_0) = 0$, then $\widetilde{H}$ is a versal deformation of $(h_t^T)_{v_0}$.*

Proof.

$$\widetilde{H}(s, v, u) = H_t^T(s, v) + u(s - s_0)^2 = -v_0 x_0'(s) + v_1 x_1'(s) + v_2 x_2'(s) + v_3 x_3'(s) + u(s - s_0)^2,$$

where $v = (v_0, v_1, v_2, v_3)$, $t(s) = (x_0'(s), x_1'(s), x_2'(s), x_3'(s))$ and $v_0 = \sqrt{1 + v_1^2 + v_2^2 + v_3^2}$.

Therefore,

$$\frac{\partial\widetilde{H}}{\partial v_i}(s, v, 0) = x_i'(s) - \frac{v_i}{v_0}x_0'(s),$$

for $i = 1, 2, 3$. Therefore, the 2-jet of $\dfrac{\partial \widetilde{H}}{\partial v_i}(s, v, 0)$ at s_0 is

$$x_i'(s_0) - \frac{v_i}{v_0}x_0'(s_0) + \left(x_i''(s_0) - \frac{v_i}{v_0}x_0''(s_0)\right)(s - s_0) + \frac{1}{2}\left(x_i'''(s_0) - \frac{v_i}{v_0}x_0'''(s_0)\right)(s - s_0)^2,$$

and the 2-jet of $\dfrac{\partial \widetilde{H}}{\partial u}(s, v, 0)$ at s_0 is $(s - s_0)^2$.

We assume $(h_t^T)_v$ has an A_3-singularity at $s = s_0$, and it is enough to show

$$\operatorname{rank}\begin{pmatrix} x_1'(s_0) - \frac{v_1}{v_0}x_0'(s_0) & x_2'(s_0) - \frac{v_2}{v_0}x_0'(s_0) & x_3'(s_0) - \frac{v_3}{v_0}x_0'(s_0) & 0 \\ x_1''(s_0) - \frac{v_1}{v_0}x_0''(s_0) & x_2''(s_0) - \frac{v_2}{v_0}x_0''(s_0) & x_3''(s_0) - \frac{v_3}{v_0}x_0''(s_0) & 0 \\ x_1'''(s_0) - \frac{v_1}{v_0}x_0'''(s_0) & x_2'''(s_0) - \frac{v_2}{v_0}x_0'''(s_0) & x_3'''(s_0) - \frac{v_3}{v_0}x_0'''(s_0) & 1 \end{pmatrix}$$

$$= \operatorname{rank}\begin{pmatrix} 0 & 0 & 1 \\ x_1'(s_0) - \frac{v_1}{v_0}x_0'(s_0) & x_1''(s_0) - \frac{v_1}{v_0}x_0''(s_0) & 0 \\ x_2'(s_0) - \frac{v_2}{v_0}x_0'(s_0) & x_2''(s_0) - \frac{v_2}{v_0}x_0''(s_0) & 0 \\ x_3'(s_0) - \frac{v_3}{v_0}x_0'(s_0) & x_3''(s_0) - \frac{v_3}{v_0}x_0''(s_0) & 0 \end{pmatrix} = 3.$$

The rank of the last matrix has the same value of the rank of

$$\begin{pmatrix} 1 & 0 & 1 \\ x_1'(s_0) - \frac{v_1}{v_0}x_0'(s_0) & x_1''(s_0) - \frac{v_1}{v_0}x_0''(s_0) & 0 \\ x_2'(s_0) - \frac{v_2}{v_0}x_0'(s_0) & x_2''(s_0) - \frac{v_2}{v_0}x_0''(s_0) & 0 \\ x_3'(s_0) - \frac{v_3}{v_0}x_0'(s_0) & x_3''(s_0) - \frac{v_3}{v_0}x_0''(s_0) & 0 \end{pmatrix}.$$

Let us consider

$$a(s_0) = \left(1, x_1'(s_0) - \frac{v_1}{v_0}x_0'(s_0), x_2'(s_0) - \frac{v_2}{v_0}x_0'(s_0), x_3'(s_0) - \frac{v_3}{v_0}x_0'(s_0)\right),$$

$$b(s_0) = \left(0, x_1''(s_0) - \frac{v_1}{v_0}x_0''(s_0), x_2''(s_0) - \frac{v_2}{v_0}x_0''(s_0), x_3''(s_0) - \frac{v_3}{v_0}x_0''(s_0)\right)$$

and $c(s_0) = (1, 0, 0, 0)$. $a(s_0)$, $b(s_0)$, $c(s_0)$ are linearly independent. Indeed, if $a(s_0)$, $b(s_0)$, $c(s_0)$ are linearly dependent, then $x_1'(s_0) = \dfrac{v_1}{v_0}x_0'(s_0)$, $x_2'(s_0) = \dfrac{v_2}{v_0}x_0'(s_0)$ and $x_3'(s_0) = \dfrac{v_3}{v_0}x_0'(s_0)$, that is, $t(s_0)$ and v are parallel, which leads to a contradiction, since t is spacelike and v is timelike. □

The cuspidal beaks are defined to be a germ of surface diffeomorphic to $CBK = \{(x_1, x_2, x_3) | x_1 = v, x_2 = -2u_3 + v_2u, x_3 = 3u_4 - v_2u_2\}$ (see picture in [6]). Using Theorem 2.2, Propositions 3.3 and 3.4, we can obtain the diffeomorphism type of the hyperbolic surface in the following theorem.

Theorem 3.5. *Let $\gamma : I \to M$ be a unit speed curve with $k_g(s) \neq 0$, $(k_n\tau_2 + k_g\tau_g)(s) \neq 0$ and $k_g^2(s) > k_n^2(s)$. Let S_γ be the hyperbolic surface of γ. Then*

(1) *S_γ is singular at (s_0, θ_0) if and only if*

$$\tanh\theta_0 = \frac{k_g k_n' + k_g^2\tau_1 - k_n^2\tau_1 - k_n k_g'}{\sqrt{k_g^2 - k_n^2}(k_n\tau_2 + k_g\tau_g)}(s_0).$$

The singular points of the hyperbolic surface are given by $S_\gamma(s) = S_\gamma(s, \theta(s))$, where $\tanh\theta(s)$ satisfies the above equation.

