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THE TORSION GENERATING SET OF THE EXTENDED MAPPING CLASS GROUPS IN LOW GENUS CASES

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Abstract

We prove that for genus $g = 3, 4$, the extended mapping class group $\mathrm{Mod}^{\pm}(S_g)$ can be generated by two elements of finite orders. But for $g = 1$, $\mathrm{Mod}^{\pm}(S_1)$ cannot be generated by two elements of finite orders.

1. Introduction

Let S_g be a connected oriented closed surface of genus g. We denote by $\mathrm{Mod}(S_g)$ the mapping class group of S_g, the group of isotopy classes of orientation-preserving diffeomorphisms on S_g. We also denote by $\mathrm{Mod}^{\pm}(S_g)$ the extended mapping class group of S_g, the group of isotopy classes of all orientation-preserving and orientation-reversing diffeomorphisms on S_g.

Korkmaz has proved that the mapping class group $\mathrm{Mod}(S_g)$ can be generated by two elements of finite orders in [4]. Using the notation that $\langle m, n\rangle$ (m, n are integers) to mean a group can be generated by two elements whose orders are m and n respectively, Korkmaz's result shows the orders of the generators are as in Table 1.

Table 1.

$\mathrm{Mod}(S_g)$	torsion generating set consisting of two elements
$g = 1$	$\langle 4, 6\rangle$
$g = 2$	$\langle 6, 10\rangle$
$g \geq 3$	$\langle 4g + 2, 4g + 2\rangle$

It is an open problem listed in [5] that whether the extended mapping class group $\mathrm{Mod}^{\pm}(S_g)$ can be generated by two torsion elements. In [1], the author partially solved such a problem when the genus $g \geq 5$. In this paper, we deal with $g = 1, 3, 4$.

When $g = 3, 4$, the method and idea in the process of calculation in this paper are mostly the same as those in [1] and [4]. The reason for $g = 3$ and $g = 4$ should be treated separately is as the follow. When the genus is high, there will be plenty of space to find a simple closed curve satisfying two conditions: (1) it lies in the periodic orbit; (2) it does not intersect with some given curves. When the genus is less than 5, we cannot do this. So we use other

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treatment carefully. When $g = 1$, we use the presentation of $GL(2, \mathbb{Z})$ to prove it cannot be generated by two elements of finite orders. So we can summarize the result as in Table 2.

Table 2.

$\mathrm{Mod}^{\pm}(S_g)$	torsion generating set consisting of two elements
$g = 1$	impossible
$g = 2$	still unknown
$g \geq 3$	$\langle 2, 4g + 2\rangle$

2. Preliminary

2.1. Notations. (a) We use the convention of functional notation, namely, elements of the mapping class group are applied right to left, i.e. the composition FG means that G is applied first.

(b) On an oriented surface, for each explicit two-sided simple closed curve, a *Dehn twist* means a right-handed Dehn twist along such a curve according to the orientation of the surface, and a left-handed Dehn twist is the inverse of a right-handed Dehn twist.

(c) We denote the curves by lower case letters a, b, c, d (possibly with subscripts) and the Dehn twists about them by the corresponding capital letters A, B, C, D. Notationally we do not distinguish a diffeomorphism/curve and its isotopy class.

2.2. Basic relations between Dehn twists. We recall the following results (see, for instance, section 3.3, 5.1, 7.5 of [2]):

Lemma 2.1. *For any $\varphi \in Mod(S_g)$ and any isotopy classes a, b of simple closed curves in S_g satisfying $\varphi(a) = b$, we have:*

$$B = \varphi A \varphi^{-1}.$$

Lemma 2.2. *For any $\varphi \in Mod^{\pm}(S_g) \setminus Mod(S_g)$ and any isotopy classes a, b of simple closed curves in S_g satisfying $\varphi(a) = b$, we have:*

$$B^{-1} = \varphi A \varphi^{-1}.$$

Lemma 2.3. *Let a, b be two simple closed curves on S_g. If a is disjoint from b, then*

$$AB = BA.$$

Lemma 2.4 (Lantern relation)**.** *Let a, b, c, d, x, y, z be the curves showed in* Figure 1 *on a genus zero surface with four boundaries. Then*

$$ABCD = XYZ.$$

In other words, since a, b, c are disjoint from x, y, z, we have

$$D = (XA^{-1})(YB^{-1})(ZC^{-1}).$$

2.3. Humphries generators and the $(4g+2)$-gon. Humphries have proved the following theorem ([3]).

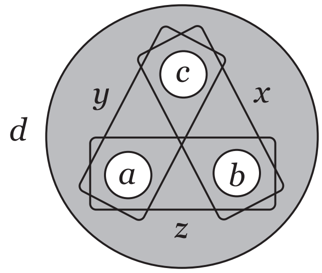

Fig. 1

Theorem 2.5. *Let $a_1, a_2, \ldots, a_{2g}, b_0$ be the curves as on the left-hand side of* Figure 2*. Then the mapping class group $Mod(S_g)$ is generated by A_i's and B_0.*

