A CHARACTERIZATION OF PROJECTIVE SPACES FROM THE MORI THEORETIC VIEWPOINT

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Abstract

We give a characterization of projective spaces for quasi-log canonical pairs from the Mori theoretic viewpoint.

Contents

1. Introduction

In this paper, we give a characterization of projective spaces for quasi-log canonical pairs from the Mori theoretic viewpoint. We are mainly interested in singular varieties which naturally appear in the minimal model theory of higher-dimensional complex projective varieties. Although we could not find it explicitly in the literature, the following theorem is more or less well known to the experts.

Theorem 1.1. *Let (X, Δ) be a projective kawamata log terminal pair such that $-(K_X + \Delta)$ is ample. Assume that $-(K_X + \Delta) \equiv rH$ for some Cartier divisor H on X with $r > n = \dim X$. Then X is isomorphic to $\mathbb{P}^n$ with $\mathcal{O}_X(H) \simeq \mathcal{O}_{\mathbb{P}^n}(1)$.*

In his lectures on Fano manifolds in Osaka, Kento Fujita explained the above theorem and asked if it could be generalized. The following theorem is an answer to Fujita's question.

Theorem 1.2. *Let $[X, \omega]$ be a projective quasi-log canonical pair such that X is connected. Assume that ω is not nef and that $\omega \equiv rD$ for some Cartier divisor D on X with $r > n = \dim X$. Then X is isomorphic to $\mathbb{P}^n$ with $\mathcal{O}_X(D) \simeq \mathcal{O}_{\mathbb{P}^n}(-1)$. Moreover, there are no qlc centers of $[X, \omega]$.*

By combining Theorem 1.2 with [6, Theorem 1.1], we obtain the following corollary.

Corollary 1.3. *Let* (X, Δ) *be a projective semi-log canonical pair such that X is connected. Assume that $K_X + \Delta$ is not nef and that $K_X + \Delta \equiv rD$ for some Cartier divisor D on X with $r > n = \dim X$. Then X is isomorphic to $\mathbb{P}^n$ with $\mathcal{O}_X(D) \simeq \mathcal{O}_{\mathbb{P}^n}(-1)$ and (X, Δ) is kawamata log terminal.*

Just after we put this paper on arXiv, Stéphane Druel and Yoshinori Gongyo pointed out that Theorem 1.1 was already generalized as follows:

Theorem 1.4 ([2, Theorem 1.1]). *Let X be a normal projective variety and let Δ be an effective $\mathbb{R}$-divisor on X such that $K_X + \Delta$ is $\mathbb{R}$-Cartier. Assume that $-(K_X + \Delta) \equiv rH$ for some ample Cartier divisor H on X with $r > n = \dim X$. Then $n < r \leq n + 1$, $(X, \mathcal{O}_X(H)) \simeq (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$, and $\deg \Delta = n + 1 - r$. In particular, (X, Δ) has only kawamata log terminal singularities.*

We note that there are no assumptions on singularities of (X, Δ) in Theorem 1.4. On the other hand, in Theorem 1.2 and Corollary 1.3, we relax the assumption that $-(K_X + \Delta)$ is ample in Theorem 1.1, although we still require some assumptions on singularities of pairs.

We summarize the contents of this paper. In Section 2, we give a sketch of proof of Theorem 1.2 for log canonical pairs in order to make our main result more accessible. In Section 3, we collect some basic definitions of the minimal model theory of higher-dimensional algebraic varieties and the theory of quasi-log schemes. In Section 4, we prepare three important lemmas on quasi-log schemes for the proof of Theorem 1.2. Section 5 is devoted to the proof of Theorem 1.2 and Corollary 1.3.

We will work over $\mathbb{C}$, the complex number field, throughout this paper. In this paper, a *scheme* means a separated scheme of finite type over $\mathbb{C}$. We will use the theory of quasi-log schemes discussed in [7, Chapter 6].

