Calviño-Louzao, E., García-Río, E., Gutiérrez-Rodríguez, I. and Vázquez-Lorenzo, R.
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HOMOGENEOUS CONFORMAL C-SPACES IN DIMENSION FOUR

E. CALVIÑO-LOUZAO, E. GARCÍA-RÍO, I. GUTIÉRREZ-RODRÍGUEZ and R. VÁZQUEZ-LORENZO

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Abstract

We classify four-dimensional homogeneous conformal C-spaces and show that they are conformally Cotton-flat.

1. Introduction

A metric is said to be Einstein if its Ricci curvature is proportional to the metric. Einstein metrics, being critical for the Hilbert-Einstein functional, are central not only in Mathematics but also in Physics. Three-dimensional Einstein manifolds are of constant curvature but non-trivial examples exist in dimension four, where some topological obstructions to their existence are known to exist. The Einstein condition is known to be very rigid under some assumptions like homogeneity (see [11]). Various generalizations of Einstein metrics are important and have been extensively investigated. From a conformal point of view, the existence of Einstein, or more generally Cotton-flat, representatives of a given conformal class motivated some conformal generalizations of the Einstein condition (see for example [8, 9, 13]).

Let (M, g) be a Riemannian manifold and denote by ρ and $\tau = \operatorname{tr}_g \rho$ the Ricci tensor and the scalar curvature, respectively. Let $S = \rho - \frac{\tau}{2(n-1)}\, g$ denote the Schouten tensor of (M, g) and let $C_{ijk} = (\nabla_i S)_{jk} - (\nabla_j S)_{ik}$ be the *Cotton tensor*. The Schouten tensor of any Einstein manifold is a scalar multiple of the metric and thus parallel. Hence Cotton-flatness (equivalently harmonic Weyl tensor) is a necessary condition to be Einstein since $\operatorname{div}_4 W = -\frac{n-3}{n-2}\, C$. It follows from the work in [15] that homogeneous Cotton-flat manifolds are symmetric in dimension four.

The necessary and sufficient conditions for a metric to be conformally Einstein were established by Brinkmann [4] in terms of the existence of positive solutions to the differential equation

$$(n-2)\operatorname{Hes}_\varphi + \varphi\,\rho - \frac{1}{n}\{(n-2)\Delta\varphi + \varphi\,\tau\} g = 0\,, \tag{1}$$

where $\operatorname{Hes}_\varphi = \nabla d\varphi$ is the Hessian tensor and $\Delta\varphi = \operatorname{tr}_g \operatorname{Hes}_\varphi$ denotes the Laplacian. Any four-dimensional conformally Einstein manifold satisfies (see, for example [13])

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$$\text{(2)} \qquad \text{(i)} \quad \mathfrak{B} = 0, \qquad \text{(ii)} \quad C(\cdot,\cdot,\cdot) - W(\cdot,\cdot,\cdot,\nabla\sigma) = 0,$$

where the conformal Einstein metric is given by $\widetilde{g} = e^{2\sigma}g$, $\sigma = -2\log\varphi$, and $\mathfrak{B} = \operatorname{div}_1\operatorname{div}_4 W + \frac{1}{2}W[\rho]$ is the Bach tensor. Moreover, it was shown in [12] that conditions in Equation (2) are also sufficient to be conformally Einstein if (M,g) is *weakly-generic* (i.e., the Weyl tensor viewed as a map $TM \to \bigotimes^3 TM$ is injective).

Four-dimensional Bach-flat homogeneous manifolds are either symmetric (and hence Einstein or locally conformally flat) or homothetic to one of the Lie groups determined by the following Lie algebras, where $\{e_i\}$ is an orthonormal basis (see [5]):

(a) The Lie algebra $\mathfrak{g}_\alpha = \mathbb{R}e_4 \ltimes \mathfrak{r}^3$ given by

$$[e_4,e_1] = e_1, \quad [e_4,e_2] = \frac{1}{4}e_2 + \alpha e_3, \quad [e_4,e_3] = -\alpha e_2 + \frac{1}{4}e_3\,.$$

(b) The Lie algebra $\mathfrak{g}_\alpha = \mathbb{R}e_4 \ltimes \mathfrak{h}^3$ given by

$$[e_1,e_2] = e_3 \quad [e_4,e_1] = e_1 - \alpha e_2, \quad [e_4,e_2] = \alpha e_1 + e_2, \quad [e_4,e_3] = 2e_3\,.$$

(c) The Lie algebra $\mathfrak{g}_\alpha = \mathbb{R}e_4 \ltimes \mathfrak{r}^3$ given by

$$[e_4,e_1] = e_1, \quad [e_4,e_2] = (\alpha+1)^2\,e_2, \quad [e_4,e_3] = \alpha^2\,e_3, \qquad \alpha > 1\,.$$

(d) The Lie algebra $\mathfrak{g} = \mathbb{R}e_4 \ltimes \mathfrak{e}(1,1)$ given by

$$\begin{aligned} [e_2,e_3] &= e_1, & [e_1,e_3] &= (2+\sqrt{3})\,e_2,\\ [e_4,e_1] &= \sqrt{6+3\sqrt{3}}\,e_1, & [e_4,e_2] &= \sqrt{6+3\sqrt{3}}\,e_2. \end{aligned}$$

(e) The Lie algebra $\mathfrak{g} = \mathbb{R}e_4 \ltimes \mathfrak{h}^3$ given by

$$\begin{aligned} [e_1,e_2] &= e_3, & [e_4,e_1] &= \tfrac{1}{4}\sqrt{7-3\sqrt{5}}\,e_1,\\ [e_2,e_4] &= \tfrac{1}{4}\sqrt{7+3\sqrt{5}}\,e_2, & [e_3,e_4] &= \tfrac{\sqrt{5}}{2\sqrt{2}}\,e_3\,. \end{aligned}$$

It follows from the work in [5] that Lie groups corresponding to cases (a), (b) and (c) are conformally Einstein, while the Lie gropus determined by (d) and (e) fail to satisfy condition (ii) in Equation (2) (see [1]).

A Riemannian manifold (M,g) is said to be *conformally Cotton* if there is a Cotton-flat representative of the conformal class $[g]$. This is achieved if there is a smooth function such that Equation (2)-(ii) is satisfied since the Weyl tensor of the conformal metric satisfies $\widetilde{\operatorname{div}}\widetilde{W} = \operatorname{div} W + (n-3)\iota_{\nabla\sigma}W$. More generally, (M,g) is said to be a *conformal C-space* if there is a (not necessarily gradient) vector field ξ on M so that $C - i_\xi W = 0$ (see [9] for more information on conformal C-spaces). It was shown in [9, Theorem 1.2] that any compact conformal C-space is conformally Einstein if and only if it is Bach-flat, independently of any weakly-genericity assumption. It follows from the work in [1, 5] that a conformal C-space is conformally Einstein if and only if it is Bach-flat in the homogeneous setting as well.

Our main purpose in this work is to prove Theorem 1.1 below, which provides a complete description of homogeneous conformal C-spaces in dimension four. Since any symmetric space is Cotton-flat, we exclude these trivial cases (which corresponds to Einstein metrics and products of the form $\mathbb{R}\times N(c)$ or $N_1(c_1)\times N_2(c_2)$ where $N(c)$ denotes a space of constant curvature). Furthermore, observe that two conformally related homogeneous spaces are either locally conformally flat or homothetic. Hence we work modulo homotheties to show the existence of a one-parameter family of homothetical classes of homogeneous conformal C-spaces which are not conformally Einstein.

Theorem 1.1. *Let (M, g) be a four-dimensional complete and simply connected homogeneous manifold which is a conformal C-space. Then (M, g) is Bach-flat or otherwise it is homothetic to a Lie group determined by the solvable Lie algebra $\mathfrak{g}_\alpha = \mathbb{R}e_4 \ltimes \mathfrak{r}^3$ given by*

$$[e_1, e_4] = e_1, \quad [e_2, e_4] = e_2, \quad [e_3, e_4] = \alpha e_3,$$

where $\alpha \notin \{0, 1, 4\}$ and $\{e_1, \ldots, e_4\}$ is an orthonormal basis. Here, $\xi = -6e_4$ and thus (M, g) is indeed conformally Cotton-flat.

It was shown in [9, Theorem 1.1] that any compact conformal C-space is indeed conformally Cotton. It immediately follows from Theorem 1.1 that the same result holds true in the homogeneous case.

By a result of Bérard-Bergery [3], a complete and simply connected homogeneous four-manifold is either symmetric or a Lie group. In particular, either (M, g) is isometric to one of the groups $\widetilde{SL(2,\mathbb{R})} \times \mathbb{R}$, $SU(2) \times \mathbb{R}$ or it is a solvable Lie group whose Lie algebra is an extension of the three-dimensional unimodular Lie algebras: the abelian Lie algebra $\mathfrak{r}^3$, the Heisenberg algebra $\mathfrak{h}^3$, the Poincaré algebra $\mathfrak{e}(1,1)$ and the Euclidean algebra $\mathfrak{e}(2)$. Since symmetric spaces are Cotton-flat, the analysis of the conformal C-space condition is considered separately for the different four-dimensional Lie groups through Sections 2–5. Determining the left-invariant conformal C-space metrics on Lie groups equals to solve some rather complicated polynomial systems. We make use of Gröbner bases theory [6, 7] to achive the results. Finally, the proof of Theorem 1.1 and some remarks are given in Section 6.

2. Left-invariant metrics on $\mathbb{R}e_4 \ltimes \mathbb{R}^3$

Let $\mathfrak{g} = \mathbb{R} \ltimes \mathfrak{r}^3$ be a semi-direct extension of the Abelian Lie algebra $\mathfrak{r}^3$. Let $\langle \cdot, \cdot \rangle$ be an inner product on $\mathfrak{g}$ and $\langle \cdot, \cdot \rangle_3$ its restriction to $\mathfrak{r}^3$. The algebra of all derivations $\mathfrak{D}$ of $\mathfrak{r}^3$ is $\mathfrak{gl}(3, \mathbb{R})$. If we fix $\mathfrak{D} \in \mathfrak{gl}(3, \mathbb{R})$, there exists a $\langle \cdot, \cdot \rangle_3$-orthonormal basis $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ of $\mathfrak{r}^3$ where $\mathfrak{D}$ decomposes as a sum of a diagonal matrix and a skew-symmetric matrix. Hence

$$\operatorname{der}(\mathfrak{r}^3) = \left\{ \begin{pmatrix} a & -b & -c \\ b & f & -h \\ c & h & p \end{pmatrix}; a, b, c, f, h, p \in \mathbb{R} \right\}.$$

Now, the corresponding semi-direct product $\mathfrak{g} = \mathbb{R} \ltimes \mathfrak{r}^3$ is given by

$$\begin{aligned}
&[\mathbf{v}_1, \mathbf{v}_2] = 0, \qquad [\mathbf{v}_1, \mathbf{v}_3] = 0, \quad [\mathbf{v}_2, \mathbf{v}_3] = 0, \\
&[\mathbf{v}_4, \mathbf{v}_1] = a\mathbf{v}_1 + b\mathbf{v}_2 + c\mathbf{v}_3, \qquad [\mathbf{v}_4, \mathbf{v}_2] = -b\mathbf{v}_1 + f\mathbf{v}_2 + h\mathbf{v}_3, \\
&[\mathbf{v}_4, \mathbf{v}_3] = -c\mathbf{v}_1 - h\mathbf{v}_2 + p\mathbf{v}_3,
\end{aligned}$$

with respect to some basis $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ so that $\mathfrak{g} = \mathbb{R}\mathbf{v}_4 \oplus \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. Since $\mathbb{R}\mathbf{v}_4$ needs not to be orthogonal to $\mathfrak{r}^3$, we set $k_i = \langle \mathbf{v}_i, \mathbf{v}_4 \rangle$, for $i = 1, 2, 3$. Let $\hat{e}_4 = \mathbf{v}_4 - \sum_i k_i \mathbf{v}_i$ and normalize it to get an orthonormal basis $\{e_1, \ldots, e_4\}$ of $\mathfrak{g} = \mathbb{R} \oplus \mathfrak{r}^3$ so that

$$(3)\qquad \begin{aligned} &[e_4, e_1] = \tfrac{1}{R}\{ae_1 + be_2 + ce_3\}, \qquad [e_4, e_2] = \tfrac{1}{R}\{-be_1 + fe_2 + he_3\},\\ &[e_4, e_3] = \tfrac{1}{R}\{-ce_1 - he_2 + pe_3\}, \quad R > 0. \end{aligned}$$

Lemma 2.1. *The group $\mathbb{R}e_4 \ltimes \mathbb{R}^3$ admits a non-Bach-flat left-invariant conformal C-space metric if and only if it corresponds to a Lie group determined by one of the following solvable Lie algebras:*

(i) $[e_1, e_4] = -\frac{a}{R}e_1 - \frac{b}{R}e_2$, $\quad [e_2, e_4] = \frac{b}{R}e_1 - \frac{a}{R}e_2$, $\quad [e_3, e_4] = -\frac{p}{R}e_3$,
where $p \notin \{0, a, 4a\}$ and $a \neq 0$. Here, $\xi = \frac{6a}{R}e_4$.

(ii) $[e_1, e_4] = -\frac{a}{R}e_1 - \frac{c}{R}e_3$, $\quad [e_2, e_4] = -\frac{f}{R}e_2$, $\quad [e_3, e_4] = \frac{c}{R}e_1 - \frac{a}{R}e_3$,
where $f \notin \{0, a, 4a\}$ and $a \neq 0$. Here, $\xi = \frac{6a}{R}e_4$.

