

WEIGHTED SHIFTS ON DIRECTED TREES WITH ONE BRANCHING VERTEX: n -CONTRACTIVITY AND HYPONORMALITY

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Abstract

Let S_λ be a weighted shift on a rooted directed tree with one branching vertex \tilde{u} , η branches ($2 \leq \eta < \infty$) and positive weight sequence λ . We define a collection of (classical) weighted shifts, the so-called “the i -th branching weighted shifts” $W^{(i)}$ for $0 \leq i \leq \eta$, whose weights are derived from those of S_λ . In this note we discuss the relationships between n -contractivity, n -hypercontractivity and hyponormality of S_λ and these properties for the $W^{(i)}$ ($0 \leq i \leq \eta$).

1. Introduction and preliminaries

Let \mathcal{H} be an infinite dimensional complex Hilbert space and let $\mathcal{B}(\mathcal{H})$ be the algebra of bounded linear operators on \mathcal{H} . The class of weighted shifts on directed trees (these and other definitions reviewed below) has been an important generalization of the classical unilateral weighted shifts and has provided a much broader class of operators to study (see [11]). One feature of that study is consideration of what properties of such a shift (for example, subnormality or its lack) may be deduced from certain classical shifts naturally associated with it (see, for example, [8]). In this paper we consider this basic motivating question for what has been a particularly useful family of these shifts on directed trees and for some “weak subnormalities” in two senses: for n -contractivity (and the associated n -hypercontractivity), and for hyponormality.

The organization of this paper is as follows. In the remainder of this section we give definitions and set notation, recall some fundamental known facts concerning subnormality of a weighted shift S_λ on the particular sort of directed tree we consider, and recall as well the definitions of certain classical weighted shifts associated with S_λ . In Section 2 we take up the motivating question in the case of n -contractivity and n -hypercontractivity, and in Section 3 consider the analogous question in the case of hyponormality.

An operator T in $\mathcal{B}(\mathcal{H})$ is *subnormal* if it is (unitarily equivalent to) the restriction of a normal operator to an invariant subspace, and *hyponormal* if $T^*T \geq TT^*$. For an operator T and $n \in \mathbb{N}$ set

$$(1.1) \quad A_n(T) := \sum_{k=0}^n (-1)^k \binom{n}{k} T^{*k} T^k,$$

where \mathbb{N} is the set of positive integers and $\binom{n}{k}$ is the usual binomial coefficient. Recall from

[2] that T is contractive subnormal if and only if $A_n(T) \geq 0$ for all $n \in \mathbb{N}$. For a fixed $n \in \mathbb{N}$, an operator $T \in \mathcal{B}(\mathcal{H})$ is n -contractive [resp., n -hypercontractive] if $A_n(T) \geq 0$ [resp., $A_k(T) \geq 0$ for all $k = 1, \dots, n$]. Observe that “1-contractive” is simply “contractive.” Obviously, certain implications hold: contractive subnormal $\Rightarrow \dots \Rightarrow 2$ -hypercontractive $\Rightarrow 1$ -hypercontractive. However, such implications in the case of n -contractivity rather than n -hypercontractivity do not hold in general; for example, see [1], [5], [6], [7]. Also it is well known that the reverse implications, either for the contractivity or hypercontractivity conditions, do not hold (see [1], [4], [5], [6], [7]). The reader is referred to [1], [5], [7], [9], [10] for more information on n -contractive operators.

It is also known that if an operator T is subnormal then it is hyponormal, and also (from [7]) that if T is hyponormal then it is 2-contractive. In addition, we recall more information about relationships concerning n -contractivity.

- For odd $n \in \mathbb{N}$, if T is n -contractive, then T is $(n-1)$ -contractive ([9, Corollary 3.1]).
- Let W_α be a weighted shift with only finitely many weights in the weight sequence α larger than 1 and let $n \geq 2$ be arbitrary. If W_α is n -contractive, then it is $(n-1)$ -contractive (and hence contractive) ([6, Proposition 0.3]).

We introduce some notation which will be used in this paper. We set \mathbb{R}_+ [\mathbb{C} , \mathbb{Z}_+ , resp.] to the set of nonnegative real numbers [complex numbers, nonnegative integers, resp.]. For $k \in \mathbb{N}$, we let $J_k = \{1, \dots, k\}$ and $\mathbb{N}_k = \{k, k+1, \dots\}$. For a subset \mathcal{M} of \mathcal{H} , $\vee \mathcal{M}$ is the span of \mathcal{M} .

In 2012, Jabłoński-Jung-Stochel [11] developed the theory of a weighted shift on a directed tree which generalizes the classical weighted shift. We give some terminology to describe the trees considered in this paper. Let $\mathcal{T} = (V, E)$ be a directed tree, where V and E are the set of vertices and the set of edges, respectively. Set $\text{Chi}(u) = \{v \in V : (u, v) \in E\}$ for the children of $u \in V$.

