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# FINITE TIME BLOW-UP FOR A VISCOELASTIC WAVE EQUATION WITH WEAK-STRONG DAMPING AND POWER NONLINEARITY

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**Abstract**

We consider a viscoelastic wave equation with weak, strong damping and power nonlinearity. We have already obtained a global solution and its decay rate in [8]. In this paper, we apply the concavity method in order to show that the solution blows up in finite time under non-classic constraint on $\mu$.

## 1. Introduction

One of the most important terms, from mathematical point of view, of equation to study in this article is the viscoelasticity ($\int_0^t \mu(t-s)\Delta v(s)\,ds$), which also includes weak ($v_t$) and strong damping ($\Delta v_t$). Real materials dissipate energy when subjected to deformation. These environments are the seat of intrinsic dissipation phenomena which cause a decrease in energy and an exponential attenuation of the amplitude of the waves during their propagation. We are interested in the modeling of this phenomenon by the introduction of semilinear viscoelastic model, which is well suited to the description of a large class of dissipative phenomena. It requires knowledge not only of current values of stresses and deformations but also of past values, which are said to be memory materials. To begin with, we consider the problem

$$v_{tt} + av_t - \Theta(x)\left(\Delta v + \omega\Delta v_t - \int_0^t \mu(t-s)\Delta v(s)\,ds\right) = v\,|v|^{p-1} \tag{1}$$

for $x \in \mathbb{R}^n$ and $t > 0$ with

$$\begin{cases} v(x,0) = v_0(x) & \text{for } x \in \mathbb{R}^n, \\ v_t(x,0) = v_1(x) & \text{for } x \in \mathbb{R}^n, \end{cases}$$

where $a \in \mathbb{R}$, $\omega > 0$, $p > 1$, $n \geq 3$, the density function $\Theta(x) > 0$ for all $x \in \mathbb{R}^n$ and its inverse $(\Theta(x))^{-1} = 1/\Theta(x) \equiv \theta(x)$ satisfies

$$\theta \in L^r(\mathbb{R}^n) \quad \text{with} \quad r = \frac{2n}{2n - qn + 2q} \quad \text{for} \quad 2 \leq q \leq \frac{2n}{n-2}. \tag{2}$$

The novelty of our work lies primarily in the dispense with the use of the new condition between the weights of weak and strong damping (7) taken in [8]. The constant $\lambda_1$ being the first eigenvalue of the operator $-\Delta$. We also proposed an algebraic nonlinearity in sources

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which make the problem very interesting in the application point of view. In order to compensate the lack of classical Poincare's inequality in $\mathbb{R}^n$, we used the weighted function to use the generalized Poincare's one. The main contribution located in Theorem 2, where we obtained sufficient conditions on the kernel (related with a convex function) and on the nonlinearity to guarantee the nonexistence of solutions. Of course, this result completes our study in [8] concerning the global existence in time.
We assume that the kernel function $\mu \in C^1(\mathbb{R}^+, \mathbb{R}^+)$ satisfying

$$l \equiv 1 - \overline{\mu} > 0 \quad \text{for} \quad \overline{\mu} = \int_0^{+\infty} \mu(s)\, ds, \tag{3}$$

where $\mathbb{R}^+ = \{\kappa \mid \kappa \geq 0\}$. Furthermore we assume that there is a function $\Xi \in C^1(\mathbb{R}^+, \mathbb{R}^+)$ such that

$$\mu'(t) + \Xi(\mu(t)) \leq 0, \quad \Xi(0) = 0, \quad \Xi'(0) > 0 \quad \text{and} \quad \Xi''(\xi) \geq 0 \tag{4}$$

for any $\xi \geq 0$. Under the similar assumptions, many researchers have studied the problem

$$v_{tt} - \Theta(x)\left(\Delta v + \omega \Delta v_t - \int_0^t \mu(t-s)\Delta v(s)\, ds\right) = 0.$$