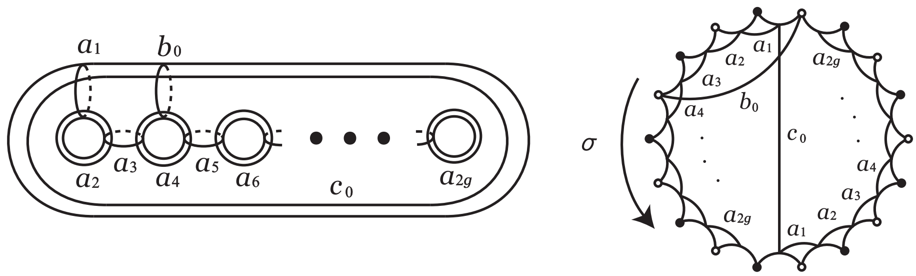

Fig. 2

The genus g surface can be looked as a $(4g + 2)$-gon, whose opposite edges are glued together in pairs. $(4g + 2)$ vertices of the $(4g + 2)$-gon are glued to be two vertices.

We can also draw the curves $a_1, a_2, \ldots, a_{2g}, b_0$ on the $(4g + 2)$-gon as the right-hand side of Figure 2. There is a natural rotation σ of the $(4g + 2)$-gon that sends a_i to a_{i+1}. In this paper, we will use the curve c_0 as Figure 2 shows. Denote $b_i = \sigma^i(b_0), c_i = \sigma^i(c_0)$. They are also used in this paper.

We need the intersection numbers between the curves a_j, b_k, c_l. Consider the indexes i, j, k in modulo $4g + 2$ classes. The intersection numbers between a_j, b_k, c_l are listed as follow:

(1) $i(a_j, a_k) = 0$ if and only if $|j - k| \neq 1$.
(2) $i(a_j, a_k) = 1$ if and only if $|j - k| = 1$.
(3) $i(b_j, b_k) = 0$ if and only if $|j - k| \notin \{1, 2, 3, 2g - 2, 2g\}$.
(4) $i(b_j, b_k) = 1$ if and only if $|j - k| \in \{1, 3, 2g - 2, 2g\}$.
(5) $i(b_j, b_k) = 2$ if and only if $|j - k| = 2$.
(6) $i(c_j, c_k) = 0$ if and only if $j = k$.
(7) $i(c_j, c_k) = 1$ if and only if $j \neq k$.
(8) $i(a_j, b_k) = 0$ if and only if $j - k \notin \{0, 4\}$.
(9) $i(a_j, b_k) = 1$ if and only if $j - k \in \{0, 4\}$.
(10) $i(a_j, c_k) = 0$ if and only if $k - j \notin \{-1, 0\}$.
(11) $i(a_j, c_k) = 1$ if and only if $k - j \in \{-1, 0\}$.
(12) $i(b_j, c_k) = 0$ if and only if $k - j \notin \{0, 1, 2, 3\}$.
(13) $i(b_j, c_k) = 1$ if and only if $k - j \in \{0, 1, 2, 3\}$.

Except (3), (4) and (5), the above intersection numbers can be verified directly on the $(4g+2)$-gon, as shown by the right picture of Figure 2. For (3), (4) and (5), we can verify them from the Figure 3 in [1].

Remark 2.6. In the calculation of (3), (4) and (5), when viewing these curves in the $(4g+2)$-gon, we need to be careful. Sometimes though two such curves meet at the vertex of the $(4g+2)$-gon, They do not really intersect. We can perturb them a little to cancel the intersection point.

2.4. Some torsion elements. Obviously we have $\sigma^{4g+2} = 1$. Take the reflection τ of the regular $(4g+2)$-gon satisfying $\tau(b_0) = b_0$. We can check $(\tau B_0)^2 = 1$. See Figure 3.

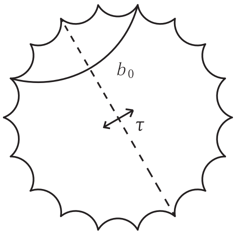

Fig. 3

In [1] we know $\mathrm{Mod}^{\pm}(S_g) = \langle \sigma, \tau B_0 \rangle$ for $g \geq 5$. We will see it is also true for $g = 3, 4$.

