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FELLER GENERATORS AND STOCHASTIC DIFFERENTIAL EQUATIONS WITH SINGULAR (FORM-BOUNDED) DRIFT

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Abstract

We consider the problem of constructing weak solutions to the Itô and to the Stratonovich stochastic differential equations having critical-order singularities in the drift and critical-order discontinuities in the dispersion matrix.

1. Introduction

The present paper is concerned with the problem of existence of a (unique) weak solution to the stochastic differential equation (SDE)

$$(SDE_I)\qquad X(t) = x - \int_0^t b(X(s))ds + \sqrt{2}\int_0^t \sigma(X(s))dW(s), \quad x \in \mathbb{R}^d,$$
$$W(t) \text{ is a } d\text{-dimensional Brownian motion}, \quad d \geq 3,$$

with drift $b : \mathbb{R}^d \to \mathbb{R}^d$ that is in general locally unbounded, and dispersion matrix $\sigma \in L^\infty(\mathbb{R}^d, \mathbb{R}^d \otimes \mathbb{R}^d)$ that can be discontinuous.

The search for the largest class(es) of admissible b and σ is of fundamental importance and has long history. The first principal result is due to N. I. Portenko [27]: If $|b| \in L^p \equiv L^p(\mathbb{R}^d)$ for a $p > d$, and $a = \sigma\sigma^\top$ is Hölder continuous, then there exists a unique in law weak solution to (SDE_I); the weak solution can be constructed using either an analytic approach or the Girsanov transform. The result in [27] was extended (in the case $a = I$) by R. Bass-Z.-Q. Chen [2] to b in the standard Kato class $\mathbf{K}_0^{d+1}$, see definition and more detailed discussion below. Since $\mathbf{K}_0^{d+1}$ contains, for every $\varepsilon > 0$, vector fields b such that $|b| \notin L^{1+\varepsilon}_{\rm loc}$, the use of Girsanov transform to construct a weak solution becomes problematic. In recent papers [18, 19], N. V. Krylov established weak existence and uniqueness in law for a general measurable uniformly elliptic σ and $b \in L^d(\mathbb{R}^d, \mathbb{R}^d)$ (both σ and b can be time-dependent); it is easily seen that $L^d(\mathbb{R}^d, \mathbb{R}^d) - \mathbf{K}_0^{d+1} \neq \varnothing$.

(With regard to the existence and pathwise uniqueness of strong solutions to (SDE_I), the corresponding result for $\sigma = I$, $|b| \in L^p$, $p > d$ is due to N. V. Krylov-M. Röckner [21], and for $|\nabla\sigma| \in L^p$, $|b| \in L^p$, $p > d$, due to X. Zhang [31]; both these results allow time-dependent coefficients. Let us also note that imposing additional assumption on the structure of b (integrability condition on the negative part of $\operatorname{div} b$) allows to prove weak existence and uniqueness for (SDE_I) with $\sigma = I$, $|b| \in L^p$ for some $p < d$, see X. Zhang-G. Zhao [32].)

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See Section 3 below for further discussion.

In this paper we establish existence and uniqueness (in appropriate sense) of weak solution to (SDE_I), not assuming additional structure of the drift b such as radial symmetry or differentiability, under the following assumptions on b and σ:

Condition ($\mathbf{C_1}$) The vector field b is form-bounded, i.e. $|b| \in L^2_{\rm loc}$ and there exist constants $\delta > 0$ and $\lambda = \lambda_\delta > 0$ such that

$$\||b|(\lambda - \Delta)^{-\frac{1}{2}}\|_{2\to 2} \leqslant \sqrt{\delta}$$

(write $b \in \mathbf{F}_\delta$). Here and below, $\|\cdot\|_{p\to q} := \|\cdot\|_{L^p\to L^q}$. Equivalently, condition $b \in \mathbf{F}_\delta$ can be stated as the quadratic form inequality

$$\|b\varphi\|_2^2 \leq \delta\|\nabla\varphi\|_2^2 + c_\delta\|\varphi\|_2^2, \quad \varphi \in W^{1,2},$$

for a constant c_δ $(= \lambda\delta)$. The constant δ is called the form-bound of b.

Clearly,

$$b_1 \in \mathbf{F}_{\delta_1}, b_2 \in \mathbf{F}_{\delta_2} \quad \Rightarrow \quad b_1 + b_2 \in \mathbf{F}_\delta, \qquad \sqrt{\delta} := \sqrt{\delta_1} + \sqrt{\delta_2}.$$

EXAMPLES. Let us list some sub-classes of $\mathbf{F}_\delta$ defined in *elementary terms*.

1. The class $\mathbf{F}_\delta$ contains vector fields b $(= b_1 + b_2)$ in $L^p(\mathbb{R}^d, \mathbb{R}^d) + L^\infty(\mathbb{R}^d, \mathbb{R}^d)$, $p > d$ (by Hölder's inequality) and in $L^d(\mathbb{R}^d, \mathbb{R}^d) + L^\infty(\mathbb{R}^d, \mathbb{R}^d)$ (by Sobolev's inequality) with form-bound δ that can be chosen arbitrarily small.

2. The class $\mathbf{F}_\delta$ also contains vector fields having critical-order singularities, such as

$$b(x) = \sqrt{\delta}\frac{d-2}{2}|x|^{-2}x$$

(by the Hardy-Rellich inequality $\||x|^{-1}\varphi\|_2^2 \leq (\frac{2}{d-2})^2\|\nabla\varphi\|_2^2$, $\varphi \in W^{1,2}$). More generally, the class $\mathbf{F}_\delta$ contains vector fields b with $|b|$ in $L^{d,\infty}$ (the weak L^d space) $\supsetneq L^d$. Recall that a measurable function $h : \mathbb{R}^d \to \mathbb{R}$ is in $L^{d,\infty}$ if $\|h\|_{d,\infty} := \sup_{s>0} s|\{x \in \mathbb{R}^d : |h(x)| > s\}|^{1/d} < \infty$. By the Strichartz inequality with sharp constants [16, Prop. 2.5, 2.6, Cor. 2.9], if $|b|$ in $L^{d,\infty}$, then

$$\begin{aligned} b \in \mathbf{F}_{\delta_1} \quad \text{with } \sqrt{\delta_1} &= \||b|(\lambda - \Delta)^{-\frac{1}{2}}\|_{2\to 2} \\ &\leq \|b\|_{d,\infty}\Omega_d^{-\frac{1}{d}}\||x|^{-1}(\lambda - \Delta)^{-\frac{1}{2}}\|_{2\to 2} \\ &\leq \|b\|_{d,\infty}\Omega_d^{-\frac{1}{d}}2^{-1}\frac{\Gamma(\frac{d-2}{4})}{\Gamma(\frac{d+2}{4})} = \|b\|_{d,\infty}\Omega_d^{-\frac{1}{d}}\frac{2}{d-2}. \end{aligned}$$

where $\Omega_d = \pi^{\frac{d}{2}}\Gamma(\frac{d}{2} + 1)$ is the volume of the unit ball in $\mathbb{R}^d$.

3. Furthermore, $\mathbf{F}_\delta$ contains vector fields in the Campanato-Morrey class and the Chang-Wilson-Wolff class with δ depending on the respective norm of the vector field in these classes, see [6]. The class $\mathbf{F}_\delta$ contains b with $|b|^2$ in the Kato class of potentials $\{V \in L^1_{\rm loc} \mid \|(\lambda - \Delta)^{-1}|V|\|_\infty \leq \sqrt{\delta}$ for some $\lambda = \lambda_\delta > 0\}$ (by interpolation).

