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DISCRETENESS OF HYPERBOLIC ISOMETRIES BY TEST MAPS

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Abstract

Let $\mathbb{F} = \mathbb{R}, \mathbb{C}$ or the Hamilton's quaternions $\mathbb{H}$. Let $\mathbf{H}^n_{\mathbb{F}}$ denote the n-dimensional $\mathbb{F}$-hyperbolic space. Let $\mathrm{U}(n, 1; \mathbb{F})$ be the linear group that acts by the isometries of $\mathbf{H}^n_{\mathbb{F}}$. A subgroup G of $\mathrm{U}(n, 1; \mathbb{F})$ is called *Zariski dense* if it does not fix a point on $\mathbf{H}^n_{\mathbb{F}} \cup \partial\mathbf{H}^n_{\mathbb{F}}$ and neither it preserves a totally geodesic subspace of $\mathbf{H}^n_{\mathbb{F}}$. We prove that a Zariski dense subgroup G of $\mathrm{U}(n, 1; \mathbb{F})$ is discrete if for every loxodromic element $g \in G$, the two generator subgroup $\langle f, g\rangle$ is discrete, where $f \in \mathrm{U}(n, 1; \mathbb{F})$ is a test map not necessarily from G.

1. Introduction

Let $\mathbb{F} = \mathbb{R}, \mathbb{C}$ or the Hamilton's quaternions $\mathbb{H}$. Let $\mathbf{H}^n_{\mathbb{F}}$ be the n-dimensional hyperbolic space over $\mathbb{F}$. Let $\mathrm{U}(n, 1; \mathbb{F})$ the unitary group that acts on $\mathbf{H}^n_{\mathbb{F}}$ by isometries. For simplicity of notations, $\mathrm{U}(n, 1; \mathbb{R})$ will be considered as the identity component of the full isometry group. Following standard notations, we denote $\mathrm{U}(n, 1; \mathbb{R}) = \mathrm{PO}(n, 1)$, $\mathrm{U}(n, 1; \mathbb{C}) = \mathrm{U}(n, 1)$, $\mathrm{U}(n, 1; \mathbb{H}) = \mathrm{Sp}(n, 1)$.

The Jørgensen inequality is an important result on discreteness of subgroups in two and three dimensional real hyperbolic geometry. It was developed by Jørgensen and later generalized to arbitrary dimension by Martin [20] and Waterman [24] using different approaches. Abikoff and Haas [1] proved that a Zariski-dense subgroup G of $\mathrm{PO}(n, 1)$ is discrete if and only if every two-generator subgroup of G is discrete, also see [20], [10], [19], [23]. Following this theme, Chen, in [9], has obtained a discreteness criterion that uses a fixed 'test map' to check discreteness of a subgroup. Chen proved that a Zariski-dense subgroup G of $\mathrm{PO}(n, 1)$ is discrete if for each g in G, the group $\langle g, h\rangle$ is discrete, where h is a fixed non-trivial element from $\mathrm{PO}(n, 1)$, not necessarily from G, such that h is either of infinite order but not an irrational rotation, or if having finite order, it does not pointwise fix the minimal sphere containing the limit set of G.

Chen's work suggests that the discreteness is not completely an internal property of a subgroup G, and one may detect it by performing discreteness of the two-generator subgroups having a fixed generator that might also be an element in the complement of G. Such a generator is called a 'test map'. The action of $\mathrm{SL}(2, \mathbb{C})$ on the Riemann sphere by the linear fractional transformations provides an identification of $\mathrm{PO}(2, 1)$ with $\mathrm{PSL}(2, \mathbb{C})$. In [26],

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[25] and [7], refined versions of discreteness criteria in $\mathrm{SL}(2,\mathbb{C})$ using test maps have been obtained. A generalization of the complex linear fractional transformations are the quaternionic linear fractional transformations that can be identified with the group $\mathrm{PSL}(2,\mathbb{H})$. Here $\mathrm{SL}(2,\mathbb{H})$, the 2×2 quaternionic matrices with Dieudonné determinant 1, acts by the linear fractional transformations on the boundary of the 5-dimensional hyperbolic space. In [14], also see [18], some Jørgensen type inequalities for two generator subgroups of $\mathrm{SL}(2,\mathbb{H})$ were obtained. In [13], these inequalities are used to prove that the discreteness of a Zariski-dense subgroup G of $\mathrm{SL}(2,\mathbb{H})$ is determined by the two generator subgroups $\langle f, g\rangle$, where f is a certain test map from $\mathrm{SL}(2,\mathbb{H})$ and g is a loxodromic element of G.

In this short note, we generalise the above results to $\mathrm{U}(n,1;\mathbb{F})$ to show that the discreteness of a subgroup G is determined by a test map and the loxodromic elements of G. We also provide some quantitative bounds for the test maps.

Recall that an element f in $\mathrm{U}(n,1;\mathbb{F})$ is called *elliptic* if it has a fixed point on $\mathbf{H}^n_{\mathbb{F}}$, it is *parabolic*, resp. *loxodromic* (or hyperbolic) if it has exactly one, resp. two fixed points on $\partial\mathbf{H}^n_{\mathbb{F}}$ and no fixed point on $\mathbf{H}^n_{\mathbb{F}}$. An elliptic element is called *regular* if it has a unique fixed point on $\mathbf{H}^n_{\mathbb{F}}$. This type of isometries exist in all dimensions over $\mathbb{F}=\mathbb{C},\mathbb{H}$. However, regular elliptic isometries of $\mathbf{H}^n_{\mathbb{R}}$ exist if and only if n is even. When n odd, every elliptic isometry of $\mathbf{H}^n_{\mathbb{R}}$ has at least two fixed points on the boundary $\partial\mathbf{H}^n_{\mathbb{R}}$. By abuse of notation, an elliptic isometry of $\mathbf{H}^n_{\mathbb{R}}$ will be called *regular* if it has at most two boundary fixed points. A subgroup G of $\mathrm{U}(n,1;\mathbb{F})$ is called *Zariski-dense* or *irreducible* if it does not have a global fixed point on $\overline{\mathbf{H}_{\mathbb{F}}}^n=\mathbf{H}^n_{\mathbb{F}}\cup\partial\mathbf{H}^n_{\mathbb{F}}$ and neither it preserves a proper totally geodesic subspace of $\mathbf{H}^n_{\mathbb{F}}$.