3. The main result and the proof

Theorem 3.1. *Let τ, σ, B_0 as before. For $g = 3, 4$, $Mod^{\pm}(S_g) = \langle \sigma, \tau B_0 \rangle$.*

Proof. Denote the subgroup generated by τB_0 and σ as G. We will prove that G includes all the elements in $\mathrm{Mod}^{\pm}(S_g)$. Similar to [1], The proof of the theorem has 4 steps.

Step 1. For every i, k, we prove $B_i B_k^{-1}$ is in G.

Step 2. For every i, k, we prove $B_i A_k^{-1}$ is in G.

Step 3. Using lantern relation, we prove that for every i, A_i is in G.

Step 4. $G = \mathrm{Mod}^{\pm}(S_g)$.

The motivation of step 2 and step 3 is as follow. There is a lantern on the surface where the curves in the lantern relation appear as a_1, a_3, a_5, b_0, b_2, e, f showed on the upper side of Figure 4. The lantern relation $B_0 B_2 E = A_1 A_3 A_5 F$ can be also written as $A_1 = (B_0 A_3^{-1})(B_2 A_5^{-1})(EF^{-1})$. So one Dehn twist can be decomposed into the product of pairs of Dehn twists. Draw the lantern in the $(4g+2)$-gon as on the lower side of Figure 4. We will find some of the pairs of Dehn twists we use can be expressed as the form $B_k A_i^{-1}$. When the $g \leq 2$, we cannot find a lantern on the surface.

The proof of Step 1:

We can check $\sigma^j(\tau B_0)\sigma^j(\tau B_0) = B_j^{-1} B_0$. Choosing j such that j is coprime to $4g+2$ and b_j do not intersect with b_0, we have B_j commutes with B_0, hence $B_0 B_j^{-1}$ is in G. For every i, by conjugating $B_0 B_j^{-1}$ with σ^i, we have $B_i B_{i+j}^{-1}$ is in G. Since j is coprime to $4g+2$, we have $B_i B_k^{-1}$ is in G for every i, k.

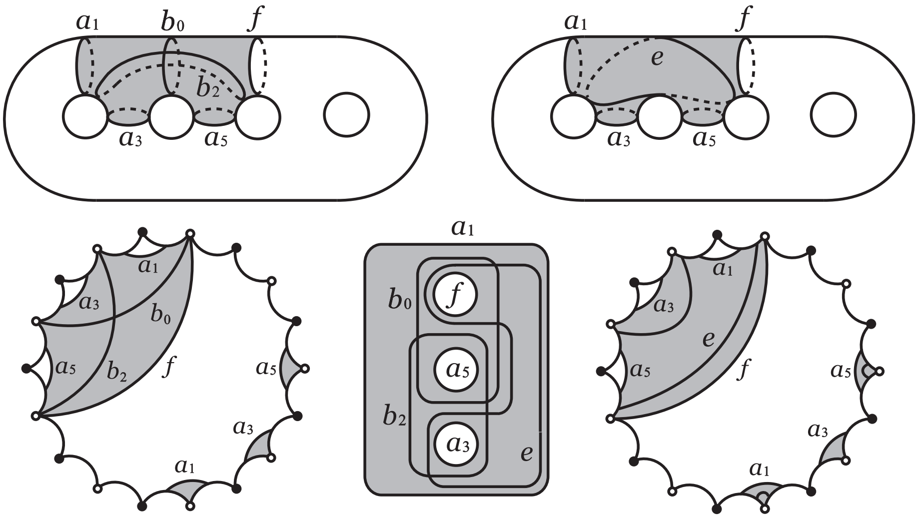

Fig.4

The proof of step 2:

Suppose the genus $g = 4$.

We already know b_{11} does not intersect with b_0 or b_6. So $B_{11}B_6^{-1}$ maps the pair of curves (b_{11}, b_0) to the pair of curves $(b_{11}, B_6^{-1}(b_0))$. Since $B_{11}B_0^{-1}$ is in G, $B_{11}(B_6^{-1}B_0^{-1}B_6)$ is in G. We also have for every k, $B_k(B_6^{-1}B_0^{-1}B_6) = (B_kB_{11}^{-1})\,(B_{11}(B_6^{-1}B_0^{-1}B_6))$ is in G. See Figure 5.