We consider here the particular directed tree with one branching vertex which is the main model of this paper. Given $\eta \in \mathbb{N}_2, \kappa \in \mathbb{Z}_+$, define the directed tree $\mathcal{T}_{\eta, \kappa} = (V_{\eta, \kappa}, E_{\eta, \kappa})$ by (see Figure 1)

$$\begin{aligned} V_{\eta, \kappa} &= \{-k : k \in J_\kappa\} \sqcup \{0\} \sqcup \{(i, j) : i \in J_\eta, j \in \mathbb{N}\}^1, \\ E_{\eta, \kappa} &= E_\kappa \sqcup \{(0, (i, 1)) : i \in J_\eta\} \sqcup \{((i, j), (i, j+1)) : i \in J_\eta, j \in \mathbb{N}\}, \\ E_\kappa &= \{(-k, -k+1) : k \in J_\kappa\}. \end{aligned}$$

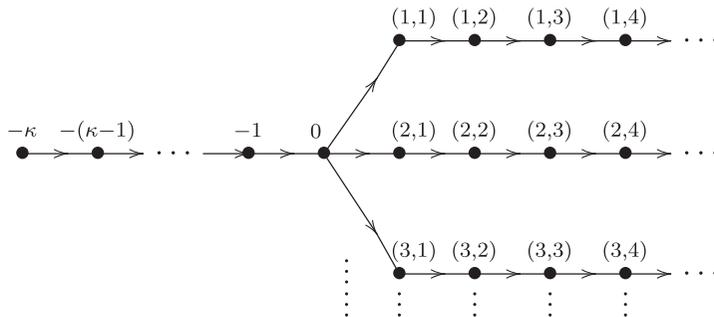


Fig. 1. The directed tree $\mathcal{T}_{\eta, \kappa}$ for $\eta \in \mathbb{N}_2, \kappa \in \mathbb{Z}_+$.

¹The symbol “ \sqcup ” denotes the disjoint union.

Let $\ell^2(V_{\eta,\kappa})$ be the Hilbert space of all square summable complex functions on $V_{\eta,\kappa}$ equipped with the standard inner product. The system $\{e_u\}_{u \in V_{\eta,\kappa}}$ defined by

$$e_u(v) = \begin{cases} 1 & \text{if } v = u, \\ 0 & \text{otherwise,} \end{cases} \quad v \in V_{\eta,\kappa},$$

is an orthonormal basis of $\ell^2(V_{\eta,\kappa})$. Set $V_{\eta,\kappa}^\circ = V_{\eta,\kappa} \setminus \{-\kappa\}$. For a system $\lambda = \{\lambda_v\}_{v \in V_{\eta,\kappa}^\circ} \subseteq \mathbb{C}$ of weights, in [11] there is a general definition of the weighted shift S_λ on a directed tree suitable even for unbounded shifts; in this paper, we consider only shifts $S_\lambda \in \mathcal{B}(\ell^2(V_{\eta,\kappa}))$, in which case we may take S_λ as defined by

$$S_\lambda e_u = \sum_{v \in \text{Chi}(u)} \lambda_v e_v.$$

As well, for our questions of interest we may and do take the weights λ_v to be positive (see [11, Theorem 3.2.1] and surrounding discussion). Since S_λ is bounded, we often assume that it is a contraction.

The weighted shifts S_λ on the directed trees $\mathcal{T}_{\eta,\kappa}$ in Figure 1 have provided several interesting results and exotic examples related to subnormality since 2012 (see e.g., [3], [8], [11], [12], [13]). It is thus worth considering whether properties of such S_λ (such as n -contractivity or n -hypercontractivity) can be detected directly from the properties of certain associated classical weighted shifts.

The following characterization of subnormality of weighted shifts on $\mathcal{T}_{\eta,\kappa}$ will be used subsequently.

Lemma 1.1. ([11, Corollary 6.2.2]) *Suppose $\eta \in \mathbb{N}_2$ and $\kappa \in \mathbb{Z}_+$ are given. Let $S_\lambda \in \mathcal{B}(\ell^2(V_{\eta,\kappa}))$ be a weighted shift with positive weights $\lambda = \{\lambda_v\}_{v \in V_{\eta,\kappa}^\circ}$. Then the following assertions hold.*

(i) *If $\kappa = 0$, then S_λ is subnormal if and only if there exist Borel probability measures $\{\mu_i\}_{i=1}^\eta$ on \mathbb{R}_+ such that*

$$(1.2) \quad \int_0^\infty s^n d\mu_i(s) = \prod_{j=2}^{n+1} \lambda_{i,j}^2, \quad n \in \mathbb{N}, i \in J_\eta,$$

$$\sum_{i=1}^\eta \lambda_{i,1}^2 \int_0^\infty \frac{1}{s} d\mu_i(s) \leq 1.$$

(ii) *If $\kappa \in \mathbb{N}$, then S_λ is subnormal if and only if one of the following two equivalent conditions holds:*