1.1. Discreteness in $\mathrm{PO}(n,1)$**.** For $\mathrm{PO}(n,1)$ we use the Clifford algebraic formalism that was initiated by Ahlfors in [3], [2]. Waterman gave an alternative formulation of this approach in [24] and proved its equivalence to Ahlfors's formalism. In this approach the Clifford group $\mathrm{SL}(2,C_n)$, $n\geq 0$, acts by the orientation-preserving isometries of $\mathbf{H}^{n+2}_{\mathbb{R}}$, $n\geq 0$. The action is by the familiar looking linear fractional transformations. The group $\mathrm{SL}(2,C_n)$ consists of the 2×2 invertible matrices over Clifford numbers with 'Clifford determinant' one. Waterman obtained Jørgensen type inequalities for two-generator subgroups of $\mathrm{SL}(2,C_n)$ in [24].

Cao and Waterman extended Waterman's inequalities using conjugacy invariants in [6]. Given an isometry f of $\mathbf{H}^{n+2}_{\mathbb{R}}$, one can associate 'rotation angles' to it, and the rotation angles may be chosen to be elements of $(-\pi,\pi]$. The rotation angles are conjugacy invariants of an element and one can further classify dynamical types of elements in $\mathrm{SL}(2,C_n)$ using the rotation angles and translation lengths, see [12]. For a non-elliptic isometry f, let τ_f denotes the translation length of f between the fixed points. $\tau_f=0$ if and only if f is parabolic. The conjugacy invariant $\beta(f)$ used by Cao and Waterman can be defined as follows.

Definition 1. Let f be an element in $\mathrm{SL}(2,C_n)$. Let $\theta_1,\ldots,\theta_k\in(-\pi,\pi]$ be rotation angles of f (counted with multiplicities). Let $\Theta=\max_{1\leq i\leq k}|\theta_i|$.
If f is elliptic or parabolic, then $\beta(f)=4\sin^2(\Theta/2)$.
If f is loxodromic, then $\beta(f)=4\sinh^2(\tau_f/2)+4\sin^2(\Theta/2)$.

We apply the Jørgensen type inequalities of Cao and Waterman to obtain discreteness criteria of a Zariski-dense subgroup G of $\mathrm{SL}(2, C_n)$ using test maps. We prove the following.

Theorem 1.1. *Let G be a Zariski-dense subgroup of* $\mathrm{SL}(2, C_n)$.

(1) *Let f be a loxodromic element in* $\mathrm{SL}(2, C_n)$, *not necessarily in G, such that* $0 < \beta(f) < 1$. *If the two generator subgroup* $\langle f, g\rangle$ *is discrete for every loxodromic element g in G, then G is discrete.*

(2) *Let f be a non-elliptic isometry in* $\mathrm{SL}(2, C_n)$, *not necessarily in G, such that*

$$0 < 2\cosh(\tau_f/2)\sqrt{\beta(f)} < 1.$$

If the two generator subgroup $\langle f, g\rangle$ *is discrete for every loxodromic element g in G, then G must be discrete.*

(3) *Let f be an elliptic element in* $\mathrm{SL}(2, C_n)$, *not necessarily in G, such that* $0 < \beta(f) < 4\sin^2(\pi/10)$. *If the two generator subgroup* $\langle f, g\rangle$ *is discrete for every loxodromic element g in G, then G is discrete.*

The following theorem also follows using similar methods as in the proof of the above theorem.

Theorem 1.2. *Let G be a Zariski-dense subgroup of* $\mathrm{SL}(2, C_n)$.

(1) *Let f be a loxodromic element in* $\mathrm{SL}(2, C_n)$, *not necessarily in G, such that* $0 < \beta(f) < 1$. *If the two generator subgroup* $\langle f, gfg^{-1}\rangle$ *is discrete for every loxodromic element g in G, then G is discrete.*

(2) *Let f be a non-elliptic isometry in* $\mathrm{SL}(2, C_n)$, *not necessarily in G, such that*

$$0 < \rho = 2\cosh(\tau_f/2)\sqrt{\beta(f)} < 1.$$

If the two generator subgroup $\langle f, gfg^{-1}\rangle$ *is discrete for every loxodromic element g in G, then G is discrete.*

(3) *Let f be a regular elliptic element in* $\mathrm{SL}(2, C_n)$, *not necessarily in G, such that* $0 < \beta(f) < 4\sin^2(\pi/10)$. *If the two generator subgroup* $\langle f, gfg^{-1}\rangle$ *is discrete for every loxodromic element g in G, then G is discrete.*

1.2. Discreteness in $\mathrm{U}(n, 1; \mathbb{F})$**.** It is natural to ask for extending the above results to isometries of the complex and the quaternionic hyperbolic spaces. Some discreteness criteria in $\mathrm{SU}(n, 1)$ are available in the literature, eg. [11], [17], [21]. However, not much attention has been given to $\mathrm{Sp}(n, 1)$, partly because it lacks conjugacy invariants (unlike the complex case) due to non-commutativity of the quaternions. In the following we note a version of Theorem 1.1 in this set up.