2. Sketch of Proof

In order to make Theorem 1.2 more accessible, we give a sketch of proof of the following very special case of Theorem 1.2 and Corollary 1.3. We note that $[X, K_X + \Delta]$ naturally becomes a quasi-log canonical pair when (X, Δ) is a log canonical pair. In this section, we will freely use some standard results of the minimal model theory for log canonical pairs (see [5]).

Theorem 2.1 (Theorem 1.2 for log canonical pairs). *Let (X, Δ) be a projective log canonical pair with $\dim X = n$. Assume that $K_X + \Delta$ is not nef and that $-(K_X + \Delta) \equiv rH$ for some Cartier divisor H on X with $r > n$. Then $X \simeq \mathbb{P}^n$ with $\mathcal{O}_X(H) \simeq \mathcal{O}_{\mathbb{P}^n}(1)$.*

Sketch of Proof of Theorem 2.1. Since $K_X + \Delta$ is not nef, we have a $(K_X + \Delta)$-negative extremal contraction $\varphi \colon X \to W$ by the cone and contraction theorem for log canonical pairs (see [5, Theorem 1.1]).

Case 1 ($\dim W \geq 1$). We can take an effective $\mathbb{R}$-Cartier divisor B on W with the following properties:

    (i) $(X, \Delta + \varphi^* B)$ is log canonical outside finitely many points, and

    (ii) there exists a log canonical center C of $(X, \Delta + \varphi^* B)$ such that $\varphi(C)$ is a point with $\dim C \geq 1$.

In this situation, we obtain that

$$-(K_X + \Delta + \varphi^* B)|_C \equiv rH|_C$$

and $H|_C$ is ample since $\varphi(C)$ is a point. Therefore, by the vanishing theorem for quasi-log schemes (see Lemma 4.2 below), we obtain

$$\chi(C, \mathcal{O}_C(tH)) \equiv 0.$$

This is a contradiction since $H|_C$ is ample. This means that $\dim W \geq 1$ does not happen.

CASE 2 ($\dim W = 0$). Since $\varphi \colon X \to W$ is a $(K_X + \Delta)$-negative extremal contraction, we see that H is ample. We can explicitly determine

$$\chi(X, \mathcal{O}_X(tH))$$

by $-(K_X + \Delta) \equiv rH$ with $r > n$ and the vanishing theorem for log canonical pairs (see [5, Theorem 8.1]). Then we get $H^n = 1$ and

$$\dim_{\mathbb{C}} H^0(X, \mathcal{O}_X(H)) = n + 1.$$

Therefore,

$$\Delta(X, H) = n + H^n - \dim_{\mathbb{C}} H^0(X, \mathcal{O}_X(H)) = 0$$

holds, where $\Delta(X, H)$ is Fujita's Δ-genus of (X, H). This implies $X \simeq \mathbb{P}^n$ with $\mathcal{O}_X(H) \simeq \mathcal{O}_{\mathbb{P}^n}(1)$ (see [9, Theorem 2.1] or [13, Theorem 1.1]).

This is a sketch of the proof of Theorem 2.1. $\qquad\qquad\square$

When (X, Δ) is toric, Theorem 2.1 was already established by the first author in [4, Theorem 1.2], which is an easy direct consequence of [3, Theorem 0.1]. In [3] and [4], the sharp estimate of lengths of extremal rational curves plays a crucial role. On the other hand, we will use some vanishing theorems for quasi-log schemes in this paper.

3. Preliminaries

In this section, we collect some basic definitions of the minimal model program and the theory of quasi-log schemes. For the details, see [5] and [7].

Let us recall singularities of pairs.