We note that for every $\varepsilon > 0$ one can find $b \in \mathbf{F}_\delta$ such that $|b| \notin L^{2+\varepsilon}_{\rm loc}(\mathbb{R}^d, \mathbb{R}^d)$, e.g. consider

$$|b|^2(x) = C\frac{\mathbf{1}_{B(0,1+\alpha)} - \mathbf{1}_{B(0,1-\alpha)}}{\big||x| - 1\big|^{-1}(-\ln\big||x| - 1\big|)^\beta}, \quad \beta > 1, \quad 0 < \alpha < 1.$$

Another example is: if $h \in L^2(\mathbb{R})$, $T : \mathbb{R}^d \to \mathbb{R}$ is a linear map, then the vector field $b(x) = h(Tx)e$, where $e \in \mathbb{R}^d$, is in $\mathbf{F}_\delta$ with appropriate δ (but $|b|$ may not be in $L^{d,\infty}_{\rm loc}$).

We refer to [12, sect. 4] for a more detailed discussion on class $\mathbf{F}_\delta$.

Condition $(\mathbf{C_2})$ The diffusion matrix $a := \sigma\sigma^\top$ satisfies $a \geq \nu I$, $\nu > 0$ and (write $\nabla_i \equiv \partial_{x_i}$)

$$(\nabla_r a_{i\ell})_{i=1}^d \in \mathbf{F}_{\gamma_{r\ell}}, \qquad 1 \leq r, \ell \leq d,$$

for some $\gamma_{r\ell} > 0$.

For example, a matrix a with entries in $W^{1,d}$ satisfies $(\mathbf{C}_2)$ with $\gamma_{r\ell}$ that can be chosen arbitrarily small. The model example of a matrix a satisfying $(\mathbf{C}_2)$ and having a critical discontinuity is

$$a(x) = I + c\frac{x \otimes x}{|x|^2}, \qquad \text{the constant } c > -1$$

(in fact, $\nabla_r a_{i\ell} = c\mathbf{1}_{r=i}\frac{x_\ell}{|x|^2} + c\mathbf{1}_{r=\ell}\frac{x_i}{|x|^2} + cx_i x_\ell \frac{2x_r}{|x|^4}$, so $|(\nabla_r a_{i\ell})_{i=1}^d| \leq 2|c||x|^{-1} \Rightarrow (\nabla_r a_{i\ell})_{i=1}^d \in \mathbf{F}_{\gamma_{r\ell}}$, $\gamma_{r\ell} = (4c)^2/(d-2)^2$ by the Hardy-Rellich inequality). Another example is

$$a(x) = I + c(\sin\log(|x|))^2 e \otimes e, \quad e \in \mathbb{R}^d, |e| = 1$$

(indeed, $\nabla_r a_{i\ell} = 2c(\sin\log|x|)(\cos\log|x|)|x|^{-2}x_r\, e_i e_\ell$; now, use example 2 above). More generally, we can consider an infinite sum of these two matrices with their points of discontinuity constituting e.g. a dense subset of $\mathbb{R}^d$.

We note that the class $(\mathbf{C}_2)$ contains matrices $a \notin$ VMO class, see details below.

Intuitively, the form-bounds $\gamma_{r\ell}$ can be viewed as measures of discontinuity of (differentiable) matrix a. (To illustrate this, we note that if $a_{ij} \in W^{1,p}$, $p > d$, then $\gamma_{r\ell}$ can be chosen arbitrarily small, while a_{ij} are Hölder continuous by the Sobolev Embedding Theorem. On the other hand, in the previous example of a discontinuous a, form-bounds $\gamma_{r\ell} > 0$ are determined by c.)

Denote $C_\infty := \{g \in C(\mathbb{R}^d) \mid \lim_{x\to\infty} g(x) = 0\}$ (with the sup-norm). The central analytic object in this paper is positivity preserving contraction C_0 semigroup $e^{-t\Lambda_{C_\infty}(a,b)}$ on C_∞ (Feller semigroup) whose generator $-\Lambda_{C_\infty}(a, b)$ is an operator realization in C_∞ of the formal operator

$$(\nabla \cdot a \cdot \nabla - b \cdot \nabla)f(x) = \sum_{i,j=1}^d \nabla_i \left(a_{ij}(x)\nabla_j f(x)\right) - \sum_{j=1}^d b_j(x)\nabla_j f(x).$$

We construct $e^{-t\Lambda_{C_\infty}(a,b)}$ under assumptions $(\mathbf{C}_1)$, $(\mathbf{C}_2)$. The construction, based on a $W^{1,\frac{qd}{d-2}}$ estimate on solutions to the corresponding elliptic equation in L^q and a $L^r \to L^\infty$ iteration procedure, is the main analytic result of this paper.

We note that the condition $b \in \mathbf{F}_\delta$, $\delta < 1$ is known in the literature first of all as the condition ensuring that the sesquilinear form $\mathfrak{t}[u, v] := \langle \nabla u \cdot a \cdot \nabla \bar{v}\rangle + \langle b \cdot \nabla u, v\rangle$ with a general uniformly elliptic a, on $u, v \in W^{1,2}$, where

$$\langle h\rangle := \int_{\mathbb{R}^d} h(x)dx, \quad \langle h, g\rangle := \langle h\bar{g}\rangle,$$

is m-sectorial; then $\mathfrak{t}[u, v] = \langle \Lambda_2(a, b)u, v\rangle$, $u \in D(\Lambda_2) \subset W^{1,2}$, $v \in W^{1,2}$, where operator

$-\Lambda_2(a,b)$ (a realization of the formal operator $\nabla \cdot a \cdot \nabla - b \cdot \nabla$ in L^2) is the generator of a quasi contraction C_0 semigroup on L^2 [10, Ch.VI] ("form-method" of constructing C_0 semigroups). Below we construct C_0 semigroup in C_∞, a space having (locally) stronger topology than L^2, under the same assumption $b \in \mathbf{F}_\delta$, for $\delta < c_d$ with appropriate constant $0 < c_d < 1$, at expense of requiring that a satisfies $(\mathbf{C}_2)$ with $\gamma_{r\ell} < c'_d$, $c'_d = c'_d(\delta) > 0$.

The operator behind (SDE_I) is the non-divergence form operator

$$-a \cdot \nabla^2 + b \cdot \nabla = -\sum_{i,j=1}^{d} a_{ij}(x)\nabla_i\nabla_j + \sum_{j=1}^{d} b_j(x)\nabla_j.$$

We re-write it as

$$-a \cdot \nabla^2 + b \cdot \nabla = -\nabla \cdot a \cdot \nabla + (\nabla a + b) \cdot \nabla, \tag{1}$$

where the vector field ∇a is defined by $(\nabla a)_k := \sum_{i=1}^{d}(\nabla_i a_{ik})$. By $(\mathbf{C}_2)$, ∇a is in $\mathbf{F}_{\delta_a}$ with $\delta_a \leq \sum_{r,\ell=1}^{d} \gamma_{r\ell}$, and so $\nabla a + b$ is an admissible drift. Thus, under appropriate assumptions on the values of form-bounds δ, δ_a, $\gamma_{r\ell}$, the Feller generator $-\Lambda_{C_\infty}(a, \nabla a + b)$ is well defined (Theorem 1(i)). In Theorem 1(ii), (iii), we show that the probability measures on the space of continuous trajectories determined by the Feller semigroup $e^{-t\Lambda_{C_\infty}(a,\nabla a+b)}$ admit description as weak solutions to (SDE_I).

If we could only handle drifts in $L^p(\mathbb{R}^d, \mathbb{R}^d)$, $p > d$, and thus in order to use (1) would have to require $\nabla_r a_{\cdot\ell} \in L^p(\mathbb{R}^d, \mathbb{R}^d)$, $p > d$, then by the Sobolev Embedding Theorem a would have to be Hölder continuous. It is the fact that we can handle critical-order singularities in the drift that allows us to consider diffusion matrices a with critical discontinuities.