A loxodromic element in $\mathrm{Sp}(n, 1)$ is conjugate to a matrix of the form

$$f = diag(\lambda_1, \bar{\lambda}_1^{-1}, \lambda_3 \dots, \lambda_{n+1}), \tag{1.1}$$

where $|\lambda_1| > 1$, and $|\lambda_i| = 1$ for $i = 3, \dots, n+1$. Cao and Parker defined the following conjugacy invariant in [5]:

$$\delta_{cp}(f) = \max\{|\lambda_i - 1| \ : \ i = 3, \dots, n+1\},$$

$$M_f = 2\delta_{cp}(f) + |\lambda_1 - 1| + |\bar{\lambda}_1^{-1} - 1|.$$

An eigenvalue λ of a matrix in $\mathrm{Sp}(n,1)$ is called negative-type or positive-type according as the Hermitian length of the corresponding eigenvector is negative or positive. An elliptic element in $\mathrm{Sp}(n,1)$ is conjugate to a matrix of the form

$$f = diag(\lambda_1, \dots, \lambda_{n+1}), \tag{1.2}$$

where for all i, $|\lambda_i| = 1$, and we choose the underlying Hermitian form so that λ_1 is a negative-type eigenvalue and all others are positive-type eigenvalues. In [15], we defined the following invariant, cf. [4],

$$\delta(f) = \max\{|\lambda_i - 1| + |\lambda_1 - 1| \ : \ i = 2, \dots, n+1\}. \tag{1.3}$$

Clearly, $\delta(f)$ is an invariant of the conjugacy class of the elliptic element f.

Let $T_{s,\zeta}$ be a unipotent parabolic element in $\mathrm{Sp}(n,1)$. We shall call such element in $\mathrm{Sp}(n,1)$ or $\mathrm{SU}(n,1)$ as Heisenberg translation. We may assume (see [9, p. 70]) that up to conjugacy,

$$T_{s,\zeta} = \begin{pmatrix} 1 & 0 & 0 \\ s & 1 & \zeta^* \\ \zeta & 0 & I \end{pmatrix}, \tag{1.4}$$

where $Re(s) = \frac{1}{2}|\zeta|^2$.

Theorem 1.3. *Let G be Zariski dense in* $\mathrm{Sp}(n,1)$.

(1) *Let $f \in \mathrm{Sp}(n,1)$ be a loxodromic element such that $M_f < 1$. If $\langle f, g\rangle$ is discrete for every loxodromic element $g \in G$, then G is discrete.*
(2) *Let $f \in \mathrm{Sp}(n,1)$ be a Heisenberg translation such that $|\zeta| < \frac{1}{2}$. If $\langle f, g\rangle$ is discrete for every loxodromic element g in G, then G is discrete.*
(3) *Let $f \in \mathrm{Sp}(n,1)$ be a regular elliptic element such that $\delta(f) < 1$. If $\langle f, g\rangle$ is discrete for every loxodromic element $g \in G$, then G is discrete.*

As a by-product of the proof of the above theorem, we have the following result for subgroups in $\mathrm{SU}(n,1)$. A version of this result was obtained by Qin and Jiang in [21].

Corollary 1.4. *Let G be Zariski dense in* $\mathrm{SU}(n,1)$.

(1) *Let $f \in \mathrm{SU}(n,1)$ be a loxodromic element such that $M_f < 1$. If $\langle f, g\rangle$ is discrete for every loxodromic element $g \in G$, then G is discrete.*
(2) *Let $f \in \mathrm{SU}(n,1)$ be a Heisenberg translation such that $|\zeta| < \frac{1}{2}$. If $\langle f, g\rangle$ is discrete for every loxodromic element g in G, then G is discrete.*
(3) *Let $f \in \mathrm{SU}(n,1)$ be a regular elliptic element such that $\delta(f) < 1$. If $\langle f, g\rangle$ is discrete for every loxodromic element $g \in G$, then G is discrete.*

After discussing some background materials in Section 2, we prove Theorem 1.1 and Theorem 1.2 in Section 3. We prove Theorem 1.3 in Section 4.

2. Preliminaries

2.1. Clifford Algebra. The *Clifford algebra* C_n, $n \geq 0$, is the real associative algebra which has been generated by n symbols $i_1, i_2, \cdots, i_n$ subject to the following relations:

$$i_t i_s = -i_s i_t \text{ , for } t \neq s \text{ and } i_t^2 = -1 \text{ .}$$

Let us define $i_0 = 1$ and then every element of C_n can be expressed uniquely in the form $a = \sum a_I I$, where the sum is over all products $I = i_{v_1} i_{v_2} \cdots i_{v_k}$,with $1 \leq v_1 < v_2 < \cdots < v_k \leq n$ and $a_I \in \mathbb{R}$. Here the null product is permitted and identified with the real number 1 . We equip C_n with the Euclidean norm. Thus $C_0 = \mathbb{R}$, $C_1 = \mathbb{C}$, $C_2 = \mathbb{H}$ etc. The following are involutions in C_n:

$*$: In $a \in C_n$ as above, replace in each $I = i_{v_1} i_{v_2} \cdots i_{v_k}$ by $i_{v_k} \cdots i_{v_1}$. $a \mapsto a^*$ is an anti-automorphism.

$'$: Replace i_k by $-i_k$ in a to obtain a'.

The conjugate $\bar{a}$ of a is now defined as: $\bar{a} = (a^*)' = (a')^*$.