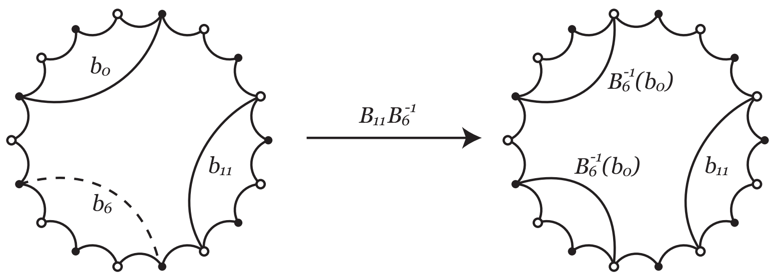

Fig.5

We know b_1 does not intersect with b_5. We can check $B_1B_5^{-1}B_6^{-1}(b_0) = a_5$. So B_5^{-1} maps the pair of curves $(b_5, B_6^{-1}(b_0))$ to the pair of curves $(b_5, B_5^{-1}B_6^{-1}(b_0))$, B_1 maps the pair of curves $(b_5, B_5^{-1}B_6^{-1}(b_0))$ to the pair of curves (b_5, a_5). This means $B_1B_5^{-1}$ maps the pair of curves $(b_5, B_6^{-1}(b_0))$ to the pair of curves (b_5, a_5). See Figure 6.

Hence $B_5A_5^{-1}$ is in G. After conjugating some power of σ and multiplying some $B_iB_j^{-1}$, we have for every i, j, $B_iA_j^{-1}$ is in G.

Suppose the genus $g = 3$.

We know that b_9 does not intersect with b_0 or b_4. So $B_9B_4^{-1}$ maps the pair of curves (b_9, b_0) to the pair of curves $(b_9, B_4^{-1}(b_0))$. We can also check when the genus is 3, $c_0 = B_4^{-1}(b_0)$. So $B_9C_0^{-1}$ is in G. See Figure 7.

After conjugating with some power of σ and multiplying some $B_iB_j^{-1}$, we have for every i, j, $B_iC_j^{-1}$ and $C_iB_j^{-1}$ are in G. We also have for every i, j, $C_iC_j^{-1}$ is in G.

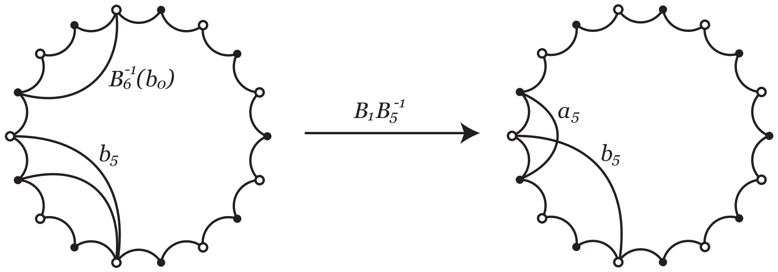

Fig. 6

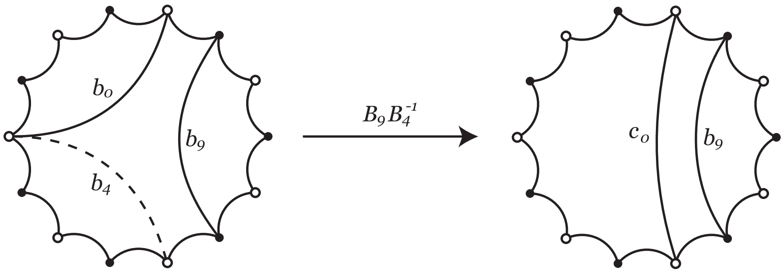

Fig. 7

We know c_0 does not intersect with b_1 or b_2. So $B_2C_0^{-1}$ maps the pair of curves (c_0, b_1) to the pair of curves $(c_0, B_2(b_1))$. Then $C_0(B_2B_1^{-1}B_2^{-1})$ is in G. For every i, $C_i(B_2B_1^{-1}B_2^{-1})$ is also in G. See Figure 8.

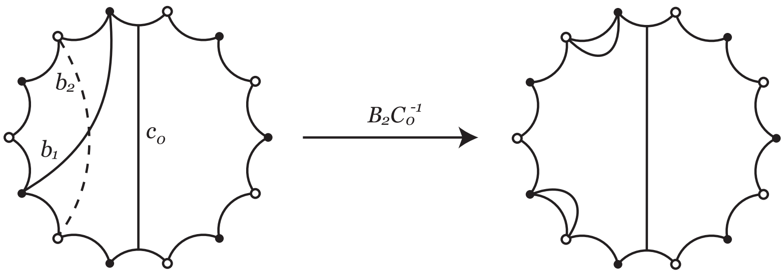

Fig. 8

We know c_4 does not intersect with b_6 or $B_2(b_1)$. So $C_4B_6^{-1}$ maps the pair of curves $(c_4, B_2(b_1))$ to the pair of curves $(c_4, B_6^{-1}B_2(b_1))$. Then $C_0(B_6^{-1}B_2B_1^{-1}B_2^{-1}B_6)$ is in G. See Figure 9.