(2) *The germ of S_γ at (s_0, θ_0) is diffeomorphic to a cuspidal edge if*

$$\tanh\theta_0 = \frac{k_g k_n' + k_g^2\tau_1 - k_n^2\tau_1 - k_n k_g'}{(k_n\tau_2 + k_g\tau_g)\sqrt{k_g^2 - k_n^2}}(s_0) \text{ and } \rho(s_0) \neq 0.$$

(3) *The germ of S_γ at (s_0, θ_0) is diffeomorphic to a swallowtail if*

$$\tanh\theta_0 = \frac{k_g k_n' + k_g^2\tau_1 - k_n^2\tau_1 - k_n k_g'}{(k_n\tau_2 + k_g\tau_g)\sqrt{k_g^2 - k_n^2}}(s_0), \ \lambda_0(s_0) \neq 0, \ \rho(s_0) = 0 \text{ and } \rho'(s_0) \neq 0.$$

(4) *The germ of S_γ at (s_0, θ_0) is diffeomorphic to cuspidal beaks if*

$$\lambda_0(s_0) = 0, \ \lambda_1(s_0) \neq 0, \ \rho(s_0) = 0 \text{ and } \rho'(s_0) \neq 0.$$

(5) *Cuspidal lips do not appear.*

Proof. Let us consider the hyperbolic surface

$$S_\gamma(s, \theta) = \frac{\cosh\theta}{\sqrt{k_g^2(s) - k_n^2(s)}}\Big(k_g(s)n_\gamma(s) + k_n(s)n_1(s)\Big) + \sinh\theta n_2(s).$$

Therefore, we have

$$\begin{aligned}\frac{\partial S_\gamma}{\partial s}(s,\theta) = &\left(\frac{\cosh\theta(-k_g'k_n^2 + k_g k_n k_n' + k_n\tau_1 k_g^2 - k_n^3\tau_1) + \sinh\theta\tau_2(k_g^2 - k_n^2)\sqrt{k_g^2 - k_n^2}}{(k_g^2 - k_n^2)\sqrt{k_g^2 - k_n^2}}\right)(s)n_\gamma(s)\\ &+ \left(\frac{\cosh\theta(k_g^3\tau_1 - k_g\tau_1 k_n^2 + k_n' k_g^2 - k_n k_g k_g') - \sinh\theta\tau_g(k_g^2 - k_n^2)\sqrt{k_g^2 - k_n^2}}{(k_g^2 - k_n^2)\sqrt{k_g^2 - k_n^2}}\right)(s)n_1(s)\\ &+ \left(\frac{\cosh\theta(k_g\tau_2 + k_n\tau_g)}{\sqrt{k_g^2 - k_n^2}}\right)(s)n_2(s) \quad \text{and}\end{aligned}$$

$$\frac{\partial S_\gamma}{\partial\theta}(s,\theta) = \frac{\sinh\theta k_g(s)}{\sqrt{k_g^2(s) - k_n^2(s)}}n_\gamma(s) + \frac{\sinh\theta k_n(s)}{\sqrt{k_g^2(s) - k_n^2(s)}}n_1(s) + \cosh\theta n_2(s).$$

Therefore, the vectors $\left\{\frac{\partial S_\gamma}{\partial s}(s_0,\theta_0), \frac{\partial S_\gamma}{\partial\theta}(s_0,\theta_0)\right\}$ are linearly dependent if and only if

$\tanh\theta_0 = \dfrac{k_g k_n' + k_g^2\tau_1 - k_n^2\tau_1 - k_n k_g'}{(k_n\tau_2 + k_g\tau_g)\sqrt{k_g^2 - k_n^2}}(s_0)$ and assertion (1) holds.

By Corollary 3.2, the discriminant set $\mathcal{D}_{H_t^T}$ of the family of timelike tangential height functions H_t^T of γ is the hyperbolic surface S_γ. It also follows from assertions (4) and (5) of Proposition 3.1 that $(h_t^T)_{v_0}$ has an A_2-singularity (respectively, an A_3-singularity) at $s = s_0$ if and only if

$$\tanh\theta_0 = \frac{k_g k_n' + k_g^2\tau_1 - k_n^2\tau_1 - k_n k_g'}{(k_n\tau_2 + k_g\tau_g)\sqrt{k_g^2 - k_n^2}}(s_0) \ \text{ and } \ \rho(s_0) \neq 0$$

(respectively, $\tanh\theta_0 = \dfrac{k_g k_n' + k_g^2\tau_1 - k_n^2\tau_1 - k_n k_g'}{(k_n\tau_2 + k_g\tau_g)\sqrt{k_g^2 - k_n^2}}(s_0)$, $\rho(s_0) = 0$ and $\rho'(s_0) \neq 0$). Therefore, by Proposition 3.3, we have assertions (2) and (3).

By Proposition 7.5 in [6] and previous Proposition 3.4, H_t^T is a Morse family of hypersurfaces.

Calculating $\varphi = (\partial^2 H_t^T/\partial s^2)|\mathcal{D}_{H_t^T}$, we have

$$\begin{aligned}\frac{\partial^2 H_t^T}{\partial s^2}(s,\theta) &= \left\langle t''(s), \frac{\cosh\theta}{\sqrt{k_g^2(s) - k_n^2(s)}}\Big(k_g(s)n_\gamma(s) + k_n(s)n_1(s)\Big) + \sinh\theta n_2(s)\right\rangle \\ &= \frac{-\cosh\theta}{\sqrt{k_g^2(s) - k_n^2(s)}}(k_g k_n' + k_g^2\tau_1 - k_n^2\tau_1 - k_n k_g')(s) + \sinh\theta(k_n\tau_2 + k_g\tau_g)(s).\end{aligned}$$

The Hessian matrix of $\varphi(s,\theta) = \dfrac{-\cosh\theta}{\sqrt{k_g^2(s) - k_n^2(s)}}(k_g k_n' + k_g^2\tau_1 - k_n^2\tau_1 - k_n k_g')(s) + \sinh\theta(k_n\tau_2 + k_g\tau_g)(s)$ is

$$\mathrm{Hess}(\varphi)(s_0, 0) = \begin{pmatrix} \dfrac{\partial^2\varphi}{\partial s^2}(s_0,0) & \lambda_1(s_0) \\ \lambda_1(s_0) & 0 \end{pmatrix}.$$