Let us identify $\mathbb{R}^{n+1}$ with the $(n+1)-$dimensional subspace of C_n formed by the Clifford numbers of the form

$$v = a_0 + a_1 i_1 + \ldots + a_n i_n.$$

These numbers are known as *vectors*. The products of non-zero vectors form a multiplicative group, denoted by Γ_n. For a vector v, $v^{-1} = \bar{v}/|v|^2$.

A Clifford matrix of dimension n is a 2×2 matrix $T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that

(i) $a, b, c, d \in \Gamma_n - \{0\}$;
(ii) the Clifford determinant $\Delta(T) = ad^* - bc^* = 1$, and,
(iii) $ab^*, cd^*, c^*a, d^*b \in \mathbb{R}^{n+1}$.

The group of all Clifford matrices is denoted by $\mathrm{SL}(2, C_n)$. In [24], Waterman showed that $\mathrm{SL}(2, C_n)$ is same as the group of all invertible 2×2 matrices over C_n with Clifford determinant 1.

The group $\mathrm{SL}(2, C_n)$ acts on $\mathbb{S}^{n+1} = \mathbb{R}^{n+1} \cup \{\infty\}$ by the action:

$$A : v \mapsto (av + b)(cv + d)^{-1}.$$

This action extends by Poincaré extension to $\mathbf{H}_{\mathbb{R}}^{n+2}$. The group $\mathrm{SL}(2, C_n)$ acts as the orientation-preserving isometry group of $\mathbf{H}_{\mathbb{R}}^{n+2}$. For more details we refer to [3], [2], [24], [6].

2.2. Classification of elements of $\mathrm{SL}(2, C_n)$ **:** We recall that, see [24], a parabolic f element in $\mathrm{SL}(2, C_n)$ is conjugate to

$$\begin{pmatrix} \lambda & \mu \\ 0 & \lambda^{*-1} \end{pmatrix}, \ |\lambda| = 1, \ \mu \neq 0.$$

If $\lambda = 1$, then f is called a *translation*.

Up to conjugacy in $\mathrm{SL}(2, C_n)$, a loxodromic element f is given by

$$f = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{*-1} \end{pmatrix},$$

where $\lambda \in \Gamma_n$, $|\lambda| \neq 1$. If $|\lambda| = 1$, then it is a non-regular elliptic element.

Suppose f is regular elliptic in $\mathrm{SL}(2, C_n)$, where n is even. Note that $\mathrm{SL}(2, C_n)$ has a natural inclusion in $\mathrm{SL}(2, C_{n+1})$ as a closed subgroup. We shall consider the inclusion of f in $\mathrm{SL}(2, C_{n+1})$, and assume that f fixes at least two points on the boundary $\partial\mathbf{H}_{\mathbb{R}}^{n+3}$. Otherwise, we can choose two fixed points of f on $\partial\mathbf{H}_{\mathbb{R}}^{n+2}$. So, up to conjugacy in $\mathrm{SL}(2, C_{n+1})$, f is of the form

$$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{*-1} \end{pmatrix}, \ |\lambda| = 1.$$

The diagonal element λ depends on the rotation angles of f, for details see [24, Section 4].

2.3. Clifford Cross Ratio. As in the complex analysis, Clifford cross ratios are defined similarly. Let $z_1, z_2, z_3, z_4 \in \partial\mathbf{H}_{\mathbb{R}}^{n+2}$ be any four distinct points. Let $z_1 \neq \infty$. The Clifford cross ratio of (z_1, z_2, z_3, z_4) is given by

$$\begin{aligned} [z_1, z_2, z_3, z_4] &= (z_1 - z_3)(z_1 - z_2)^{-1}(z_2 - z_4)(z_3 - z_4)^{-1}, \text{ if } z_2, z_3, z_4 \neq \infty; \\ &= (z_1 - z_3)(z_3 - z_4)^{-1}, \text{ if } z_2 = \infty; \\ &= (z_1 - z_2)^{-1}(z_2 - z_4), \text{ if } z_3 = \infty; \\ &= (z_1 - z_3)(z - z_2)^{-1}, \text{ if } z_4 = \infty. \end{aligned}$$

One can easily prove that for any $f = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, C_n)$, we have

$$[fz_1, fz_2, fz_3, fz_4] = (cz_3 + d)^{*-1}[z_1, z_2, z_3, z_4](cz_3 + d)^*.$$

Thus $\|[z_1, z_2, z_3, z_4]\|$ and $\mathrm{Re}[z_1, z_2, z_3, z_4]$ are invariants of Möbius maps in $\mathrm{SL}(2, C_n)$. We have the following basic properties of cross ratios, see [6] for details.

(1) $[z_1, z_2, z_3, z_4] + [z_2, z_1, z_3, z_4] = 1$.
(2) $[z_1, z_2, z_3, z_4][z_4, z_2, z_3, z_1] = 1$.
(3) $\|[z_1, z_2, z_3, z_4]\| = \|[z_2, z_1, z_4, z_3]\|$.
(4) $\|[z_1, z_2, z_3, z_4]\| = \|[z_3, z_4, z_1, z_2]\|$.

2.4. Cao-Waterman Jørgensen inequality. We need call the following results which are important Jørgensen type inequalities for two-generator subgroups of $\mathrm{SL}(2, C_n)$ when one of the generators is either elliptic or loxodromic.