We know c_4 does not intersect with b_5 or $B_6^{-1}B_2(b_1)$. So $C_4B_5^{-1}$ maps the pair of curves $(c_4, B_6^{-1}B_2(b_1))$ to the pair of curves $(c_4, B_5^{-1}B_6^{-1}B_2(b_1))$. Then C_4 $(B_5^{-1}B_6^{-1}B_2B_1^{-1}B_2^{-1}B_6B_5)$ is in G. See Figure 10.

We can check that $B_5^{-1}B_6^{-1}B_2(b_1) = a_2$. So $C_4A_2^{-1} = C_4$ $(B_5^{-1}B_6^{-1}B_2B_1^{-1}B_2^{-1}B_6B_5)$ is in G. Conjugating with some power of σ and multiplying $C_jC_k^{-1}$, we have for every j, k, $C_jA_k^{-1}$ is in G. Multiplying it by $B_iC_j^{-1}$, we have for every i, k, $B_iA_k^{-1}$ is in G.

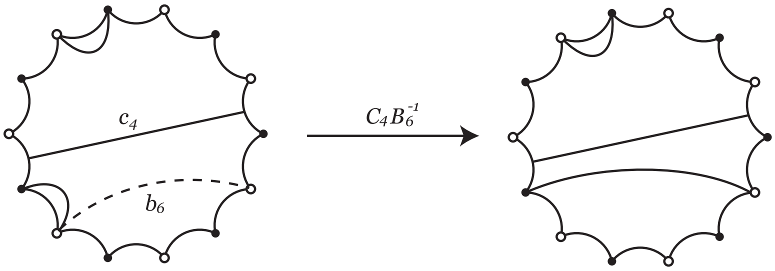

Fig. 9

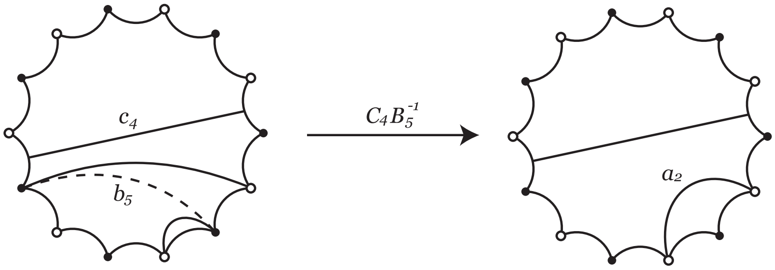

Fig. 10

The proof of step 3:

We want to show for every i, A_i is in G.

Recall lantern relation, we have $B_0B_2E = A_1A_3A_5F$, or $A_1 = (B_0A_3^{-1})\ (B_2A_5^{-1})\ (EF^{-1})$, where e and f are the curves showed in Figure 4. By the result of step 2, $B_0A_3^{-1}$ and $B_2A_5^{-1}$ are in G. What we need is to prove EF^{-1} is also in G. Notice $EF^{-1} = (EB_i^{-1})(B_iB_j^{-1})(B_jF^{-1})$. We only need to show there exist some i, j such that EB_i^{-1} and B_jF^{-1} are in G.

Suppose $g = 4$.

We can check that $f = B_3^{-1}A_6A_5A_4(b_0)$. We also know b_7 does not intersect with a_4, a_5, a_6, b_3. So $(B_7B_3^{-1})(A_6B_7^{-1})(A_5B_7^{-1})(A_4B_7^{-1})$ maps (b_7, b_0) to (b_7, f). Hence B_7F^{-1} is in G. See Figure 11.

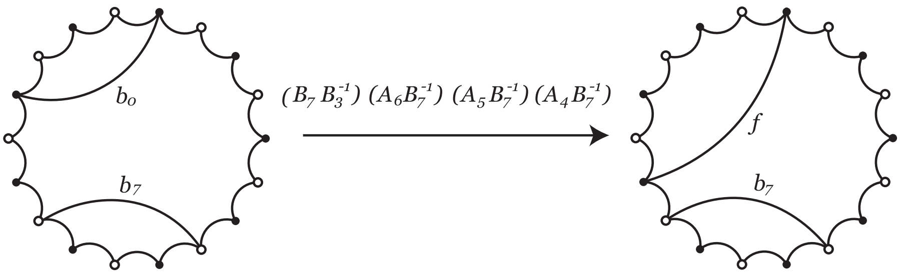

Fig. 11

We can check $e = A_2A_1A_4^{-1}B_1(a_5)$. Since b_{12} does not intersect with a_1, a_2, a_4, a_5, b_1, $(A_2B_{12}^{-1})\ (A_1B_{12}^{-1})\ (B_{12}A_4^{-1})\ (B_1B_{12}^{-1})$ maps (a_5, b_{12}) to (e, b_{12}). Hence EB_{12}^{-1} is in G. See Figure 12.