DEFINITION 3.1 (SINGULARITIES OF PAIRS). A *normal pair* (X, Δ) consists of a normal variety X and an $\mathbb{R}$-divisor Δ on X such that $K_X + \Delta$ is $\mathbb{R}$-Cartier. Let $f \colon Y \to X$ be a projective birational morphism from a normal variety Y. Then we can write

$$K_Y = f^*(K_X + \Delta) + \sum_E a(E, X, \Delta)E$$

with

$$f_* \left(\sum_E a(E, X, \Delta)E \right) = -\Delta,$$

where E runs over prime divisors on Y. We call $a(E, X, \Delta)$ the *discrepancy* of E with respect to (X, Δ). Note that we can define the discrepancy $a(E, X, \Delta)$ for any prime divisor E over X by taking a suitable resolution of singularities of X. If $a(E, X, \Delta) \geq -1$ (resp. > -1) for every prime divisor E over X, then (X, Δ) is called *sub log canonical* (resp. *sub kawamata log terminal*). We further assume that Δ is effective. Then (X, Δ) is called *log canonical* and *kawamata log terminal* if it is sub log canonical and sub kawamata log terminal, respectively.

Let (X, Δ) be a normal pair. If there exist a projective birational morphism $f \colon Y \to X$ from a normal variety Y and a prime divisor E on Y such that (X, Δ) is sub log canonical in a neighborhood of the generic point of $f(E)$ and that $a(E, X, \Delta) = -1$, then $f(E)$ is called a *log canonical center* of (X, Δ).

DEFINITION 3.2 (OPERATIONS FOR $\mathbb{R}$-DIVISORS). Let V be an equidimensional reduced scheme. An $\mathbb{R}$-*divisor* D on V is a finite formal sum

$$\sum_{i=1}^{l} d_i D_i$$

where D_i is an irreducible reduced closed subscheme of V of pure codimension one with $D_i \neq D_j$ for $i \neq j$ and d_i is a real number for every i. We put

$$D^{<1} = \sum_{d_i < 1} d_i D_i, \quad D^{=1} = \sum_{d_i = 1} D_i, \quad \text{and} \quad D^{>1} = \sum_{d_i > 1} d_i D_i.$$

For every real number x, $\lceil x \rceil$ is the integer defined by $x \leq \lceil x \rceil < x + 1$. Then we put

$$\lceil D \rceil = \sum_{i=1}^{l} \lceil d_i \rceil D_i \quad \text{and} \quad \lfloor D \rfloor = -\lceil -D \rceil.$$

DEFINITION 3.3 ($\sim_{\mathbb{R}}$ AND $\equiv$). Let B_1 and B_2 be $\mathbb{R}$-Cartier divisors on a scheme X. Then $B_1 \sim_{\mathbb{R}} B_2$ means that B_1 is $\mathbb{R}$-*linearly equivalent* to B_2, that is, $B_1 - B_2$ is a finite $\mathbb{R}$-linear combination of principal Cartier divisors. When X is complete, $B_1 \equiv B_2$ means that B_1 is *numerically equivalent* to B_2.

In order to define quasi-log schemes, we need the notion of globally embedded simple normal crossing pairs.

DEFINITION 3.4 (GLOBALLY EMBEDDED SIMPLE NORMAL CROSSING PAIRS, SEE [7, Definition 6.2.1]). Let Y be a simple normal crossing divisor on a smooth variety M and let D be an $\mathbb{R}$-divisor on M such that $\mathrm{Supp}(D + Y)$ is a simple normal crossing divisor on M and that D and Y have no common irreducible components. We put $B_Y = D|_Y$ and consider the pair (Y, B_Y). We call (Y, B_Y) a *globally embedded simple normal crossing pair* and M the *ambient space* of (Y, B_Y). A *stratum* of (Y, B_Y) is a log canonical center of $(M, Y + D)$ that is contained in Y.

Let us recall the definition of quasi-log schemes.

DEFINITION 3.5 (QUASI-LOG SCHEMES, SEE [7, Definition 6.2.2]). A *quasi-log scheme* is a scheme X endowed with an $\mathbb{R}$-Cartier divisor (or $\mathbb{R}$-line bundle) ω on X, a proper closed subscheme $X_{-\infty} \subset X$, and a finite collection $\{C\}$ of reduced and irreducible subschemes of

X such that there is a proper morphism $f\colon (Y, B_Y) \to X$ from a globally embedded simple normal crossing pair satisfying the following properties:

(1) $f^*\omega \sim_{\mathbb{R}} K_Y + B_Y$.