Theorem 2.1. [6] *Let* $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, C_n)$ *be any element and* $f \in \mathrm{SL}(2, C_n)$ *be a loxodromic element having two fixed points* u, v *in* $\partial\mathbf{H}_{\mathbb{R}}^{n+2}$ *satisfying that* $\{gu, gv\}$ *is not equal to* $\{u, v\}$. *If* $\langle f, g\rangle$ *generate a discrete subgroup in* $\mathrm{SL}(2, C_n)$, *then*

$$\beta(f)(1 + \|[u, v, gu, gv]\|) \geq 1.$$

Theorem 2.2. [6] *If* $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, C_n)$ *any element and* $f \in \mathrm{SL}(2, C_n)$ *be an elliptic element such that* $\langle f, g\rangle$ *forms a non-elementary discrete subgroup in* $\mathrm{SL}(2, C_n)$, *then we have*

$$\beta(f)\Big(\frac{1}{4\sin^2(\pi/10)} + \|[u, v, gu, gv]\|\Big) \geq 1,$$

where u, v are any two boundary fixed points of f.

The Jørgensen type inequality for non-elliptic isometries fixing the boundary point ∞ is given by the following.

Theorem 2.3. [6] $f = \begin{pmatrix} \lambda & \mu \\ 0 & \lambda^{*-1} \end{pmatrix} \in \mathrm{SL}(2, C_n)$ *be a non-elliptic isometry that fixes the boundary point ∞. Let* $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, C_n)$ *be any element in* $\mathrm{SL}(2, C_n)$ *such that* $0 < \rho = 2\cosh(\tau_f/2)\sqrt{\beta(f)} < 1$*, and* $fix(f) \cap fix(g) = \emptyset$*. If* $\langle f, g\rangle$ *generate a discrete subgroup in* $\mathrm{SL}(2, C_n)$*, then*

$$|tr^2(fgf^{-1})[fg(\infty), fg^{-1}(\infty), g(\infty), g^{-1}(\infty)]| \geq \frac{1 - \rho + \sqrt{(1-\rho)^2 - 4\beta(f)}}{2}.$$

Moreover, if f is a translation, i.e. $\lambda = 1$, then we have

$$|c|^2|\mu|^2 \geq \frac{1 - \rho + \sqrt{(1-\rho)^2 - 4\beta(f)}}{2}.$$

2.5. Useful Results. Let $\mathcal{L}$ be the set of loxodromic elements in $\mathrm{U}(n, 1; \mathbb{F})$. It is well known that $\mathcal{L}$ is an open subset of $\mathrm{U}(n, 1; \mathbb{F})$. This fact will be crucial for our proofs.

Let $\mathcal{E}$ be the set of all regular elliptic elements in $\mathrm{U}(n, 1; \mathbb{F})$. When $\mathbb{F} = \mathbb{C}, \mathbb{H}$, $\mathcal{E} \neq \emptyset$. When $\mathbb{F} = \mathbb{R}$, note that $\mathcal{E} \neq \emptyset$ if and only if n is even. For n odd, an elliptic f in $\mathrm{U}(n, 1; \mathbb{R})$ has at least two fixed points on $\partial\mathbf{H}^n_{\mathbb{R}}$. It is known that $\mathcal{E}$ is an open subset of $\mathrm{U}(n, 1; \mathbb{F})$.

The following theorem will also be useful for our purpose.

Theorem 2.4. [8] *Let G be a subgroup of* $\mathrm{U}(n, 1; \mathbb{F})$ *such that there is no point in* $\overline{\mathbf{H}^n_{\mathbb{F}}}$ *or proper totally geodesic submanifold in* $\mathbf{H}^n_{\mathbb{F}}$ *which is invariant under G. Then G is either discrete or dense in* $\mathrm{U}(n, 1; \mathbb{F})$.

2.6. Limit set. Let $L(G)$ be the limit set of a subgroup G of $\mathrm{U}(n, 1; \mathbb{F})$. The limit set $L(G)$ is a closed G-invariant subset of $\partial\mathbf{H}^n_{\mathbb{F}}$. The group G is elementary if $L(G)$ is finite. If G is elementary, $L(G)$ consists of at most two points. If G is non-elementary, then $L(G)$ is an infinite set and every non-empty, closed G-invariant subset of $\partial\mathbf{H}^n_{\mathbb{F}}$ contains $L(G)$. We note the following lemma, for proof see [22, Chapter 12].

Lemma 2.5. *Let $a \in \partial\mathbf{H}^n_{\mathbb{F}}$ be fixed by a non-elliptic element of a subgroup G of* $\mathrm{U}(n, 1; \mathbb{F})$*, then a is a limit point of G.*

3. Proof of Theorem 1.1

Let $Fix(f)$ be subset of $\overline{\mathbf{H}_{\mathbb{R}}}^{n+2}$ that is pointwise fixed by f. Let O_f be the stabilizer subgroup of $Fix(f)$ in $\mathrm{SL}(2, C_n)$. Clearly, O_f is a closed subgroup of $\mathrm{SL}(2, C_n)$.

If possible suppose G is not discrete. Since G is Zariski-dense and assumed to be non-discrete, by Theorem 2.4, G is dense in $\mathrm{SL}(2, C_n)$. Let f be a 'test map'. Then there exists a sequence $\{g_n\}$ of distinct loxodromic elements such that $g_n \to f$. We may further assume that $Fix(g_n) \cap Fix(f) = \emptyset$. Clearly, there is such a sequence g'_n in $\mathrm{SL}(2, C_n)$. Since G is dense in $\mathrm{SL}(2, C_n)$, we can choose g_n sufficiently close to g'_n in the open neighbourhood $\mathcal{L} \setminus O_f$.