Since $\lambda_1(s_0) \neq 0$, $\det \mathrm{Hess}(\varphi)(s_0, 0) \neq 0$. By Lemma 7.7 in [6], H_t^T is P-$\mathcal{K}$-equivalent to $t^4 \pm v_1^2 t^2 + v_2 t + v^3$ (the notion of generating families, Legendrian equivalence and P-$\mathcal{K}$-equivalent are given in [6] page 30). The singular set of $\mathcal{D}_{H_t^T}$ is given by $\varphi(s,\theta) = 0$. Therefore it consists of two curves that transversally intersect at $(s_0, 0)$. Therefore, the normal form is $t^4 - v_1^2 t^2 + v_2 t + v^3$, the surface is diffeomorphic to cuspidal beaks, and we have assertions (4) and (5). □

We have three types of models of surfaces in M, which are given by intersections of M with hyperplanes in $\mathbb{R}^4_1$. We call a surface $M \cap HP(v,c)$ a *timelike slice* if v is spacelike, a *spacelike slice* if v is timelike, or a *lightlike slice* if v is lightlike.

In the following proposition, the curve γ of the hyperbolic surface is related to the invariant ρ and a slice surface. In this case, the singular locus of the hyperbolic surface of γ is a point.

Proposition 3.6. *Let $\gamma : I \to M$ be a unit speed curve, such that $k_g(s) \neq 0$, $(k_n\tau_2 + k_g\tau_g)(s) \neq 0$ and $k_g^2(s) > k_n^2(s)$ for any $s \in I$. Let $S_\gamma(s, \theta(s))$ be the singular points of the hyperbolic surface of γ. Then, the following conditions are equivalent:*

(1) *$S_\gamma(s, \theta(s))$ is a constant timelike vector;*

(2) *$\rho(s) \equiv 0$;*

(3) *there exist a timelike vector v and a real number c, such that $Im(\gamma) \subset M \cap HP(v, c)$.*

Proof. By definition

$$S_\gamma(s,\theta(s)) = \frac{\cosh\theta(s)}{\sqrt{(k_g^2 - k_n^2)(s)}}\left((k_g n_\gamma)(s) + (k_n n_1)(s) + \frac{(k_g k_n' + k_g^2\tau_1 - k_n^2\tau_1 - k_n k_g')(s)}{(k_n\tau_2 + k_g\tau_g)(s)} n_2(s)\right).$$

Thus,

$$\frac{dS_\gamma(s,\theta(s))}{ds} = \left(\frac{\cosh\theta(s)}{\sqrt{k_g^2(s) - k_n^2(s)}}\right)'\left(k_g(s)n_\gamma(s) + k_n(s)n_1(s) + \frac{(k_g k_n' + k_g^2\tau_1 - k_n^2\tau_1 - k_n k_g')}{(k_n\tau_2 + k_g\tau_g)}(s)n_2(s)\right)$$
$$+ \left(\frac{\cosh\theta(s)}{\sqrt{k_g^2(s) - k_n^2(s)}}\right)\left(k_g(s)n_\gamma(s) + k_n(s)n_1(s) + \frac{(k_g k_n' + k_g^2\tau_1 - k_n^2\tau_1 - k_n k_g')}{(k_n\tau_2 + k_g\tau_g)}(s)n_2(s)\right)'.$$

Furthermore,

$$\theta'(s) = \frac{X(s)}{\sqrt{(k_g^2 - k_n^2)(s)}((k_g^2 - k_n^2)(k_n\tau_2 + k_g\tau_g)^2 - (k_g k_n' + k_g^2\tau_1 - k_n^2\tau_1 - k_n k_g')^2)(s)},$$

where $X(s) = (k_g k_n'' + 2k_g k_g'\tau 1 + k_g^2\tau_1' - 2k_n k_n'\tau_1 - k_n^2\tau_1' - k_n k_g'')(k_g^2 - k_n^2)(k_n\tau_2 + k_g\tau_g) - (k_g k_n' + k_g^2\tau_1 - k_n^2\tau_1 - k_n k_g')((k_g k_g' - k_n k_n')(k_n\tau_2 + k_g\tau_g) + (k_g^2 - k_n^2)(k_n'\tau_2 + k_n\tau_2' + k_g'\tau_g + k_g\tau_g'))(s)$.

Using the Frenet-Serret type formulae, replacing $\theta'(s)$ in the previous expression of the derivative and performing some calculations, we have

$$\frac{dS_\gamma(s,\theta(s))}{ds} = \frac{-\cosh\theta(an_\gamma + bn_1 + cn_2)\rho}{\sqrt{k_g^2 - k_n^2}(k_n\tau_2 + k_g\tau_g)((k_g^2 - k_n^2)(k_n\tau_2 + k_g\tau_g)^2 - (k_g k_n' + k_g^2\tau_1 - k_n^2\tau_1 - k_n k_g')^2)}(s),$$

where $a(s) = k_g(k_n' k_g + k_g^2\tau_1 - k_n^2\tau_1 - k_g' k_n)(s)$, $b(s) = k_n(k_n' k_g + k_g^2\tau_1 - k_n^2\tau_1 - k_g' k_n)(s)$, $c(s) = (k_g^2 - k_n^2)(k_n\tau_2 + k_g\tau_g)(s)$ and $\rho(s)$ is the invariant.

Therefore, $\dfrac{dS_\gamma}{ds} \equiv 0$ if and only if $\rho(s) \equiv 0$. Therefore, statements (1) and (2) are equivalent. We now assume statement (1) holds and has

$$\langle\gamma(s), S_\gamma(s,\theta(s))\rangle = \frac{\cosh\theta}{\sqrt{k_g^2 - k_n^2}}\left(k_g\langle\gamma, n_\gamma\rangle + k_n\langle\gamma, n_1\rangle + \frac{(k_g k_n' + k_g^2\tau_1 - k_n^2\tau_1 - k_n k_g')}{k_n\tau_2 + k_g\tau_g}\langle\gamma, n_2\rangle\right)(s).$$