(iii) $[e_1, e_4] = -\frac{a}{R}e_1$, $\quad [e_2, e_4] = -\frac{f}{R}e_2 - \frac{h}{R}e_3$, $\quad [e_3, e_4] = \frac{h}{R}e_2 - \frac{f}{R}e_3$,
where $a \notin \{0, f, 4f\}$ and $f \neq 0$. Here, $\xi = \frac{6f}{R}e_4$.

Proof. Let $\mathfrak{C} = C - \iota_\xi W$. A long but straightforward calculation shows that the components $\mathfrak{C}_{ijk} = C_{ijk} - \sum_\alpha \xi_\alpha W_{ijk\alpha}$, considering the structure constants in Equation (3), are given by

$$(4)\qquad \begin{array}{llll}
\mathfrak{C}_{211} = \frac{1}{6R^2}\mathfrak{P}_{211}, & \mathfrak{C}_{212} = \frac{1}{6R^2}\mathfrak{P}_{212}, & \mathfrak{C}_{213} = \frac{1}{2R^2}\mathfrak{P}_{213}, & \mathfrak{C}_{214} = \frac{1}{R^3}\mathfrak{P}_{214},\\
\mathfrak{C}_{311} = \frac{1}{6R^2}\mathfrak{P}_{311}, & \mathfrak{C}_{312} = \frac{1}{2R^2}\mathfrak{P}_{312}, & \mathfrak{C}_{313} = \frac{1}{6R^2}\mathfrak{P}_{313}, & \mathfrak{C}_{314} = \frac{1}{R^3}\mathfrak{P}_{314},\\
\mathfrak{C}_{321} = \frac{1}{2R^2}\mathfrak{P}_{321}, & \mathfrak{C}_{322} = \frac{1}{6R^2}\mathfrak{P}_{322}, & \mathfrak{C}_{323} = \frac{1}{6R^2}\mathfrak{P}_{323}, & \mathfrak{C}_{324} = \frac{1}{R^3}\mathfrak{P}_{324},\\
\mathfrak{C}_{411} = \frac{1}{6R^3}\mathfrak{P}_{411}, & \mathfrak{C}_{412} = \frac{1}{2R^3}\mathfrak{P}_{412}, & \mathfrak{C}_{413} = \frac{1}{2R^3}\mathfrak{P}_{413}, & \mathfrak{C}_{414} = \frac{1}{6R^2}\mathfrak{P}_{414},\\
\mathfrak{C}_{421} = \frac{1}{2R^3}\mathfrak{P}_{421}, & \mathfrak{C}_{422} = \frac{1}{6R^3}\mathfrak{P}_{422}, & \mathfrak{C}_{423} = \frac{1}{2R^3}\mathfrak{P}_{423}, & \mathfrak{C}_{424} = \frac{1}{6R^2}\mathfrak{P}_{424},\\
\mathfrak{C}_{431} = \frac{1}{2R^3}\mathfrak{P}_{431}, & \mathfrak{C}_{432} = \frac{1}{2R^3}\mathfrak{P}_{432}, & \mathfrak{C}_{433} = \frac{1}{6R^3}\mathfrak{P}_{433}, & \mathfrak{C}_{434} = \frac{1}{6R^2}\mathfrak{P}_{434},
\end{array}$$

where the polynomials $\mathfrak{P}_{ijk}$'s correspond to:

$\mathfrak{P}_{211} = ((a-f)^2 - 2p^2 + (a+f)p)\xi_2 - 3(f-p)h\,\xi_3,$

$\mathfrak{P}_{212} = -((a-f)^2 - 2p^2 + (a+f)p)\xi_1 + 3(a-p)c\,\xi_3,$

$\mathfrak{P}_{213} = (f-p)h\,\xi_1 - (a-p)c\,\xi_2,$

$\mathfrak{P}_{214} = -(a-f)^2 b,$

$\mathfrak{P}_{311} = -3(f-p)h\,\xi_2 + (a^2 - 2f^2 + p^2 + (f-2p)a + fp)\xi_3,$

$\mathfrak{P}_{312} = (f-p)h\,\xi_1 - (a-f)b\,\xi_3,$

$\mathfrak{P}_{313} = -(a^2 - 2f^2 + p^2 + (f-2p)a + fp)\xi_1 + 3(a-f)b\,\xi_2,$

$\mathfrak{P}_{314} = -(a-p)^2 c,$

$\mathfrak{P}_{321} = (a-p)c\,\xi_2 - (a-f)b\,\xi_3,$

$\mathfrak{P}_{322} = -3(a-p)c\,\xi_1 - (2a^2 - (f-p)^2 - (f+p)a)\xi_3,$

$\mathfrak{P}_{323} = 3(a-f)b\,\xi_1 + (2a^2 - (f-p)^2 - (f+p)a)\xi_2,$

$\mathfrak{P}_{324} = -(f-p)^2 h,$

$\mathfrak{P}_{411} = -(2a^2 - (f-p)^2 - (f+p)a)R\,\xi_4$

$$
\begin{aligned}
&\quad - 6((2b^2 + 2c^2 + f^2 + p^2)a - (f + p)a^2 - 2(b^2 f + c^2 p)),\\
\mathfrak{P}_{412} &= (a - f)Rb\,\xi_4 - 2(2a^2 b - (f + p)bf + (f - 2p)ch + (ch - (f - p)b)a),\\
\mathfrak{P}_{413} &= (a - p)Rc\,\xi_4 - 2(2a^2 c + (2f - p)bh - (f + p)cp - (bh - (f - p)c)a),\\
\mathfrak{P}_{414} &= (2a^2 - (f - p)^2 - (f + p)a)\xi_1 - 3(a - f)b\,\xi_2 - 3(a - p)c\,\xi_3,\\
\mathfrak{P}_{421} &= (a - f)Rb\,\xi_4 - 2(a^2 b - (2f + p)bf + (f - 2p)ch + (ch + (f + p)b)a),\\
\mathfrak{P}_{422} &= (a^2 - 2f^2 + p^2 + (f - 2p)a + fp)R\,\xi_4\\
&\quad - 6(a^2 f + 2b^2 f - (2b^2 + f^2)a + (f - p)(2h^2 - fp)),\\
\mathfrak{P}_{423} &= (f - p)Rh\,\xi_4 + 2((2bc - (f - p)h)a - (f + p)bc - (f - p)(2f + p)h),\\
\mathfrak{P}_{424} &= -3(a - f)b\,\xi_1 - (a^2 + (f - 2p)a - (f - p)(2f + p))\xi_2 - 3(f - p)h\,\xi_3,\\
\mathfrak{P}_{431} &= (a - p)Rc\,\xi_4 - 2(a^2 c - 2cp^2 - abh - cfp + (f + p)ac + (2f - p)bh),\\
\mathfrak{P}_{432} &= (f - p)Rh\,\xi_4 + 2((2bc - (f - p)h)a - (f + p)bc - (f - p)(f + 2p)h),\\
\mathfrak{P}_{433} &= ((a - f)^2 - 2p^2 + (a + f)p)R\,\xi_4\\
&\quad + 6(a + f)p^2 - 6(a^2 + 2c^2 + f^2 + 2h^2)p + 12(ac^2 + fh^2),\\
\mathfrak{P}_{434} &= -3(a - p)c\,\xi_1 - 3(f - p)h\,\xi_2 - ((a - f)^2 - 2p^2 + (a + f)p)\xi_3\,.
\end{aligned}
$$

Hence, $\mathbb{R}e_4 \ltimes \mathbb{R}^3$ admits a left-invariant conformal C-space metric if and only if the structure constants in Equation (3) satisfy the equations $\{\mathfrak{P}_{ijk} = 0\}$. Since

$$\mathfrak{P}_{214} = -(a - f)^2 b, \quad \mathfrak{P}_{314} = -(a - p)^2 c, \quad \mathfrak{P}_{324} = -(f - p)^2 h,$$

we are led to the following cases:

(1) $f = a, p = a,$	(4) $f \neq a, b = 0, p \neq a, c = 0, p = f,$
(2) $f = a, p \neq a, c = h = 0,$	(5) $f \neq a, b = 0, p \neq a, c = 0, p \neq f, h = 0\,,$
(3) $f \neq a, b = 0, p = a, h = 0,.$	

Case (1): $f = a$, $p = a$. If $f = p = a$ then a direct calculation shows that the corresponding Lie group given by Equation (3) is locally conformally flat and therefore a symmetric manifold [16] and trivially Bach-flat.

Case (2): $f = a$, $p \neq a$, $c = h = 0$. In this case, from Equation (4) we get

$$\mathfrak{P}_{212} = -2(a - p)p\,\xi_1, \quad \mathfrak{P}_{211} = 2(a - p)p\,\xi_2, \quad \mathfrak{P}_{311} = -(a - p)p\,\xi_3\,.$$

If $p = 0$ then a direct calculation shows that the corresponding Lie group given by Equation (3) is locally conformally flat and therefore a symmetric manifold [16] and trivially Bach-flat. If $p \neq 0$ then necessarily $\xi_1 = \xi_2 = \xi_3 = 0$ and Equation (4) implies that

$$\mathfrak{P}_{411} = (a - p)p(6a - R\,\xi_4)$$

and therefore $\xi_4 = \frac{6a}{R}$. Now, a direct calculation shows that the manifold is a conformal C-space which is Bach-flat if and only if $p = 4a$, thus corresponding to Assertion (i) in Lemma 2.1.

Case (3): $f \neq a$, $b = 0$, $p = a$, $h = 0$. We use Equation (4) to get

$$\mathfrak{P}_{212} = (a - f) f\, \xi_1, \quad \mathfrak{P}_{211} = -(a - f) f\, \xi_2, \quad \mathfrak{P}_{311} = 2(a - f) f\, \xi_3 .$$

If $f = 0$ then a direct calculation shows that the corresponding Lie group given by Equation (3) is locally conformally flat and therefore a symmetric manifold [16] and trivially Bach-flat. If $f \neq 0$ then $\xi_1 = \xi_2 = \xi_3 = 0$ and Equation (4) implies that

$$\mathfrak{P}_{411} = (a - f) f (6a - R\, \xi_4)$$

and hence $\xi_4 = \frac{6a}{R}$. A direct calculation shows that the manifold is a conformal C-space which is Bach-flat if and only if $f = 4a$, thus corresponding to Assertion (ii) in Lemma 2.1.

Case (4): $f \neq a$, $b = 0$, $p \neq a$, $c = 0$, $p = f$. In this case, Equation (4) implies

$$\mathfrak{P}_{212} = -(a - f) a\, \xi_1, \quad \mathfrak{P}_{211} = (a - f) a\, \xi_2, \quad \mathfrak{P}_{311} = (a - f) a\, \xi_3 .$$

If $a = 0$ then a direct calculation shows that the corresponding Lie group given by Equation (3) is locally conformally flat and therefore a symmetric manifold [16] and trivially Bach-flat. If $a \neq 0$ then necessarily $\xi_1 = \xi_2 = \xi_3 = 0$ and from Equation (4) we get

$$\mathfrak{P}_{411} = 2(a - f) a (6f - R\, \xi_4)$$

and therefore $\xi_4 = \frac{6f}{R}$. Now, a direct calculation shows that the manifold is a conformal C-space which is Bach-flat if and only if $a = 4f$, thus corresponding to Assertion (iii) in Lemma 2.1.

Case (5): $f \neq a$, $b = 0$, $p \neq a$, $c = 0$, $p \neq f$, $h = 0$. With these conditions, we use Equation (4) to get

$$\begin{aligned}
\mathfrak{P}_{212} &= -((a - f)^2 - 2p^2 + (a + f)p)\xi_1, \\
\mathfrak{P}_{313} &= -(a^2 - 2f^2 + p^2 + fp + (f - 2p)a)\xi_1, \\
\mathfrak{P}_{211} &= ((a - f)^2 - 2p^2 + (a + f)p)\xi_2, \\
\mathfrak{P}_{323} &= (2a^2 - (f - p)^2 - (f + p)a)\xi_2, \\
\mathfrak{P}_{311} &= (a^2 - 2f^2 + p^2 + fp + (f - 2p)a)\xi_3, \\
\mathfrak{P}_{322} &= (-2a^2 + (f - p)^2 + (f + p)a)\xi_3 .
\end{aligned}$$

A direct calculation shows that

$$\begin{aligned}
\mathfrak{P}_{212} - \mathfrak{P}_{313} &= 3(f - p)(a - f - p)\xi_1, \\
\mathfrak{P}_{212} + 2\,\mathfrak{P}_{313} &= -3(a - f)(a + f - p)\xi_1, \\
\mathfrak{P}_{211} + \mathfrak{P}_{323} &= 3(a - p)(a - f + p)\xi_2, \\
\mathfrak{P}_{211} - 2\,\mathfrak{P}_{323} &= -3(a - f)(a + f - p)\xi_2, \\
\mathfrak{P}_{311} - \mathfrak{P}_{322} &= 3(a - f)(a + f - p)\xi_3, \\
\mathfrak{P}_{311} + 2\,\mathfrak{P}_{322} &= -3(a - p)(a - f + p)\xi_3 .
\end{aligned}$$