(1) Let f be loxodromic. Upto conjugacy, assume f fixes 0 and ∞, that is,

$$f = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{*-1} \end{pmatrix}, \quad |\lambda| \neq 1. \tag{3.1}$$

Let

$$g_n = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix} \tag{3.2}$$

It can be seen that $[0, \infty, g_n(0), g_n(\infty)] = -b_n c_n^*$. By Lemma 2.5, the subgroup $\langle f, g_n \rangle$ has more than two limit points, so it is non-elementary, also discrete by hypothesis. Thus using Theorem 2.1 and by the hypothesis,

$$\begin{aligned} &\beta(f)(1 + |b_n c_n|) \geq 1 \\ \Rightarrow \quad & |b_n c_n| \geq -1 + \frac{1}{\beta(f)} > 0. \end{aligned}$$

But we have $b_n c_n \to 0$ as $n \to \infty$. This leads to a contradiction.

(2) Let f be non-elliptic. Applying suitable conjugation, without loss of generality we may assume that one of the fixed point of f be ∞ which leaves f in the form $f = \begin{pmatrix} \lambda & \mu \\ 0 & \lambda^{*-1} \end{pmatrix}$. By Lemma 2.5 and hypothesis, for large n, the subgroup $\langle f, g_n \rangle$ is non-elementary and discrete. Then using Theorem 2.3 we must have

$$|tr^2(f g_n f^{-1})[f g_n(\infty), f g_n^{-1}(\infty), g_n(\infty), g_n^{-1}(\infty)]| \geq \frac{1 - \rho + \sqrt{(1-\rho)^2 - 4\beta(f)}}{2}.$$

By calculation, we see that the left hand side of the above inequality will be same as the left hand side of the following inequality:

$$|\lambda|^{-2} |c_n|^2 |f(a_n c_n^{-1}) - (a_n c_n^{-1})|.|f(-c_n^{-1} d_n) - (-c_n^{-1} d_n)| \geq \frac{1 - \rho + \sqrt{(1-\rho)^2 - 4\beta(f)}}{2},$$

i.e.

$$k_n = |c_n|^2 |f(a_n c_n^{-1}) - (a_n c_n^{-1})|.|f(-c_n^{-1} d_n) - (-c_n^{-1} d_n)| \geq \frac{|\lambda|^2 \left(1 - \rho + \sqrt{(1-\rho)^2 - 4\beta(f)}\right)}{2}.$$

Since f and g_n does not have a common fixed point, we must have $c_n \neq 0$. Also since $0 < \rho < 1$, hence, $\frac{1-\rho+\sqrt{(1-\rho)^2-4\beta(f)}}{2}$ is a positive real number. So, $|f(a_n c_n^{-1}) - (a_n c_n^{-1})|$ and $|f(-c_n^{-1} d_n) - (-c_n^{-1} d_n)|$ are non-zero. Thus for all n, k_n is bounded above by a positive real number. But $k_n \to 0$ as $n \to \infty$. This is a contradiction.

(3) Let f be elliptic as given. Recall that in the case when n is even and f has no fixed points on $\partial \mathbf{H}_{\mathbb{R}}^{n+2}$, we use inclusion to view f as an element in $\mathrm{SL}(2, C_{n+1})$ and assume $0, \infty$ to be points on $\partial \mathbf{H}_{\mathbb{R}}^{n+3}$, and thus

$$f = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{*-1} \end{pmatrix}, \quad |\lambda| = 1. \tag{3.3}$$

By hypothesis, $\langle f, g_n \rangle$ is discrete. We claim that $\langle f, g_n \rangle$ is non-elementary. If not, then it must keep the fixed points of g_n invariant. Since f does not have a common fixed point with g_n, it much swipes the fixed points of g_n. That would imply that f must have a rotation angle π. But then $\beta(f)$ would be more than $4\sin^2(\pi/10)$, which is not possible by assumption.

Let g_n be of the form (4.1). Since $b_n c_n \to 0$, for large n,

$$\beta(f)\Big(\frac{1}{4\sin^2(\pi/10)} + |b_n c_n|\Big) < 1.$$

This is a contradiction to Theorem 2.2.

This proves the theorem.

3.1. Proof of Theorem 1.2.

As above, given a test map f we choose a sequence of loxodromic elements g_n such that $g_n \to f$ and $Fix(g_n) \cap Fix(f) = \emptyset$. Let $L_n = g_n f g_n^{-1} = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix}$. Note that $Fix(L_n) = g_n(Fix(f))$.

(1) Let f be of the form (3.1). Since g_n does not fix the boundary fixed points of f, $Fix(L_n)$ would be disjoint from $Fix(f)$. Thus $\langle f, L_n \rangle$ is non-elementary as the limit set contains $Fix(f) \cup Fix(L_n)$, and it is discrete by hypothesis. Hence by Theorem 2.1, we have

$$|b_n c_n| \geq -1 + \frac{1}{\beta(f)} > 0.$$

But as $L_n \to f$, we have $b_n c_n \to 0$ as $n \to \infty$. This leads to a contradiction.

(2) In this case, we follow the similar arguments as in Theorem 1.1, and we get by Theorem 2.3 inequality that,

$$|c_n|^2 |f(a_n c_n^{-1}) - (a_n c_n^{-1})|.|f(-c_n^{-1} d_n) - (-c_n^{-1} d_n)| \geq \frac{|\lambda|^2 (1 - \rho + \sqrt{(1-\rho)^2 - 4\beta(f)}}{2}.$$

But since $L_n \to f$, so $c_n \to 0$, and hence $k_n \to 0$ as $n \to \infty$. This leads to a contradiction.

(3) Let f be a regular elliptic. Let f be of the form (4.2). We claim that $\langle f, L_n \rangle$ is non-elementary. If not then, it either fixes a point or keeps a two point set $\{a, b\}$ on the boundary invariant. If $\langle f, L_n \rangle$ fixes a point p on $\mathbf{H}^n_{\mathbb{R}}$, then f fixes the geodesic l joining p and $g_n^{-1}(p)$. Consequently f fixes the boundary points of l. But that would imply, f must preserve $g_n^{-1}(l)$. If g_n does not preserve l, this would imply that f must have another boundary fixed point or a rotation angle π, both not possible by assumption. So g_n must keep l invariant. This is again not possible.