(ii-a) *there exist Borel probability measures $\{\mu_i\}_{i=1}^\eta$ on \mathbb{R}_+ which satisfy (1.2) and the following requirements:*

$$\sum_{i=1}^\eta \lambda_{i,1}^2 \int_0^\infty \frac{1}{s} d\mu_i(s) = 1,$$

$$\sum_{i=1}^\eta \lambda_{i,1}^2 \int_0^\infty \frac{1}{s^{k+1}} d\mu_i(s) = \frac{1}{\prod_{j=0}^{k-1} \lambda_{-j}^2}, \quad k \in J_{\kappa-1},$$

$$\sum_{i=1}^{\eta} \lambda_{i,1}^2 \int_0^{\infty} \frac{1}{s^{\kappa+1}} d\mu_i(s) \leq \frac{1}{\prod_{j=0}^{\kappa-1} \lambda_{-j}^2},$$

(ii-b) there exist Borel probability measures $\{\mu_i\}_{i=1}^{\eta}$ and ν on \mathbb{R}_+ which satisfy (1.2) and the equations below

$$\int_0^{\infty} s^n d\nu(s) = \begin{cases} \prod_{j=\kappa-n}^{\kappa-1} \lambda_{-j}^2 & \text{if } n \in J_{\kappa}, \\ (\prod_{j=0}^{\kappa-1} \lambda_{-j}^2)(\sum_{i=1}^{\eta} \prod_{j=1}^{n-\kappa} \lambda_{i,j}^2) & \text{if } n \in \mathbb{N} \setminus J_{\kappa}. \end{cases}$$

The idea of the following definition, which produces classical weighted shifts associated to some S_{λ} , comes from the study of the subnormal completion problem for weighted shifts on directed trees in [8].

DEFINITION 1.2. Suppose S_{λ} is a weighted shift on $\mathcal{T}_{\eta,\kappa} = (V_{\eta,\kappa}, E_{\eta,\kappa})$ with positive weights $\lambda = \{\lambda_v\}_{v \in V_{\eta,\kappa}^{\circ}}$. In what follows we assume $\kappa \in \mathbb{Z}_+$ and $\eta \in \mathbb{N}_2$. We associate to S_{λ} some classical weighted shifts as follows: let $W^{(i)}$ be the classical weighted shift with the weight sequence

$$\alpha^{(i)} : \lambda_{i,2}, \lambda_{i,3}, \lambda_{i,4}, \lambda_{i,5}, \dots, \quad i \in J_{\eta},$$

under the order of branches as in Figure 2. We will say $W^{(i)}$ is the i -th branching shift with weight sequence $\alpha^{(i)}$. As well, let $W^{(0)}$ be the classical weighted shift with the weight sequence $\widehat{\lambda} = \{\widehat{\lambda}_i\}_{i=-\kappa+1}^{\infty}$ given by

$$\begin{aligned} \widehat{\lambda}_i &= \lambda_i, \quad -\kappa + 1 \leq i \leq 0, \\ \widehat{\lambda}_1 &:= \sqrt{\sum_{i=1}^{\eta} \lambda_{i,1}^2}, \quad \widehat{\lambda}_{j+1} := \sqrt{\frac{\sum_{i=1}^{\eta} \prod_{k=1}^{j+1} \lambda_{i,k}^2}{\sum_{i=1}^{\eta} \prod_{k=1}^j \lambda_{i,k}^2}}, \quad j \in \mathbb{N}. \end{aligned} \tag{1.3}$$

We say that $W^{(0)}$ is the basic branching shift with weight sequence $\widehat{\lambda}$. For convenience of language, we say that “ $W^{(i)}$ is the i -th branching shift of S_{λ} for $i \in J_{\eta} \cup \{0\}$.”

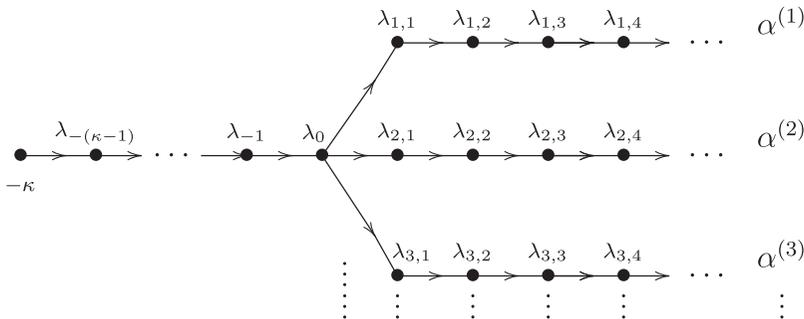


Fig. 2. The weights on $\mathcal{T}_{\eta,\kappa}$ for $\eta \in \mathbb{N}_2, \kappa \in \mathbb{Z}_+$.

2. n -contractivity and n -hypercontractivity

We consider the following question concerning n -contractivity and n -hypercontractivity in this section.

Q1. Is it true that S_λ is n -contractive [resp., n -hypercontractive] if and only if every i -th branching shift of S_λ is n -contractive [resp., n -hypercontractive] for $i \in J_\eta \cup \{0\}$?

This question is motivated by considering Lemma 1.1 (ii-b): we may confirm that

S_λ is subnormal if and only if every i -th branching shift is subnormal for all $i \in J_\eta \cup \{0\}$.

This result can be improved to the following theorem, which answers Q1 affirmatively.