Since $f \neq a$, $p \neq a$ and $p \neq f$, it follows that necessarily $\xi_1 = \xi_2 = \xi_3 = 0$. Now, Equation (4) reduces to

$$
\begin{aligned}
\mathfrak{P}_{411} &= 6((f+p)a^2-(f^2+p^2)a)-(2a^2-(f-p)^2-(f+p)a)R\,\xi_4,\\
\mathfrak{P}_{433} &= 6((a+f)p^2-(a^2+f^2)p)-(2p^2-(a-f)^2-(a+f)p)R\,\xi_4,\\
\mathfrak{P}_{422} &= -\mathfrak{P}_{411}-\mathfrak{P}_{433},
\end{aligned}
\tag{5}
$$

and we compute

$$
\frac{\mathfrak{P}_{411}+2\,\mathfrak{P}_{433}}{6(f-p)}-\frac{\mathfrak{P}_{411}-\mathfrak{P}_{433}}{6(a-p)}=(a-f)^2-3(a+f)p+pR\,\xi_4=0\,.
$$

Note that $p=0$ is not possible since $f\neq a$. Then,

$$
\xi_4=\frac{-(a-f)^2+3(a+f)p}{pR}
$$

and hence Equation (5) transforms into

$$
\begin{aligned}
\mathfrak{P}_{411} &= \tfrac{1}{p}(a-f)(2a+f-3p)(p^2-2(a+f)p+(a-f)^2),\\
\mathfrak{P}_{433} &= -\tfrac{1}{p}(a-f)^2(p^2-2a(a+f)+(a-f)^2),\\
\mathfrak{P}_{422} &= -\mathfrak{P}_{411}-\mathfrak{P}_{433}\,.
\end{aligned}
$$

Thus

$$
p^2-2(a+f)p+(a-f)^2=0,
$$

so we finally conclude that the manifold is a conformal C-space if and only if

$$
p=a+f\pm 2\sqrt{af}\quad\text{and}\quad \xi_4=\frac{2(a+f\pm\sqrt{af})}{R}
$$

and, in such a case, a straightforward calculation shows that the manifold is Bach-flat. This finishes the proof. □

3. Left-invariant metrics on $\mathbb{R}e_4 \ltimes H^3$

Let $\mathfrak{g}=\mathbb{R}\ltimes\mathfrak{h}^3$ be a semi-direct extension of the Heisenberg algebra $\mathfrak{h}^3$. Let $\langle\cdot,\cdot\rangle$ be an inner product on $\mathfrak{g}$ and $\langle\cdot,\cdot\rangle_3$ its restriction to $\mathfrak{h}^3$. Then, there exists an orthonormal basis $\{\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3\}$ of $\mathfrak{h}^3$ such that (see [14])

$$
[\mathbf{v}_3,\mathbf{v}_2]=0,\qquad [\mathbf{v}_3,\mathbf{v}_1]=0,\qquad [\mathbf{v}_1,\mathbf{v}_2]=\lambda_3\mathbf{v}_3,
\tag{6}
$$

where $\lambda_3\neq 0$ is a real number. The algebra of all derivations of $\mathfrak{h}^3$ is given with respect to the basis $\{\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3\}$ by

$$
\operatorname{der}(\mathfrak{h}^3)=\left\{\begin{pmatrix}\alpha_{11}&\alpha_{12}&0\\ \alpha_{21}&\alpha_{22}&0\\ \hat h&\hat f&\alpha_{11}+\alpha_{22}\end{pmatrix};\ \alpha_{ij},\hat f,\hat h\in\mathbb{R}\right\}.
$$

We rotate the basis elements $\{\mathbf{v}_1,\mathbf{v}_2\}$ so that the matrix $A=(\alpha_{ij})$ decomposes as the sum of a diagonal matrix and a skew-symmetric matrix. Hence

$$
\operatorname{der}(\mathfrak{h}^3)=\left\{\begin{pmatrix}a&c&0\\ -c&d&0\\ h&f&a+d\end{pmatrix};\ a,c,d,f,h\in\mathbb{R}\right\},
$$

and consider the Lie algebra $\mathfrak{g} = \mathbb{R}\mathbf{v}_4 \oplus \mathfrak{h}^3$ given by

$$\begin{array}{lll} [\mathbf{v}_3, \mathbf{v}_2] = 0, & [\mathbf{v}_3, \mathbf{v}_1] = 0, & [\mathbf{v}_1, \mathbf{v}_2] = \gamma \mathbf{v}_3, \\ [\mathbf{v}_4, \mathbf{v}_1] = a\mathbf{v}_1 - c\mathbf{v}_2 + h\mathbf{v}_3, & [\mathbf{v}_4, \mathbf{v}_2] = c\mathbf{v}_1 + d\mathbf{v}_2 + f\mathbf{v}_3, & [\mathbf{v}_4, \mathbf{v}_3] = (a+d)\mathbf{v}_3. \end{array}$$

Since $\mathbb{R}\mathbf{v}_4$ needs not to be orthogonal to $\mathfrak{h}^3$, we set $k_i = \langle \mathbf{v}_i, \mathbf{v}_4 \rangle$, for $i = 1, 2, 3$. Let $\hat{e}_4 = \mathbf{v}_4 - \sum_i k_i \mathbf{v}_i$ and normalize it to get an orthonormal basis $\{e_1, \ldots, e_4\}$ of $\mathfrak{g} = \mathbb{R} \oplus \mathfrak{h}^3$ so that

$$\begin{array}{lll} [e_1, e_2] = \gamma e_3, & [e_4, e_1] = \frac{1}{R}\{ae_1 - ce_2 + (h + k_2\gamma)e_3\}, & \\ [e_4, e_3] = \frac{1}{R}(a+d)e_3, & [e_4, e_2] = \frac{1}{R}\{ce_1 + de_2 + (f - k_1\gamma)e_3\}, & R > 0. \end{array} \tag{7}$$

Lemma 3.1. *The group $\mathbb{R}e_4 \ltimes H^3$ does not admit any non-Bach-flat left-invariant conformal C-space metric.*

Proof. In order to simplify the expressions we use the notation $F = f - k_1\gamma$ and $H = h + k_2\gamma$. Moreover, since the structure constant of $\mathfrak{h}^3$ satisfies $\gamma \neq 0$, one may work with a homothetic basis $\widetilde{e}_k = \frac{1}{\gamma}e_k$ so that we may assume $\gamma = 1$. Let $\mathfrak{C} = C - \iota_\xi W$. A long but straightforward calculation shows that the components $\mathfrak{C}_{ijk} = C_{ijk} - \sum_\alpha \xi_\alpha W_{ijk\alpha}$, considering the structure constants in Equation (7), are given by

$$\begin{array}{llll} \mathfrak{C}_{211} = \frac{1}{12R^2}\mathfrak{P}_{211}, & \mathfrak{C}_{212} = \frac{1}{12R^2}\mathfrak{P}_{212}, & \mathfrak{C}_{213} = \frac{1}{4R^2}\mathfrak{P}_{213}, & \mathfrak{C}_{214} = \frac{1}{4R^3}\mathfrak{P}_{214}, \\ \mathfrak{C}_{311} = \frac{1}{12R^2}\mathfrak{P}_{311}, & \mathfrak{C}_{312} = \frac{1}{4R^2}\mathfrak{P}_{312}, & \mathfrak{C}_{313} = \frac{1}{12R^2}\mathfrak{P}_{313}, & \mathfrak{C}_{314} = \frac{1}{2R^3}\mathfrak{P}_{314}, \\ \mathfrak{C}_{321} = \frac{1}{4R^2}\mathfrak{P}_{321}, & \mathfrak{C}_{322} = \frac{1}{12R^2}\mathfrak{P}_{322}, & \mathfrak{C}_{323} = \frac{1}{12R^2}\mathfrak{P}_{323}, & \mathfrak{C}_{324} = \frac{1}{2R^3}\mathfrak{P}_{324}, \\ \mathfrak{C}_{411} = \frac{1}{12R^3}\mathfrak{P}_{411}, & \mathfrak{C}_{412} = \frac{1}{4R^3}\mathfrak{P}_{412}, & \mathfrak{C}_{413} = \frac{1}{4R^3}\mathfrak{P}_{413}, & \mathfrak{C}_{414} = \frac{1}{12R^2}\mathfrak{P}_{414}, \\ \mathfrak{C}_{421} = \frac{1}{4R^3}\mathfrak{P}_{421}, & \mathfrak{C}_{422} = \frac{1}{12R^3}\mathfrak{P}_{422}, & \mathfrak{C}_{423} = \frac{1}{4R^3}\mathfrak{P}_{423}, & \mathfrak{C}_{424} = \frac{1}{12R^2}\mathfrak{P}_{424}, \\ \mathfrak{C}_{431} = \frac{1}{4R^3}\mathfrak{P}_{431}, & \mathfrak{C}_{432} = \frac{1}{4R^3}\mathfrak{P}_{432}, & \mathfrak{C}_{433} = \frac{1}{12R^3}\mathfrak{P}_{433}, & \mathfrak{C}_{434} = \frac{1}{12R^2}\mathfrak{P}_{434}, \end{array} \tag{8}$$

where the polynomials $\mathfrak{P}_{ijk}$'s correspond to:

$\mathfrak{P}_{211} = 2(F^2 + H^2 - 4ad - 2R^2)\xi_2 + 3((a+2d)F + cH)\xi_3 + 6RH\xi_4 + 9cF - 3(4a+9d)H,$

$\mathfrak{P}_{212} = -(2(F^2 + H^2) - 8ad - 4R^2)\xi_1 + (3cF - 3(2a+d)H)\xi_3 + 6RF\xi_4 - 3((9a+4d)F + 3cH),$

$\mathfrak{P}_{213} = -((a+2d)F + cH)\xi_1 - (cF - (2a+d)H)\xi_2 + 2(a+d)R\xi_4 + 4(F^2 + H^2 - (a+d)^2 + R^2),$

$\mathfrak{P}_{214} = -2R^2H\xi_1 - 2R^2F\xi_2 - 2(a+d)R^2\xi_3 + cF^2 + cH^2 + 3(a-d)FH + 4(a-d)^2c,$

$\mathfrak{P}_{311} = 3((a+2d)F + cH)\xi_2 - 2(2F^2 - H^2 - 2ad - R^2)\xi_3 - 6(a-d)c,$

$\mathfrak{P}_{312} = -((a+2d)F + cH)\xi_1 + 2(FH + (a-d)c)\xi_3 + 2dR\xi_4 + 2(F^2 + H^2 - 2(a+d)d + R^2),$

$\mathfrak{P}_{313} = 2(2F^2 - H^2 - 2ad - R^2)\xi_1 - 6(FH + (a-d)c)\xi_2 - 6RF\xi_4 + 3((5a+4d)F + cH),$

$\mathfrak{P}_{314} = -dR^2\xi_2 + R^2F\xi_3 - (a-2d)cF - (F^2 + H^2 + d^2 + R^2)H,$

$\mathfrak{P}_{321} = (cF - (2a+d)H)\xi_2 + 2(FH + (a-d)c)\xi_3 - 2aR\xi_4 - 2(F^2 + H^2 + R^2) + 4(a+d)a,$

$\mathfrak{P}_{322} = -(3cF - 3(2a+d)H)\xi_1 + 2(F^2 - 2H^2 + 2ad + R^2)\xi_3 + 6(a-d)c,$

$\mathfrak{P}_{323} = -6(FH + (a-d)c)\xi_1 - 2(F^2 - 2H^2 + 2ad + R^2)\xi_2 + 6RH\xi_4 + 3(cF - (4a+5d)H),$

$\mathfrak{P}_{324} = aR^2\xi_1 - R^2H\xi_3 - a^2F - (2a-d)cH - (F^2 + H^2 + R^2)F,$

$\mathfrak{P}_{411} = 6R^2H\xi_2 + 2(F^2 - 2H^2 + 2ad + R^2)R\xi_4$
$\quad + 3(9dH^2 - 7cFH - 2(F^2 - 3H^2 + 4c^2 + 4d^2 - R^2)a + 8c^2d),$

$\mathfrak{P}_{412} = -2R^2H\xi_1 - 2aR^2\xi_3 - 2(FH + (a-d)c)R\xi_4$

$$
\begin{aligned}
&\quad + 4cH^2 + (10a + 7d)FH - (3F^2 - 4(a - d)(3a + 2d))c,\\
\mathfrak{P}_{413} &= 2aR^2\xi_2 + (cF - (2a + d)H)R\,\xi_4\\
&\quad + 4a^2H - 2(5cF - 7dH)a - 2(cdF + c^2H + 2(F^2 + H^2 - d^2 + R^2)H),\\
\mathfrak{P}_{414} &= -2(F^2 - 2H^2 + 2ad + R^2)\xi_1 + 6(FH + (a - d)c)\xi_2 - (3cF - 3(2a + d)H)\xi_3 + 6(2aF + cH),\\
\mathfrak{P}_{421} &= 2R^2F\,\xi_2 + 2dR^2\xi_3 - 2(FH + (a - d)c)R\,\xi_4\\
&\quad + 3cH^2 + (7a + 10d)FH - 4(F^2 - 2a^2 + 3d^2 - ad)c,\\
\mathfrak{P}_{422} &= -6R^2F\,\xi_1 - 2(2F^2 - H^2 - 2ad - R^2)R\,\xi_4\\
&\quad + 3(7cFH + (9F^2 + 8c^2)a + 2(3F^2 - H^2 + R^2)d - 8(a^2 + c^2)d),\\
\mathfrak{P}_{423} &= -2dR^2\xi_1 - ((a + 2d)F + cH)R\,\xi_4\\
&\quad + 2(2a^2 - c^2)F + 10cdH + 2(7dF + cH)a - 4(F^2 + H^2 - d^2 + R^2)F,\\
\mathfrak{P}_{424} &= 6(FH + (a - d)c)\xi_1 + 2(2F^2 - H^2 - 2ad - R^2)\xi_2 + 3((a + 2d)F + cH)\xi_3 + 6(cF - 2dH),\\
\mathfrak{P}_{431} &= 2(a + d)R^2\xi_2 - 2R^2F\,\xi_3 + (cF - (2a + d)H)R\,\xi_4\\
&\quad + 4a^2H - 2a(4cF - 7dH) - 2(3cdF + c^2H + (F^2 + H^2 - 3d^2 + R^2)H),\\
\mathfrak{P}_{432} &= -2(a + d)R^2\xi_1 + 2R^2H\,\xi_3 - ((a + 2d)F + cH)R\,\xi_4\\
&\quad + (6a^2 - 2c^2 + 14ad + 4d^2)F + 2(3a + 4d)cH - 2(F^2 + H^2 + R^2)F,\\
\mathfrak{P}_{433} &= 6R^2F\,\xi_1 - 6R^2H\,\xi_2 + 2(F^2 + H^2 - 4ad - 2R^2)R\,\xi_4\\
&\quad - 3((7a + 6d)F^2 + (6a + 7d)H^2 - 8(a + d)ad + 2(a + d)R^2),\\
\mathfrak{P}_{434} &= 3((2a + d)H - cF)\xi_1 + 3((a + 2d)F + cH)\xi_2 - 2(F^2 + H^2 - 4ad - 2R^2)\xi_3\,.
\end{aligned}
$$