They obtained the global existence, decay rate and blow-up of solution. For instance, see [1, 3], where the question of the decay estimates of solutions for the linear problem were discussed from different perspectives and angles. The Kirchhoff type problem

$$v_{tt} - \Theta(x)\left(m_0 + m_1 \int_{\mathbb{R}^n} |\nabla v|^2\, dx\right)\Delta v + \Theta(x) \int_0^t \mu(t-s)\Delta v(s)\, ds = 0$$

for $m_0 > 0$ and $m_1 > 0$ is also investigated in [10, 11]. For (1), the global existence, decay rate and blow-up of solutions are studied. The authors consider the $\mu = 0$ case in [7], the $\omega = 0$, $\mu = 0$ case in [2], the $a > 0$, $\omega = 0$ case in [12] and the $a = 1$, $\theta \equiv 1$ case with a bounded domain in [5], respectively. Recently, in [8], the authors give a simplified computation of decay rate from the convexity. The aim of this paper is to prove that the solution blows up in finite time under certain conditions. To introduce the theorem, we define the function spaces $\mathcal{H}$ as the closure of $C_0^\infty(\mathbb{R}^n)$ with respect to the norm $\|v\|_{\mathcal{H}} = (v, v)_{\mathcal{H}}^{1/2}$ for the inner product

$$(v, w)_{\mathcal{H}} = \int_{\mathbb{R}^n} \nabla v \cdot \nabla w\, dx$$

and $L^2_\theta(\mathbb{R}^n)$ as that to the norm $\|v\|_{L^2_\theta} = (v, v)^{1/2}_{L^2_\theta}$ for

$$(v, w)_{L^2_\theta} = \int_{\mathbb{R}^n} \theta v w\, dx,$$

respectively. As mentioned in [9], we have

$$\mathcal{H} = \{u \in L^{\frac{2n}{n-2}}(\mathbb{R}^n) \mid \nabla u \in L^2(\mathbb{R}^n)^n\}.$$

For general $q \in [1, +\infty)$, $L^q_\theta(\mathbb{R}^n)$ is the weighted $L^q$ space under a weighted norm

$$\|v\|_{L^q_\theta} = \left(\int_{\mathbb{R}^n} \theta\, |v|^q\, dx\right)^{\frac{1}{q}}.$$

To distinguish the usual $L^q$ space from the weighted one, we denote the standard $L^q$ norm by

$$\|v\|_q = \left(\int_{\mathbb{R}^n} |v|^q \, dx\right)^{\frac{1}{q}}.$$

The main tool to obtain necessary estimates is a decreasing energy which is defined by

$$E(t) = \frac{1}{2}\|v_t\|_{L^2_\theta}^2 + \frac{1}{2}\left(1 - \int_0^t \mu(s)\, ds\right)\|v\|_{\mathcal{H}}^2 + \frac{1}{2}(\mu \circ v) - \frac{1}{p+1}\|v\|_{L_\theta^{p+1}}^{p+1}$$

for $(v, v_t) \in \mathcal{H} \times L^2_\theta(\mathbb{R}^n)$, where

$$(\mu \circ u)(t) = \int_0^t \mu(t-s)\,\|u(t) - u(s)\|_{\mathcal{H}}^2 \, ds$$

for any $u \in \mathcal{H}$. We denote an eigenpair $\left\{(\lambda_j, w_j)\right\}_{j\in\mathbb{N}} \subset \mathbb{R} \times \mathcal{H}$ of

$$-\Theta(x)\Delta w_j = \lambda_j w_j \quad \text{for } x \in \mathbb{R}^n$$

for any $j \in \mathbb{N}$. Then according to [9],

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_j \leq \cdots \uparrow +\infty$$

holds and $\left\{w_j\right\}$ is a complete orthonormal system in $\mathcal{H}$. In this setting, we can establish a local solution

$$v \in \mathcal{X}_{T_{v_0}} \equiv C([0, T_{v_0}); \mathcal{H}) \cap C^1([0, T_{v_0}); L^2_\theta(\mathbb{R}^n)),$$

where $T_{v_0} > 0$ is a maximal existing time of a local solution for the initial value $v_0$. For the proof, see [6, 7, 8, 12]. Then, we introduce the results of the existence of the global solution and its convergence rate obtained in [8].

Definition 1. The functions $v$ is said to be a weak solution to (1) on $[0, T]$ if it satisfies $v \in L^2([0,T); \mathcal{H})$, $v_t \in L^2([0,T); L^2_\theta(\mathbb{R}^n))$, $v_{tt} \in L^2([0,T); \mathcal{H}')$,

$$\int_{\mathbb{R}^n} v_{tt}\psi \, dx + a\int_{\mathbb{R}^n} v_t\psi \, dx \tag{5}$$

$$= -\int_{\mathbb{R}^n} \nabla\left(v + \omega v_t - \int_0^t \mu(t-s)v(s)\, ds\right).\nabla(\Theta(x)\psi)\, dx + \int_{\mathbb{R}^n} v\,|v|^{p-1}\,\psi\, dx$$

for all test function $\psi \in \mathcal{H}$ for almost all $t \in [0, T]$, $v(x, 0) = v_0$ in $\mathcal{H}$ and $v_t(x, 0) = v_1$ in $L^2_\theta(\mathbb{R}^n)$, where $\mathcal{H}'$ denotes the dual space of $\mathcal{H}$.