(2) The natural map $\mathcal{O}_X \to f_*\mathcal{O}_Y(\lceil -(B_Y^{<1})\rceil)$ induces an isomorphism

$$\mathcal{I}_{X_{-\infty}} \xrightarrow{\simeq} f_*\mathcal{O}_Y(\lceil -(B_Y^{<1})\rceil - \lfloor B_Y^{>1}\rfloor),$$

where $\mathcal{I}_{X_{-\infty}}$ is the defining ideal sheaf of $X_{-\infty}$.

(3) The collection of reduced and irreducible subschemes $\{C\}$ coincides with the images of (Y, B_Y)-strata that are not included in $X_{-\infty}$.

We simply write $[X, \omega]$ to denote the above data

$$(X, \omega, f\colon (Y, B_Y) \to X)$$

if there is no risk of confusion. Note that a quasi-log scheme $[X, \omega]$ is the union of $\{C\}$ and $X_{-\infty}$. The reduced and irreducible subschemes C are called the *qlc strata* of $[X, \omega]$, $X_{-\infty}$ is called the *non-qlc locus* of $[X, \omega]$, and $f\colon (Y, B_Y) \to X$ is called a *quasi-log resolution* of $[X, \omega]$. We sometimes use $\mathrm{Nqlc}(X, \omega)$ to denote $X_{-\infty}$. If a qlc stratum C of $[X, \omega]$ is not an irreducible component of X, then it is called a *qlc center* of $[X, \omega]$.

Definition 3.6 (Quasi-log canonical pairs, see [7, Definition 6.2.9]). Let

$$(X, \omega, f\colon (Y, B_Y) \to X)$$

be a quasi-log scheme. If $X_{-\infty} = \emptyset$, then it is called a *quasi-log canonical pair*.

The following example is very important. Example 3.7 shows that we can treat log canonical pairs as quasi-log canonical pairs.

Example 3.7 ([7, 6.4.1]). Let (X, Δ) be a normal pair such that Δ is effective. Let $f\colon Y \to X$ be a resolution of singularities such that

$$K_Y + B_Y = f^*(K_X + \Delta)$$

and that $\mathrm{Supp}\, B_Y$ is a simple normal crossing divisor on Y. We put $\omega = K_X + \Delta$. Then

$$(X, \omega, f\colon (Y, B_Y) \to X)$$

becomes a quasi-log scheme. By construction, (X, Δ) is log canonical if and only if $[X, \omega]$ is quasi-log canonical. We note that C is a log canonical center of (X, Δ) if and only if C is a qlc center of $[X, \omega]$.

For the basic properties of quasi-log schemes, see [7, Chapter 6].

4. Lemmas

In this section, we prepare three lemmas on quasi-log schemes for the proof of Theorem 1.2. The first one is an easy consequence of Fujita's theory of Δ-genus (see [9], [10], [11] and [12, Chapter 3]) and the theory of quasi-log schemes (see [7, Chapter 6]).

Lemma 4.1. *Let $[X, \omega]$ be a projective quasi-log canonical pair such that X is irreducible with $\dim X = n \geq 1$. Let H be an ample Cartier divisor on X. Assume that $-\omega \equiv rH$ for some $r > n$. Then $X \simeq \mathbb{P}^n$, $\mathcal{O}_X(H) \simeq \mathcal{O}_{\mathbb{P}^n}(1)$, $r \leq n + 1$, and there are no qlc centers of $[X, \omega]$.*

Proof. We will use Fujita's theory of Δ-genus (see [9], [10], [11, Chapter I], and [12, Chapter 3]) and the theory of quasi-log schemes (see [7, Chapter 6]).