We emphasize that there are $b \in \mathbf{F}_\delta$ so singular that they destroy the Gaussian upper (and lower) bound on the heat kernel of $-\nabla \cdot a \cdot \nabla + b \cdot \nabla$, $-a \cdot \nabla^2 + b \cdot \nabla$ (e.g. for $a = I$, $b(x) = \frac{d-2}{2}\sqrt{\delta}|x|^{-2}x$, see [22, 23], see also [24]).

The following example shows that the existence of a weak solution to (SDE_I) must depend on the value of the form-bound of b.

EXAMPLE 1. Consider the SDE ($d \geq 3$)

$$X(t) = -\int_0^t b(X(s))ds + \sqrt{2}W(t), \quad t \geq 0,$$

where

$$b(x) := \sqrt{\delta}\frac{d-2}{2}|x|^{-2}x \in \mathbf{F}_\delta.$$

If $\sqrt{\delta} < 1 \wedge \frac{2}{d-2}$, then by Theorem 1 below this equation has a weak solution.

If $\sqrt{\delta} \geq \frac{2d}{d-2}$, then an elementary argument (see e.g. [15, Example 1]) shows that the SDE does not have a weak solution. In this sense, the singularity of b is of critical order.

In Section 4 we consider the Stratonovich SDE

$$X(t) = x - \int_0^t b(X(s))ds + \sqrt{2}\int_0^t \sigma(X(s)) \circ dW(s), \quad x \in \mathbb{R}^d, \tag{SDE_S}$$

assuming that $(\nabla_r \sigma_{ij})_{i=1}^d \in \mathbf{F}_{\delta_{rj}}$ for some $\delta_{rj} > 0$. We put (SDE_S) in Itô form without losing the class of singularities of the drift or the class of discontinuities of the dispersion matrix (although imposing somewhat more restrictive assumptions on the values of form-bounds δ and $\gamma_{r\ell}$). From the analytic point of view, imposing conditions on $\nabla_r\sigma_{ij}$ seems to be pertinent to the subject matter since it provides an operator behind (SDE_S).

We prove that the weak solution to (SDE_I) or (SDE_S) is unique among all weak solutions that can be constructed using reasonable approximations of a, b, i.e. the ones that keep the values of form-bounds intact, see remark 2 below.

Since in our construction the weak solutions to (SDE_I), (SDE_S) are determined from the very beginning by a Feller semigroup, we do not need the uniqueness in law in order to prove that the associated process is strong Markov. Concerning a possible proof of the uniqueness in law we note that, under the assumptions $(\mathbf{C}_1)$, $(\mathbf{C}_2)$, in general $|\nabla u| \notin L^\infty$, $u = (\mu + \Lambda_q(a, \nabla a + b))^{-1} f$, even if $f \in C_c^\infty$ and $a = I$.

Let us also note that $v(t,\cdot) = e^{-t\Lambda_q(a,\nabla a+b)} f(\cdot)$ is a unique weak solution to Cauchy problem for the corresponding parabolic equation in L^q, cf. remark 4 below.

The results of this paper are new even if $b = 0$ or $\sigma = I$.

Notation. We denote by $\mathcal{B}(X, Y)$ the space of bounded linear operators between Banach spaces $X \to Y$, endowed with the operator norm $\|\cdot\|_{X\to Y}$. Set $\mathcal{B}(X) := \mathcal{B}(X, X)$. We write $T = s\text{-}X\text{-}\lim_n T_n$ for $T, T_n \in \mathcal{B}(X)$ if $Tf = \lim_n T_n f$ in X for every $f \in X$.

$\bar{\mathbb{R}}^d := \mathbb{R}^d \cup \{\infty\}$ is the one-point compactification of $\mathbb{R}^d$.

$\bar{\Omega}_D := D([0,\infty[, \bar{\mathbb{R}}^d)$ the set of all right-continuous functions $X : [0,\infty[\to \bar{\mathbb{R}}^d$ having the left limits (càdlàg functions), such that $X(t) = \infty$, $t > s$, whenever $X(s) = \infty$ or $X(s-) = \infty$.

$\mathcal{F}_t = \sigma\{X(s) \mid 0 \le s \le t, X \in \bar{\Omega}_D\}$ the minimal σ-algebra containing all cylindrical sets $\{X \in \bar{\Omega}_D \mid (X(s_1), \dots, X(s_n)) \in A, A \subset (\bar{\mathbb{R}}^d)^n \text{ is open}\}_{0\le s_1\le\dots\le s_n\le t}$. Set $\mathcal{F}_\infty = \sigma\{X(s) \mid 0 \le s < \infty, X \in \bar{\Omega}_D\}$.

$\Omega := C([0,\infty[, \mathbb{R}^d)$ denotes the set of all continuous functions $X : [0,\infty[\to \mathbb{R}^d$.

$\mathcal{G}_t := \sigma\{X(s) \mid 0 \le s \le t, X \in \Omega\}$, $\mathcal{G}_\infty := \sigma\{X(s) \mid 0 \le s < \infty, X \in \Omega\}$.

$C^{0,\alpha} = C^{0,\alpha}(\mathbb{R}^d)$ is the space of Hölder continuous functions with exponent $0 < \alpha < 1$.

The results of the present paper were announced in [13] and [14].

2. Itô diffusion

Without loss of generality, we assume from now on that $a \ge I$.

We fix the following smooth approximation of the matrix a and the vector field b:

$$a_n := I + e^{\epsilon_n\Delta}(\eta_n(a - I)), \quad \epsilon_n \downarrow 0,$$

where $\eta_n(x) = 1$ if $|x| < n$, $\eta_n(x) = n + 1 - |x|$ if $n \le |x| \le n + 1$, $\eta_n(x) = 0$ if $|x| > n + 1$, and

$$b_n := e^{\varepsilon_n\Delta}(\mathbf{1}_n b), \quad \varepsilon_n \downarrow 0,$$

where $\mathbf{1}_n$ is the indicator of $\{x \in \mathbb{R}^d \mid \ |x| \le n, |b(x)| \le n\}$.

By a standard result, given a Feller semigroup T^t on C_∞, there exist probability measures $\{\mathbb{P}_x\}_{x\in\mathbb{R}^d}$ on $\mathcal{F}_\infty$ such that $(\bar{\Omega}_D, \mathcal{F}_t, \mathcal{F}_\infty, \mathbb{P}_x)$ is a Markov process and

$$\mathbb{E}_{\mathbb{P}_x}[f(X(t))] = T^t f(x), \quad X \in \bar{\Omega}_D, \quad f \in C_\infty, \quad x \in \mathbb{R}^d$$

(see e.g. [3, Ch.I.9]). In the next theorem we prove that these probability measures are, in fact, concentrated on finite continuous trajectories $(\Omega, \mathcal{G}_\infty)$.

Theorem 1 (Main result)**.** *Let $d \geq 3$. Assume that conditions $(\mathbf{C}_1)$, $(\mathbf{C}_2)$ are satisfied, i.e. the vector fields $b \in \mathbf{F}_\delta$, $\nabla a \in \mathbf{F}_{\delta_a}$, $(\nabla_r a_{i\ell})_{i=1}^d \in \mathbf{F}_{\gamma_{r\ell}}$, with the form-bounds δ, δ_a, $\gamma_{r\ell}$ satisfying, for some $q > d-2$ if $d \geq 4$ or $q \geq 2$ if $d = 3$,*

$$(2)\quad \begin{cases} 1 - \frac{q}{4}(\sqrt{\gamma} + \|a - I\|_\infty \sqrt{\delta + \delta_a}) > 0, \quad \text{where } \gamma := \sum_{r,\ell=1}^d \gamma_{r\ell}, \\ (q-1)(1 - \frac{q\sqrt{\gamma}}{2}) - (\sqrt{\delta + \delta_a}\sqrt{\delta_a} + \delta + \delta_a)\frac{q^2}{4} - (q-2)\frac{q\sqrt{\delta+\delta_a}}{2} - \|a - I\|_\infty \frac{q\sqrt{\delta+\delta_a}}{2} > 0. \end{cases}$$

The following is true:

(i) *The limit*

$$(\star)\qquad s\text{-}C_\infty\text{-}\lim e^{-t\Lambda_{C_\infty}(a_n, \nabla a_n + b_n)} \quad (\text{loc. uniformly in } t \geq 0),$$

where $\Lambda_{C_\infty}(a_n, \nabla a_n + b_n) := -a_n \cdot \nabla^2 + b_n \cdot \nabla$, $D(\Lambda_{C_\infty}(a_n, \nabla a_n + b_n)) := (1 - \Delta)^{-1} C_\infty$, exists and thus determines a Feller semigroup on C_∞, which we denote by $e^{-t\Lambda_{C_\infty}(a, \nabla a + b)}$. Its generator $-\Lambda_{C_\infty}(a, \nabla a + b)$ is an appropriate operator realization of the formal operator $a \cdot \nabla^2 - b \cdot \nabla$ on C_∞. (We explain the choice of notation $\Lambda_{C_\infty}(a, \nabla a + b)$ below.)