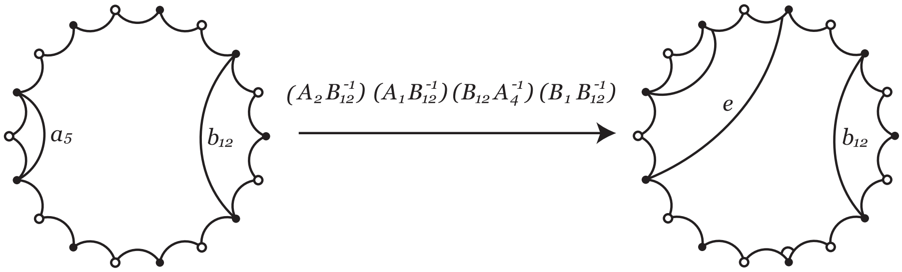

Fig. 12

Suppose $g = 3$.

The fact $f = B_3^{-1}A_6A_5A_4(b_0)$ still holds. When $g = 3$ we cannot find some b_i that does not intersect with a_4, a_5, a_6, b_3 simultaneously. We use some curves c_i instead.

At first we find c_6 does not intersect with a_4, a_5, b_0. So $(A_5C_6^{-1})\,(A_4C_6^{-1})$ maps (c_6, b_0) to $(c_6, A_5A_4(b_0))$, $C_6(A_5A_4B_0A_4^{-1}A_5^{-1})^{-1}$ is in G. See Figure 13.

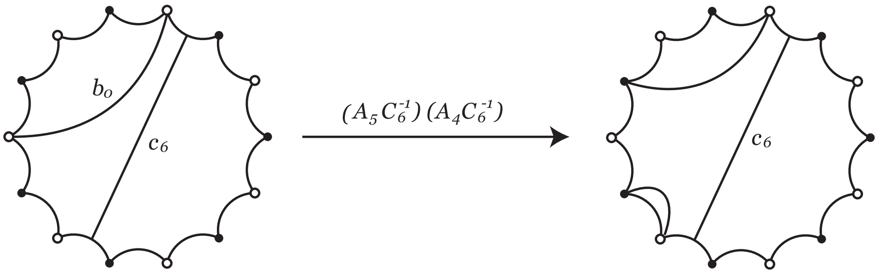

Fig. 13

$B_8(A_5A_4B_0A_4^{-1}A_5^{-1})^{-1} = (B_8C_6^{-1})(C_6(A_5A_4B_0A_4^{-1}A_5^{-1})^{-1})$ is also in G. Then we find b_8 does not intersect with a_6, b_3 or $A_5A_4(b_0)$. So $(B_8B_3^{-1})(B_8^{-1}A_6)$ maps $(b_8, A_5A_4(b_0))$ to $(b_8, B_3^{-1}A_6A_5A_4(b_0)) = (b_8, f)$. Hence B_8F^{-1} is in G. See Figure 14.

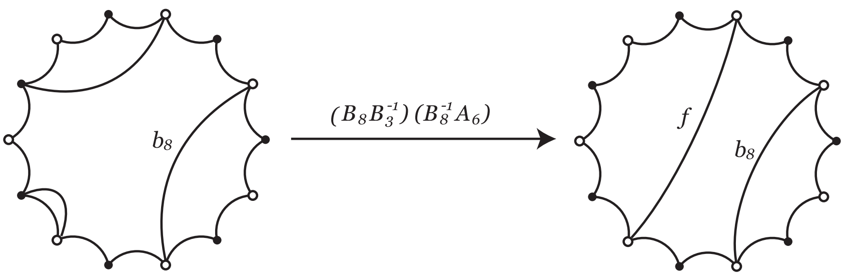

Fig. 14

Similarly, The fact $e = A_2A_1A_4^{-1}B_1(a_5)$ still holds. When $g = 3$, we can find c_i does not intersect with a_1, a_2, a_4, a_5, b_1. So $(A_2C_6^{-1})\,(A_1C_6^{-1})\,(C_6A_4^{-1})\,(B_1C_6^{-1})$ maps (a_5, c_6) to (e, c_6). Hence EC_6^{-1} is in G. And then multiply $C_6B_i^{-1}$, we have EB_i^{-1} in G. See Figure 15.