Let $g(s) = \langle \gamma(s), S_\gamma(s,\theta(s))\rangle$. Deriving, by Frenet-Serret type formulae and making long calculations, we show

$$g'(s) = g_1(s)\langle \gamma(s), n_\gamma(s)\rangle + g_2(s)\langle \gamma(s), n_1(s)\rangle + g_3(s)\langle \gamma(s), n_2(s)\rangle,$$

where $g_1(s) = \dfrac{A(s)\cosh\theta(s)}{D(s)}$, $g_2(s) = \dfrac{B(s)\cosh\theta(s)}{D(s)}$ and $g_3(s) = \dfrac{C(s)\cosh\theta(s)}{D_1(s)}$ with

$$\begin{aligned}A(s) =\Bigg(& k_g(k_gk_n' + k_g^2\tau_1 - k_n^2\tau_1 - k_nk_g')\Big[(k_gk_n'' + 2k_gk_g'\tau_1 + k_g^2\tau_1' - 2k_nk_n'\tau_1 - k_n^2\tau_1' - k_nk_g'')\\
&(k_g^2 - k_n^2)(k_n\tau_2 + k_g\tau_g) - (k_gk_n' + k_g^2\tau_1 - k_n^2\tau_1 - k_nk_g')\Big((k_gk_g' - k_nk_n')(k_n\tau_2 + k_g\tau_g)\\
&+ (k_g^2 - k_n^2)(k_n'\tau_2 + k_n\tau_2' + k_g'\tau_g + k_g\tau_g')\Big)\Big] - k_g(k_gk_g' - k_nk_n')(k_n\tau_2 + k_g\tau_g)^3(k_g^2 - k_n^2)\\
&+ k_g(k_gk_g' - k_nk_n')(k_n\tau_2 + k_g\tau_g)(k_gk_n' + k_g^2\tau_1 - k_n^2\tau_1 - k_nk_g')^2 + \Big((k_n\tau_2 + k_g\tau_g)(k_g'+\\
&k_n\tau_1) + \tau_2k_gk_n' + \tau_2k_g^2\tau_1 - \tau_2k_n^2\tau_1 - \tau_2k_nk_g'\Big)(k_g^2 - k_n^2)^2(k_n\tau_2 + k_g\tau_g)^2 - \Big((k_g' + k_n\tau_1)\\
&(k_n\tau_2 + k_g\tau_g) + \tau_2k_gk_n' + \tau_2k_g^2\tau_1 - \tau_2k_n^2\tau_1 - \tau_2k_nk_g'\Big)(k_g^2 - k_n^2)(k_gk_n' + k_g^2\tau_1 - k_n^2\tau_1-\\
&k_nk_g')^2\Bigg)(s),\end{aligned}$$

$$D(s) =\left((k_n\tau_2 + k_g\tau_g)\sqrt{(k_g^2 - k_n^2)^3}\Big((k_g^2 - k_n^2)(k_n\tau_2 + k_g\tau_g)^2 - (k_gk_n' + k_g^2\tau_1 - k_n^2\tau_1 - k_nk_g')^2\Big)\right)(s),$$

$$\begin{aligned}B(s) =\Bigg(& k_n(k_gk_n' + k_g^2\tau_1 - k_n^2\tau_1 - k_nk_g')\Big[(k_gk_n'' + 2k_gk_g'\tau_1 + k_g^2\tau_1' - 2k_nk_n'\tau_1 - k_n^2\tau_1' - k_nk_g'')\\
&(k_g^2 - k_n^2)(k_n\tau_2 + k_g\tau_g) - (k_gk_n' + k_g^2\tau_1 - k_n^2\tau_1 - k_nk_g')\Big((k_gk_g' - k_nk_n')(k_n\tau_2 + k_g\tau_g)\\
&+ (k_g^2 - k_n^2)(k_n'\tau_2 + k_n\tau_2' + k_g'\tau_g + k_g\tau_g')\Big)\Big] - k_n(k_gk_g' - k_nk_n')(k_n\tau_2 + k_g\tau_g)^3(k_g^2 - k_n^2)\\
&+ k_n(k_gk_g' - k_nk_n')(k_n\tau_2 + k_g\tau_g)(k_gk_n' + k_g^2\tau_1 - k_n^2\tau_1 - k_nk_g')^2 + (k_gk_n\tau_1\tau_2 + k_nk_n'\tau_2\\
&+ \tau_gk_n^2\tau_1 + \tau_gk_nk_g')(k_g^2 - k_n^2)^2(k_n\tau_2 + k_g\tau_g) - (k_gk_n\tau_1\tau_2 + k_nk_n'\tau_2 + \tau_gk_n^2\tau_1 + \tau_gk_nk_g')\\
&(k_g^2 - k_n^2)(k_gk_n' + k_g^2\tau_1 - k_n^2\tau_1 - k_nk_g')^2\Bigg)(s),\end{aligned}$$

$$\begin{aligned}C(s) =\Bigg(& - (k_n\tau_2 + k_g\tau_g)(k_gk_n' + k_g^2\tau_1 - k_n^2\tau_1 - k_nk_g')^2(k_g\tau_2 + k_n\tau_g) - (k_gk_n' + k_g^2\tau_1 - k_n^2\tau_1\\
&- k_nk_g')(k_n\tau_2 + k_g\tau_g)^2(k_gk_g' - k_nk_n') + (k_gk_n' + k_g^2\tau_1 - k_n^2\tau_1 - k_nk_g')(k_n'\tau_2 + k_n\tau_1\tau_g+\\
&k_g'\tau_g + k_g\tau_1\tau_2)(k_g^2 - k_n^2)(k_n\tau_2 + k_g\tau_g)\Bigg)(s),\end{aligned}$$

$$D_1(s) =\left(\sqrt{k_g^2 - k_n^2}(k_n\tau_2 + k_g\tau_g)\Big((k_n^2 - k_g^2)(k_n\tau_2 + k_g\tau_g)^2 - (k_gk_n' + k_g^2\tau_1 - k_n^2\tau_1 - k_nk_g')^2\Big)\right)(s).$$

Furthermore, reorganising the calculations in $A(s)$, $B(s)$ and $C(s)$, we show $A(s) = B(s) = C(s) = 0$ for all $s \in I$, therefore, $g_i(s) = 0$, $i = 1,2,3$ for all $s \in I$, (i.e., $g'(s) = 0$ for all $s \in I$), so that g is constant and the statement (3) follows. For the converse, we assume $\langle \gamma(s), v\rangle = c$ for a constant vector v and a real number c, therefore, $\langle \gamma'(s), v\rangle = 0$, that is, $(h_t^T)_v(s) = 0$ for all s, and $(h_t^T)_v(s) = (h_t^T)_v'(s) = (h_t^T)_v''(s) = (h_t^T)_v'''(s) = 0$ for all s. By

Proposition 3.1, $v = S_\gamma(s, \theta(s))$ and $\rho(s) = 0$ for all s, and (1) follows. □

In Proposition 3.6, the invariant $\rho \equiv 0$ means the curve γ is part of a spacelike slice surface. For the next result, we assume $\rho \not\equiv 0$, i.e., γ is not part of any spacelike slice surface $M \cap HP(v_0, c)$.