We have

$$(e^{-t\Lambda_{C_\infty}(a, \nabla a + b)} \upharpoonright L^q \cap C_\infty)^{\text{clos}}_{L^q \to C_\infty} \in \mathcal{B}(L^q, C_\infty), \qquad t > 0.$$

Also, there exists a constant $\mu_0 = \mu_0(d, q, \delta, \delta_a, \gamma) > 0$ such that $u = (\mu + \Lambda_{C_\infty}(a, \nabla a + b))^{-1} f$, $\mu > \mu_0$, $f \in L^q \cap C_\infty$, is in $C^{0,\alpha}$, possibly after change on a measure zero set, with the Hölder continuity exponent $\alpha = 1 - \frac{d-2}{q}$.

Let $\mathbb{P}_x$ be determined by $T^t := e^{-t\Lambda_{C_\infty}(a, \nabla a + b)}$.

Then for every $x \in \mathbb{R}^d$:

(ii) *The trajectories of the process are $\mathbb{P}_x$ a.s. finite and continuous on $0 \leq t < \infty$.*

We denote $\mathbb{P}_x \upharpoonright (\Omega, \mathcal{G}_\infty)$ again by $\mathbb{P}_x$.

(iii) *$\mathbb{E}_{\mathbb{P}_x} \int_0^t |b(X(s))| ds < \infty$, $X \in \Omega$, and, for any selection of $f \in C_c^\infty$, $f(y) := y_i$, or $f(y) := y_i y_j$, $1 \leq i, j \leq d$, the process*

$$M^f(t) := f(X(t)) - f(x) + \int_0^t (-a \cdot \nabla^2 f + b \cdot \nabla f)(X(s)) ds, \quad t \geq 0,$$

is a continuous martingale relative to $(\Omega, \mathcal{G}_t, \mathbb{P}_x)$; the latter thus determines a weak solution to (SDE_I) on an extension of $(\Omega, \mathcal{G}_t, \mathbb{P}_x)$.

REMARK 1. Clearly, condition (2) is trivially satisfied if δ and γ $(\geq \delta_a)$ are sufficiently small; for example, if $a_{ij} \in W^{1,d}$ and $|b| \in L^d$, then δ and γ can be made arbitrarily small, see examples above. If $a = I$, then this condition reduces to $\delta < 1 \wedge (\frac{2}{d-2})^2$.

Since our assumptions on δ, δ_a, $\gamma_{r\ell}$ involve only strict inequalities, we may and will assume that $\epsilon_n, \varepsilon_n \downarrow 0$ in the definition of a_n, b_n are chosen so that

$$(\nabla_r (a_n)_{i\ell})_{i=1}^d \in \mathbf{F}_{\hat{\gamma}_{r\ell}} \ (1 \leq r, \ell \leq d), \quad \nabla a_n \in \mathbf{F}_{\hat{\delta}_a}, \quad b_n \in \mathbf{F}_{\hat{\delta}}$$

with form-bounds $\hat{\delta}$, $\hat{\delta}_a$, $\hat{\gamma}_{r\ell}$ (with $\lambda \neq \lambda(n)$) satisfying (2). Below, without loss of generality, $\hat{\delta} = \delta$, $\hat{\delta}_a = \delta_a$, $\hat{\gamma}_{r\ell} = \gamma_{r\ell}$.

REMARK 2. The solution to the martingale problem of (iii) is unique in the following sense. In addition to the hypothesis of Theorem 1, assume that $\|a - I\|_\infty + \delta < 1$. If $\{\mathbb{Q}_x\}_{x\in\mathbb{R}^d}$ is another solution to the martingale problem such that

$$\mathbb{Q}_x = w\text{-}\lim_n \mathbb{P}_x(\tilde{a}_n, \tilde{b}_n) \quad \text{for every } x \in \mathbb{R}^d,$$

for some smooth $\tilde{a}_n$, $\tilde{b}_n$ whose form-bounds $\tilde{\delta}$, $\tilde{\delta}_a$, $\tilde{\gamma}_{r\ell}$ (provided that $\lambda \neq \lambda(n)$) satisfy (2), then $\{\mathbb{Q}_x\}_{x\in\mathbb{R}^d} = \{\mathbb{P}_x\}_{x\in\mathbb{R}^d}$. See Appendix B for the proof.

3. Comments

1. The Feller semigroup in (i) is the principal analytic object; the other assertions (ii), (iii) of the theorem follow from the regularity properties of the resolvent $(\mu + \Lambda_{C_\infty}(a, \nabla a + b))^{-1}$, cf. Lemmas 7 and 8 below.

2. Even if $a = I$, $C_c^\infty \not\subset D(\Lambda_{C_\infty}(I, b))$ (already for $b \in L^\infty(\mathbb{R}^d, \mathbb{R}^d) - C_b(\mathbb{R}^d, \mathbb{R}^d)$). An attempt to find a complete description of $D(\Lambda_{C_\infty}(I, b))$ in elementary terms for a general $b \in \mathbf{F}_\delta$ is doomed.

3. To prove Theorem 1(i), we first construct a Feller semigroup $e^{-\Lambda_{C_\infty}(a,b)}$ associated to the divergence form operator $-\nabla \cdot a \cdot \nabla + b \cdot \nabla$ by showing that if the form-bounds δ, δ_a, $\gamma_{r\ell}$ satisfy, for some $q > d - 2$ if $d \geq 4$ or $q \geq 2$ if $d = 3$

$$(3) \qquad \begin{cases} 1 - \frac{q}{4}(\sqrt{\gamma} + \|a - I\|_\infty \sqrt{\delta}) > 0, \\ (q-1)(1 - \frac{q\sqrt{\gamma}}{2}) - (\sqrt{\delta}\sqrt{\delta_a} + \delta)\frac{q^2}{4} - (q-2)\frac{q\sqrt{\delta}}{2} - \|a - I\|_\infty \frac{q\sqrt{\delta}}{2} > 0 \end{cases}$$

(the role of q will be explained below), then the limit

$$(\star\star) \qquad s\text{-}C_\infty\text{-}\lim e^{-t\Lambda_{C_\infty}(a_n,b_n)} \quad \text{(loc. uniformly in } t \geq 0),$$

where $\Lambda_{C_\infty}(a_n, b_n) := -\nabla \cdot a_n \cdot \nabla + b_n \cdot \nabla$, $D(\Lambda_{C_\infty}(a_n, b_n)) := (1 - \Delta)^{-1} C_\infty$, exists and thus determines a Feller semigroup, $e^{-t\Lambda_{C_\infty}(a,b)}$. Then, since

$$-a_n \cdot \nabla^2 + b_n \cdot \nabla = -\nabla \cdot a_n \cdot \nabla + (\nabla a_n + b_n) \cdot \nabla,$$

and $\nabla a + b \in \mathbf{F}_{\delta_a+\delta}$, $\nabla a_n + b_n \in \mathbf{F}_{\delta_a+\delta}$ with $\lambda \neq \lambda(n)$, we obtain $(\star)$ from $(\star\star)$ upon replacing b by $\nabla a + b$. (In particular, (2) is (3) with δ replaced by $\delta_a + \delta$.)