If $\langle f, L_n \rangle$ keeps $\{a, b\}$ invariant, then f keeps $g_n^{-1}(l)$ invariant, where l is the geodesic joining a and b. Thus f either fixes a, b or swipes them. If f swipes them, it must have a rotation angle π which is not possible given the value of $\beta(f)$. If f fixes a and b, then $\{a, b\}$ must be $\{0, \infty\}$. Since L_n also preserves l, g_n must preserve l joining 0 and ∞. This is not possible because g_n and f do not have the same fixed points, and if g_n swipes them, it must have a fixed point on $\mathbf{H}^n_{\mathbb{R}}$, which is again impossible. Hence $\langle f, L_n \rangle$ must be non-elementary, and also discrete by hypothesis. Now the result follows similarly as in the proof of Theorem 1.1(3).

This proves the theorem.

4. Proof of Theorem 1.3

Recall that

$$\mathrm{Sp}(n,1) = \{A \in \mathrm{GL}(n+1,\mathbb{H}) : A^* J_2 A = J_2\},$$

where

$$J_2 = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & I_{n-1} \end{pmatrix}.$$

Equivalently, one may also use the Hermitian form given by the following matrix wherever convenient.

$$J_1 = \begin{pmatrix} -1 & 0 \\ 0 & I_n \end{pmatrix}.$$

An element $g \in \mathrm{Sp}(n,1)$ acts on $\overline{\mathbf{H}_{\mathbb{H}}}^n = \mathbf{H}^n_{\mathbb{H}} \cup \partial\mathbf{H}^n_{\mathbb{H}}$ by projective transformations. Thus the isometry group of $\mathbf{H}^n_{\mathbb{H}}$ is given by $\mathrm{PSp}(n,1) = \mathrm{Sp}(n,1)/\{I,-I\}$. For a matrix (or a vector) T over $\mathbb{H}$, let $T^* = \bar{T}^t$. Let A be an element in $\mathrm{Sp}(n,1)$. Then one can choose A to be of the following form.

$$A = \begin{pmatrix} a & b & \gamma^* \\ c & d & \delta^* \\ \alpha & \beta & U \end{pmatrix}, \tag{4.1}$$

where a, b, c, d are scalars, $\gamma, \delta, \alpha, \beta$ are column matrices in $\mathbb{H}^{n-1}$ and U is an element in $M(n-1,\mathbb{H})$. Then, it is easy to compute that

$$A^{-1} = \begin{pmatrix} \bar{d} & \bar{b} & -\beta^* \\ \bar{c} & \bar{a} & -\alpha^* \\ -\delta & -\gamma & U^* \end{pmatrix}.$$

Let $o, \infty \in \partial\mathbf{H}^n_{\mathbb{H}}$ stand for the vectors $(0,1,\ldots,0)^t$ and $(1,0,\ldots,0)^t \in \mathbb{H}^{n+1}$ under the projection map respectively.

4.1. Quaternionic hyperbolic Jørgensen inequalities. For two generator subgroups of $\mathrm{Sp}(n,1)$ with an elliptic generator, one has the following, see [4], [15]. For elliptic elements, we use the form J_1 to represent $\mathrm{Sp}(n,1)$.

Theorem 4.1. [4] *Let g and h be elements of* $\mathrm{Sp}(n,1)$. *Suppose that g is a regular elliptic element with fixed point* $0 = (0,\ldots,0)^t \in \mathbf{H}^n_{\mathbb{H}}$, *i.e. g is of the form*

$$g = \begin{pmatrix} \lambda_1 & 0 \\ 0 & L \end{pmatrix}, \tag{4.2}$$

where $L = diag(\lambda_2,\ldots,\lambda_{n+1})$. Let

$$h = (a_{i,j})_{i,j=1,\ldots,n+1} = \begin{pmatrix} a_{1,1} & \beta \\ \alpha & A \end{pmatrix},$$

be an arbitrary element in $\mathrm{Sp}(n,1)$, *where $a_{1,1}$ is a scalar, α,β column vectors and $A \in M(n,\mathbb{H})$. If*

$$|a_{1,1}|\delta(g) < 1,$$

then the group $\langle g, h\rangle$ generated by g and h is either elementary or non-discrete.

For representing parabolic and loxodromic elements, we shall use the Hermitian form J_2. In [16, Appendix], Hersonsky and Paulin proved a version of Shimizu's lemma for subgroups in $\mathrm{SU}(n,1)$. The following quaternionic version of [16, Proposition A.1] is a straight-forward adaption of the proof of Hersonsky and Paulin.

Theorem 4.2. *Suppose $T_{s,\zeta}$ be an Heisenberg translation in* $\mathrm{Sp}(n,1)$ *of the form* (1.4)*, and A be an element in* $\mathrm{Sp}(n,1)$ *of the form* (4.1)*. Set*

$$t = \mathrm{Sup}\{|b|, |\beta|, |\gamma|, |U - I|\},\ M = |s| + 2|\zeta|. \tag{4.3}$$

If

$$Mt + 2|\zeta| < 1, \tag{4.4}$$

then the group generated by A and $T_{s,\zeta}$ is either non-discrete or fixes o.

For two generator subgroups with a loxodromic element, we have the following version of the Jørgensen inequality from the work of Cao and Parker [5]. Up to conjugacy, a loxodrmic element has fixed points o and ∞, and it is conjugate to a matrix of the form (1.1).