We now consider the hyperbolic curve CH_γ of γ, defined in Section 3. We have defined $C(2,3,4) = \{(t^2, t^3, t^4) \mid t \in \mathbb{R}\}$, which is called a $(2,3,4)$-*cusp*, and obtained the following result.

Proposition 3.7. *Let $\gamma : I \to M$ be a unit speed curve, such that $k_g(s) \neq 0$, $(k_n\tau_2 + k_g\tau_g)(s) \neq 0$ and $k_g^2(s) > k_n^2(s)$ for any $s \in I$. Let $v_0 = S_\gamma(s_0, \theta_0)$ and $c = \langle \gamma(s_0), v_0 \rangle$. Then we have*

(1) *γ and the spacelike slice surface $M \cap HP(v_0, c)$ have contact of at least order 3 at s_0 if and only if $(h_t^T)_{v_0}$ has A_k-singularity at s_0, $k \geq 2$. Furthermore, if γ and the spacelike slice surface $M \cap HP(v_0, c)$ have contact of order exactly 3 at s_0, then the hyperbolic curve CH_γ of γ is, at s_0, locally diffeomorphic to a line.*

(2) *γ and the spacelike slice surface $M \cap HP(v_0, c)$ have contact of order 4 at s_0 if and only if $(h_t^T)_{v_0}$ has A_3-singularity at s_0. In this case, if $\lambda_0(s_0) \neq 0$ then, the hyperbolic curve CH_γ of γ is, at s_0, locally diffeomorphic to the* $(2,3,4)$-cusp $C(2,3,4)$.

Proof. Let us consider $v_0 = S_\gamma(s_0, \theta_0)$ and $c = \langle \gamma(s_0), v_0 \rangle$ and $D_{v_0} : M \to \mathbb{R}$ a function defined by $D_{v_0}(x) = \langle x, v_0 \rangle - c$. Then, $D_{v_0}^{-1}(0) = M \cap HP(v_0, c)$, which is a spacelike slice surface. Furthermore, $D_{v_0}^{-1}(0)$ and γ have contact of at least order 3 at s_0 if and only if the function $g(s) = D_{v_0} \circ \gamma(s) = \langle \gamma(s_0), v_0 \rangle - c$ satisfies $g(s_0) = g'(s_0) = g''(s_0) = g'''(s_0) = 0$. Such conditions are equivalent to $g(s_0) = (h_t^T)_v(s) = (h_t^T)'_v(s) = (h_t^T)''_v(s) = 0$. By Proposition 3.1, they are equivalent to condition

$$v_0 = \frac{\cosh\theta_0}{\sqrt{k_g^2(s_0) - k_n^2(s_0)}}\Big(k_g(s_0)n_\gamma(s_0) + k_n(s_0)n_1(s_0)\Big) + \sinh\theta_0 n_2(s_0),$$

$\tanh\theta_0 = \dfrac{k_gk'_n + k_g^2\tau_1 - k_n^2\tau_1 - k_nk'_g}{(k_n\tau_2 + k_g\tau_g)\sqrt{k_g^2 - k_n^2}}(s_0)$. If γ and the spacelike slice surface $M \cap HP(v_0, c)$ have contact of order 3 at s_0, then

$$v_0 = \frac{\cosh\theta_0}{\sqrt{k_g^2(s_0) - k_n^2(s_0)}}\Big(k_g(s_0)n_\gamma(s_0) + k_n(s_0)n_1(s_0)\Big) + \sinh\theta_0 n_2(s_0),$$

$\tanh\theta_0 = \dfrac{k_gk'_n + k_g^2\tau_1 - k_n^2\tau_1 - k_nk'_g}{(k_n\tau_2 + k_g\tau_g)\sqrt{k_g^2 - k_n^2}}(s_0)$ and $\rho(s_0) \neq 0$. Furthermore, by Theorem 3.5, the germ of the image of the hyperbolic surface S_γ at (s_0, θ_0) is locally diffeomorphic to the cuspidal edge. Since the locus of the singularities of cuspidal edge is locally diffeomorphic to a line, assertion (1) holds.

The first part of (2) follows from assertions (4) and (5) of Proposition 3.1. For the second part, if γ and the spacelike slice surface $M \cap HP(v_0, c)$ have contact of order 4 at s_0, then

$$v_0 = \frac{\cosh\theta_0}{\sqrt{k_g^2(s_0) - k_n^2(s_0)}}\Big(k_g(s_0)n_\gamma(s_0) + k_n(s_0)n_1(s_0)\Big) + \sinh\theta_0 n_2(s_0),$$

$\tanh\theta_0 = \dfrac{k_g k_n' + k_g^2\tau_1 - k_n^2\tau_1 - k_n k_g'}{(k_n\tau_2 + k_g\tau_g)\sqrt{k_g^2 - k_n^2}}(s_0)$, $\rho(s_0) = 0$ and $\rho'(s_0) \neq 0$. Furthermore, we have the assumption $\lambda_0(s_0) \neq 0$. By Theorem 3.5, the germ of the image of the hyperbolic surface S_γ at (s_0, θ_0) is locally diffeomorphic to the swallowtail surface. Since the locus of singularities of the swallowtail surface is locally diffeomorphic to $C(2, 3, 4)$, assertion (2) holds. □

4. Examples

This section provides two examples of curves on spacelike hypersurface M in $\mathbb{R}^4_1$, namely $M = \mathbb{R}^3$ and $M = H^3(-1)$, which is the hyperbolic space.