4. The proof of existence of the limit $(\star\star)$ goes as follows.

By Theorem A.1 below, there exists a positivity preserving L^∞ contraction quasi contraction C_0 semigroup $e^{-t\Lambda_r(a,b)}$ in L^r, $r > \frac{2}{2-\sqrt{\delta}}$

$$(4) \qquad e^{-t\Lambda_r(a,b)} := s\text{-}L^r\text{-}\lim e^{-t\Lambda_r(a_n,b_n)} \quad \text{(loc. uniformly in } t \geq 0),$$

where $\Lambda_r(a_n, b_n) := -\nabla \cdot a_n \cdot \nabla + b_n \cdot \nabla$, $D(\Lambda_q(a_n, b_n)) = W^{2,r}$. Note that L^r has a (locally) weaker topology than C_∞, so it is easier to prove convergence there.

At the next step, we prove a priori bound on $u_n := (\mu + \Lambda_q(a_n, b_n))^{-1} f$, $\mu > \mu_0$, $f \in L^q$, for $q \geq 2$ satisfying (3):

$$(\star\star\star) \qquad \|\nabla u_n\|_{\frac{qd}{d-2}} \leq C\|f\|_q,$$

(Lemma 1). This bound for $q > d - 2$ if $d \geq 4$ or $q \geq 2$ for $d = 3$, allows us to run an iteration procedure $L^r \to L^\infty$ that yields for a $r_0 > \frac{2}{2-\sqrt{\delta}}$

$$\|u_n - u_m\|_\infty \leq B\|u_n - u_m\|_{r_0}^\gamma, \quad n, m \geq 1,$$

where constants $\gamma > 0$, B do not depend on n, m (Section 5.2). Thus, since u_n converge in L^{r_0} by (4), it follows that u_n converge in C_∞. The latter allows us to apply the Trotter Approximation Theorem, which yields existence of the limit ($\star\star$).

5. Let us comment more on the existing literature on stochastic differential equations with singular drifts.

In [2], the authors prove existence and uniqueness in law of weak solution to the SDE

$$X(t) = x - \int_0^t b(X(s))ds + \sqrt{2}W(t), \quad x \in \mathbb{R}^d, \tag{5}$$

for b in $\mathbf{K}_0^{d+1} := \cap_{\delta>0}\mathbf{K}_\delta^{d+1}$, where the Kato class $\mathbf{K}_\delta^{d+1} = \{|b| \in L^1_{\rm loc} \mid \|(\lambda - \Delta)^{-\frac{1}{2}}|b|\|_\infty \leq \sqrt{\delta}$ for some $\lambda = \lambda_\delta\}$ (in fact, [2] allow b to be a measure). We note that

$$\mathbf{K}_0^{d+1} - \mathbf{F}_\delta \neq \varnothing, \quad \mathbf{F}_\delta - \mathbf{K}_{\delta_1}^{d+1} \neq \varnothing$$

(already $L^d(\mathbb{R}^d, \mathbb{R}^d) \not\subset \mathbf{K}_{\delta_1}^{d+1}$).

In [17], the authors construct a Feller semigroup on C_∞ associated to the operator $-\Delta + b \cdot \nabla$, $b \in \mathbf{F}_\delta$. The a priori bound ($\star\star\star$) and the $L^r \to L^\infty$ iteration procedure developed in the present paper extend the corresponding results in [17] (see also [12, sect. 4]).

In [15, Theorem 1], we proved an analogue of Theorem 1(ii),(iii) for SDE (5) with b in the class of weakly form-bounded vector fields

$$\mathbf{F}_\delta^{1/2} := \{|b| \in L^1_{\rm loc} \mid \||b|^{\frac{1}{2}}(\lambda - \Delta)^{-\frac{1}{4}}\|_{2\to 2} \leq \sqrt{\delta} \text{ for some } \lambda = \lambda_\delta\}$$

that contains both the Kato class $\mathbf{K}_\delta^{d+1}$ and $\mathbf{F}_{\delta^2}$ as its proper subclasses (as thus the sums of vector fields in these two classes). The corresponding Feller semigroup was constructed in [11], see also [12, sect. 5], but using a different technique that depends crucially on the pointwise estimate $|\nabla(\mu-\Delta)^{-1}(x,y)| \leq c(\kappa\mu-\Delta)^{-\frac{1}{2}}(x,y)$. This estimate holds for $-\Delta$ replaced by $-\nabla \cdot a \cdot \nabla$ but only for Hölder continuous a. In the present paper, we include matrices a having critical discontinuities at expense of restricting the class of admissible drifts b from $\mathbf{F}_\delta^{1/2}$ to $\mathbf{F}_\delta$.

We note that Theorem 1 in the case $a = I$ is not a special case of [15, Theorem 1] since it admits larger values of the form bound of b.

6. The last assertion of Theorem 1(i), i.e. that $u \in C^{0,\alpha}$, $\alpha = 1 - \frac{d-2}{q}$, follows easily from the a priori bound ($\star\star\star$) via the Sobolev Embedding Theorem. This result captures quantitative dependence of the Hölder continuity of u on the values of form-bounds δ, δ_a and $\gamma_{r\ell}$. (Let us note that we can appropriately normalize the coefficients of $-\nabla \cdot a \cdot \nabla + b \cdot \nabla$, $-a \cdot \nabla^2 + b \cdot \nabla$ so that δ, δ_a and $\gamma_{r\ell}$ become constant multiples of the corresponding coefficients).

7. The proof of Theorem 1 does not use $W^{2,p}$ bounds on u_n. In fact, even if $a = I$, such bounds do not exist for large p for a general $b \in \mathbf{F}_\delta$.

For less singular drifts, there is an extensive literature on $W^{2,p}$ bounds on solutions to the corresponding non-divergence form elliptic and parabolic equations. In particular, such estimates exist for matrices a with entries in the VMO class (which includes $a_{ij} \in W^{1,d}$

considered in [25]), see [9], see also [8] where the VMO class appeared for the first time in the context of $W^{2,p}$ bounds for non-divergence form equations. Moreover, $W^{1,p}$ and $W^{2,p}$ bounds exist for both divergence and non-divergence form equations for a locally small in the BMO norm, see [4, 5]. Note that already our model example $a(x) = I + c\frac{x\otimes x}{|x|^2}$, $c > -1$, is not in the VMO class. Although this a is in BMO, we do not require in Theorem 1 local smallness of its BMO norm (≡ smallness of c, i.e. smallness of $\gamma_{r\ell}$). For other admissible classes of a, however requiring control over geometry of the set of discontinuities, see [7, 20] and references therein. See also [31] where $W^{2,p}$ estimates are obtained assuming $|\nabla a|$, $|b| \in L^p$, $p > d$ (in fact, a, b can be time-dependent).

We note that the operator $-a\cdot\nabla^2$ with $\nabla_k a_{ij} \in L^{d,\infty}$ has been studied in L^2 in [1].