Hence, $\mathbb{R}e_4 \ltimes H^3$ admits a left-invariant conformal C-space metric if and only if the structure constants in Equation (7) satisfy the equations $\{\mathfrak{P}_{ijk} = 0\}$. We consider separately the following cases:

$$
(1)\, d = 0, \qquad (2)\, d \neq 0, H = 0, \qquad (3)\, d \neq 0, H \neq 0\,.
$$

Case (1): $d = 0$. Let $\mathcal{I}_1 \subset \mathbb{R}[a, c, d, F, H, R, \xi_1, \xi_2, \xi_3, \xi_4]$ be the ideal generated by the polynomials $\{\mathfrak{P}_{ijk}\} \cup \{d\}$. We compute a Gröbner basis $\mathcal{G}_1$ of $\mathcal{I}_1$ with respect to the lexicographical order and a detailed analysis of the Gröbner basis shows that the polynomial

$$
\mathbf{g}_1 = (F^2 + H^2 + R^2)R^2
$$

belongs to $\mathcal{G}_1$. Since the zero sets of $\{\mathfrak{P}_{ijk} = 0, d = 0\}$ and $\mathcal{I}_1 = \langle \mathfrak{P}_{ijk}, d\rangle = \langle \mathcal{G}_1\rangle$ coincide, and $R > 0$, we conclude that there is no solution in this case.

Case (2): $d \neq 0$, $H = 0$. Let $\mathcal{I}_2 \subset \mathbb{R}[\xi_4, F, H, d, a, c, R, \xi_1, \xi_2, \xi_3]$ be the ideal generated by the polynomials $\{\mathfrak{P}_{ijk}\} \cup \{H\}$. We compute a Gröbner basis $\mathcal{G}_2$ of $\mathcal{I}_2$ with respect to the lexicographical order and a detailed analysis of the Gröbner basis shows that the polynomials

$$
\mathbf{g}_2 = (F^2 + R^2)F, \qquad \mathbf{g}_2' = (\xi_2^2 + \xi_3^2)dR^2, \qquad \mathbf{g}_2'' = F^2d + (d - a)R^2
$$

belong to $\mathcal{G}_2$. As a consequence, $F = \xi_2 = \xi_3 = 0$ and $d = a$. Now, Equation (8) implies that

$$
\mathfrak{P}_{212} = 4(2a^2 + R^2)\xi_1
$$

and therefore $\xi_1 = 0$. Using again Equation (8) we get

$$
\mathfrak{P}_{213} = 4(aR\,\xi_4 - 4a^2 + R^2), \qquad \mathfrak{P}_{411} = (4a^2 + 2R^2)R\,\xi_4 - 24a^3 + 6aR^2\,.
$$

Thus, necessarily $a \neq 0$ and $\xi_4 = \frac{4a^2 - R^2}{aR}$, while $a = \pm R$ or $a = \pm\frac{R}{2}$. Now, a straightforward calculation shows that, in any case, the manifold is Bach-flat.

Case (3): $d \neq 0$, $H \neq 0$. In this last case, we are not able to get a Gröbner basis using the initial polynomials $\mathfrak{P}_{ijk}$. Thus, our strategy consists in reducing the number of variables as follows. Since $dH \neq 0$, and also $R \neq 0$, we can use Equation (8) to get expressions for ξ_2, ξ_3 and ξ_4. In particular, from $\mathfrak{P}_{324}$ we obtain

$$\xi_3 = \frac{1}{HR^2}\{aR^2\xi_1 - a^2F - (2a-d)cH - (F^2+H^2+R^2)F\},$$

from $\mathfrak{P}_{314}$ we get

$$\xi_2 = \frac{1}{dR^2}\{FR^2\xi_3 - (a-2d)cF - (d^2+F^2+H^2+R^2)H\}$$

and, finally, using $\mathfrak{P}_{323}$ we get

$$\xi_4 = \frac{1}{6HR}\{6(FH+(a-d)c)\xi_1 + 2(F^2-2H^2+2ad+R^2)\xi_2 - 3(cF-(4a+5d)H)\}\,.$$

Thus, we can eliminate the variables ξ_2, ξ_3 and ξ_4 from the polynomials $\mathfrak{P}_{ijk}$ in Equation (8). Let us denote by $\mathfrak{Q}'_{ijk}$ the expressions obtained from the polynomials $\mathfrak{P}_{ijk}$ after substituting ξ_2, ξ_3 and ξ_4 with the expressions above. These expressions $\mathfrak{Q}'_{ijk}$ are not directly polynomials in $\mathbb{R}[a,c,d,F,H,R,\xi_1]$ since they contain powers of d, H and R with negative exponents. We avoid this problem considering $\mathfrak{P}'_{ijk}$ given by

$$\begin{array}{llll}
\widetilde{\mathfrak{P}}'_{211} = dHR^2\,\mathfrak{Q}'_{211}, & \widetilde{\mathfrak{P}}'_{212} = dH^2R^2\,\mathfrak{Q}'_{212}, & \widetilde{\mathfrak{P}}'_{213} = 3dH^2R^2\,\mathfrak{Q}'_{213}, & \widetilde{\mathfrak{P}}'_{214} = dH\,\mathfrak{Q}'_{214},\\
\widetilde{\mathfrak{P}}'_{311} = dHR^2\,\mathfrak{Q}'_{311}, & \widetilde{\mathfrak{P}}'_{312} = 3H^2R^2\,\mathfrak{Q}'_{312}, & \widetilde{\mathfrak{P}}'_{313} = dH^2R^2\,\mathfrak{Q}'_{313}, & \widetilde{\mathfrak{P}}'_{314} = 0,\\
\widetilde{\mathfrak{P}}'_{321} = 3dH^2R^2\,\mathfrak{Q}'_{321}, & \widetilde{\mathfrak{P}}'_{322} = HR^2\,\mathfrak{Q}'_{322}, & \widetilde{\mathfrak{P}}'_{323} = 0, & \widetilde{\mathfrak{P}}'_{324} = 0,\\
\widetilde{\mathfrak{P}}'_{411} = 3dH^2R^2\,\mathfrak{Q}'_{411}, & \widetilde{\mathfrak{P}}'_{412} = 3dH^2R^2\,\mathfrak{Q}'_{412}, & \widetilde{\mathfrak{P}}'_{413} = 6dH^2R^2\,\mathfrak{Q}'_{413}, & \widetilde{\mathfrak{P}}'_{414} = dHR^2\,\mathfrak{Q}'_{414},\\
\widetilde{\mathfrak{P}}'_{421} = 3dH^2R^2\,\mathfrak{Q}'_{421}, & \widetilde{\mathfrak{P}}'_{422} = 3dH^2R^2\,\mathfrak{Q}'_{422}, & \widetilde{\mathfrak{P}}'_{423} = 6dH^2R^2\,\mathfrak{Q}'_{423}, & \widetilde{\mathfrak{P}}'_{424} = dHR^2\,\mathfrak{Q}'_{424},\\
\widetilde{\mathfrak{P}}'_{431} = 6dH^2R^2\,\mathfrak{Q}'_{431}, & \widetilde{\mathfrak{P}}'_{432} = 6dH^2R^2\,\mathfrak{Q}'_{432}, & \widetilde{\mathfrak{P}}'_{433} = 3dH^2R^2\,\mathfrak{Q}'_{433}, & \widetilde{\mathfrak{P}}'_{434} = dHR^2\,\mathfrak{Q}'_{434},
\end{array}$$

which are polynomials in $\mathbb{R}[a,c,d,F,H,R,\xi_1]$. Now, Let $\mathcal{I}_3 \subset \mathbb{R}[a,c,d,F,H,R,\xi_1]$ be the ideal generated by the polynomials $\mathfrak{P}'_{ijk}$. We compute a Gröbner basis $\mathcal{G}_3$ of $\mathcal{I}_3$ with respect to the graded reverse lexicographical order and a detailed analysis of the Gröbner basis shows that the polynomial

$$\mathbf{g}_3 = d^4H^5R^2$$

belongs to $\mathcal{G}_3$. Since we are assuming $dH \neq 0$, and moreover $R > 0$, we conclude that there is no solution in this case. This finishes the proof. □

4. Left-invariant metrics on $\mathbb{R}e_4 \ltimes E(1,1)$ and $\mathbb{R}e_4 \ltimes E(2)$

Let $\mathfrak{g} = \mathbb{R} \ltimes \mathfrak{g}_3$ be a semi-direct extension of the unimodular Lie algebra $\mathfrak{g}_3 = \mathfrak{e}(1,1)$ or $\mathfrak{g}_3 = \mathfrak{e}(2)$. Let $\langle\cdot,\cdot\rangle$ be an inner product on $\mathfrak{g}$ and $\langle\cdot,\cdot\rangle_3$ its restriction to $\mathfrak{g}_3$. Following the work of Milnor [14], there exists an orthonormal basis $\{\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3\}$ of $\mathfrak{g}_3$ such that

$$[\mathbf{v}_2,\mathbf{v}_3] = \lambda_1\mathbf{v}_1, \qquad [\mathbf{v}_3,\mathbf{v}_1] = \lambda_2\mathbf{v}_2, \qquad [\mathbf{v}_1,\mathbf{v}_2] = 0, \tag{9}$$

where $\lambda_1,\lambda_2 \in \mathbb{R}$ and $\lambda_1\lambda_2 \neq 0$. Moreover, the associated Lie group corresponds to $E(2)$ (resp., $E(1,1)$) if $\lambda_1\lambda_2 > 0$ (resp., $\lambda_1\lambda_2 < 0$). The algebra of derivations of $\mathfrak{g}_3$ is given by

$$\operatorname{der}(\mathfrak{g}_3) = \left\{ \begin{pmatrix} b & a & c \\ -\frac{\lambda_2}{\lambda_1} a & b & d \\ 0 & 0 & 0 \end{pmatrix} ; a, b, c, d \in \mathbb{R} \right\}.$$

Let $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ be a basis of $\mathfrak{g}$ such that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ are given by Equation (9) and $\mathfrak{g} = \mathbb{R}\mathbf{v}_4 \oplus \mathfrak{g}_3$. Since $\mathbb{R}\mathbf{v}_4$ needs not to be orthogonal to $\mathfrak{g}_3$, we set $k_i = \langle \mathbf{v}_i, \mathbf{v}_4 \rangle$, for $i = 1, 2, 3$. Let $\hat{e}_4 = \mathbf{v}_4 - \sum_i k_i \mathbf{v}_i$ and normalize it to get an orthonormal basis $\{e_1, \ldots, e_4\}$ of $\mathfrak{g} = \mathbb{R} \oplus \mathfrak{g}_3$ so that

$$\begin{aligned} &[e_2, e_3] = \lambda_1 e_1, && [e_3, e_1] = \lambda_2 e_2, \\ &[e_4, e_1] = \tfrac{1}{R}\{b e_1 - \lambda_2(\tfrac{a}{\lambda_1} + k_3) e_2\}, && [e_4, e_2] = \tfrac{1}{R}\{(a + k_3\lambda_1) e_1 + b e_2\}, \\ &[e_4, e_3] = \tfrac{1}{R}\{(c - k_2\lambda_1) e_1 + (d + k_1\lambda_2) e_2\}, \quad R > 0. && \end{aligned} \tag{10}$$

Lemma 4.1. *The groups $\mathbb{R}e_4 \ltimes E(1,1)$ and $\mathbb{R}e_4 \ltimes E(2)$ do not admit any non-Bach-flat left-invariant conformal C-space metric.*

Proof. Unlike the cases $\mathbb{R}e_4 \ltimes \mathbb{R}^3$ and $\mathbb{R}e_4 \ltimes H^3$, for $\mathbb{R}e_4 \ltimes E(1,1)$ and $\mathbb{R}e_4 \ltimes E(2)$ we are not able to get the left-invariant conformal C-space metrics using Equation (2)-(ii) only. It was shown in [8] that a Riemannian four-manifold with non-vanishing Weyl tensor is a conformal C-space if and only if (observe that our convention for the Cotton tensor is different from the one in [8])

$$|W|^2\, C_{kji} - 4\, W^{dabc} C_{cba} W_{dijk} = 0\,. \tag{11}$$

In what follows, we combine Equation (2)-(ii) and Equation (11) to get the appropriate Gröbner bases.