Example 4.1. We consider $M = \mathbb{R}^3 = \{x = (x_0, x_1, x_2, x_3) \in \mathbb{R}^4_1 \mid x_0 = 0\}$. For $\gamma : I \to \mathbb{R}^3$, we have $n_\gamma = e_0$, $t(s) = \gamma'(s)$, $n_1(s) = n(s)$ and $n_2(s) = b(s)$. Here $\{t, n, b\}$ is the ordinary Frenet frame, and $k_n = \tau_1 = \tau_2 = 0$, $k_g = k$ and $\tau_g = \tau$. The Frenet-Serret type formulae are the original Frenet-Serret formulae (see [1]):

$$\begin{cases} e_0'(s) = 0, \\ t'(s) = k(s)\,n(s), \\ n'(s) = -k(s)\,t(s) + \tau(s)\,b(s), \\ b'(s) = -\tau(s)\,n(s). \end{cases}$$

The hyperbolic surface of γ in $H^3(-1) \subset \mathbb{R}^4_1$ is given by

$$S_\gamma(s, \theta) = \cosh\theta e_0 + \sinh\theta b(s)$$

and the hyperbolic curve of γ is given by $CH_\gamma(s) = e_0$, which is a constant point.

Example 4.2. Let us consider $M = H^3(-1)$. For $\gamma : I \to H^3(-1)$, we have $n_\gamma(s) = \gamma(s)$, $t(s) = \gamma'(s)$, $n_1(s)$ and $n_2(s)$. Here $\{\gamma, t, n_1, n_2\}$ is the pseudo orthonormal frame, and $k_n(s) = 1$, $\tau_1(s) = \tau_2(s) = 0$, $k_g(s) = k_h(s)$ and $\tau_g(s) = \tau_h(s)$.

$$\begin{cases} \gamma'(s) = t(s), \\ t'(s) = \gamma(s) + k_h(s)\,n_1(s), \\ n_1'(s) = -k_h(s)\,t(s) + \tau_h(s)\,n_2(s), \\ n_2'(s) = -\tau_h(s)\,n_1(s). \end{cases}$$

Therefore, for $k_h^2(s) > 1$, the hyperbolic surface of γ is given by

$$S_\gamma(s, \theta) = \frac{\cosh\theta}{\sqrt{k_h^2(s) - 1}}(k_h(s)\gamma(s) + n_1(s)) + \sinh\theta n_2(s).$$

Therefore, the hyperbolic surface is precisely the hyperbolic focal surface of γ given in [3].

5. Spacelike tangential height functions

This section introduces the family of spacelike tangential height functions on a curve in a spacelike hypersurface M and addresses the definition and a study of the de Sitter surface, given by the discriminant set of the family. The arguments and results are analogous to those of Section 3, therefore the detailed arguments are not presented.

We define a family of functions on a curve, $\gamma : I \to M \subset \mathbb{R}^4_1$ as follows:

$$H_t^S : I \times S_1^3 \to \mathbb{R}; \quad (s, v) \mapsto \langle t(s), v \rangle.$$

We call H_t^S the *family of spacelike tangential height functions of* γ, and denote $(h_t^S)_v(s) = H_t^S(s, v)$ for any fixed $v \in S_1^3$. The family H_t^S measures the contact of the curve t with timelike hyperplanes in $\mathbb{R}^4_1$, which, generically, can be of order k, $k = 1, 2, 3$.

The conditions that characterise A_k-singularities, $k = 1, 2, 3$, can be obtained in Proposition 5.1.

We assume $k_n^2(s) > k_g^2(s)$ for $s \in I$, and towards avoiding more complicated situations, $(k_n\tau_2 + k_g\tau_g)(s) \neq 0$ for any $s \in I$.

Proposition 5.1. *Let $\gamma : I \to M$ be a unit speed curve, such that $k_g(s) \neq 0$, $(k_n\tau_2 + k_g\tau_g)(s) \neq 0$ and $k_n^2(s) > k_g^2(s)$. Then,*

(1) $(h_t^S)_v(s) = 0$ *if and only if there exist μ, λ, $\eta \in \mathbb{R}$, such that $-\mu^2 + \lambda^2 + \eta^2 = 1$ and $v = \mu n_\gamma(s) + \lambda n_1(s) + \eta n_2(s)$.*

(2) $(h_t^S)_v(s) = (h_t^S)'_v(s) = 0$ *if and only if there exists $\theta \in \mathbb{R}$, such that*

$$v = \frac{\cos\theta}{\sqrt{k_n^2(s) - k_g^2(s)}} \Big(k_g(s) n_\gamma(s) + k_n(s) n_1(s) \Big) + \sin\theta n_2(s).$$

(3) $(h_t^S)_v(s) = (h_t^S)'_v(s) = (h_t^S)''_v(s) = 0$ *if and only if*

$$v = \frac{\cos\theta}{\sqrt{k_n^2(s) - k_g^2(s)}} \Big(k_g(s) n_\gamma(s) + k_n(s) n_1(s) \Big) + \sin\theta n_2(s),$$

$$\tan\theta = \frac{k_g k_n' + k_g^2\tau_1 - k_n^2\tau_1 - k_n k_g'}{(k_n\tau_2 + k_g\tau_g)\sqrt{k_n^2 - k_g^2}}(s).$$

(4) $(h_t^S)_v(s) = (h_t^S)'_v(s) = (h_t^S)''_v(s) = (h_t^S)'''_v(s) = 0$ *if and only if*

$$v = \frac{\cos\theta}{\sqrt{k_n^2(s) - k_g^2(s)}} \Big(k_g(s) n_\gamma(s) + k_n(s) n_1(s) \Big) + \sin\theta n_2(s),$$

$$\tan\theta = \frac{k_g k_n' + k_g^2\tau_1 - k_n^2\tau_1 - k_n k_g'}{(k_n\tau_2 + k_g\tau_g)\sqrt{k_n^2 - k_g^2}}(s) \text{ and } \rho(s) = 0, \text{ where}$$

$$\rho(s) = \Big((-k_g k_n'' - k_g k_n \tau_2^2 - 2k_g k_g' \tau_1 - k_g^2 \tau_1' - k_g^2 \tau_g \tau_2 + 2k_n k_n' \tau_1 + k_n^2 \tau_1' - k_n^2 k_g \tau_2 + k_g'' k_n - k_g k_n \tau_g^2)(k_n\tau_2 + k_g\tau_g) + (k_g k_n' + k_g^2\tau_1 - k_n^2\tau_1 - k_n k_g')(2k_n'\tau_2 + k_n\tau_1\tau_g + k_n\tau_2' + 2k_g'\tau_g + k_g\tau_1\tau_2 + k_g\tau_g') \Big)(s).$$