8. For the divergence form operator, the iteration procedure depends on the a priori bound $(\star\star\star)$, more precisely, on

(6)
$$\sup_n \|\nabla u_n\|^2_{\frac{qd}{d-2}} < \infty, \quad u_n := (\mu+\Lambda_q(a_n,b_n))^{-1}f, \quad f\in L^1\cap L^\infty, \quad q \in \Big]\frac{2}{2-\sqrt{\delta}} \vee (d-2), \frac{2}{\sqrt{\delta}}\Big[.$$

Other than that, the iteration procedure works for an arbitrary uniformly elliptic matrix a:

$$(H_u) \qquad \begin{array}{l} a = a^* : \mathbb{R}^d \to \mathbb{R}^d\otimes\mathbb{R}^d \quad \text{is measurable,} \\ \sigma I \le a(x) \le \xi I \quad \text{for a.e. } x\in\mathbb{R}^d \text{ for some } 0<\sigma<\xi<\infty, \end{array}$$

see Section 5.2 for details. In this regard, we note another instance where a priori bound (6) is valid. Let $a \in (H_u)$. Without loss of generality, $\sigma = 1$. Let $\sigma I \le a_n \le \xi I$ be a smooth approximation of a as defined above. Set $A := -\nabla\cdot a_n\cdot\nabla$. Then for $p > 2$, by the N. Meyers Embedding Theorem [26],

$$(\mu + A)^{-1} \in \mathcal{B}(\mathcal{W}^{-1,p}, \mathcal{W}^{1,p}), \quad \mu > 0, \tag{7}$$

$$\|\nabla(\mu+A)^{-1}(\mu-\Delta)^{\frac12}\|_{p\to p} \le C_p, \quad C_p \ne C_p(\mu), \tag{8}$$

provided that

$$\|\nabla(\mu-\Delta)^{-\frac12}\|^2_{p\to p} \le \frac{\xi}{\xi-\sigma},$$

see also [12, Theorem G.1]. Thus, assuming that ξ and σ are sufficiently close to each other, we can select $p = qj$, where $q \in]d-2,\infty[$, to obtain a priori bound (6) for $u_n := (\mu+A)^{-1}f$. The latter allows us to run the iteration procedure, which then yields associated to $-\nabla\cdot a\cdot\nabla$ Feller semigroup. (The convergence of u_n in L^{r_0}, $r_0 > 2$, follows e.g. from Theorem A.1.)

Let $b = b' + b'' \in L^d(\mathbb{R}^d,\mathbb{R}^d) + L^\infty(\mathbb{R}^d,\mathbb{R}^d)$, in which case $b \in \mathbf{F}_\delta$ with δ that can be chosen arbitrarily small. We can further perturb A by $b\cdot\nabla$ arguing as in [12, sect. 4]. Namely, let $T := b_n\cdot\nabla(\mu+A)^{-1}$. Then ($p = \frac{dq}{d-q}$, $f \in C_c$)

$$\begin{aligned} \|Tf\|_q &\le \|b'\|_d\|\nabla(\mu+A)^{-1}(\mu-\Delta)^{\frac12}\|_{p\to p}\|(\mu-\Delta)^{-\frac12}f\|_p \\ &\quad + \|b''\|_\infty\|\nabla(\mu+A)^{-1}(\mu-\Delta)^{\frac12}\|_{q\to q}\|(\mu-\Delta)^{-\frac12}f\|_q \\ &\quad \text{(we use (8))} \\ &\le (\|b'\|_d C_s C_p + \mu^{-\frac12}C_q\|b''\|_\infty)\|f\|_q \end{aligned}$$

for all $q \in]2, 2+\varepsilon]$, where $0 < \varepsilon < d-2$ depends on ξ and σ. Without loss of generality, we may assume that $\|b'\|_d$ is sufficiently small (at expense of increasing $\|b''\|_\infty$) Thus, for $\mu > 0$ sufficiently large so that $\|T\|_{q\to q} < 1$, we have

$$(\mu + A + b_n \cdot \nabla)^{-1} = (\mu + A)^{-1}(1+T)^{-1}$$

and thus

$$\begin{aligned} &\|\nabla(\mu + A + b_n \cdot \nabla)^{-1} f\|_p \\ &\leq \|\nabla(\mu + A)^{-1}(1-\Delta)^{\frac{1}{2}}\|_{p\to p}\|(1-\Delta)^{-\frac{1}{2}}(1+T)^{-1} f\|_p \\ &\text{(we apply (7) and the Sobolev Embedding Theorem)} \\ &\leq C\|(1+T)^{-1} f\|_q \leq C_1\|f\|_q. \end{aligned}$$

Above we can replace $p = \frac{dq}{d-q}$ by $q \leq p \leq \frac{qd}{d-q}$. Take $p = \frac{dq}{d-2}$. If ξ and σ are close to each other, we can select $q \in]2 \vee (d-2), d[$, thus arriving at a priori bound

$$\sup_n \|\nabla u_n\|_p^2 < \infty, \quad u_n := (\mu + A + b_n \cdot \nabla)^{-1} f, \quad f \in L^1 \cap L^\infty,$$

needed to run the iteration procedure. (The convergence of u_n in L^{r_0}, $r_0 > 2$, follows from Theorem A.1.) The latter allows to prove: *Let $d \geq 3$, $a \in (H_u)$, $b \in L^d(\mathbb{R}^d, \mathbb{R}^d) + L^\infty(\mathbb{R}^d, \mathbb{R}^d)$. If ξ and σ are sufficiently close to each other, then the limit*

$$s\text{-}C_\infty\text{-}\lim e^{-t\Lambda_{C_\infty}(a_n, b_n)} \quad (\textit{loc. uniformly in } t \geq 0),$$

where $\Lambda_{C_\infty}(a_n, b_n) := -\nabla \cdot a_n \cdot \nabla + b_n \cdot \nabla$, $D(\Lambda_{C_\infty}(a_n, b_n)) := (1-\Delta)^{-1} C_\infty$, exists and determines a Feller semigroup on C_∞. We note that this result can not be achieved on the basis of the De Giorgi-Nash theory.

9. The method of proof of the a priori bound ($\star\star\star$) is rather general.

We could have considered matrices a of the form

$$a = I + c\,\mathsf{f} \otimes \mathsf{f}, \quad c > -1 \quad \text{(or a sum of such matrices)},$$

assuming that

$$\mathsf{f} \in L^\infty(\mathbb{R}^d, \mathbb{R}^d) \cap W^{1,2}_{\rm loc}(\mathbb{R}^d, \mathbb{R}^d), \quad \|\mathsf{f}\|_\infty = 1,$$

$$\nabla_i \mathsf{f} \in \mathbf{F}_{\delta^i}, \quad \delta^i > 0, \quad i = 1, 2, \dots, d, \quad \delta_{\mathsf{f}} := \sum_{i=1}^d \delta^i,$$

in which case $\nabla a = c[(\mathrm{div}\mathsf{f})\mathsf{f} + \mathsf{f} \cdot \nabla \mathsf{f}] \in \mathbf{F}_{\delta_a}$ with $\delta_a \leq |c|^2(\sqrt{d}+1)^2\delta_{\mathsf{f}}$.

Then the condition (3) is replaced with: $\delta < 1 \wedge (\frac{2}{d-2})^2$, and, for some $q > d-2$ if $d \geq 4$ or $q \geq 2$ if $d = 3$,

$$0 < c < (q-1-Q)\begin{cases} [(q-1)\frac{q\sqrt{\delta_{\mathsf{f}}}}{2} + \frac{q^2(\sqrt{\delta_{\mathsf{f}}}+\sqrt{\delta})^2}{16} + (q-2)\frac{q^2\delta_{\mathsf{f}}}{16}]^{-1} & \text{if} \quad 1 - \frac{q\sqrt{\delta_{\mathsf{f}}}}{4} - \frac{q\sqrt{\delta}}{4} \geq 0, \\ (\frac{q^2\sqrt{\delta_{\mathsf{f}}}}{2} + (q-2)\frac{q^2\delta_{\mathsf{f}}}{16} + \frac{q\sqrt{\delta}}{2} - 1)^{-1} & \text{if} \quad 0 \leq 1 - \frac{q\sqrt{\delta_{\mathsf{f}}}}{4} < \frac{q\sqrt{\delta}}{4}, \\ [(q-1)(q\sqrt{\delta_{\mathsf{f}}} - 1) + \frac{q\sqrt{\delta}}{2}]^{-1} & \text{if} \quad 1 - \frac{q\sqrt{\delta_{\mathsf{f}}}}{4} < 0, \end{cases}$$

where $Q := \frac{q\sqrt{\delta}}{2}[q-2+(\sqrt{\delta_a}+\sqrt{\delta})\frac{q}{2}]$, or

$$-(q-1-Q)\Big[(q-1)(1+q\sqrt{\delta_{\mathrm{f}}})+\frac{q\sqrt{\delta}}{2}\Big]^{-1}<c<0.$$

4. Stratonovich diffusion

We replace condition $(\mathbf{C}_2)$ of the introduction by

$(\mathbf{C}_2')$ $a \geq \nu I$, $\nu > 0$ and, for each $1 \leq r, j \leq d$,

$$(\nabla_r \sigma_{ij})_{i=1}^d \in \mathbf{F}_{\delta_{rj}}$$

for some $\delta_{rj} > 0$.