In order to simplify the expressions, from now on we use the notation $A = \frac{a}{\lambda_1} + k_3$, $C = c - k_2\lambda_1$ and $D = d + k_1\lambda_2$. The components $\mathfrak{C}_{ijk} = C_{ijk} - \sum_\alpha \xi_\alpha W_{ijk\alpha}$ and $\widetilde{\mathfrak{C}}_{ijk} = |W|^2\, C_{kji} - 4\, W^{dabc} C_{cba} W_{dijk}$ of the $(0,3)$-tensor fields associated to Equations (2)-(ii) and (11) determine polynomials $\mathfrak{P}_{ijk}$ and $\widetilde{\mathfrak{P}}_{ijk}$ as follows:

$$\begin{aligned} &\mathfrak{C}_{211} = \tfrac{1}{12R^2}\mathfrak{P}_{211}, && \mathfrak{C}_{212} = \tfrac{1}{12R^2}\mathfrak{P}_{212}, && \mathfrak{C}_{213} = \tfrac{1}{4R^2}\mathfrak{P}_{213}, && \mathfrak{C}_{214} = \tfrac{1}{4R^3}\mathfrak{P}_{214}, \\ &\mathfrak{C}_{311} = \tfrac{1}{12R^2}\mathfrak{P}_{311}, && \mathfrak{C}_{312} = \tfrac{1}{4R^2}\mathfrak{P}_{312}, && \mathfrak{C}_{313} = \tfrac{1}{12R^2}\mathfrak{P}_{313}, && \mathfrak{C}_{314} = \tfrac{1}{4R^3}\mathfrak{P}_{314}, \\ &\mathfrak{C}_{321} = \tfrac{1}{4R^2}\mathfrak{P}_{321}, && \mathfrak{C}_{322} = \tfrac{1}{12R^2}\mathfrak{P}_{322}, && \mathfrak{C}_{323} = \tfrac{1}{12R^2}\mathfrak{P}_{323}, && \mathfrak{C}_{324} = \tfrac{1}{4R^3}\mathfrak{P}_{324}, \\ &\mathfrak{C}_{411} = \tfrac{1}{12R^3}\mathfrak{P}_{411}, && \mathfrak{C}_{412} = \tfrac{1}{4R^3}\mathfrak{P}_{412}, && \mathfrak{C}_{413} = \tfrac{1}{4R^3}\mathfrak{P}_{413}, && \mathfrak{C}_{414} = \tfrac{1}{12R^2}\mathfrak{P}_{414}, \\ &\mathfrak{C}_{421} = \tfrac{1}{4R^3}\mathfrak{P}_{421}, && \mathfrak{C}_{422} = \tfrac{1}{12R^3}\mathfrak{P}_{422}, && \mathfrak{C}_{423} = \tfrac{1}{4R^3}\mathfrak{P}_{423}, && \mathfrak{C}_{424} = \tfrac{1}{12R^2}\mathfrak{P}_{424}, \\ &\mathfrak{C}_{431} = \tfrac{1}{4R^3}\mathfrak{P}_{431}, && \mathfrak{C}_{432} = \tfrac{1}{4R^3}\mathfrak{P}_{432}, && \mathfrak{C}_{433} = \tfrac{1}{12R^3}\mathfrak{P}_{433}, && \mathfrak{C}_{434} = \tfrac{1}{12R^2}\mathfrak{P}_{434}, \end{aligned} \tag{12}$$

and

$$
(13)\qquad
\begin{aligned}
&\widetilde{\mathfrak{C}}_{121} = \tfrac{1}{12R^6}\widetilde{\mathfrak{P}}_{121},\ \widetilde{\mathfrak{C}}_{131} = \tfrac{1}{12R^6}\widetilde{\mathfrak{P}}_{131},\ \widetilde{\mathfrak{C}}_{132} = \tfrac{1}{12R^6}\widetilde{\mathfrak{P}}_{132},\ \widetilde{\mathfrak{C}}_{141} = \tfrac{1}{12R^7}\widetilde{\mathfrak{P}}_{141},\\
&\widetilde{\mathfrak{C}}_{142} = \tfrac{1}{12R^7}\widetilde{\mathfrak{P}}_{142},\ \widetilde{\mathfrak{C}}_{143} = \tfrac{1}{12R^7}\widetilde{\mathfrak{P}}_{143},\ \widetilde{\mathfrak{C}}_{221} = \tfrac{1}{12R^6}\widetilde{\mathfrak{P}}_{221},\ \widetilde{\mathfrak{C}}_{231} = \tfrac{1}{12R^6}\widetilde{\mathfrak{P}}_{231},\\
&\widetilde{\mathfrak{C}}_{232} = \tfrac{1}{12R^6}\widetilde{\mathfrak{P}}_{232},\ \widetilde{\mathfrak{C}}_{241} = \tfrac{1}{12R^7}\widetilde{\mathfrak{P}}_{241},\ \widetilde{\mathfrak{C}}_{242} = \tfrac{1}{12R^7}\widetilde{\mathfrak{P}}_{242},\ \widetilde{\mathfrak{C}}_{243} = \tfrac{1}{12R^7}\widetilde{\mathfrak{P}}_{243},\\
&\widetilde{\mathfrak{C}}_{321} = \tfrac{1}{12R^6}\widetilde{\mathfrak{P}}_{321},\ \widetilde{\mathfrak{C}}_{331} = \tfrac{1}{12R^6}\widetilde{\mathfrak{P}}_{331},\ \widetilde{\mathfrak{C}}_{332} = \tfrac{1}{12R^6}\widetilde{\mathfrak{P}}_{332},\ \widetilde{\mathfrak{C}}_{341} = \tfrac{1}{12R^7}\widetilde{\mathfrak{P}}_{341},\\
&\widetilde{\mathfrak{C}}_{342} = \tfrac{1}{12R^7}\widetilde{\mathfrak{P}}_{342},\ \widetilde{\mathfrak{C}}_{343} = \tfrac{1}{12R^7}\widetilde{\mathfrak{P}}_{343},\ \widetilde{\mathfrak{C}}_{421} = \tfrac{1}{12R^7}\widetilde{\mathfrak{P}}_{421},\ \widetilde{\mathfrak{C}}_{431} = \tfrac{1}{12R^7}\widetilde{\mathfrak{P}}_{431},\\
&\widetilde{\mathfrak{C}}_{432} = \tfrac{1}{12R^7}\widetilde{\mathfrak{P}}_{432},\ \widetilde{\mathfrak{C}}_{441} = \tfrac{1}{12R^6}\widetilde{\mathfrak{P}}_{441},\ \widetilde{\mathfrak{C}}_{442} = \tfrac{1}{12R^6}\widetilde{\mathfrak{P}}_{442},\ \widetilde{\mathfrak{C}}_{443} = \tfrac{1}{12R^6}\widetilde{\mathfrak{P}}_{443}\,.
\end{aligned}
$$

The expressions of the polynomials $\mathfrak{P}_{ijk}$ and $\widetilde{\mathfrak{P}}_{ijk}$ are very lengthy, so we do not include them here for the sake of clarity. They can be obtained after a long but straightforward calculation using the Weyl curvature tensor and the Cotton tensor of $\mathbb{R}e_4 \ltimes E(1,1)$ and $\mathbb{R}e_4 \ltimes E(2)$. In particular, the Weyl curvature tensor is determined by

$$
\begin{aligned}
6R^2W_{1212} &= 6R^2W_{3434} = (A^2+R^2)(\lambda_1-\lambda_2)^2 - 2(C^2+D^2),\\
4R^2W_{1213} &= -4R^2W_{2434} = AC(2\lambda_1-\lambda_2) + bD,\\
4R^2W_{1223} &= 4R^2W_{1434} = -AD(\lambda_1-2\lambda_2) - bC,\\
6R^2W_{1313} &= 6R^2W_{2424} = -(A^2(2\lambda_1+\lambda_2) - R^2(\lambda_1+2\lambda_2))(\lambda_1-\lambda_2) + C^2 + D^2,\\
2R^2W_{1323} &= -2R^2W_{1424} = Ab(\lambda_1-\lambda_2),\\
6R^2W_{1414} &= 6R^2W_{2323} = (A^2(\lambda_1+2\lambda_2) - R^2(2\lambda_1+\lambda_2))(\lambda_1-\lambda_2) + C^2 + D^2,
\end{aligned}
$$

while the Cotton tensor is given by

$$
\begin{aligned}
4R^2C_{211} &= -bC(5\lambda_1-3\lambda_2),\\
4R^2C_{212} &= bD(3\lambda_1-5\lambda_2),\\
4R^2C_{213} &= -2(A^2+R^2)(\lambda_1-\lambda_2)^2(\lambda_1+\lambda_2) - C^2(2\lambda_1-\lambda_2) + D^2(\lambda_1-2\lambda_2),\\
4R^3C_{214} &= (2(A^2+R^2)(\lambda_1-\lambda_2)^2(\lambda_1+\lambda_2) + C^2(2\lambda_1-\lambda_2) - D^2(\lambda_1-2\lambda_2))A,\\
4R^2C_{311} &= 4Ab(\lambda_1^2-\lambda_2^2) - CD(\lambda_1+4\lambda_2),\\
4R^2C_{312} &= 2(A^2+R^2)(\lambda_1^3-2\lambda_2^3+\lambda_1\lambda_2^2) - (4b^2-2C^2-D^2)\lambda_1 + 4(b^2-D^2)\lambda_2,\\
4R^2C_{313} &= AC(\lambda_1+\lambda_2)\lambda_2 - 3bD(\lambda_1-3\lambda_2),\\
4R^3C_{314} &= 2(A^2+R^2)C\lambda_1^2 - A^2C\lambda_2^2 + AbD\lambda_1 - 3AbD\lambda_2 - A^2C\lambda_1\lambda_2 + 2(b^2+C^2+D^2)C,\\
4R^2C_{321} &= 2(A^2+R^2)(2\lambda_1^2+\lambda_2^2+\lambda_1\lambda_2)(\lambda_1-\lambda_2) - 4b^2(\lambda_1-\lambda_2) + C^2(4\lambda_1-\lambda_2) - 2D^2\lambda_2,\\
4R^2C_{322} &= -4Ab(\lambda_1^2-\lambda_2^2) + CD(4\lambda_1+\lambda_2),\\
4R^2C_{323} &= AD(\lambda_1+\lambda_2)\lambda_1 - 3bC(3\lambda_1-\lambda_2),\\
4R^3C_{324} &= -A^2D\lambda_1^2 + 2D(A^2+R^2)\lambda_2^2 + 3AbC\lambda_1 - AbC\lambda_2 - A^2D\lambda_1\lambda_2 + 2(b^2+C^2+D^2)D,\\
4R^3C_{411} &= -2b(3A^2+R^2)(\lambda_1^2-\lambda_2^2) + ACD(\lambda_1+4\lambda_2) - (7C^2+2D^2)b,\\
4R^3C_{412} &= -2A(A^2+R^2)(\lambda_1^3-2\lambda_2^3+\lambda_1\lambda_2^2) + (4b^2-2C^2-D^2)A\lambda_1 - 4(b^2-D^2)A\lambda_2 - 5bCD,\\
4R^3C_{413} &= -2(A^2+R^2)C\lambda_1^2 + CR^2\lambda_2^2 + 2AbD\lambda_1 - 10AbD\lambda_2 + CR^2\lambda_1\lambda_2 + 2(3b^2-C^2-D^2)C,\\
4R^2C_{414} &= -(AC(\lambda_1+\lambda_2) + 4bD)\lambda_2,\\
4R^3C_{421} &= -A(A^2+R^2)(4\lambda_1^3-2\lambda_2^3-2\lambda_1^2\lambda_2) + 4A(b^2-C^2)\lambda_1 - (4b^2-C^2-2D^2)A\lambda_2 - 5bCD,\\
4R^3C_{422} &= 2b(3A^2+R^2)(\lambda_1^2-\lambda_2^2) - ACD(4\lambda_1+\lambda_2) - (2C^2+7D^2)b,\\
4R^3C_{423} &= DR^2\lambda_1^2 - 2(A^2+R^2)D\lambda_2^2 + DR^2\lambda_1\lambda_2 + AbC(10\lambda_1-2\lambda_2) + (6b^2-2(C^2+D^2))D,
\end{aligned}
$$

$$
\begin{aligned}
4R^2C_{424} &= -(AD(\lambda_1+\lambda_2)-4bC)\lambda_1,\\
4R^3C_{431} &= -C(A^2+R^2)(4\lambda_1^2-\lambda_2^2-\lambda_1\lambda_2)+AbD(\lambda_1-7\lambda_2)+4(b^2-C^2-D^2)C,\\
4R^3C_{432} &= D(A^2+R^2)(\lambda_1^2-4\lambda_2^2+\lambda_1\lambda_2)+AbC(7\lambda_1-\lambda_2)+4(b^2-C^2-D^2)D,\\
4R^3C_{433} &= 3ACD(\lambda_1-\lambda_2)+9(C^2+D^2)b,\\
4R^2C_{434} &= 3CD(\lambda_1-\lambda_2)\,.
\end{aligned}
$$

Hence, $\mathbb{R}e_4 \ltimes E(1,1)$ or $\mathbb{R}e_4 \ltimes E(2)$ admit a left-invariant conformal C-space metric if and only if the structure constants in Equation (10) satisfy the equations $\{\mathfrak{P}_{ijk}=0\}$ or, equivalently, $\{\widetilde{\mathfrak{P}}_{ijk}=0\}$.