**Theorem 1** (Theorem 2 in [8] )**.** *Let*

$$1 < p \leq \frac{n+2}{n-2} \quad \textit{and} \quad n \geq 3. \tag{6}$$

*Under the assumptions* (2), (3) *and* (4)*, suppose that*

$$a + \lambda_1\omega > 0. \tag{7}$$

*For sufficiently small* $(v_0, v_1) \in \mathcal{H} \times L^2_\theta(\mathbb{R}^n)$, (1) *admits a unique global solution u in the* *space*

$$v \in \mathcal{X} \equiv C([0, +\infty); \mathcal{H}) \cap C^1([0, +\infty); L^2_\theta(\mathbb{R}^n)).$$

*Furthermore, there exists* $t_0 > 0$ *depending only on* $\mu$, $a$, $\omega$, $n$ *and* $\Xi'(0)$ *such that*

$$0 \le E(t) < E(t_0) \exp\left( -\int_{t_0}^{t} \frac{\mu(s)}{1 - \int_0^s \mu(p)\, dp}\, ds \right)$$

*holds for all* $t \ge t_0$.

The main theorem in this paper is concerned with the blow-up.

**Theorem 2.** *Under the assumptions* (2), (3), (6) *and* (7), *suppose that* $\mu'(t) \le 0$ *holds for any* $t \ge 0$. *Let* $(v_0, v_1) \in \mathcal{H} \times L^2_\theta(\mathbb{R}^n)$. *If either of the following conditions is satisfied, then the local solution blows up in finite time in* $\mathcal{H}$.

$$(i) \quad E(0) < 0, \quad p^2 l - 1 \ge 0.$$

$$(ii) \quad 0 \le E(0) < \frac{p^2 l - 1}{(p^2 - 1)\, l} E_0, \quad \|v_0\|_{L^{p+1}_\theta} > \lambda_0,$$

*where* $\lambda_0$ *and* $E_0$ *are positive constants depending only on* $n$, $p$, $\theta$ *and* $\mu$ *to be defined in* Section 3.

This paper is composed of 3 sections. In section 2, we introduce several important facts such as embedding inequalities, decreasing energy, inner product and ODE theory. In section 3, we prove that the solution blows up in finite time under the conditions in Theorem 2.

## 2. Preliminaries

First, we introduce Sobolev embedding inequalities.

**Lemma 1** (Lemma 2.2 in [2]). *Let* $\theta$ *satisfy* (2). *Then there is a positive constant* $C_S > 0$ *which depends only on* $n$ *and* $\theta$ *such that*

$$\|v\|_{\frac{2n}{n-2}} \le C_S \|v\|_{\mathcal{H}}$$

*and*

$$\|v\|_{L^q_\theta} \le C_q \|v\|_{\mathcal{H}}$$

*for* $v \in \mathcal{H}$, *where*

$$C_q = C_S \|\theta\|_s^{\frac{1}{q}} \quad \textit{and} \quad s = 2n/(2n - qn + 2q)$$

*for* $1 \le q \le 2n/(n-2)$.

Next we define the inner product and the corresponding norm by

$$(v, w)_* = \omega \int_{\mathbb{R}^n} \nabla v \cdot \nabla w\, dx + a \int_{\mathbb{R}^n} \theta v w\, dx \quad \text{and} \quad \|v\|_* = \sqrt{(v, v)_*}$$

for any $v, w \in \mathcal{H}$, respectively.

Third, we introduce the energy function which plays an important role in obtaining the estimates.