Then we can re-write (SDE_S) as

$$(SDE_S') \quad X(t) = x - \int_0^t b(X(s))ds + \int_0^t c(X(s))ds + \sqrt{2}\int_0^t \sigma(X(s))dW(s), \qquad x \in \mathbb{R}^d,$$

where

$$c := (c^i)_{i=1}^d, \quad c^i := \frac{1}{\sqrt{2}}\sum_{r,j=1}^d (\nabla_r \sigma_{ij})\sigma_{rj}.$$

By $(\mathbf{C}_2')$,

$$c \in \mathbf{F}_{\delta_c}, \quad \delta_c \leq \frac{1}{2}\|\sigma\|_\infty^2 \sum_{r,j=1}^d \delta_{rj}$$

(here $\|\sigma\|_\infty = \|(\sum_{r,j=1}^d \sigma_{rj}^2)^{\frac{1}{2}}\|_\infty$). We note that $(\mathbf{C}_2')$ yields $(\mathbf{C}_2)$:

$$(\nabla_r a_{i\ell})_{i=1}^d \in \mathbf{F}_{\gamma_{r\ell}}, \quad \gamma_{r\ell} \leq \Big[\|\sigma_{\cdot\ell}\|_\infty\Big(\sum_{j=1}^d \delta_{rj}\Big)^{\frac{1}{2}} + \|\sigma\|_\infty \delta_{r\ell}^{\frac{1}{2}}\Big]^2.$$

We fix the following approximation of σ by smooth matrices: $\sigma_n = I + e^{\epsilon_n \Delta}(\eta_n(\sigma - I))$ (η_n were defined earlier). Then we may assume that a_n, b_n and c_n satisfy

$$(\nabla_r (a_n)_{i\ell})_{i=1}^d \in \mathbf{F}_{\gamma_{r\ell}} \ (1 \leq r, \ell \leq d), \quad \nabla a_n \in \mathbf{F}_{\delta_a}, \quad c_n \in \mathbf{F}_{\delta_c}, \quad \nabla a_n - c_n + b_n \in \mathbf{F}_{\delta_a+\delta_c+\delta}$$

with $\lambda \neq \lambda(n)$.

The next result is an immediate consequence of Theorem 1.

Theorem 2. *Let $d \geq 3$. Assume that conditions $(\mathbf{C}_1)$, $(\mathbf{C}_2')$ are satisfied, with δ, δ_a, δ_c, γ satisfying* (3) *for some $q > 2 \vee (d-2)$ with δ replaced by $\delta + \delta_a + \delta_c$. Then:*

(i) *The limit*

$$e^{-t\Lambda_{C_\infty}(a,\nabla a+b-c)} := s\text{-}C_\infty\text{-}\lim e^{-t\Lambda_{C_\infty}(a_n,\nabla a_n+b_n-c_n)} \quad (\textit{loc. uniformly in } t \geq 0),$$

where $\Lambda_{C_\infty}(a_n, \nabla a_n + b_n - c_n) := -a_n \cdot \nabla^2 + (b_n - c_n)\cdot\nabla$, $D(\Lambda_{C_\infty}(a_n, \nabla a_n + b_n - c_n)) := (1-\Delta)^{-1}C_\infty$, exists and determines a Feller semigroup.

Let $(\bar{\Omega}_D, \mathcal{F}_t, \mathcal{F}_\infty, \mathbb{P}_x)$ be the Feller process determined by $e^{-t\Lambda_{C_\infty}(a,\nabla a+b-c)}$. The following is true for every $x \in \mathbb{R}^d$:

(ii) *The trajectories of the process are $\mathbb{P}_x$ a.s. finite and continuous on $0 \leq t < \infty$.*

We denote $\mathbb{P}_x \upharpoonright (\Omega, \mathcal{G}_\infty)$ again by $\mathbb{P}_x$.

(iii) $\mathbb{E}_{\mathbb{P}_x} \int_0^t |b(X(s))| ds < \infty$, *$X \in \Omega$, and for any selection of $f \in C_c^\infty$, $f(y) := y_i$, or $f(y) := y_i y_j$, $1 \leq i, j \leq d$, the process*

$$M^f(t) := f(X(t)) - f(x) + \int_0^t (-a \cdot \nabla^2 f + (b - c) \cdot \nabla f)(X(s)) ds, \quad t \geq 0,$$

is a continuous martingale relative to $(\Omega, \mathcal{G}_t, \mathbb{P}_x)$; the latter thus determines a weak solution to (SDE'_S) on an extension of $(\Omega, \mathcal{G}_t, \mathbb{P}_x)$.

Remark 2 also applies to Theorem 2, provided that $\|a - I\|_\infty + \delta + \delta_c < 1$.

5. Proof of Theorem 1(i): Construction of Feller semigroup

As we explained in the introduction, it suffices to construct the Feller semigroup $e^{-t\Lambda_{C_\infty}(a,b)}$ corresponding to the divergence form operator $-\nabla \cdot a \cdot \nabla + b \cdot \nabla$:

Proposition 1. *Assume that $b \in \mathbf{F}_\delta$, $(\nabla_r a_{i\ell})_{i=1}^d \in \mathbf{F}_{\gamma_{r\ell}}$ and $\nabla a \in \mathbf{F}_{\delta_a}$, with $\gamma_{r\ell}$, δ, δ_a satisfy, for some $q > d - 2$ if $d \geq 4$ or $q \geq 2$ if $d = 3$,*

$$(*) \qquad \begin{cases} 1 - \frac{q}{4}(\sqrt{\gamma} + \|a - I\|_\infty \sqrt{\delta}) > 0, \quad \text{where } \gamma := \sum_{r,\ell=1}^d \gamma_{r\ell}, \\ (q-1)(1 - \frac{q\sqrt{\gamma}}{2}) - (\sqrt{\delta}\sqrt{\delta_a} + \delta)\frac{q^2}{4} - (q-2)\frac{q\sqrt{\delta}}{2} - \|a - I\|_\infty \frac{q\sqrt{\delta}}{2} > 0. \end{cases}$$

Then the limit

$$s\text{-}C_\infty\text{-}\lim e^{-t\Lambda_{C_\infty}(a_n, b_n)} \quad (\text{loc. uniformly in } t \geq 0),$$

where $\Lambda_{C_\infty}(a_n, b_n) := -\nabla \cdot a_n \cdot \nabla + b_n \cdot \nabla$, $D(\Lambda_{C_\infty}(a_n, b_n)) := (1 - \Delta)^{-1} C_\infty$ exists, and determines a contraction C_0 semigroup, $e^{-t\Lambda_{C_\infty}(a,b)}$.

We have

$$(e^{-t\Lambda_{C_\infty}(a,b)} \upharpoonright L^q \cap C_\infty)^{\text{clos}}_{L^q \to C_\infty} \in \mathcal{B}(L^q, C_\infty), \qquad t > 0.$$

Also, $u = (\mu + \Lambda_{C_\infty}(a, b))^{-1} f$, $\mu > \mu_0$, $f \in L^q \cap C_\infty$ is in $C^{0,\alpha}$, $\alpha = 1 - \frac{d-2}{q}$.