Note that since the structure constants of $\mathfrak{g}_3$ satisfy $\lambda_1\lambda_2\neq 0$, one may work with a homothetic basis $\widetilde{e}_k=\frac{1}{\lambda_1}e_k$ so that we may assume $\lambda_1=1$ in the rest of the proof. First, we start working with the polynomials $\widetilde{\mathfrak{P}}_{ijk}$ given by Equation (13). Let $\widetilde{\mathcal{I}}\subset\mathbb{R}[A,b,C,D,\lambda_2,R]$ be the ideal generated by the polynomials $\widetilde{\mathfrak{P}}_{ijk}$. We compute a Gröbner basis $\widetilde{\mathcal{G}}$ of $\widetilde{\mathcal{I}}$ with respect to the graded reverse lexicographical order and a detailed analysis of the Gröbner basis shows that the polynomial

$$\widetilde{\mathbf{g}}=b^3CD(C-D)^2(C+D)^2(C^2+D^2)R^2$$

belongs to $\widetilde{\mathcal{G}}$. Since the zero sets of $\{\widetilde{\mathfrak{P}}_{ijk}=0\}$ and $\widetilde{\mathcal{I}}=\langle\widetilde{\mathfrak{P}}_{ijk}\rangle=\langle\widetilde{\mathcal{G}}\rangle$ coincide, and $R>0$, we are led to the following cases:

$$(1)\,C=0,\quad (2)\,D=0,\quad (3)\,b=0,\quad (4)\,D=\pm C\,.$$

Next we analyze each one of these cases by separate using the polynomials $\mathfrak{P}_{ijk}$ given by Equation (12).

Case (1): $C=0$. Let $\mathcal{I}_1\subset\mathbb{R}[A,b,C,D,R,\lambda_2,\xi_1,\xi_2,\xi_3,\xi_4]$ be the ideal generated by the polynomials $\{\mathfrak{P}_{ijk}\}\cup\{C\}$. We compute a Gröbner basis $\mathcal{G}_1$ of $\mathcal{I}_1$ with respect to the graded reverse lexicographical order and a detailed analysis of the Gröbner basis shows that the polynomials

$$
\begin{aligned}
\mathbf{g}_1 &= D(8A^2+5D^2+8R^2),\\
\mathbf{g}_1' &= (A^2+R^2)R^2(\lambda_2-1)\xi_1,\\
\mathbf{g}_1'' &= (A^2+R^2)R^2(\lambda_2-1)\xi_2,\\
\mathbf{g}_1''' &= (9AD\xi_1+4(A^2+R^2)(\lambda_2-1)\xi_3)R^2
\end{aligned}
$$

belong to $\mathcal{G}_1$. Since the zero sets of $\{\mathfrak{P}_{ijk}=0, C=0\}$ and $\mathcal{I}_1=\langle\mathfrak{P}_{ijk}\rangle=\langle\mathcal{G}_1\rangle$ coincide, and $R>0$, it follows that necessarily $D=0$ and, moreover, either $\lambda_2=1$ or $\xi_1=\xi_2=\xi_3=0$. If $\lambda_2=1$ then a straightforward calculation shows that the corresponding Lie group given by Equation (10) is locally conformally flat and therefore a symmetric manifold [16] and trivially Bach-flat. Now, if $\lambda_2\neq 1$ and $\xi_1=\xi_2=\xi_3=0$, then we compute

$$\mathfrak{P}_{433}=2(A^2+R^2)R(\lambda_2-1)^2\xi_4,$$

which shows that there is no solution in this case.

Case (2): $D = 0$. Let $\mathcal{I}_2 \subset \mathbb{R}[A, b, C, D, R, \lambda_2, \xi_1, \xi_2, \xi_3, \xi_4]$ be the ideal generated by the polynomials $\{\mathfrak{P}_{ijk}\} \cup \{D\}$. We compute a Gröbner basis $\mathcal{G}_2$ of $\mathcal{I}_2$ with respect to the graded reverse lexicographical order and a detailed analysis of the Gröbner basis shows that the polynomial

$$\mathbf{g}_2 = C(C^2 + R^2)(A^2 + C^2 + R^2)$$

belongs to $\mathcal{G}_2$. Thus necessarily $C = 0$, which corresponds to Case (1).

Case (3): $b = 0$. Let $\mathcal{I}_3 \subset \mathbb{R}[A, b, C, D, R, \lambda_2, \xi_1, \xi_2, \xi_3, \xi_4]$ be the ideal generated by the polynomials $\{\mathfrak{P}_{ijk}\} \cup \{b\}$. We compute a Gröbner basis $\mathcal{G}_3$ of $\mathcal{I}_3$ with respect to the graded reverse lexicographical order and a detailed analysis of the Gröbner basis shows that the polynomial

$$\mathbf{g}_3 = D^2(C^2 + D^2)$$

belongs to $\mathcal{G}_3$. Thus necessarily $D = 0$, which corresponds to Case (2).

Case (4): $D = \varepsilon C$ ($\varepsilon = \pm 1$). Let $\mathcal{I}_4 \subset \mathbb{R}[A, b, C, D, R, \lambda_2, \xi_1, \xi_2, \xi_3, \xi_4]$ be the ideal generated by the polynomials $\{\mathfrak{P}_{ijk}\} \cup \{D - \varepsilon C\}$. We compute a Gröbner basis $\mathcal{G}_4$ of $\mathcal{I}_4$ with respect to the graded reverse lexicographical order and a detailed analysis of the Gröbner basis shows that the polynomial

$$\mathbf{g}_4 = D(4A^2 + 5D^2 + 4R^2)R$$

belongs to $\mathcal{G}_4$. Hence $D = 0$, which corresponds to Case (2). This finishes the proof. □

5. Left-invariant metrics on $\widetilde{SL(2,\mathbb{R})} \times \mathbb{R}$ and $SU(2) \times \mathbb{R}$

Let $\mathfrak{g} = \mathfrak{g}_3 \times \mathbb{R}$ be a direct extension of the unimodular Lie algebra $\mathfrak{g}_3 = \mathfrak{sl}(2, \mathbb{R})$ or $\mathfrak{g}_3 = \mathfrak{su}(2)$. Let $\langle \cdot, \cdot \rangle$ be an inner product on $\mathfrak{g}$ and let $\langle \cdot, \cdot \rangle_3$ denote its restriction to $\mathfrak{g}_3$. Following the work of Milnor [14], there exists an orthonormal basis $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ of $\mathfrak{g}_3$ such that

$$[\mathbf{v}_2, \mathbf{v}_3] = \lambda_1 \mathbf{v}_1, \qquad [\mathbf{v}_3, \mathbf{v}_1] = \lambda_2 \mathbf{v}_2, \qquad [\mathbf{v}_1, \mathbf{v}_2] = \lambda_3 \mathbf{v}_3, \tag{14}$$

where $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$ and $\lambda_1 \lambda_2 \lambda_3 \neq 0$. Moreover, the associated Lie group corresponds to $SU(2)$ (resp., $SL(2, \mathbb{R})$) if $\lambda_1, \lambda_2, \lambda_3$ are all positive (resp., if any of $\lambda_1, \lambda_2, \lambda_3$ is negative).

Let $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ be a basis of $\mathfrak{g}$ such that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ are given by Equation (14) and $\mathfrak{g} = \mathfrak{g}_3 \oplus \mathbb{R}\mathbf{v}_4$. Since $\mathbb{R}\mathbf{v}_4$ needs not to be orthogonal to $\mathfrak{g}_3$, we set $k_i = \langle \mathbf{v}_i, \mathbf{v}_4 \rangle$, for $i = 1, 2, 3$. Let $\hat{e}_4 = \mathbf{v}_4 - \sum_i k_i \mathbf{v}_i$ and normalize it to get an orthonormal basis $\{e_1, \ldots, e_4\}$ of $\mathfrak{g} = \mathfrak{g}_3 \oplus \mathbb{R}$ so that

$$\begin{aligned} &[e_1, e_2] = \lambda_3 e_3, \qquad [e_2, e_3] = \lambda_1 e_1, \qquad [e_3, e_1] = \lambda_2 e_2, \\ &[e_1, e_4] = \tfrac{1}{R}(k_3 \lambda_2 e_2 - k_2 \lambda_3 e_3), \qquad [e_2, e_4] = \tfrac{1}{R}(k_1 \lambda_3 e_3 - k_3 \lambda_1 e_1), \\ &[e_3, e_4] = \tfrac{1}{R}(k_2 \lambda_1 e_1 - k_1 \lambda_2 e_2), \qquad R > 0. \end{aligned} \tag{15}$$

Lemma 5.1. *The groups $\widetilde{SL(2,\mathbb{R})} \times \mathbb{R}$ and $SU(2) \times \mathbb{R}$ do not admit any non-Bach-flat left-invariant conformal C-space metric.*

Proof. As in the cases $\mathbb{R}e_4 \ltimes E(1,1)$ and $\mathbb{R}e_4 \ltimes E(2)$, for $\widetilde{SL(2,\mathbb{R})} \times \mathbb{R}$ and $SU(2) \times \mathbb{R}$ we are not able to get the left-invariant conformal C-space metrics using Equation (2)-(ii) only. We proceed as in Lemma 4.1 combining Equation (2)-(ii) and Equation (11) to get the appropriate Gröbner bases. The components $\mathfrak{C}_{ijk} = C_{ijk} - \sum_\alpha \xi_\alpha W_{ijk\alpha}$ and $\widetilde{\mathfrak{C}}_{ijk} = |W|^2\, C_{kji} - 4\, W^{dabc} C_{cba} W_{dijk}$ of the $(0,3)$-tensor fields associated to Equations (2)-(ii) and (11) determine polynomials $\mathfrak{P}_{ijk}$ and $\widetilde{\mathfrak{P}}_{ijk}$ with exactly the same expressions than in Lemma 4.1 (Equations (12) and (13)). The expressions of the polynomials $\mathfrak{P}_{ijk}$ and $\widetilde{\mathfrak{P}}_{ijk}$ are very lengthy, so we do not include them here for the sake of clarity. They can be obtained after a long but straightforward calculation using the Weyl curvature tensor and the Cotton tensor of $\widetilde{SL(2,\mathbb{R})} \times \mathbb{R}$ and $SU(2) \times \mathbb{R}$. The Weyl curvature tensor is determined by

$$\begin{aligned}
6R^2 W_{1212} = {}& 6R^2 W_{3434} = -(\lambda_2 - \lambda_3)(2\lambda_2 + \lambda_3)k_1^2 + (\lambda_1 - \lambda_2)^2 k_3^2 \\
& - (2\lambda_1^2 - \lambda_3^2 - \lambda_1\lambda_3)k_2^2 + ((\lambda_1 - \lambda_2)^2 - 2\lambda_3^2 + (\lambda_1 + \lambda_2)\lambda_3)R^2, \\
4R^2 W_{1213} = {}& -4R^2 W_{2434} = -(2\lambda_1 - \lambda_2 - \lambda_3)\lambda_1 k_2 k_3, \\
4R^2 W_{1223} = {}& 4R^2 W_{1434} = -(\lambda_1 - 2\lambda_2 + \lambda_3)\lambda_2 k_1 k_3, \\
6R^2 W_{1313} = {}& 6R^2 W_{2424} = (\lambda_2 - \lambda_3)(\lambda_2 + 2\lambda_3)k_1^2 + (\lambda_1 - \lambda_3)^2 k_2^2 \\
& - (2\lambda_1^2 - \lambda_2^2 - \lambda_1\lambda_2)k_3^2 + (\lambda_1^2 - 2\lambda_2^2 + \lambda_3^2 + \lambda_1(\lambda_2 - 2\lambda_3) + \lambda_2\lambda_3)R^2, \\
4R^2 W_{1323} = {}& -4R^2 W_{1424} = (\lambda_1 + \lambda_2 - 2\lambda_3)\lambda_3 k_1 k_2, \\
6R^2 W_{1414} = {}& 6R^2 W_{2323} = (\lambda_2 - \lambda_3)^2 k_1^2 + (\lambda_1 - \lambda_3)(\lambda_1 + 2\lambda_3)k_2^2 \\
& + (\lambda_1 - \lambda_2)(\lambda_1 + 2\lambda_2)k_3^2 - (2\lambda_1^2 - (\lambda_2 - \lambda_3)^2 - \lambda_1(\lambda_2 + \lambda_3))R^2,
\end{aligned}$$