**Lemma 2.** *For a local solution $v(t) \in \mathcal{X}_{T_{v_0}}$ of* (1)*,*

$$E(t) = \frac{1}{2}\|v_t\|_{L^2_\theta}^2 + \frac{1}{2}\left(1 - \int_0^t \mu(s)\,ds\right)\|v\|_{\mathcal{H}}^2 + \frac{1}{2}(\mu \circ v) - \frac{1}{p+1}\|v\|_{L^{p+1}_\theta}^{p+1}$$

*is a decreasing energy for* (1)*.*

Proof. We have

$$\frac{d}{dt}E(t) = -\|v_t\|_*^2 - \frac{1}{2}\mu(t)\|v\|_{\mathcal{H}}^2 + \frac{1}{2}(\mu' \circ v)$$

for all $0 \le t < T_{v_0}$ and

$$E(t_2) = E(t_1) - \int_{t_1}^{t_2}\left(\|v_t\|_*^2 + \frac{1}{2}\mu(s)\|v\|_{\mathcal{H}}^2 - \frac{1}{2}(\mu' \circ v)\right)ds \tag{8}$$

for all $0 \le t_1 < t_2 < T_{v_0}$. □

Finally, we introduce the following lemma, which leads the local solution to the blow-up:

**Lemma 3** ([4])**.** *Let $G(t) \in C^2(\mathbb{R}^+)$, $G(t) > 0$ and $G'(t) > 0$ for $t \in \mathbb{R}^+$. If*

$$\{G'(t)\}^2 - (1-\alpha)\,G''(t)G(t) < 0$$

*holds for some $\alpha \in (0,1)$, then $G(t)$ blows up as $t \to T \le (1-\alpha)\,G(0)/(\alpha G'(0))$.*

## 3. Blow-up

First, we prepare several lemmas. Next, we show that the local solution blows up in finite time. For the sake of the proof, we define

$$G(t) = \|v\|_{L^2_\theta}^2 + \int_0^t \|v(s)\|_*^2\,ds + (T_0 - t)\|v_0\|_*^2 + \beta\,(T_1 + t)^2, \tag{9}$$

where $T_0$, $T_1$ and $\beta$ are the positive constants to be chosen later. We define positive constants $\lambda_0$ and $E_0$ by

$$\lambda_0 \equiv \left(\frac{l}{(C_{p+1})^2}\right)^{\frac{1}{p-1}} \quad \text{and} \quad E_0 \equiv \frac{p-1}{2(p+1)}\left(\frac{l}{(C_{p+1})^2}\right)^{\frac{p+1}{p-1}},$$

respectively, where $C_{p+1}$ is a constant defined in Lemma 1. Let

$$\gamma \equiv 2\beta - 2\,(p+1)\,E(0).$$

**Lemma 4.** *If $\mu'(t) \le 0$ holds for any $t \ge 0$, a local solution $v(t) \in \mathcal{X}_{T_{v_0}}$ of* (1) *satisfies*

$$\begin{aligned}\|v(t)\|_{L^{p+1}_\theta}^{p+1} \ \ge\ & (p+1)\left\{-E(0) + \frac{1}{2}\|v_t\|_{L^2_\theta}^2 + \frac{1}{2}\left(1 - \int_0^t \mu(s)\,ds\right)\|v\|_{\mathcal{H}}^2\right.\\ & \left. + \int_0^t \left(\|v_t(s)\|_*^2 + \frac{1}{2}\mu(s)\|v(s)\|_{\mathcal{H}}^2\right)ds + \frac{1}{2}(\mu \circ v)\right\}\end{aligned}$$

*for all $t \in [0, T_{v_0})$.*

Proof. The conclusion follows from the definition of $E(t)$ and (8). □

**Lemma 5.** *A local solution $v(t) \in \mathcal{X}_{T_{v_0}}$ of (1) satisfies*

$$2\int_0^t \int_{\mathbb{R}^n} \mu(t-s)\nabla v(s)\cdot\nabla v(t)\,dx\,ds \geq -(p+1)(\mu\circ v)(t) + \frac{2p+1}{p+1}\int_0^t \mu(s)\,ds\,\|v(t)\|_{\mathcal{H}}^2$$