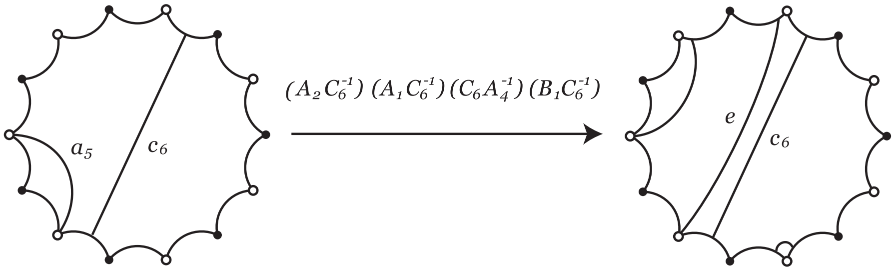

Fig. 15

The proof of step 4:

Since both $B_iA_j^{-1}$ and A_j are in G, by Humphries's result, G contains the mapping class group $\mathrm{Mod}(S_g)$. Now $\tau B_0 \in G$ is an orientation reversing element. $\mathrm{Mod}(S_g)$ is an index 2 subgroup of $\mathrm{Mod}^{\pm}(S_g)$. So $G = \mathrm{Mod}^{\pm}(S_g)$ for $g = 3, 4$. □

Theorem 3.2. *For $g = 1$, Mod$^{\pm}(S_1)$ is $GL(2,\mathbb{Z})$. It cannot be generated by two elements of finite orders.*

Proof. We only need to prove that $PGL(2,\mathbb{Z})$ cannot be generated by two elements of finite orders. The idea of the proof is:

(1) use a presentation of $PGL(2,\mathbb{Z})$ whose generators are elements of finite orders;
(2) list the possible conjugacy classes of all finite order elements in $PGL(2,\mathbb{Z})$;
(3) give a homomorphism from $PGL(2,\mathbb{Z})$ to a finite group $D_6 \times \mathbb{Z}_2$;
(4) check all the possible generating set of $D_6 \times \mathbb{Z}_2$ consisting of two generators;
(5) find the conjugacy classes of the pre-image of the generators of $PGL(2,\mathbb{Z})$ in (4), prove either they cannot be the conjugacy classes of finite orders elements or they cannot generate $PGL(2,\mathbb{Z})$.

It is well known that $PGL(2,\mathbb{Z}) \cong D_6 *_{\mathbb{Z}_2} D_4$, where D_6 and D_4 are the dihedral group of order 6 and order 4 respectively (see, for example, [6] or [7]). It has a presentation as

$$PGL(2,\mathbb{Z}) = \langle a, b, t \mid a^3 = t^2 = b^2 = 1, at = ta^2, bt = tb\rangle.$$

Every element α in $PGL(2,\mathbb{Z})$ can be written as a reduced form in one of the following 8 types:

$$\begin{array}{ll}
(1)\ a^{i_1}b^{j_1}\dots a^{i_k}b^{j_k}, & (2)\ b^{j_1}a^{i_1}\dots b^{j_k}a^{i_k},\\
(3)\ a^{i_0}b^{j_1}a^{i_1}\dots b^{j_k}a^{i_k}, & (4)\ b^{j_0}a^{i_1}b^{j_1}\dots a^{i_k}b^{j_k},\\
(5)\ a^{i_1}b^{j_1}\dots a^{i_k}b^{j_k}t, & (6)\ b^{j_1}a^{i_1}\dots b^{j_k}a^{i_k}t,\\
(7)\ a^{i_0}b^{j_1}a^{i_1}\dots b^{j_k}a^{i_k}t, & (8)\ b^{j_0}a^{i_1}b^{j_1}\dots a^{i_k}b^{j_k}t.
\end{array}$$

Here each $i_n \in \{1, 2\}$ and each $j_n = 1$.

- For an element in type (1), (2), (5) or (6), since its power will have a larger word length, it must not be of finite order.
- For an element in type (3) or (4), it can be conjugated to an element in (1) or (2), or it can be conjugated to an element in (4) or (3) with a shorter word length.
- For type (7), $a^{i_0}b^{j_1}a^{i_1}\dots b^{j_k}a^{i_k}t$ is conjugated to $b^{j_1}a^{i_1}\dots b^{j_k}a^{i_k+2i_0}t$, which is in type (8) with a shorter word length or in type (6).

- For type (8), $b^{j_0}a^{i_1}b^{j_1}\dots b^{j_{k-1}}a^{i_k}b^{j_k}t$ is conjugated to $a^{i_1}b^{j_1}\dots b^{j_{k-1}}a^{i_k}t$, which is in type (7) with a shorter word length.