while the Cotton tensor is given by

$$\begin{aligned}
4R^2 C_{211} = {}& -(\lambda_1^2 - \lambda_1(3\lambda_2 - 4\lambda_3) - \lambda_2(\lambda_2 + \lambda_3))\lambda_3 k_1 k_3, \\
4R^2 C_{212} = {}& (\lambda_1^2 + \lambda_1(3\lambda_2 + \lambda_3) - \lambda_2(\lambda_2 + 4\lambda_3))\lambda_3 k_2 k_3, \\
4R^2 C_{213} = {}& (\lambda_1(\lambda_2 + \lambda_3) - 2(\lambda_2^2 + 2\lambda_3^2 + \lambda_2\lambda_3))(\lambda_2 - \lambda_3)k_1^2 \\
& - (2\lambda_1^3 - \lambda_1^2\lambda_2 + 2\lambda_1\lambda_3^2 + (\lambda_2 - 4\lambda_3)\lambda_3^2)k_2^2 \\
& - 2(\lambda_1 - \lambda_2)^2(\lambda_1 + \lambda_2)k_3^2 \\
& - 2((\lambda_1 - \lambda_2)^2(\lambda_1 + \lambda_2) + (\lambda_1 + \lambda_2)\lambda_3^2 - 2\lambda_3^3)R^2, \\
4R^3 C_{214} = {}& \{-(\lambda_1(\lambda_2 + 2\lambda_3) - \lambda_2(2\lambda_2 + \lambda_3))(\lambda_2 - \lambda_3)k_1^2 \\
& + (2\lambda_1^2 - \lambda_1(\lambda_2 - \lambda_3) - 2\lambda_2\lambda_3)(\lambda_1 - \lambda_3)k_2^2 \\
& + 2(\lambda_1 - \lambda_2)^2(\lambda_1 + \lambda_2)k_3^2 \\
& + 2(\lambda_1 - \lambda_2)^2(\lambda_1 + \lambda_2)R^2\}k_3, \\
4R^2 C_{311} = {}& (\lambda_1^2 + \lambda_1(4\lambda_2 - 3\lambda_3) - (\lambda_2 + \lambda_3)\lambda_3)\lambda_2 k_1 k_2, \\
4R^2 C_{312} = {}& (\lambda_1(\lambda_2 + \lambda_3) - 2(2\lambda_2^2 + \lambda_2\lambda_3 + \lambda_3^2))(\lambda_2 - \lambda_3)k_1^2 \\
& + 2(\lambda_1 - \lambda_3)^2(\lambda_1 + \lambda_3)k_2^2 \\
& + (2\lambda_1^3 + 2\lambda_1\lambda_2^2 - \lambda_1^2\lambda_3 - \lambda_2^2(4\lambda_2 - \lambda_3))k_3^2 \\
& + 2(\lambda_1^3 - 2\lambda_2^3 - \lambda_1^2\lambda_3 + \lambda_2^2\lambda_3 + \lambda_3^3 + \lambda_1(\lambda_2^2 - \lambda_3^2))R^2, \\
4R^2 C_{313} = {}& -(\lambda_1^2 + \lambda_1(\lambda_2 + 3\lambda_3) - (4\lambda_2 + \lambda_3)\lambda_3)\lambda_2 k_2 k_3, \\
4R^3 C_{314} = {}& \{-(\lambda_1(2\lambda_2 + \lambda_3) - (\lambda_2 + 2\lambda_3)\lambda_3)(\lambda_2 - \lambda_3)k_1^2 \\
& - 2(\lambda_1 - \lambda_3)^2(\lambda_1 + \lambda_3)k_2^2 \\
& - (\lambda_1(2\lambda_1 + \lambda_2) - (\lambda_1 + 2\lambda_2)\lambda_3)(\lambda_1 - \lambda_2)k_3^2 \\
& - 2(\lambda_1 - \lambda_3)^2(\lambda_1 + \lambda_3)R^2\}k_2,
\end{aligned}$$

$$\begin{aligned}
4R^2C_{321} &= -2(\lambda_2-\lambda_3)^2(\lambda_2+\lambda_3)k_1^2\\
&\quad + (4\lambda_1^3-\lambda_1^2(\lambda_2+2\lambda_3)+(\lambda_2-2\lambda_3)\lambda_3^2)k_2^2\\
&\quad + (4\lambda_1^3-\lambda_1^2(2\lambda_2+\lambda_3)-(2\lambda_2-\lambda_3)\lambda_2^2)k_3^2\\
&\quad + 2(2\lambda_1^3-\lambda_1^2(\lambda_2+\lambda_3)-(\lambda_2-\lambda_3)^2(\lambda_2+\lambda_3))R^2,\\
4R^2C_{322} &= (\lambda_3^2-(4\lambda_1+\lambda_2)\lambda_2+(\lambda_1+3\lambda_2)\lambda_3)\lambda_1k_1k_2,\\
4R^2C_{323} &= (\lambda_2^2-\lambda_3^2+\lambda_1(\lambda_2-4\lambda_3)+3\lambda_2\lambda_3)\lambda_1k_1k_3,\\
4R^3C_{324} &= \{2(\lambda_2-\lambda_3)^2(\lambda_2+\lambda_3)k_1^2\\
&\quad - ((\lambda_1+2\lambda_2)\lambda_2-(2\lambda_1+\lambda_2)\lambda_3)(\lambda_1-\lambda_2)k_3^2\\
&\quad + (\lambda_1(2\lambda_2-\lambda_3)+(\lambda_2-2\lambda_3)\lambda_3)(\lambda_1-\lambda_3)k_2^2\\
&\quad + 2(\lambda_2-\lambda_3)^2(\lambda_2+\lambda_3)R^2\}k_1,\\
4R^3C_{411} &= -(\lambda_1^2+4\lambda_1(\lambda_2+\lambda_3)+3\lambda_2\lambda_3)(\lambda_2-\lambda_3)k_1k_2k_3,\\
4R^3C_{412} &= \{(-(\lambda_1-4\lambda_2)\lambda_2^2-(2\lambda_1+\lambda_2)\lambda_3^2+(\lambda_1-\lambda_2)\lambda_2\lambda_3)k_1^2\\
&\quad - 2(\lambda_1^3+\lambda_1\lambda_2\lambda_3-2\lambda_2\lambda_3^2)k_2^2\\
&\quad - 2(\lambda_1^3+\lambda_1\lambda_2^2-2\lambda_2^3)k_3^2\\
&\quad - (2(\lambda_1^2+\lambda_1\lambda_2+2\lambda_2^2)-(\lambda_1+\lambda_2)\lambda_3-\lambda_3^2)(\lambda_1-\lambda_2)R^2\}k_3,\\
4R^3C_{413} &= \{(2\lambda_1\lambda_2^2+(\lambda_1+\lambda_2)\lambda_3^2-(\lambda_1-\lambda_2)\lambda_2\lambda_3-4\lambda_3^3)k_1^2\\
&\quad + 2(\lambda_1^3+\lambda_1\lambda_3^2-2\lambda_3^3)k_2^2\\
&\quad + 2(\lambda_1^3+(\lambda_1-2\lambda_2)\lambda_2\lambda_3)k_3^2\\
&\quad + (2\lambda_1^2-\lambda_1\lambda_2-\lambda_2^2+2\lambda_1\lambda_3-\lambda_2\lambda_3+4\lambda_3^2)(\lambda_1-\lambda_3)R^2\}k_2,\\
4R^2C_{414} &= (\lambda_1^2+\lambda_1(\lambda_2+\lambda_3)-3\lambda_2\lambda_3)(\lambda_2-\lambda_3)k_2k_3,\\
4R^3C_{421} &= \{2(\lambda_2^3+\lambda_1\lambda_2\lambda_3-2\lambda_1\lambda_3^2)k_1^2\\
&\quad - 2(2\lambda_1^3-\lambda_1^2\lambda_2-\lambda_2^3)k_3^2\\
&\quad - (4\lambda_1^3-\lambda_1^2(\lambda_2+\lambda_3)+\lambda_1(\lambda_2-\lambda_3)\lambda_3-2\lambda_2\lambda_3^2)k_2^2\\
&\quad - (4\lambda_1^2+\lambda_1(2\lambda_2-\lambda_3)+(\lambda_2-\lambda_3)(2\lambda_2+\lambda_3))(\lambda_1-\lambda_2)R^2\}k_3,\\
4R^3C_{422} &= (4\lambda_1\lambda_2+3\lambda_1\lambda_3+\lambda_2^2+4\lambda_2\lambda_3)(\lambda_1-\lambda_3)k_1k_2k_3,\\
4R^3C_{423} &= \{-2(\lambda_2^3+\lambda_2\lambda_3^2-2\lambda_3^3)k_1^2\\
&\quad - (\lambda_1^2(2\lambda_2+\lambda_3)-\lambda_1(\lambda_2-\lambda_3)\lambda_3+(\lambda_2-4\lambda_3)\lambda_3^2)k_2^2\\
&\quad + 2(\lambda_1(2\lambda_1-\lambda_2)\lambda_3-\lambda_2^3)k_3^2\\
&\quad + (\lambda_1^2+\lambda_1(\lambda_2+\lambda_3)-2(\lambda_2^2+\lambda_2\lambda_3+2\lambda_3^2))(\lambda_2-\lambda_3)R^2\}k_1,\\
4R^2C_{424} &= -(\lambda_1(\lambda_2-3\lambda_3)+\lambda_2(\lambda_2+\lambda_3))(\lambda_1-\lambda_3)k_1k_3,\\
4R^3C_{431} &= \{(2\lambda_1\lambda_2(2\lambda_2-\lambda_3)-2\lambda_3^3)k_1^2\\
&\quad + (4\lambda_1^3-2\lambda_1^2\lambda_3-2\lambda_3^3)k_2^2\\
&\quad + (4\lambda_1^3-\lambda_1^2(\lambda_2+\lambda_3)-\lambda_1\lambda_2(\lambda_2-\lambda_3)-2\lambda_2^2\lambda_3)k_3^2\\
&\quad + (4\lambda_1^2-\lambda_1(\lambda_2-2\lambda_3)-(\lambda_2-\lambda_3)(\lambda_2+2\lambda_3))(\lambda_1-\lambda_3)R^2\}k_2,\\
4R^3C_{432} &= \{-2(2\lambda_2^3-\lambda_2^2\lambda_3-\lambda_3^3)k_1^2\\
&\quad - 2(2\lambda_1^2\lambda_2-\lambda_1\lambda_2\lambda_3-\lambda_3^3)k_2^2\\
&\quad + (\lambda_1^2(\lambda_2+2\lambda_3)+\lambda_1\lambda_2(\lambda_2-\lambda_3)-\lambda_2^2(4\lambda_2-\lambda_3))k_3^2\\
&\quad + (\lambda_1^2+\lambda_1(\lambda_2+\lambda_3)-2(2\lambda_2^2+\lambda_2\lambda_3+\lambda_3^2))(\lambda_2-\lambda_3)R^2\}k_1,\\
4R^3C_{433} &= -(3\lambda_1\lambda_2+4(\lambda_1+\lambda_2)\lambda_3+\lambda_3^2)(\lambda_1-\lambda_2)k_1k_2k_3,\\
4R^2C_{434} &= -(\lambda_1(3\lambda_2-\lambda_3)-(\lambda_2+\lambda_3)\lambda_3)(\lambda_1-\lambda_2)k_1k_2\,.
\end{aligned}$$

Hence, $\widetilde{SL(2,\mathbb{R})}\times\mathbb{R}$ or $SU(2)\times\mathbb{R}$ admit a left-invariant conformal C-space metric if and only if the structure constants in Equation (15) satisfy the equations $\{\mathfrak{P}_{ijk}=0\}$ or, equivalently, $\{\widetilde{\mathfrak{P}}_{ijk} = 0\}$.

Note that since the structure constants of $\mathfrak{g}_3$ satisfy $\lambda_1\lambda_2\lambda_3 \neq 0$, one may work with a homothetic basis $\widetilde{e}_k = \frac{1}{\lambda_1}e_k$ so that we may assume $\lambda_1 = 1$ in the rest of the proof. We consider separately the following cases:

$$(1)\, k_1k_2k_3 \neq 0, \qquad (2)\, k_1k_2k_3 = 0\,.$$

Case (1): $k_1k_2k_3 \neq 0$. In this first case we work with the polynomials $\widetilde{\mathfrak{P}}_{ijk}$. A key observation is that, in this case, most of the polynomials $\widetilde{\mathfrak{P}}_{ijk}$ can be further decomposed. In particular,

$$\begin{array}{llll}
\widetilde{\mathfrak{P}}_{121} = k_1k_3\lambda_3\widetilde{\mathfrak{P}}'_{121}, & \widetilde{\mathfrak{P}}_{131} = k_1k_2\lambda_2\widetilde{\mathfrak{P}}'_{131}, & \widetilde{\mathfrak{P}}_{132} = \widetilde{\mathfrak{P}}'_{132}, & \widetilde{\mathfrak{P}}_{141} = k_1k_2k_3\widetilde{\mathfrak{P}}'_{141}, \\
\widetilde{\mathfrak{P}}_{142} = k_3\widetilde{\mathfrak{P}}'_{142}, & \widetilde{\mathfrak{P}}_{143} = k_2\widetilde{\mathfrak{P}}'_{143}, & \widetilde{\mathfrak{P}}_{221} = k_2k_3\lambda_3\widetilde{\mathfrak{P}}'_{221}, & \widetilde{\mathfrak{P}}_{231} = \widetilde{\mathfrak{P}}'_{231}, \\
\widetilde{\mathfrak{P}}_{232} = k_1k_2\widetilde{\mathfrak{P}}'_{232}, & \widetilde{\mathfrak{P}}_{241} = k_3\widetilde{\mathfrak{P}}'_{241}, & \widetilde{\mathfrak{P}}_{242} = k_1k_2k_3\widetilde{\mathfrak{P}}'_{242}, & \widetilde{\mathfrak{P}}_{243} = k_1\widetilde{\mathfrak{P}}'_{243}, \\
\widetilde{\mathfrak{P}}_{321} = \widetilde{\mathfrak{P}}'_{321}, & \widetilde{\mathfrak{P}}_{331} = k_2k_3\lambda_2\widetilde{\mathfrak{P}}'_{331}, & \widetilde{\mathfrak{P}}_{332} = k_1k_3\widetilde{\mathfrak{P}}'_{332}, & \widetilde{\mathfrak{P}}_{341} = k_2\widetilde{\mathfrak{P}}'_{341}, \\
\widetilde{\mathfrak{P}}_{342} = k_1\widetilde{\mathfrak{P}}'_{342}, & \widetilde{\mathfrak{P}}_{343} = k_1k_2k_3\widetilde{\mathfrak{P}}'_{343}, & \widetilde{\mathfrak{P}}_{421} = k_3\widetilde{\mathfrak{P}}'_{421}, & \widetilde{\mathfrak{P}}_{431} = k_2\widetilde{\mathfrak{P}}'_{431}, \\
\widetilde{\mathfrak{P}}_{432} = k_1\widetilde{\mathfrak{P}}'_{432}, & \widetilde{\mathfrak{P}}_{441} = k_2k_3\widetilde{\mathfrak{P}}'_{441}, & \widetilde{\mathfrak{P}}_{442} = k_1k_3\widetilde{\mathfrak{P}}'_{442}, & \widetilde{\mathfrak{P}}_{443} = k_1k_2\widetilde{\mathfrak{P}}'_{443}\,.
\end{array} \tag{16}$$