*for all $t \in [0, T_{v_0})$.*

Proof. By Young inequality, we have

$$\begin{aligned}
&2\int_0^t \int_{\mathbb{R}^n} \mu(t-s)\nabla v(s)\cdot\nabla v(t)\,dx\,ds \\
&= 2\int_0^t \int_{\mathbb{R}^n} \mu(t-s)\left(\nabla v(s)-\nabla v(t)\right)\cdot\nabla v(t)\,dx\,ds + 2\int_0^t \int_{\mathbb{R}^n} \mu(t-s)\nabla v(t)\cdot\nabla v(t)\,dx\,ds \\
&\geq -2\int_0^t \mu(t-s)\,\|v(s)-v(t)\|_{\mathcal{H}}\,\|v(t)\|_{\mathcal{H}}\;ds + 2\int_0^t \mu(t-s)\,\|v(t)\|_{\mathcal{H}}^2\;ds \\
&= -2\int_0^t \sqrt{(p+1)\mu(t-s)}\,\|v(s)-v(t)\|_{\mathcal{H}}\sqrt{\frac{\mu(t-s)}{p+1}}\,\|v(t)\|_{\mathcal{H}}\;ds + 2\int_0^t \mu(t-s)\,\|v(t)\|_{\mathcal{H}}^2\;ds \\
&\geq -(p+1)(\mu\circ v)(t) + \left(2-\frac{1}{p+1}\right)\int_0^t \mu(s)\,\|v(t)\|_{\mathcal{H}}\;ds \\
&= -(p+1)(\mu\circ v)(t) + \frac{2p+1}{p+1}\int_0^t \mu(s)\,ds\,\|v(t)\|_{\mathcal{H}}^2
\end{aligned}$$

for all $t \in [0, T_{v_0})$, which completes the proof. □

**Lemma 6** (Lemma 5 in [8])**.** *Assume that $0 \leq E(0) < E_0$.*
(i) *If $\|v_0\|_{L_\theta^{p+1}} < \lambda_0$, then a local solution $v(t) \in \mathcal{X}_{T_{v_0}}$ of (1) satisfies $\|v(t)\|_{L_\theta^{p+1}} < \lambda_0$ for all $t \in [0, T_{v_0})$.*
(ii) *If $\|v_0\|_{L_\theta^{p+1}} > \lambda_0$, then a local solution $v(t) \in \mathcal{X}_{T_{v_0}}$ of (1) satisfies $\|v(t)\|_{L_\theta^{p+1}} > \lambda_0$ for all $t \in [0, T_{v_0})$.*

Proof of Theorem 2. We have

$$\begin{aligned}
G'(t) &= 2\int_{\mathbb{R}^n} \theta v v_t\,dx + \|v(t)\|_*^2 - \|v_0\|_*^2 + 2\beta(T_1+t) \\
&= 2\int_{\mathbb{R}^n} \theta v v_t\,dx + 2\int_0^t (v, v_t)_*\;ds + 2\beta(T_1+t)
\end{aligned} \tag{10}$$

and

$$\begin{aligned}
G''(t) &= 2\int_{\mathbb{R}^n} \left(\theta v_t^2 + \theta v v_{tt} + a\theta v v_t + \omega\nabla v\cdot\nabla v_t\right)dx + 2\beta \\
&= 2\int_{\mathbb{R}^n} \theta v_t^2\,dx + 2\int_{\mathbb{R}^n} v\left(\theta v_{tt} + a\theta v_t - \omega\Delta v_t\right)\,dx + 2\beta \\
&= 2\int_{\mathbb{R}^n} \theta v_t^2\,dx - 2\int_{\mathbb{R}^n} |\nabla v|^2\;dx + 2\int_{\mathbb{R}^n} \theta\,|v|^{p+1}\;dx + 2\beta \\
&\quad +2\int_{\mathbb{R}^n}\int_0^t \mu(t-s)\nabla v(s)\cdot\nabla v(t)\,ds\,dx
\end{aligned}$$

by (1). Lemmas 4 and 5 imply that

$$\begin{aligned} G''(t) &\geq \gamma + (p+3)\,\|v_t\|^2_{L^2_\theta} + \left\{(p-1) - \frac{p^2}{p+1}\int_0^t \mu(s)\,ds\right\}\|v\|^2_{\mathcal{H}} \\ &\quad +2\,(p+1)\int_0^t \|v_t(s)\|^2_*\,ds + (p+1)\int_0^t \mu(s)\,\|v(s)\|^2_{\mathcal{H}}\,ds \qquad (11) \\ &\geq \gamma + (p+3)\,\|v_t\|^2_{L^2_\theta} + \frac{p^2 l - 1}{p+1}\,\|v\|^2_{\mathcal{H}} + (p+3)\int_0^t \|v_t(s)\|^2_*\,ds. \end{aligned}$$