Theorem 2.1. *Let S_λ be a weighted shift on $\mathcal{T}_{\eta,\kappa}$ with weights $\lambda = \{\lambda_v\}_{v \in V_{\eta,\kappa}^\circ}$. Suppose $\eta \in \mathbb{N}_2$ and $\kappa \in \mathbb{Z}_+$, and suppose $n \in \mathbb{N}$. Then S_λ is n -contractive [resp., n -hypercontractive] if and only if every i -th branching shift $W^{(i)}$ of S_λ is n -contractive [resp., n -hypercontractive] for $i \in J_\eta \cup \{0\}$.*

We begin with a computational lemma whose proof is elementary.

Lemma 2.2. *Suppose $\eta \in \mathbb{N}_2$ and $\kappa \in \mathbb{Z}_+$. Let S_λ be a weighted shift on $\mathcal{T}_{\eta,\kappa}$ with positive weights $\lambda = \{\lambda_v\}_{v \in V_{\eta,\kappa}^\circ}$ and consider the ordered basis*

$$e_{-\kappa}, e_{-\kappa+1}, \dots, e_0, e_{1,1}, e_{2,1}, \dots, e_{\eta,1}, e_{1,2}, e_{2,2}, \dots, e_{\eta,2}, \dots,$$

for $\ell^2(V_{\eta,\kappa})$. Then the following assertions hold.

(i) For $-\kappa \leq p \leq -1$,

$$(2.1) \quad \langle (S_\lambda^*)^k (S_\lambda^k) e_p, e_p \rangle = \begin{cases} \lambda_{p+1}^2 \cdots \lambda_{p+k}^2 & \text{if } k \leq |p|, \\ \lambda_{p+1}^2 \cdots \lambda_0^2 \sum_{i=1}^\eta \left(\prod_{j=1}^{k-|p|} \lambda_{i,j}^2 \right) & \text{if } k > |p|. \end{cases}$$

(ii) For $p = 0$,

$$\langle (S_\lambda^*)^k (S_\lambda^k) e_0, e_0 \rangle = \sum_{i=1}^\eta \prod_{j=1}^k \lambda_{i,j}^2.$$

(iii) For $i \in J_\eta$ and $j \in \mathbb{N}$,

$$\langle (S_\lambda^*)^k (S_\lambda^k) e_{i,j}, e_{i,j} \rangle = \lambda_{i,(j+1)}^2 \lambda_{i,(j+2)}^2 \cdots \lambda_{i,(j+k)}^2.$$

We now prove Theorem 2.1.

Proof of Theorem 2.1. For any $i \in J_\eta$, the space $\vee_{j=1}^\infty \{e_{i,j}\}$ is invariant for S_λ . Therefore, if S_λ is n -contractive, then so is $S_\lambda|_{\vee_{j=1}^\infty \{e_{i,j}\}}$, yielding that $W^{(i)}$ is appropriately n -contractive. (One must cope with the differing weight conventions for shifts on directed trees and for classical weighted shifts.) As well, set $W := W^{(0)}$ to ease the notation, so W is the basic branching shift with weight sequence as in (1.3). Suppose S_λ is n -contractive. We now claim that W is n -contractive. We need first an observation for the operators $A_n(S_\lambda)$ occurring in the n -contractive tests (1.1): in Lemma 2.2, we have that the operator $(S_\lambda^*)^k (S_\lambda^k)$ in (2.1) is diagonal with respect to the ordered basis

$$e_{-\kappa}, e_{-\kappa+1}, \dots, e_0, e_{1,1}, e_{2,1}, \dots, e_{\eta,1}, e_{1,2}, e_{2,2}, \dots, e_{\eta,2}, \dots$$

Now we view W as a weighted shift acting on some orthonormal basis

$$e_{-\kappa}, e_{-\kappa+1}, \dots, e_{-1}, e_0, f_1, f_2, f_3, \dots,$$

with weights $\lambda_{-\kappa}, \lambda_{-\kappa+1}, \dots, \lambda_0, \widehat{\lambda}_1, \widehat{\lambda}_2, \dots$, as in (1.3). Then we can easily see that

$$(2.2) \quad \langle A_n(S_\lambda)e_p, e_p \rangle = \langle A_n(W)e_p, e_p \rangle, \quad -\kappa \leq p \leq 0, n \in \mathbb{N}.$$

By the definition of W , we have for each $j, k \in \mathbb{N}$,

$$\langle W^{*k}W^k f_j, f_j \rangle = \widehat{\lambda}_{j+1}^2 \cdots \widehat{\lambda}_{j+k}^2 \langle f_{j+k}, f_{j+k} \rangle = \frac{\sum_{i=1}^{\eta} \lambda_{i,1}^2 \lambda_{i,2}^2 \cdots \lambda_{i,(j+k)}^2}{\sum_{i=1}^{\eta} \lambda_{i,1}^2 \lambda_{i,2}^2 \cdots \lambda_{i,j}^2}.$$