Step 1. Let us consider

$$\chi(X, \mathcal{O}_X(tH)) = \sum_{i=0}^{n} (-1)^i \dim_{\mathbb{C}} H^i(X, \mathcal{O}_X(tH)).$$

Since H is ample, it is a nontrivial polynomial of degree n. Since

$$tH - \omega \equiv (t + r)H$$

with $r > n$ by assumption, we have

$$H^i(X, \mathcal{O}_X(tH)) = 0$$

for $i > 0$ and $t \geq -n$ by [7, Theorem 6.3.5 (ii)]. Since

$$H^0(X, \mathcal{O}_X(tH)) = 0$$

for $t < 0$ and

$$\chi(X, \mathcal{O}_X) = \dim_{\mathbb{C}} H^0(X, \mathcal{O}_X) = 1,$$

we have

(4.1) $$\chi(X, \mathcal{O}_X(tH)) = \frac{1}{n!}(t + 1) \cdots (t + n).$$

Therefore, we obtain that $H^n = 1$ and

$$\dim_{\mathbb{C}} H^0(X, \mathcal{O}_X(H)) = \chi(X, \mathcal{O}_X(H)) = n + 1.$$

This means

$$\Delta(X, H) = n + H^n - \dim_{\mathbb{C}} H^0(X, \mathcal{O}_X(H)) = 0,$$

where $\Delta(X, H)$ is Fujita's Δ-genus of (X, H). Thus we obtain that $X \simeq \mathbb{P}^n$ and $\mathcal{O}_X(H) \simeq \mathcal{O}_{\mathbb{P}^n}(1)$ (see [9, Theorem 2.1] or [13, Theorem 1.1]).

Step 2. In this step, we will see that $r \leq n + 1$ always hold true.

We assume that $r > n + 1$ holds true. Then, by [7, Theorem 6.3.5 (ii)], we have

$$H^i(X, \mathcal{O}_X(-(n + 1)H)) = 0$$

for $i > 0$. Therefore, we obtain

$$\chi(X, \mathcal{O}_X(-(n + 1)H)) = \dim_{\mathbb{C}} H^0(X, \mathcal{O}_X(-(n + 1)H)) = 0.$$

On the other hand, by (4.1), we have

$$\chi(X, \mathcal{O}_X(-(n + 1)H)) = (-1)^n \neq 0.$$

This is a contradiction. This means that $r \leq n + 1$ always holds.

Step 3. In this step, we will see that $[X, \omega]$ has no qlc centers.

Assume that there exists a zero-dimensional qlc center P of $[X, \omega]$. Then the evaluation map

$$H^0(X, \mathcal{O}_X(-H)) \to \mathbb{C}(P)$$

is surjective since

$$H^1(X, \mathcal{I}_P \otimes \mathcal{O}_X(-H)) = 0$$

by [7, Theorem 6.3.5 (ii)], where $\mathcal{I}_P$ is the defining ideal sheaf of P on X. We note that H is ample and $-\omega \equiv rH$ with $r > \dim X \geq 1$. This means that

$$H^0(X, \mathcal{O}_X(-H)) \neq 0.$$

This is a contradiction since H is ample. Therefore, there are no zero-dimensional qlc centers of $[X, \omega]$.

Assume that there exists a qlc center C of $[X, \omega]$ with $\dim C \geq 1$. By [7, Theorem 6.3.5 (i)], $[C, \omega|_C]$ is a quasi-log canonical pair with $\dim C < \dim X$. Since

$$-\omega \equiv rH$$

with $r > n$, we have

$$-\omega|_C \equiv rH|_C$$

with $r > n \geq \dim C + 1$. This contradicts the result established in Step 2. It means that there are no qlc centers of $[X, \omega]$.

We finish the proof of Lemma 4.1. $\square$

The second one is an easy lemma on the vanishing theorem for quasi-log schemes.