**The proof of case (i).** We take $T_1 > 0$ so large that

$$\int_{\mathbb{R}^n} \theta v_0 v_1\,dx + \beta T_1 > \frac{2}{p-1}\,\|v_0\|^2_*$$

holds. Then we have

$$G'(0) = 2\left(\int_{\mathbb{R}^n} \theta v_0 v_1\,dx + \beta T_1\right) > 0 \qquad (12)$$

and choose $T_0 > 0$ sufficiently large by

$$T_0 \geq \frac{4G(0)}{(p-1)\,G'(0)} = \frac{2\left(\|v_0\|^2_{L^2_\theta} + T_0\,\|v_0\|^2_* + \beta T_1^2\right)}{(p-1)\left(\int_{\mathbb{R}^n} \theta v_0 v_1\,dx + \beta T_1\right)}.$$

Owing to $E(0) < 0$ and $\gamma = 2\beta - 2\,(p+1)\,E(0)$, we can take $\beta$ as

$$0 < \beta < \frac{\gamma}{p+3},$$

which yields

$$G''(t) > (p+3)\beta + (p+3)\,\|v_t\|^2_{L^2_\theta} + (p+3)\int_0^t \|v_t(s)\|^2_*\,ds > 0 \qquad (13)$$

by (11) and $p^2 l \geq 1$. Then we have $G(t) > 0$ and $G'(t) > 0$ for all $t \in [0, T_0)$ by (12) and (13). Owing to (9), (10) and (13), for any $\xi, \eta \in \mathbb{R}$, we have

$$G(t)\xi^2 + G'(t)\xi\eta + \frac{G''(t)}{p+3}\eta^2 > \|\xi v + \eta v_t\|^2_{L^2_\theta} + \int_0^t \|\xi v + \eta v_s\|^2_*\,ds + \beta\,\{(T_1+t)\,\xi + \eta\}^2 \geq 0$$

for any $\xi, \eta \in \mathbb{R}$ and $t \in [0, T_0)$. Hence we have

$$(G'(t))^2 - \frac{4}{p+3}G''(t)G(t) < 0$$

for any $t \in [0, T_0)$. Noting that

$$\frac{4}{p+3} = 1 - \frac{p-1}{p+3} \qquad \text{and} \qquad 0 < \frac{p-1}{p+3} < 1,$$

we can apply Lemma 3 to conclude that

$$G(t) \to +\infty \qquad \text{as} \qquad t \to T < \frac{4G(0)}{(p-1)\,G'(0)}$$

for some $T \in (0, T_0)$. By (9), we have either $\|v(t)\|_{L^2_\theta} \to +\infty$ or $\|v(t)\|_* \to +\infty$ as $t \to T$. In both cases, $\|v\|_{\mathcal{H}} \to +\infty$ follows by Lemma 1 and the definition of the inner product $(v, w)_*$.

**The proof of case (ii).** First of all, we take $T_0$ and $T_1$ in the same way as case (i). Next, since we have

$$0 \leq E(0) < \frac{p^2 l - 1}{(p^2-1)\, l} E_0 < E_0,$$

we obtain

$$\|v\|_{\mathcal{H}}^2 \geq \left( \frac{1}{C_{p+1}} \|v\|_{L_\theta^{p+1}} \right)^2 > \left( \frac{\lambda_0}{C_{p+1}} \right)^2 = \frac{1}{l} \left( \frac{l}{(C_{p+1})^2} \right)^{\frac{p+1}{p-1}} = \frac{2(p+1)}{(p-1)l} E_0$$

by Lemmas 1, 6 and the definition of $\lambda_0$ and $E_0$. By (11), we have

$$\begin{aligned} G''(t) &\geq 2\beta - 2\,(p+1)\,E(0) + (p+3)\,\|v_t\|_{L_\theta^2}^2 + \frac{2(p^2 l - 1)}{(p-1)l} E_0 + (p+3) \int_0^t \|v_t(s)\|_*^2 \, ds \\ &= 2\beta + 2\,(p+1) \left( \frac{p^2 l - 1}{(p^2-1)l} E_0 - E(0) \right) + (p+3)\,\|v_t\|_{L_\theta^2}^2 + (p+3) \int_0^t \|v_t(s)\|_*^2 \, ds. \end{aligned}$$

Hence we take $\beta$ as

$$0 < \beta < 2 \left( \frac{p^2 l - 1}{(p^2-1)l} E_0 - E(0) \right),$$

reach (13) and follow the proof of the case (i), which completes the proof of the case (ii). □

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