For $i \in J_\eta$ and $j \in \mathbb{N}$, we have

$$\begin{aligned} \langle A_n(S_\lambda)e_{i,j}, e_{i,j} \rangle &= 1 + \sum_{k=1}^n (-1)^k \binom{n}{k} \prod_{\ell=1}^k \lambda_{i,(j+\ell)}^2 \\ &= 1 - \binom{n}{1} \lambda_{i,(j+1)}^2 + \binom{n}{2} \lambda_{i,(j+1)}^2 \lambda_{i,(j+2)}^2 + \cdots + (-1)^n \binom{n}{n} \\ &\quad \times \lambda_{i,(j+1)}^2 \cdots \lambda_{i,(j+n)}^2 =: \Delta_{i,j}^{(n)}, \end{aligned}$$

and also

$$\langle A_n(W)f_j, f_j \rangle = 1 + \sum_{k=1}^n (-1)^k \binom{n}{k} \frac{\sum_{i=1}^{\eta} \prod_{\ell=1}^{j+k} \lambda_{i,\ell}^2}{\sum_{i=1}^{\eta} \prod_{\ell=1}^j \lambda_{i,\ell}^2} = \frac{1}{\sum_{i=1}^{\eta} \lambda_{i,1}^2 \cdots \lambda_{i,j}^2} \sum_{i=1}^{\eta} \lambda_{i,1}^2 \cdots \lambda_{i,j}^2 \Delta_{i,j}^{(n)}.$$

Then the following assertion holds

$$\langle A_n(S_\lambda)e_{i,j}, e_{i,j} \rangle \geq 0 \Rightarrow \langle A_n(W)f_j, f_j \rangle \geq 0, \quad j \in \mathbb{N}, i \in J_\eta.$$

Therefore S_λ n -contractive implies that the branching shifts are.

For the reverse, the equality in (2.2) guarantees positivity for $\langle A_n(S_\lambda)e_p, e_p \rangle$ ($-\kappa \leq p \leq 0$) and the positivity of any $\langle A_n(S_\lambda)e_{i,j}, e_{i,j} \rangle$ ($i \in J_\eta, j \in \mathbb{N}$) follows from that of the corresponding expression in $W^{(i)}$, as required. Finally, the case of n -hypercontractivity is obvious. Hence the proof is complete. \square

Corollary 2.3. *Suppose $\eta \in \mathbb{N}$ and $\kappa \in \mathbb{Z}_+$. Let S_λ be a weighted shift on $\mathcal{T}_{\eta,\kappa}$ with positive weights $\lambda = \{\lambda_v\}_{v \in V_{\eta,\kappa}^*}$. Suppose that $\lambda_{i,j} = \lambda_{1,j}$ ($i \in J_\eta, j \in \mathbb{N}$) (that is, the branches have the same weights). Let $n \in \mathbb{N}$. Then S_λ is n -contractive [resp., n -hypercontractive] if and only if the basic branching shift $W^{(0)}$ is n -contractive [resp., n -hypercontractive].*

Proof. It follows from the assumption $\lambda_{i,j} = \lambda_{1,j}$ for $i \in J_\eta$ and $j \in \mathbb{N}$ that

$$\langle A_n(S_\lambda)e_{i,j}, e_{i,j} \rangle = 1 + \sum_{k=1}^n (-1)^k \binom{n}{k} \prod_{\ell=1}^k \lambda_{1,(j+\ell)}^2 = \langle A_n(W)f_j, f_j \rangle, \quad n \in \mathbb{N}.$$

The rest is as in the previous proof. \square

The hypotheses of Theorem 2.1 cannot be weakened trivially: the n -contractivity of the basic branching shift $W^{(0)}$ alone is not enough to guarantee n -contractivity of S_λ ; indeed, subnormality of $W^{(0)}$ is not enough alone to guarantee subnormality of S_λ . The “reason” is that $W^{(0)}$ combines or “averages” weights of S_λ ; see the next example.

EXAMPLE 2.4. Consider a directed tree $\mathcal{T}_{2,0} = (V_{2,0}, E_{2,0})$ with weights $\lambda = \{\lambda_v\}_{v \in V_{2,0}^\circ}$ as in Figure 3. Then $W^{(0)}$ has weights

$$\sqrt{\frac{1}{2} + \frac{1}{3}} = \sqrt{\frac{5}{6}}, \frac{\sqrt{\frac{1}{2} \cdot \frac{1}{3} + \frac{1}{3} \cdot 2}}{\sqrt{\frac{1}{2} + \frac{1}{3}}} = \frac{\sqrt{\frac{5}{6}}}{\sqrt{\frac{5}{6}}} = 1, \frac{\sqrt{\frac{1}{2} \cdot \frac{1}{3} \cdot 1 + \frac{1}{3} \cdot 2 \cdot 1}}{\sqrt{\frac{1}{2} \cdot \frac{1}{3} + \frac{1}{3} \cdot 2}} = 1, \dots,$$

and this is fine for even subnormality of $W^{(0)}$ but the second branch of S_λ under the order in Figure 2 clearly fails contractivity: $S_\lambda|_{\bigvee_{j=1}^\infty \{e_{2,j}\}}$ is not contractive, nor subnormal or even hyponormal.