(5) $(h_t^S)_v(s) = (h_t^S)_v'(s) = (h_t^S)_v''(s) = (h_t^S)_v'''(s) = (h_t^S)_v^{(4)}(s) = 0$ *if and only if*

$$v = \frac{\cos\theta}{\sqrt{k_n^2(s) - k_g^2(s)}}\Big(k_g(s)n_\gamma(s) + k_n(s)n_1(s)\Big) + \sin\theta n_2(s),$$

$$\tan\theta = \frac{k_g k_n' + k_g^2\tau_1 - k_n^2\tau_1 - k_n k_g'}{(k_n\tau_2 + k_g\tau_g)\sqrt{k_n^2 - k_g^2}}(s) \text{ and } \rho(s) = \rho'(s) = 0.$$

Following Proposition 5.1, we define the invariant

$$\rho(s) = \Big((-k_g k_n'' - k_g k_n \tau_2^2 - 2k_g k_g'\tau_1 - k_g^2\tau_1' - k_g^2\tau_g\tau_2 + 2k_n k_n'\tau_1 + k_n^2\tau_1' - k_n^2 k_g\tau_2 + k_g'' k_n - k_g k_n \tau_g^2)(k_n\tau_2 + k_g\tau_g) + (k_g k_n' + k_g^2\tau_1 - k_n^2\tau_1 - k_n k_g')(2k_n'\tau_2 + k_n\tau_1\tau_g + k_n\tau_2' + 2k_g'\tau_g + k_g\tau_1\tau_2 + k_g\tau_g')\Big)(s)$$

of the curve γ, whose geometric meaning will be studied. Motivated by Proposition 5.1, we define the following surface and its singular locus. Let $\gamma : I \to M$ be a unit speed curve with $k_g(s) \neq 0$, $k_n^2(s) > k_g^2(s)$ and $(k_n\tau_2 + k_g\tau_g)(s) \neq 0$. A surface $DS_\gamma : I \times J \to S_1^3$ is defined by

$$DS_\gamma(s,\theta) = \frac{\cos\theta}{\sqrt{k_n^2(s) - k_g^2(s)}}\Big(k_g(s)n_\gamma(s) + k_n(s)n_1(s)\Big) + \sin\theta n_2(s),$$

where $J = [0, 2\pi]$. We call DS_γ a *de Sitter surface* of γ. We now define $DC_\gamma = DS_\gamma(s, \theta(s))$, where $\tan\theta(s) = \dfrac{k_g k_n' + k_g^2\tau_1 - k_n^2\tau_1 - k_n k_g'}{\sqrt{k_n^2 - k_g^2}(k_n\tau_2 + k_g\tau_g)}(s)$. We call DC_γ a *de Sitter curve* of γ. By Theorem 5.5 (1), this curve is the locus of the singular points of the de Sitter surface of γ

Corollary 5.2. *The de Sitter surface of γ is the discriminant set $\mathcal{D}_{H_t^S}$ of the family of spacelike tangential height functions H_t^S.*

Proof. The proof follows from the definition of the discriminant set given in Section 2 and Proposition 5.1 (2). □

Proposition 5.3. *Let $\gamma : I \to M$ be a unit speed curve with $k_g(s) \neq 0$ and $(k_n\tau_2 + k_g\tau_g)(s) \neq 0$.*

(a) *If $(h_t^S)_{v_0}$ has an A_2-singularity at s_0, then H_t^S is a versal deformation of $(h_t^S)_{v_0}$.*
(b) *If $(h_t^S)_{v_0}$ has an A_3-singularity at s_0 and $\lambda_0(s_0) \neq 0$ (which is a generic condition), then H_t^S is a versal deformation of $(h_t^S)_{v_0}$.*

Regarding the de Sitter surface, the result is analogous to that of Proposition 3.4, considering the deformation $\widetilde{H} : I \times S_1^3 \times \mathbb{R} \to \mathbb{R}$ by $\widetilde{H}(s, v, u) = H_t^S(s, v) + u(s - s_0)^2 = \langle t(s), v\rangle + u(s - s_0)^2$.

Proposition 5.4. *If $(h_t^S)_{v_0}$ has an A_3-singularity at s_0 and $\lambda_0(s_0) = 0$, then $\widetilde{H}$ is a versal deformation of $(h_t^S)_{v_0}$.*

Propositions 5.3 and 5.4 provided the following result.

Theorem 5.5. *Let $\gamma : I \to M$ be a unit speed curve, such that $k_g(s) \neq 0$, $k_n^2(s) > k_g^2(s)$ and $(k_n\tau_2 + k_g\tau_g)(s) \neq 0$, and DS_γ the de Sitter surface of γ. Therefore,*

(1) *DS_γ is singular at (s_0, θ_0) if and only if*

$$\tan\theta_0 = \frac{k_g k_n' + k_g^2\tau_1 - k_n^2\tau_1 - k_n k_g'}{\sqrt{k_n^2 - k_g^2}(k_n\tau_2 + k_g\tau_g)}(s_0),$$

i.e., the singular points of the de Sitter surface are given by $DS_\gamma(s) = DS_\gamma(s, \theta(s))$, where $\tan\theta(s)$ satisfies the above equation.

(2) *The germ of DS_γ at (s_0, θ_0) is locally diffeomorphic to the cuspidal edge if*

$$\tan\theta_0 = \frac{k_g k_n' + k_g^2\tau_1 - k_n^2\tau_1 - k_n k_g'}{(k_n\tau_2 + k_g\tau_g)\sqrt{k_n^2 - k_g^2}}(s_0) \ \ and\ \ \rho(s_0) \neq 0.$$

(3) *The germ of DS_γ at (s_0, θ_0) is locally diffeomorphic to the swallowtail if*

$$\tan\theta_0 = \frac{k_g k_n' + k_g^2\tau_1 - k_n^2\tau_1 - k_n k_g'}{(k_n\tau_2 + k_g\tau_g)\sqrt{k_n^2 - k_g^2}}(s_0),\ \ \lambda_0(s_0) \neq 0,\ \ \rho(s_0) = 0 \ and\ \rho'(s_0) \neq 0.$$

(4) *The germ of DS_γ at (s_0, θ_0) is diffeomorphic to cuspidal beaks if*

$$\lambda_0(s_0) = 0,\ \ \lambda_1(s_0) \neq 0,\ \ \rho(s_0) = 0\ \ and\ \ \rho'(s_0) \neq 0.$$

(5) *Cuspidal lips do not appear.*

In the next proposition, the curve γ of the de Sitter surface is related to the invariant ρ and a timelike slice surface. In this case, the singular locus of the de Sitter surface of γ is a point.