Since $k_1k_2k_3 \neq 0$ and, moreover, $\lambda_2\lambda_3 \neq 0$, the study of the equations $\{\widetilde{\mathfrak{P}}_{ijk} = 0\}$ is equivalent to $\{\widetilde{\mathfrak{P}}'_{ijk} = 0\}$. Let $\widetilde{\mathcal{I}}_1 \subset \mathbb{R}[\lambda_2, \lambda_3, k_1, k_2, k_3, R]$ be the ideal generated by the polynomials $\widetilde{\mathfrak{P}}'_{ijk}$. We compute a Gröbner basis $\widetilde{\mathcal{G}}_1$ of $\widetilde{\mathcal{I}}_1$ with respect to lexicographical order and a detailed analysis of the Gröbner basis shows that the polynomial

$$\widetilde{\mathbf{g}}_1 = \lambda_2\lambda_3^2(\lambda_3 - 1)^3(k_1^2 + k_2^2 + k_3^2 + R^2)^2R^2$$

belongs to $\mathcal{G}_1$. Since the zero sets of $\{\widetilde{\mathfrak{P}}'_{ijk} = 0\}$ and $\widetilde{\mathcal{I}}_1 = \langle\widetilde{\mathfrak{P}}'_{ijk}\rangle = \langle\widetilde{\mathcal{G}}_1\rangle$ coincide, it follows that necessarily $\lambda_3 = 1$.

Next, we compute a Gröbner basis $\widetilde{\mathcal{G}}'_1$ of the ideal generated by $\widetilde{\mathcal{G}}_1 \cup \{\lambda_3 - 1\}$ with respect to the lexicographical order and we get that the polynomial

$$\widetilde{\mathbf{g}}'_1 = (\lambda_2 - 1)^3(k_1^2 + k_3^2)(k_1^2 + k_2^2 + k_3^2 + R^2)R^2$$

belongs to $\widetilde{\mathcal{G}}'_1$. Thus, we get $\lambda_2 = 1$ and a straightforward calculation shows that the corresponding Lie group given by Equation (15) is locally conformally flat and therefore a symmetric manifold [16] and trivially Bach-flat.

Case (2): $k_1k_2k_3 = 0$. In this case we can assume without loss of generality that $k_1 = 0$ and we work with the polynomials $\mathfrak{P}_{ijk}$. Let $\mathcal{I}_2 \subset \mathbb{R}[k_1, k_2, k_3, R, \lambda_2, \lambda_3, \xi_1, \xi_2, \xi_3, \xi_4]$ be the ideal generated by the polynomials $\{\mathfrak{P}_{ijk}\} \cup \{k_1\}$. We compute a Gröbner basis $\mathcal{G}_2$ of $\mathcal{I}_2$ with respect to the graded reverse lexicographical order and a detailed analysis of the Gröbner basis shows that the polynomial

$$\mathbf{g}_2 = k_2(\lambda_3 - 1)(k_2^2 + k_3^2 + R^2)^2$$

belongs to $\mathcal{G}_2$. Thus, $\lambda_3 = 1$ or $k_2 = 0$.

If $\lambda_3 = 1$ then we compute a Gröbner basis $\mathcal{G}'_2$ of the ideal generated by $\mathcal{G}_2 \cup \{\lambda_3 - 1\}$ with respect to the lexicographical order and we get that the polynomial

$$\mathbf{g}_2' = \lambda_2^2(\lambda_2 - 1)R^4$$

belongs to $\mathcal{G}_2'$. Thus, we get $\lambda_2 = 1$. As in Case (1) the corresponding Lie group given by Equation (15) is locally conformally flat and therefore a symmetric manifold [16] and trivially Bach-flat.

If $\lambda_3 \neq 1$ and $k_2 = 0$ then we compute a Gröbner basis $\mathcal{G}_2''$ of the ideal generated by $\mathcal{G}_2 \cup \{k_2\}$ with respect to the lexicographical order and we get that the polynomial

$$\mathbf{g}_2'' = \lambda_3^4(\lambda_3 - 1)(\lambda_3^2 + \lambda_3 + 1)R^4$$

belongs to $\mathcal{G}_2''$. Thus this case is not possible and this finishes the proof. □

6. The proof of Theorem 1.1

The results obtained in the previous sections show that the only conformal C-spaces which are not Bach-flat are those given in Lemma 2.1. It is easy to see that the three cases in that lemma are equivalent. Indeed, considering $\tilde{e}_1 = e_1$, $\tilde{e}_2 = e_3$, $\tilde{e}_3 = e_2$ and $\tilde{e}_4 = e_4$, the Lie bracket in Lemma 2.1-(i) reduces to the Lie bracket in Lemma 2.1-(ii). Analogously, the equivalence between (ii) and (iii) in Lemma 2.1 follows taking $\tilde{e}_1 = e_2$, $\tilde{e}_2 = e_1$, $\tilde{e}_3 = e_3$ and $\tilde{e}_4 = e_4$ in (ii). Thus, in what follows we consider the solvable Lie algebra given by Lemma 2.1-(i). For this case, we compute the Weyl conformal tensor of type $(1, 3)$, $W_{ijk} = W(e_i, e_j)e_k$, which is determined by

$$\begin{aligned} W_{121} &= -\tfrac{(a-p)p}{3R^2}e_2, & W_{131} &= \tfrac{(a-p)p}{6R^2}e_3, & W_{141} &= \tfrac{(a-p)p}{6R^2}e_4, \\ W_{232} &= \tfrac{(a-p)p}{6R^2}e_3, & W_{242} &= \tfrac{(a-p)p}{6R^2}e_4, & W_{343} &= -\tfrac{(a-p)p}{3R^2}e_4\,. \end{aligned}$$

Since the Weyl tensor of type (1,3) does not depend on b, then it follows from the work of Hall [10] that taking $b = 0$ we get a homothetic (although not a homothetically isomorphic) Lie algebra. As a consequence, one easily checks that a solvable Lie algebra given by Lemma 2.1-(i) is homothetic to a solvable Lie algebra $\mathfrak{g}_\alpha = \mathbb{R}e_4 \ltimes \mathfrak{r}^3$ given by

$$[e_1, e_4] = e_1, \quad [e_2, e_4] = e_2, \quad [e_3, e_4] = \alpha e_3,$$

where $\alpha \notin \{0, 1, 4\}$. Here, $\xi = -6e_4$ and $\{e_1, \dots, e_4\}$ is an orthonormal basis. Finally, we show that if $\alpha \neq \beta$, then $\mathfrak{g}_\alpha$ and $\mathfrak{g}_\beta$ are not homothetic. To do this , we consider the homothetic metric $\langle\cdot,\cdot\rangle_\alpha^* = 2(\alpha^2 + 2\alpha + 3)\langle\cdot,\cdot\rangle_\alpha$ so that $\tau_\alpha^* = -1$ and we compute

$$(\|R\|_\alpha^*)^2 = \frac{\alpha^4 + 2\alpha^2 + 3}{((\alpha + 2)\alpha + 3)^2}, \qquad (\|W\|_\alpha^*)^2 = \frac{(\alpha - 1)^2\alpha^2}{3((\alpha + 2)\alpha + 3)^2}\,.$$

Hence, two metrics $\langle\cdot,\cdot\rangle_\alpha$ and $\langle\cdot,\cdot\rangle_\beta$ are homothetic if and only if

$$\begin{aligned} &(\alpha^4 + 2\alpha^2 + 3)((\beta + 2)\beta + 3)^2 = (\beta^4 + 2\beta^2 + 3)((\alpha + 2)\alpha + 3)^2, \\ &(\alpha - 1)^2\alpha^2((\beta + 2)\beta + 3)^2 = ((\alpha + 2)\alpha + 3)^2(\beta - 1)^2\beta^2, \end{aligned}$$

from where a straightforward calculation shows that necessarily $\alpha = \beta$, which finishes the proof.

Remark 6.1. Let $\{e^1, \dots, e^4\}$ be the dual basis of $\{e_1, \dots, e_4\}$. Then the structure equations corresponding to the Lie algebras in Theorem 1.1 are given by

$$de^4 = 0, \quad de^1 = -e^1 \wedge e^4, \quad de^2 = -e^2 \wedge e^4, \quad de^3 = -\alpha e^3 \wedge e^4.$$

Setting $e^4 = dt$, $e^1 = e^t dx$, $e^2 = e^t dy$, $e^3 = e^{\alpha t} dz$, one has that the manifolds in Theorem 1.1 are isometric to the doubly warped product metric $\mathbb{R} \times_{e^t} \mathbb{R}^2 \times_{e^{\alpha t}} \mathbb{R}$ on $\mathbb{R}^4$ given by

$$g = dt^2 + e^{2t}(dx^2 + dy^2) + e^{2\alpha t} dz^2.$$

Moreover, a straightforward calculation shows that the conformal metric $\hat{g} = e^{-6t} g$ is Cotton-flat. Finally, observe that the Bach tensor $\mathfrak{B}$, when expressed in the above coordinates, is diagonal and given by

$$\mathfrak{B} = \frac{1}{6}\alpha(\alpha - 1)(\alpha - 4)\,\mathrm{diag}[\alpha - 1, -(\alpha + 1)e^{2t}, -(\alpha + 1)e^{2t}, (\alpha + 3)e^{2\alpha t}].$$

Hence the Bach tensor vanishes if and only if $\alpha = 0$, $\alpha = 1$, or $\alpha = 4$. For the special cases $\alpha = 0$ or $\alpha = 1$ the underlying structure is symmetric. For $\alpha = 0$, it corresponds to a product $\mathbb{R} \times \mathbb{H}^3$ while it corresponds to the hyperbolic space $\mathbb{H}^4$ for $\alpha = 1$. The case corresponding to $\alpha = 4$ is conformally Einstein since $\hat{g} = e^{-6t} g$ is Ricci-flat.

REMARK 6.2. Let $\{e^i\}$ be the dual basis of $\{e_i\}$, and set $E_1^\pm = \frac{1}{\sqrt{2}}\left(e^1 \wedge e^2 \pm e^3 \wedge e^4\right)$, $E_2^\pm = \frac{1}{\sqrt{2}}\left(e^1 \wedge e^3 \mp e^2 \wedge e^4\right)$, and $E_3^\pm = \frac{1}{\sqrt{2}}\left(e^1 \wedge e^4 \pm e^2 \wedge e^3\right)$.

The self-dual and anti-self-dual Weyl curvature operators of any left-invariant metric in Theorem 1.1 satisfy

$$W^\pm = -\frac{1}{6}\alpha(\alpha - 1)\,\mathrm{diag}[2, -1, -1]$$

when expressed on the basis $\{E_1^+, E_2^+, E_3^+\}$ (resp., $\{E_1^-, E_2^-, E_3^-\}$) of self-dual (resp., anti-self-dual) two-forms. The distinguished eigenvalues of W^+ and W^- define one-dimensional subspaces of Λ^2_+ and Λ^2_-. The corresponding sections determine two-forms given by $E_1^\pm$. A straightforward calculation shows that $dE_1^\pm = e^4 \wedge E_1^\pm$, where $de^4 = 0$. Hence the two-forms $E_1^\pm$ are conformally symplectic and opposite conformally symplectic. Furthermore, observe that none of the corresponding almost complex structures $J^\pm$ (determined by $J^\pm e_1 = e_2$, and $J^\pm e_3 = \pm e_4$) is integrable. A straightforward calculation shows that the two-forms $\Omega_\pm = e^{-2t} E_1^\pm$ determine a symplectic pair [2] (i.e., they are non-degenerate closed two-forms such that $\Omega_+ \wedge \Omega_- = 0$, and $\Omega_+ \wedge \Omega_+ = -\Omega_- \wedge \Omega_-$).

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E. Calviño-Louzao
Consellería de Educación
Universidade e Formación Profesional
Edificio Administrativo San Caetano
15781 Santiago de Compostela
Spain
e-mail: estebcl@edu.xunta.gal

E. García-Río
Faculty of Mathematics
University of Santiago de Compostela
15782 Santiago de Compostela
Spain
e-mail: eduardo.garcia.rio@usc.es

I. Gutiérrez-Rodríguez
Department of Mathematics
University of Vigo, Campus Ourense
32004 Ourense
Spain
and
Sciences Faculty
Bernal Díaz del Castello 340
University of Colima
28045 Colima
Mexico
e-mail: ixchel.dzohara.gutierrez.rodriguez.@uvigo.es

R. Vázquez-Lorenzo
Department of Mathematics
IES de Ribadeo Dionisio Gamallo
27700 Ribadeo
Spain
e-mail: ravazlor@edu.xunta.gal