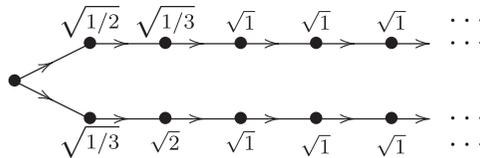


Fig.3. Weights of S_λ in Example 2.4.

3. Hyponormality of S_λ and its classical shifts

Let S_λ be a contractive weighted shift on $\mathcal{T}_{\eta,\kappa}$. Then it follows from Theorem 2.1 that S_λ is 2-contractive if and only if every i -th branching shift is 2-contractive. Recall from [7, Theorem 1.2] that if S_λ is hyponormal then S_λ is 2-contractive. So the following question is quite natural and we solve it in this section.

Q2. Is it true that S_λ is hyponormal if and only if every i -th branching shift of S_λ is hyponormal for $i \in J_\eta \cup \{0\}$?

Recall from [11, Theorem 5.1.2] that S_λ is hyponormal with nonzero weights on $\mathcal{T}_{\eta,\kappa}$ if and only if

$$(3.1) \quad \sum_{v \in \text{Chi}(u)} \frac{|\lambda_v|^2}{\|S_\lambda e_v\|^2} \leq 1, \quad u \in V_{\eta,\kappa}.$$

(Note as well that according to Theorem 2.1 and the observation above we know that if S_λ is hyponormal, then $W^{(i)}$ is 2-contractive for $i \in J_\eta \cup \{0\}$, and so a contractivity result follows from hyponormality.)

Some information is available, as shown by the following.

Theorem 3.1. *Suppose $\eta \in \mathbb{N}_2$ and $\kappa \in \mathbb{Z}_+$. Let S_λ be a weighted shift on $\mathcal{T}_{\eta,\kappa}$ with positive weights $\lambda = \{\lambda_v\}_{v \in V_{\eta,\kappa}^\circ}$. If S_λ is hyponormal, then every i -th branching shift $W^{(i)}$ is hyponormal for $i \in J_\eta \cup \{0\}$.*

Proof. By a direct computation with (3.1), we obtain

$$(3.2) \quad \lambda_{-k+1} \leq \lambda_{-k+2} \leq \cdots \leq \lambda_{-1} \leq \lambda_0,$$

$$(3.3) \quad \frac{\lambda_0^2}{\lambda_{1,1}^2 + \lambda_{2,1}^2 + \cdots + \lambda_{\eta,1}^2} \leq 1,$$

$$(3.4) \quad \frac{\lambda_{1,1}^2}{\lambda_{1,2}^2} + \frac{\lambda_{2,1}^2}{\lambda_{2,2}^2} + \cdots + \frac{\lambda_{\eta,1}^2}{\lambda_{\eta,2}^2} \leq 1,$$

$$(3.5) \quad \lambda_{i,2} \leq \lambda_{i,3} \leq \lambda_{i,4} \leq \cdots \quad (i \in J_\eta),$$

where the computations for (3.2) and (3.5) yield the inequalities in successive pairs.

Observe that (3.5) is just increasing weights for $W^{(1)}, \dots, W^{(\eta)}$, which is hyponormality for them. It remains to consider $W^{(0)}$ with the weight sequence $\widehat{\lambda} = \{\widehat{\lambda}_i\}_{i=-\kappa+1}^\infty$ in (1.3) and for hyponormality we need these increasing. In light of (3.2) and (3.3), we have that $\lambda_0 \leq \widehat{\lambda}_1$. Now we will prove the condition $\widehat{\lambda}_1 \leq \widehat{\lambda}_2$, or equivalently,

$$(3.6) \quad \left(\sum_{i=1}^{\eta} \lambda_{i,1}^2 \right)^2 \leq \sum_{i=1}^{\eta} \lambda_{i,1}^2 \lambda_{i,2}^2.$$

We will show this follows from (3.4), but it takes a little work. We will simplify the notation. Set

$$a_i = \lambda_{i,2}^2, \quad x_i = \frac{\lambda_{i,1}^2}{\lambda_{i,2}^2}, \quad i \in J_\eta.$$

In this notation (3.4) becomes

$$(3.7) \quad x_1 + \cdots + x_\eta \leq 1,$$

and our goal (3.6) becomes

$$(3.8) \quad \left(\sum_{i=1}^{\eta} a_i x_i \right)^2 \leq \sum_{i=1}^{\eta} a_i^2 x_i.$$

By applying the Cauchy-Schwartz inequality with the two vectors $(a_1 \sqrt{x_1}, \dots, a_\eta \sqrt{x_\eta})$ and $(\sqrt{x_1}, \dots, \sqrt{x_\eta})$ in \mathbb{R}^η , and using (3.7), we obtain (3.8).