Proposition 5.6. *Let $\gamma : I \to M$ be a unit speed curve, such that $k_g(s) \neq 0$, $(k_n\tau_2 + k_g\tau_g)(s) \neq 0$ and $k_n^2(s) > k_g^2(s)$ for any $s \in I$, and $DS_\gamma(s, \theta(s))$ be the singular points of the de Sitter surface of γ. The following conditions are equivalent:*

(1) *$DS_\gamma(s, \theta(s))$ is a constant spacelike vector*;
(2) $\rho(s) \equiv 0$;
(3) *there exist a spacelike vector v and a real number c, such that $Im(\gamma) \subset M \cap HP(v, c)$.*

In the previous result, the invariant $\rho \equiv 0$ means the curve γ is part of a timelike slice surface. For the next results, we have assumed $\rho \not\equiv 0$, i.e., γ is not part of any timelike slice surface $M \cap HP(v, c)$.

Proposition 5.7. *Let $\gamma : I \to M$ be a unit speed curve, such that $k_g(s) \neq 0$, $(k_n\tau_2 + k_g\tau_g)(s) \neq 0$ and $k_n^2(s) > k_g^2(s)$ for any $s \in I$, and $v_0 = DS_\gamma(s_0, \theta_0)$ and $c = \langle\gamma(s_0), v_0\rangle$. Therefore, we have*

(1) *γ and the timelike slice surface $M \cap HP(v_0, c)$ have contact of at least order 3 at s_0 if and only if $(h_t^S)_{v_0}$ has A_k-singularity at s_0, $k \geq 2$. Furthermore, if γ and the timelike slice surface $M \cap HP(v_0, c)$ have contact of order exactly 3 at s_0, then the de Sitter curve DC_γ of γ is, at s_0, locally diffeomorphic to a line at s_0.*
(2) *γ and the timelike slice surface $M \cap HP(v_0, c)$ have contact of order 4 at s_0 if and only if $(h_t^S)_{v_0}$ has A_3-singularity at s_0. In this case, if $\lambda_0(s_0) \neq 0$, then the de Sitter curve DC_γ of γ is, at s_0, locally diffeomorphic to* $(2, 3, 4)$-cusp $C(2, 3, 4)$.

6. Examples

This section provides two examples of curves on spacelike hypersurface M in $\mathbb{R}^4_1$, namely $M = \mathbb{R}^3$ and $H^3(-1)$.

EXAMPLE 6.1. We consider $M = \mathbb{R}^3$, $\gamma : I \to \mathbb{R}^3$, the Frenet frame $\{t, n, b\}$ and the Frenet-Serret formulae, as in Example 4.1.

$$\begin{cases} e_0'(s) = 0, \\ t'(s) = k(s)\, n(s), \\ n'(s) = -k(s)\, t(s) + \tau(s)\, b(s), \\ b'(s) = -\tau(s)\, n(s). \end{cases}$$

In this case, the de Sitter surface of γ in $S^3_1 \subset \mathbb{R}^4_1$ cannot be defined.

EXAMPLE 6.2. We consider $M = H^3(-1)$, $\gamma : I \to H^3(-1)$ and the pseudo orthonormal frame $\{\gamma, t, n_1, n_2\}$, as in Example 4.2.

$$\begin{cases} \gamma'(s) = t(s), \\ t'(s) = \gamma(s) + k_h(s)\, n_1(s), \\ n_1'(s) = -k_h(s)\, t(s) + \tau_h(s)\, n_2(s), \\ n_2'(s) = -\tau_h(s)\, n_1(s). \end{cases}$$

Therefore, for $k_h^2(s) < 1$, the de Sitter surface of γ is given by

$$DS_\gamma(s, \theta) = \frac{\cos\theta}{\sqrt{1 - k_h^2(s)}}(k_h(s)\gamma(s) + n_1(s)) + \sin\theta n_2(s).$$

It follows de Sitter surface is precisely the de Sitter focal surface of γ given in [3].

References

[1] J.W. Bruce and P.J. Giblin: Curves and Singularities, a geometrical introduction to singularity theory, Cambridge University Press, Cambridge, 1992.

[2] S. Hananoi, N. Ito and S. Izumiya: *Spherical Darboux images of curves on surfaces*, Beiträge zur Algebra und Geometrie/Contributions to Algebra and Geometry, (2015), 1–11.

[3] R. Hayashi, S. Izumiya and T. Sato: *Focal surfaces and evolutes of curves in hyperbolic space*, Commun. Korean Math. Soc. **32** (2017), 147–163.

[4] S. Izumiya, A.C. Nabarro and A.J. Sacramento: *Horospherical and hyperbolic dual surfaces of spacelike curves in de Sitter space*, J. Singul. **16** (2017), 180–193.

[5] S. Izumiya, A.C. Nabarro and A.J. Sacramento: *Pseudo-spherical normal Darboux images of curves on a timelike surface in three dimensional Lorentz-Minkowski space*, J. Geom. Phys. **97** (2015), 105–118.

[6] S. Izumiya, K. Saji and M. Takahashi: *Horospherical flat surfaces in hyperbolic 3-space*, J. Math. Soc. Japan **62** (2010), 789–849.

[7] S. Izumiya and T. Sato: *Lightlike hypersurface along spacelike submanifolds in Minkowski space-time*, J. Geom. Phys. **71** (2013), 30–52.

[8] A.C. Nabarro and A.J. Sacramento: *Focal set of curves in the Minkowski space near lightlike points*, Publ. Math. Debrecen **88** (2016), 487–510.

[9] J.G. Ratcliffe: Foundations of hyperbolic manifolds, Graduate texts in Mathematics **149**, Springer, New York, 2006.

[10] T. Sato: *Pseudo-spherical evolutes of curves on a spacelike surface in three dimensional, Lorentz-Minkowski space*, J. Geom **103** (2012), 319–331.

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