For an element of finite order in $PGL(2,\mathbb{Z})$, we can conjugate it to an element with shortest word length. The only seven possible choices are as follow: $a, a^2, t, at, a^2t, b, bt$. About the order of the elements in their conjugacy classes, the elements conjugating to a or a^2 have order 3, and the elements conjugating to t, at, a^2t, b or bt have order 2.

By adding a new relation $ab = ba$, we get a quotient group homomorphic to the Cartesian product $D_6 \times \mathbb{Z}_2$, which is a finite group with the presentation

$$\langle a_1, t_1, b_1 \mid a_1^3 = t_1^2 = b_1^2 = 1, a_1t_1 = t_1a_1^2, b_1t_1 = t_1b_1, a_1b_1 = b_1a_1\rangle.$$

For the convenience of calculation, we can regard $D_6 \times \mathbb{Z}_2$ as a permutation subgroup of the symmetric group Σ_5. Table 3 lists the images in Σ_5 of all elements in $D_6 \times \mathbb{Z}_2$.

Table 3.

element in $D_6 \times \mathbb{Z}_2$	permutation	element in $D_6 \times \mathbb{Z}_2$	permutation
1	()	b_1	(45)
a_1	(123)	a_1b_1	(123)(45)
a_1^2	(132)	$a_1^2b_1$	(132)(45)
t_1	(12)	b_1t_1	(12)(45)
a_1t_1	(13)	$a_1b_1t_1$	(13)(45)
$a_1^2t_1$	(23)	$a_1^2b_1t_1$	(23)(45)

As the permutations in Σ_5, both a_1 and a_1^2 have the same 3-cycle orbit type, they are in the same conjugacy class in $D_6 \times \mathbb{Z}_2$. All of $t_1, a_1t_1, a_1^2t_1$ and b_1 have the same 2-cycle orbit type. However, b_1 generates the product factor $\mathbb{Z}_2$ of $D_6 \times \mathbb{Z}_2$. Hence we can check that $t_1, a_1t_1, a_1^2t_1$ belong to the same conjugacy class, and b_1 belongs to another conjugacy class. All of $b_1t_1, a_1b_1t_1$ and $a_1^2b_1t_1$ have the same orbit type as a product of two disjoint 2-cycles. For the conjugacy classes of the possible finite order elements in $PGL(2,\mathbb{Z})$, we can check the their possible images in $D_6 \times \mathbb{Z}_2$ are as Table 4 shows.

Table 4.

Conjugacy classes in $PGL(2,\mathbb{Z})$	elements in $D_6 \times \mathbb{Z}_2$
a, a^2	a_1, a_1^2
t, at, a^2t	t_1, a_1t_1, $a_1^2t_1$
b	b_1
bt	b_1t_1, a_1b_1t, $a_1^2b_1t$

Each finite order element in $PGL(2,\mathbb{Z})$ will be mapped to an element in $D_6 \times \mathbb{Z}_2$ which conjugates with an element in $\{a_1, a_1^2, t_1, a_1t_1, a_1^2t_1, b_1, b_1t_1, a_1b_1t_1, a_1^2b_1t_1\}$.

By direct computation in the permutation subgroup of the symmetric group, we know that if two elements x, y generate $D_6 \times \mathbb{Z}_2$, there are only two possible cases:

(1) $\{x, y\}$ contain a_1b_1 or $a_1^2b_1$;

(2) both x and y are elements of order 2.

Either a_1b_1 or $a_1^2b_1$ cannot conjugate to an element in $\{a_1, a_1^2, t_1, a_1t_1, a_1^2t_1, b_1, b_1t_1, a_1b_1t_1, a_1^2b_1t_1\}$, hence their pre-image in $PGL(2,\mathbb{Z})$ cannot be elements of finite order.

The pre-images of the order 2 elements in $D_6 \times \mathbb{Z}_2$ can be conjugated to the elements in $\{t, at, a^2t, b, bt\}$ in $PGL(2,\mathbb{Z})$. They are still of order 2. But two elements of order 2 can only generate a dihedral group, not $PGL(2,\mathbb{Z})$. □

REMARK 3.3. Though $\mathrm{GL}(2,\mathbb{Z})$ cannot be generated by two torsion elements, it can be generated by two elements. In fact, since

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix},$$

we have

$$\mathrm{GL}(2,\mathbb{Z}) = \langle\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\rangle.$$

So the extended mapping class group $GL(2,\mathbb{Z})$ for $g = 1$ case: (1) can be generated by two elements; (2) cannot be generated by two elements of finite orders. This is different from the case $g \geq 3$.

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