Finally we must check that the later weights are increasing to complete the argument that $W^{(0)}$ is hyponormal, i.e. $\widehat{\lambda}_p \leq \widehat{\lambda}_{p+1}$ for all $p \geq 2$. This reduces to inequalities of the form

$$(3.9) \quad \left(\sum_{i=1}^{\eta} \prod_{j=1}^{p-1} \lambda_{i,j}^2 \right) \left(\sum_{i=1}^{\eta} \prod_{j=1}^{p+1} \lambda_{i,j}^2 \right) \geq \left(\sum_{i=1}^{\eta} \prod_{j=1}^p \lambda_{i,j}^2 \right)^2.$$

To ease the notation, set

$$a_i = \prod_{j=1}^{p-1} \lambda_{i,j}^2, \quad b_i = \lambda_{i,p}^2, \quad c_i = \lambda_{i,(p+1)}^2.$$

Then the inequality (3.9) becomes

$$\left(\sum_{i=1}^{\eta} a_i\right)\left(\sum_{i=1}^{\eta} a_i b_i c_i\right) \geq \left(\sum_{i=1}^{\eta} a_i b_i\right)^2.$$

Observe that from (3.5) we have $c_i \geq b_i$ for all $i \in J_\eta$, so it suffices to show

$$(3.10) \quad \left(\sum_{i=1}^{\eta} a_i\right)\left(\sum_{i=1}^{\eta} a_i b_i^2\right) \geq \left(\sum_{i=1}^{\eta} a_i b_i\right)^2.$$

As before, it follows from the Cauchy-Schwartz inequality that (3.10) holds. Hence $W^{(0)}$ is hyponormal and the proof is complete. \square

In general, the possible conjecture implicit in the question Q2 is not true. We first obtain a positive result under some additional assumptions, and then give a proposition and example that answer Q2 negatively.

Proposition 3.2. *Suppose $\eta \in \mathbb{N}_2$ and $\kappa \in \mathbb{Z}_+$. Let S_λ be a weighted shift on $\mathcal{T}_{\eta,\kappa}$ with positive weights $\lambda = \{\lambda_v\}_{v \in V_{\eta,\kappa}^\circ}$. Assume that $\lambda_{i,j} = \lambda_{1,j}$ ($i \in J_\eta$, $j \in \mathbb{N}_2$), that is the branches of $\mathcal{T}_{\eta,\kappa}$ have the same weights. Then S_λ is hyponormal if and only if the basic branching shift $W^{(0)}$ is hyponormal.*

Proof. For S_λ to be hyponormal under the assumption, we need the conditions

$$(3.11) \quad \lambda_{-\kappa+1} \leq \lambda_{-\kappa+2} \leq \dots \leq \lambda_{-1} \leq \lambda_0,$$

$$(3.12) \quad \frac{\lambda_0^2}{\lambda_{1,1}^2 + \lambda_{2,1}^2 + \dots + \lambda_{\eta,1}^2} \leq 1,$$

$$(3.13) \quad \frac{\lambda_{1,1}^2}{\lambda_{1,2}^2} + \frac{\lambda_{2,1}^2}{\lambda_{2,2}^2} + \dots + \frac{\lambda_{\eta,1}^2}{\lambda_{\eta,2}^2} = \frac{\lambda_{1,1}^2 + \lambda_{2,1}^2 + \dots + \lambda_{\eta,1}^2}{\lambda_{1,2}^2} \leq 1,$$

$$(3.14) \quad \lambda_{1,2} \leq \lambda_{1,3} \leq \lambda_{1,4} \leq \lambda_{1,5} \leq \dots.$$

It follows from $\lambda_{i,j} = \lambda_{1,j}$ ($i \in J_\eta$, $j \in \mathbb{N}_2$) that the weight sequence in (1.3) satisfies $\widehat{\lambda}_n = \lambda_{1,n}$ ($n \in \mathbb{N}_2$).

For hyponormality of $W^{(0)}$ we need the following conditions:

$$(3.15) \quad \lambda_{-\kappa+1} \leq \lambda_{-\kappa+2} \leq \dots \leq \lambda_{-1} \leq \lambda_0,$$

$$(3.16) \quad \lambda_0 \leq \sqrt{\lambda_{1,1}^2 + \lambda_{2,1}^2 + \dots + \lambda_{\eta,1}^2} = \widehat{\lambda}_1,$$

$$(3.17) \quad \widehat{\lambda}_1 = \sqrt{\lambda_{1,1}^2 + \dots + \lambda_{\eta,1}^2} \leq \frac{\sqrt{\lambda_{1,1}^2 \lambda_{1,2}^2 + \dots + \lambda_{\eta,1}^2 \lambda_{\eta,2}^2}}{\sqrt{\lambda_{1,1}^2 + \dots + \lambda_{\eta,1}^2}} = \widehat{\lambda}_2,$$

$$(3.18) \quad \widehat{\lambda}_2 = \frac{\sqrt{\lambda_{1,1}^2 \lambda_{1,2}^2 + \dots + \lambda_{\eta,1}^2 \lambda_{\eta,2}^2}}{\sqrt{\lambda_{1,1}^2 + \dots + \lambda_{\eta,1}^2}} \leq \frac{\sqrt{\lambda_{1,1}^2 \lambda_{1,2}^2 \lambda_{1,3}^2 + \dots + \lambda_{\eta,1}^2 \lambda_{\eta,2}^2 \lambda_{\eta,3}^2}}{\sqrt{\lambda_{1,1}^2 \lambda_{1,2}^2 + \dots + \lambda_{\eta,1}^2 \lambda_{\eta,2}^2}} = \widehat{\lambda}_3, \dots$$

It is easy to see the conditions (3.11)-(3.14) and (3.15)-(3.18) are equivalent “line by line.” \square

In general hyponormality of the classical shifts is not sufficient to guarantee hyponormality of S_λ .