Lemma 4.2 (Vanishing theorem for quasi-log schemes). *Let $[X, \omega]$ be a projective quasi-log scheme with $\dim X_{-\infty} = 0$ or $X_{-\infty} = \emptyset$. Let L be a Cartier divisor on X such that $L - \omega$ is ample. Then*

$$H^i(X, \mathcal{O}_X(L)) = 0$$

for every $i > 0$.

Proof. If $X_{-\infty} = \emptyset$, then the statement is a special case of [7, Theorem 6.3.5 (ii)]. Therefore, from now on, we may assume that $X_{-\infty} \neq \emptyset$.

Let us consider the following short exact sequence:

$$0 \to \mathcal{I}_{X_{-\infty}} \to \mathcal{O}_X \to \mathcal{O}_{X_{-\infty}} \to 0.$$

Then we obtain a long exact sequence:

(4.2) $\cdots \to H^i(X, \mathcal{I}_{X_{-\infty}} \otimes \mathcal{O}_X(L)) \to H^i(X, \mathcal{O}_X(L)) \to H^i(X, \mathcal{O}_{X_{-\infty}}(L)) \to \cdots.$

By [7, Theorem 6.3.5 (ii)], we get

$$H^i(X, \mathcal{I}_{X_{-\infty}} \otimes \mathcal{O}_X(L)) = 0$$

for every $i > 0$. Since $\dim X_{-\infty} = 0$ by assumption, we have

$$H^i(X, \mathcal{O}_{X_{-\infty}}(L)) = 0$$

for every $i > 0$. Therefore, by (4.2), we see that

$$H^i(X, \mathcal{O}_X(L)) = 0$$

holds true for every $i > 0$.   □

The final one is a somewhat technical lemma (see [8, Lemmas 3.1 and 3.2]).

Lemma 4.3. *Let $[X, \omega]$ be a quasi-log canonical pair such that X is irreducible and let $\varphi \colon X \to W$ be a proper surjective morphism onto a quasi-projective variety W with $\dim W \geq 1$. Let $P \in W$ be a closed point such that $\dim \varphi^{-1}(P) \geq 1$. Then we can construct an effective $\mathbb{R}$-Cartier divisor B on W such that $[X, \omega + \varphi^* B]$ is a quasi-log scheme with the following properties:*

  (i) *$[X, \omega + \varphi^* B]$ is quasi-log canonical outside finitely many points, and*
  (ii) *there exists a qlc center C of $[X, \omega + \varphi^* B]$ such that $\varphi(C) = P$ with $\dim C \geq 1$.*

Proof. We divide the proof into several cases.

CASE 1. In this case, we assume that there are no qlc centers of $[X, \omega]$ in $\varphi^{-1}(P)$.

Let $f \colon (Y, B_Y) \to X$ be a quasi-log resolution of $[X, \omega]$ as in Definition 3.5. We take general very ample Cartier divisors $B_1, \ldots, B_{n+1}$ on W such that $P \in \operatorname{Supp} B_i$ for every i. By [7, Proposition 6.3.1], we may further assume that

$$\left(Y, \sum_{i=1}^{n+1} (\varphi \circ f)^* B_i + \operatorname{Supp} B_Y \right)$$

is a globally embedded simple normal crossing pair (see [14, Theorem 3.35]). By [7, Lemma 6.3.13], we can take $0 < c < 1$ with the following properties:

  (a) $\left(B_Y + c \sum_{i=1}^{n+1} (\varphi \circ f)^* B_i \right)^{>1} = 0$ or $\dim f \left(\operatorname{Supp} \left(B_Y + c \sum_{i=1}^{n+1} (\varphi \circ f)^* B_i \right)^{>1} \right) = 0$, and

  (b) there exists an irreducible component G of $\left(B_Y + c \sum_{i=1}^{n+1} (\varphi \circ f)^* B_i \right)^{=1}$ such that $\dim f(G) \geq 1$.

We put $B = c \sum_{i=1}^{n+1} B_i$. Then, by construction, we see that

$$f \colon (Y, B_Y + (\varphi \circ f)^* B) \to [X, \omega + \varphi^* B]$$

gives a desired quasi-log structure on $[X, \omega + \varphi^* B]$.