Theorem 4.3. (Cao and Parker) [5] *Let $h \in$* $\mathrm{Sp}(n,1)$ *be given by* (4.1)*. Let g be a loxodromic element in* $\mathrm{Sp}(n,1)$ *with fixed points $o, \infty \in \partial\mathbf{H}^n_{\mathbb{H}}$, i.e. of the form* (1.1)*. Let $M_g < 1$. If $\langle g, h\rangle$ is non-elementary and discrete, then*

$$|ad|^{\frac{1}{2}}|bc|^{\frac{1}{2}} \geq \frac{1 - M_g}{M_g^2}. \tag{4.5}$$

4.2. Proof of Theorem 1.3. If possible suppose G is not discrete. Then G must be dense in $\mathrm{Sp}(n,1)$ by Theorem 2.4. Note that the set $\mathcal{L}$ of loxodromic elements in $\mathrm{Sp}(n,1)$ forms an open subset of $\mathrm{Sp}(n,1)$. Let $Fix(f)$ denote the fixed point set of f on $\partial\mathbf{H}^n_{\mathbb{H}}$. Let F_f be the subgroup of $\mathrm{Sp}(n,1)$ that stabilizes $Fix(f)$. The subgroup F_f is closed in $\mathrm{Sp}(n,1)$. Hence $\mathcal{L} - F_f$ is still an open subset in $\mathrm{Sp}(n,1)$.

(1) Let f be loxodromic. Up to conjugacy, assume that f is of the form (1.1). Since $f \in \bar{G}$, using similar arguments as in the proof of Theorem 1.1, there exists a sequence $\{h_n\}$ of loxodromic elements in $(\mathcal{L} - F_f) \cap G$ such that $h_n \to f$. Thus, h_n, f do not have a common fixed point, and $\langle h_n, f\rangle$ is non-elementary for each n. Let

$$h_n = \begin{pmatrix} a_n & b_n & \gamma_n^* \\ c_n & d_n & \nu_n^* \\ \alpha_n & \beta_n & U_n \end{pmatrix},$$

where a, b, c, d are scalars, $\gamma, \delta, \alpha, \beta$ are column matrices in $\mathbb{H}^{n-1}$ and U is an element in $M(n-1, \mathbb{H})$. By Theorem 4.3,

$$|a_n d_n|^{\frac{1}{2}}|b_n c_n|^{\frac{1}{2}} > \frac{1 - M_f}{M_f^2}.$$

But $b_n c_n \to 0$ as $n \to \infty$, hence

$$\frac{1 - M_f}{M_f^2} < 0,$$

which is a contradiction since $M_f < 1$.

(2) Let f be a Heisenberg translation. Without loss of generality assume it is of the form (1.4). Since, $f \in \overline{G}$, there exist a sequence of loxodromic elements $\{h_n\} \in (\mathcal{L} - F_f) \cap G$ such that

$$h_n \to f.$$

Since, f and h_n have distinct fixed points, hence $\langle f, h_n \rangle$ is discrete and non-elementary. By Theorem 4.2,

$$Mt_n + 2|\zeta| \geq 1.$$

But $t_n \to 0$ as $n \to \infty$. Thus for large n, $|\zeta| \geq \frac{1}{2}$. This is a contradiction as $|\zeta| < \frac{1}{2}$ is given.

(3) Let f be a regular elliptic. We can assume that f is of the form (4.2) with fixed point 0, up to conjugacy. Since, G is dense in $\mathrm{Sp}(n, 1)$, there is a sequence of loxodromic element $\{h_m\}$ in $\mathcal{L} \cap G$ such that $h_m \to I$. Let

$$h_m = (a_{i,j}^{(m)}) = \begin{pmatrix} a_{1,1}^{(m)} & \beta^{(m)} \\ \alpha^{(m)} & A^{(m)} \end{pmatrix}.$$

The group $\langle f, h_m \rangle$ must be non-elementary. For, if not, clearly $\langle f, h_m \rangle$ can not fix a point on $\overline{\mathbf{H}_{\mathbb{H}}^n}$ as that will contradict either regularity of f or loxodromic nature of h_m. If it keeps two points x and y on $\partial \mathbf{H}_{\mathbb{H}}^n$ invariant without fixing them, then f must swipes x and y, and hence f^2 fixes x y, and 0. Thus f^2 must have a repeated eigenvalue λ, see [8, Proposition 2.4]. This implies, g would have a repeated eigenvalue $\lambda^{1/2}$, which is a contradiction to the regularity. By our assumption $\langle f, h_m \rangle$ is also discrete for each m. Hence by Theorem 4.1,

$$|a_{1,1}^{(m)}|\, \delta(g) \geq 1.$$

But $a_{1,1}^{(m)} \to 1$ and $\delta(g) < 1$. This is a contradiction.

This proves the theorem.

Remark 4.4. The results in this paper show that in order to determine discreteness of a Zariski-dense subgroup G of $\mathrm{U}(n, 1; \mathbb{F})$, it is enough to check discreteness of the two generator subgroups of G obtained by adjoining the loxodromic elements of G to a 'test map' in $\mathrm{U}(n, 1; \mathbb{F})$. Let $\mathcal{E}$ denote the set of regular elliptic elements of $\mathrm{U}(n, 1; \mathbb{F})$. The set $\mathcal{E}$ is also an non-empty open subset of $\mathrm{U}(n, 1; \mathbb{F})$, provided n is even when $\mathbb{F} = \mathbb{R}$. Thus, if we replace the loxodromic elements g by regular elliptic elements, then versions of Theorem 1.3 and Corollary 1.4 hold true for all n, and, Theorem 1.1 goes through for all even n.

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