Proposition 3.3. *There exists a weighted shift S_λ on $\mathcal{T}_{\eta,\kappa}$ with positive weights $\lambda = \{\lambda_v\}_{v \in V_{\eta,\kappa}^\circ}$ such that*

- (i) *every i -th branching shift of S_λ is hyponormal for $i \in J_\eta \cup \{0\}$,*
- (ii) *S_λ is not hyponormal.*

Proof. See Example 3.4. □

We give an example for Proposition 3.3 below.

EXAMPLE 3.4. Let S_λ be the weighted shift on the directed tree $\mathcal{T}_{2,0}$ as in Figure 4. By

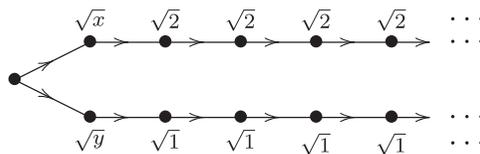


Fig.4. Weights of S_λ in Example 3.4.

using the hyponormality conditions (3.2)-(3.5) for S_λ , the relevant region in which S_λ is hyponormal is given by

$$\{(x, y) : x + 2y \leq 2, 0 < x \leq 2, 0 < y \leq 1\}.$$

Clearly $W^{(1)}$ and $W^{(2)}$ are hyponormal (even subnormal). For hyponormality of $W^{(0)}$, we require

$$\sqrt{x+y} \leq \sqrt{\frac{2x+y}{x+y}} \leq \sqrt{\frac{4x+y}{2x+y}} \leq \sqrt{\frac{8x+y}{4x+y}} \leq \dots$$

All of these except the first are automatic: we must show that for $n \geq 1$,

$$(2^{n-1}x + y)(2^{n+1}x + y) \geq (2^n x + y)^2.$$

But this is

$$2^{n-1}2^{n+1}x^2 + 2^{n-1}xy + 2^{n+1}xy + y^2 \geq 2^n 2^n x^2 + 2 \cdot 2^n xy + y^2,$$

which is immediate. So we have

$$W^{(0)} \text{ is hyponormal } \iff (x + y)^2 \leq 2x + y \quad (0 < x \leq 2, 0 < y \leq 1).$$

We can see easily that the set

$$\{(x, y) \in \mathbb{R}_+ \times \mathbb{R}_+ : W^{(0)} \text{ is hyponormal but } S_\lambda \text{ is not}\}$$

has nonempty interior. We may add to the discussion concerning the subnormality of S_λ by considering where $W^{(0)}$ is subnormal. Observe that the moment sequence for $W^{(0)}$ is

$$\gamma = \{\gamma_n\}_{n=0}^\infty : 1, x + y, 2^1x + y, 2^2x + y, 2^3x + y, \dots$$

Take $x, y \in \mathbb{R}_+ \setminus \{0\}$ with $x + 2y \leq 2$. Then the measure candidate

$$\mu(t) = \left(1 - \frac{x}{2} - y\right)\delta_0 + y\delta_1 + \frac{x}{2}\delta_2$$

works for subnormality for $W^{(0)}$, i.e.,

$$\gamma_n = \int_{\mathbb{R}_+} t^n d\mu = \begin{cases} 1, & n = 0, \\ 2^{n-1}x + y, & n \geq 1. \end{cases}$$

Since each of $W^{(1)}$ and $W^{(2)}$ is subnormal for $0 < x \leq 2$ and $0 < y \leq 1$, respectively, we deduce that

$$\begin{aligned} S_\lambda \text{ is subnormal} &\iff x + 2y \leq 2 \quad (0 < x \leq 2, 0 < y \leq 1) \\ &\iff W^{(0)} \text{ is subnormal.} \end{aligned}$$

Hence we may make a table (Table 1) for the ranges of hyponormality and subnormality for S_λ and its branching shifts.

Table 1. Description of regions for S_λ and $W^{(i)}$.

	S_λ	$W^{(0)}$	$W^{(1)}$	$W^{(2)}$
Hyponormal	$x + 2y \leq 2$ $0 < x \leq 2$ $0 < y \leq 1$	$(x + y)^2 \leq 2x + y$ $0 < x \leq 2$ $0 < y \leq 1$	$0 < x \leq 2$	$0 < y \leq 1$
Subnormal	$x + 2y \leq 2$ $0 < x \leq 2$ $0 < y \leq 1$	$x + 2y \leq 2$ $0 < x \leq 2$ $0 < y \leq 1$	$0 < x \leq 2$	$0 < y \leq 1$

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