CASE 2. In this case, we assume that there exists a qlc center C of $[X, \omega]$ in $\varphi^{-1}(P)$ with $\dim C \geq 1$.

Obviously, it is sufficient to put $B = 0$.

CASE 3. In this case, we assume that every qlc center of $[X, \omega]$ contained in $\varphi^{-1}(P)$ is zero-dimensional.

Let $f \colon (Y, B_Y) \to X$ be a quasi-log resolution of $[X, \omega]$ as in Definition 3.5. We take general very ample Cartier divisors $B_1, \ldots, B_{n+1}$ on W such that $P \in \operatorname{Supp} B_i$ for every i as

in Case 1. Let X' be the union of all qlc centers contained in $\varphi^{-1}(P)$. By [7, Proposition 6.3.1], we may assume that the union of all strata of (Y, B_Y) mapped to X' by f, which is denoted by Y', is a union of some irreducible components of Y. We put $Y'' = Y - Y'$, $K_{Y''} + B_{Y''} = (K_Y + B_Y)|_{Y''}$, and $f'' = f|_{Y''}$. We may further assume that

$$\left(Y'', \sum_{i=1}^{n+1} (\varphi \circ f'')^* B_i + \mathrm{Supp}\, B_{Y''} \right)$$

is a globally embedded simple normal crossing pair by [7, Proposition 6.3.1] and [14, Theorem 3.35]. We note that by the proof of [7, Theorem 6.3.5 (i)]

$$\mathcal{I}_{X'} = f''_* \mathcal{O}_{Y''}(\lceil -(B_{Y''}^{<1}) \rceil - Y'|_{Y''})$$

holds true, where $\mathcal{I}_{X'}$ is the defining ideal sheaf of X' on X. We also note that $B_{Y''} \geq Y'|_{Y''}$ by construction. By [7, Lemma 6.3.13], we can take $0 < c < 1$ with the following properties:

(c) $\dim f'' \left(\mathrm{Supp}\left(B_{Y''} + c \sum_{i=1}^{n+1} (\varphi \circ f'')^* B_i \right)^{>1} \right) = 0$, and

(d) there exists an irreducible component G of $\left(B_{Y''} + c \sum_{i=1}^{n+1} (\varphi \circ f'')^* B_i \right)^{=1}$ such that $\dim f''(G) \geq 1$.

We put $B = c \sum_{i=1}^{n+1} B_i$. Then, by construction, we see that

$$f'' : (Y'', B_{Y''} + (\varphi \circ f'')^* B) \to [X, \omega + \varphi^* B]$$

gives a desired quasi-log structure on $[X, \omega + \varphi^* B]$.

In any case, we got a desired effective $\mathbb{R}$-Cartier divisor B on W. We note that $[X, \omega + \varphi^* B]$ is quasi-log canonical outside $\varphi^{-1}(P)$ by construction.   □

5. Proof

In this section, we will prove Theorem 1.2 and Corollary 1.3 by using the lemmas obtained in Section 4.

Let us prove Theorem 1.2, which is the main result of this paper.

Proof of Theorem 1.2. In this proof, we put $H = -D$.

CASE 1. In this case, we assume that X is irreducible.

Since ω is not nef, we can take an ω-negative extremal contraction $\varphi : X \to W$ by the cone and contraction theorem of quasi-log canonical pairs (see [7, Theorems 6.7.3 and 6.7.4]). If $\dim W \geq 1$, then we can take an effective $\mathbb{R}$-Cartier divisor B on W satisfying the properties in Lemma 4.3. Let C be a qlc center of $[X, \omega + \varphi^* B]$ as in Lemma 4.3. We put

$$C' = C \cup \mathrm{Nqlc}(X, \omega + \varphi^* B).$$

By adjunction (see [7, Theorem 6.3.5 (i)]), $[C', (\omega + \varphi^* B)|_{C'}]$ is a quasi-log scheme. We note that there exists the following short exact sequence:

(5.1) $$0 \longrightarrow \mathrm{Ker}\, \alpha \longrightarrow \mathcal{O}_{C'} \xrightarrow{\alpha} \mathcal{O}_C \longrightarrow 0$$

such that $\mathrm{Ker}\, \alpha = 0$ or the support of $\mathrm{Ker}\, \alpha$ is zero-dimensional. We also note that

$$- (\omega + \varphi^* B)|_{C'} \equiv rH|_{C'}$$

since $\dim \varphi(C') = 0$. By Lemma 4.2 and (5.1),

$$H^i(C, \mathcal{O}_C(tH)) = H^i(C', \mathcal{O}_{C'}(tH)) = 0$$

for $i > 0$ and $t \geq -n$ since $r > n$ by assumption. Since $H|_C$ is ample, we have

$$H^0(C, \mathcal{O}_C(tH)) = 0$$

for $t < 0$. This means that

$$\chi(C, \mathcal{O}_C(tH)) = 0$$

for $t = -n, \ldots, -1$. Therefore, we get

$$\chi(C, \mathcal{O}_C(tH)) \equiv 0$$

by $n \geq \dim C + 1$. This is a contradiction since $H|_C$ is ample. This implies that $\dim W = 0$ and that H is ample. Thus we obtain that $X \simeq \mathbb{P}^n$ with $\mathcal{O}_X(D) \simeq \mathcal{O}_{\mathbb{P}^n}(-1)$ by Lemma 4.1. Moreover, there are no qlc centers of $[X, \omega]$ by Lemma 4.1.

Case 2. Let us assume now that X is not necessarily irreducible. We take an irreducible component X' of X such that $\omega' = \omega|_{X'}$ is not nef. By adjunction (see [7, Theorem 6.3.5 (i)]), $[X', \omega']$ is an irreducible quasi-log canonical pair such that $\omega' \equiv rD|_{X'}$ with $r > n \geq \dim X'$. By Case 1, we see that $H|_{X'}$ is ample. We note that $[X', \omega']$ has qlc centers if we assume $X \neq X'$ since X is connected (see [7, Theorem 6.3.11 and Theorem 6.3.5 (i)]). Therefore, by Lemma 4.1, we obtain that $X' \simeq \mathbb{P}^n$, $X' = X$, and $\mathcal{O}_X(D) \simeq \mathcal{O}_{\mathbb{P}^n}(-1)$. In particular, X is always irreducible.

We finish the proof of Theorem 1.2.   □

We prove Corollary 1.3 as an application of Theorem 1.2.

Proof of Corollary 1.3. By [6, Theorem 1.2], $[X, K_X + \Delta]$ naturally becomes a quasi-log canonical pair. Therefore, we obtain the desired statement by Theorem 1.2. We note that (X, Δ) is kawamata log terminal since there are no qlc centers of $[X, K_X + \Delta]$ (see [6, Theorem 1.2 (5)]).   □

We close this paper with a remark on Corollary 1.3.

Remark 5.1. We can prove Corollary 1.3 without using the theory of quasi-log schemes. By taking the normalization and a dlt blow-up (see [5, Theorem 10.4] and [7, Theorem 4.4.21]), we can reduce the problem to the case where (X, Δ) is a $\mathbb{Q}$-factorial dlt pair. By taking a $(K_X + \Delta)$-negative extremal contraction $\varphi : X \to W$ and decreasing the coefficients of Δ slightly, we can check that $\dim W = 0$ by using the argument in the proof of [1, Theorem 3.1] (see [1, Remark 3.1.2]). Note that some results in [1] are generalized in [8]. Then we obtain that $-(K_X + \Delta)$ is ample. This implies that $X \simeq \mathbb{P}^n$ holds (see Case 2 in Sketch of Proof of Theorem 2.1 or Step 1 in the proof of Theorem 4.1).

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