The two key ingredients of the proof of Proposition 1 are the a priori bounds and the iteration procedure.

5.1. A priori bounds.

Lemma 1. *Let $d \geq 3$. Assume that $q \geq 2$, $\gamma_{r\ell}$, δ, δ_a are such that $(*)$ is satisfied. Then there exist constants $\mu_0 = \mu_0(d, q, \delta, \delta_a, \gamma) > 0$ and $K_l = K_l(d, q, \delta, \delta_a, \gamma)$, $l = 1, 2$, such that the bounds*

$$(**) \qquad \begin{aligned} \|\nabla u_n\|_q &\leq K_1 (\mu - \mu_0)^{-\frac{1}{2}} \|h\|_q, \\ \|\nabla u_n\|_{qj} &\leq K_2 (\mu - \mu_0)^{\frac{1}{q} - \frac{1}{2}} \|h\|_q, \quad j := \tfrac{d}{d-2}, \end{aligned}$$

hold for $u_n := (\mu + \Lambda_q(a_n, b_n))^{-1} h$, $\mu > \mu_0$, $h \in L^q$, $n \geq 1$.

Proof. Since a_n, b_n are C^∞ smooth, we have for all $\mu > \frac{\lambda\delta}{2(q-1)}$

$$(\mu + \Lambda_{C_\infty}(a_n, b_n))^{-1} \upharpoonright C_\infty \cap L^q = (\mu + \Lambda_q(a_n, b_n))^{-1} \upharpoonright C_\infty \cap L^q,$$

by Theorem A.1 (to apply the theorem, we note that, by our assumptions, $b_n \in \mathbf{F}_\delta$ where $\delta < 1$).

Thus, it suffices to prove ($**$) for $(\mu + \Lambda_q(a_n, b_n))^{-1}h$, $0 \leq h \in C_c^1$.

Set $A_q^n := -\nabla \cdot a_n \cdot \nabla$, $D(A_q^n) := W^{2,q}$.

Put

$$0 \leq u_n := (\mu + \Lambda_q(a_n, b_n))^{-1}h,$$

where $\Lambda_q(a_n, b_n) = A_q^n + b_n \cdot \nabla$, $D(\Lambda_q(a_n, b_n)) = W^{2,q}$, $n \geq 1$. Clearly, since $a_n, b_n \in C^\infty$, we have $u_n \in W^{3,q}$.

For brevity, we omit index n everywhere below, and write $u \equiv u_n$, $a \equiv a_n$, $b \equiv b_n$, $A_q \equiv A_q^n$.

Set

$$w := \nabla u, \quad I_q := \sum_{r=1}^d \langle (\nabla_r w)^2 |w|^{q-2} \rangle, \quad J_q := \langle (\nabla |w|)^2 |w|^{q-2} \rangle,$$

$$I_q^a := \sum_{r=1}^d \langle (\nabla_r w \cdot a \cdot \nabla_r w) |w|^{q-2} \rangle, \quad J_q^a := \langle (\nabla |w| \cdot a \cdot \nabla |w|) |w|^{q-2} \rangle,$$

where, recall,

$$\langle h \rangle := \int_{\mathbb{R}^d} h(x) dx, \quad \langle h, g \rangle := \langle h \bar{g} \rangle.$$

Set $[F, G] := FG - GF$.

We multiply the equation $\mu u + \Lambda_q(a, b)u = h$ by the test function

$$\phi := -\nabla \cdot (w|w|^{q-2}) = -\sum_{r=1}^d \nabla_r (w_r |w|^{q-2})$$

and integrate:

$$\mu \langle |w|^q \rangle + \langle A_q w, w|w|^{q-2} \rangle + \langle [\nabla, A_q]u, w|w|^{q-2} \rangle = \langle -b \cdot \nabla u, \phi \rangle + \langle h, \phi \rangle,$$

$$\mu \langle |w|^q \rangle + I_q^a + (q-2) J_q^a + \langle [\nabla, A_q]u, w|w|^{q-2} \rangle = \langle -b \cdot \nabla u, \phi \rangle + \langle h, \phi \rangle.$$

Since by our assumption $a \geq I$, we have $I_q^a \geq I_q$, $J_q^a \geq J_q$. We thus obtain the *principal inequality*

$$(\bullet) \qquad \mu \langle |w|^q \rangle + I_q + (q-2) J_q \leq -\langle [\nabla, A_q]u, w|w|^{q-2} \rangle + \langle -b \cdot \nabla u, \phi \rangle + \langle h, \phi \rangle - R_q^1.$$

We will estimate the RHS of ($\bullet$) in terms of I_q and J_q.

First, we estimate

$$\langle [\nabla, A_q]u, w|w|^{q-2} \rangle := \sum_{r=1}^d \langle [\nabla_r, A_q]u, w_r |w|^{q-2} \rangle.$$

Set $\gamma = \sum_{r,\ell} \gamma_{r\ell}$. From now on, we omit the summation sign in repeated indices.

Claim 1.

$$|\langle [\nabla_r, A_q]u, w_r |w|^{q-2} \rangle| \leq M_1 I_q + N_1 J_q + C_1 \langle |w|^q \rangle,$$

where constants

$$M_1 := \frac{1}{4\alpha}, \qquad N_1 := \alpha\gamma\frac{q^2}{4} + (q-2)\Big[\beta\gamma\frac{q^2}{4} + \frac{1}{4\beta}\Big], \quad C_1 := (\alpha + (q-2)\beta)\lambda\gamma \qquad \alpha,\beta > 0.$$

Proof of Claim 1. Note that $[\nabla_r, A_q]u = -\nabla \cdot \nabla_r a \cdot \nabla u$. Thus,

$$\langle [\nabla_r, A_q]u, w_r|w|^{q-2}\rangle = \langle (\nabla_r a_{i\ell})w_\ell, (\nabla_i w_r)|w|^{q-2}\rangle + (q-2)\langle (\nabla_r a_{i\ell})w_\ell, w_r|w|^{q-3}\nabla_i|w|\rangle.$$

By quadratic inequalities,

$$|\langle [\nabla_r, A_q]u, w_r|w|^{q-2}\rangle| \le \alpha\langle |\nabla_r a_{\cdot\ell}|^2|w|^q\rangle + \frac{1}{4\alpha}I_q + (q-2)\Big[\beta\langle|\nabla_r a_{\cdot\ell}|^2|w|^q\rangle + \frac{1}{4\beta}J_q\Big].$$

By $(\mathbf{C}_2)$, $\nabla_r a_{\cdot\ell} \in \mathbf{F}_{\gamma_{r\ell}}$, i.e.

$$\langle |\nabla_r a_{\cdot\ell}|^2|g|^2\rangle \Big(= \sum_{r,\ell}\langle|\nabla_r a_{\cdot\ell}|^2|g|^2\rangle\Big) \le \gamma\langle|\nabla g|^2\rangle + \lambda\gamma\langle|g|^2\rangle, \quad g \in W^{1,2},$$

so that

$$\langle |\nabla_r a_{\cdot\ell}|^2|w|^q\rangle \le \gamma\frac{q^2}{4}J_q + \lambda\gamma\langle|w|^q\rangle. \tag{9}$$

The proof of Claim 1 is completed. □

We estimate the term $\langle -b\cdot w, \phi\rangle$ in (•) as follows.

Claim 2. *There exist constants C_2, C_2' such that*

$$\langle -b\cdot w, \phi\rangle \le M_2 I_q + N_2 J_q + C_2\|w\|_q^q + C_2'\|w\|_q^{q-2}\|h\|_q^2,$$

where constants

$$M_2 := \|a - I\|_\infty\frac{1}{4\alpha_1}, \qquad N_2 := (\sqrt{\delta}\sqrt{\delta_a} + \delta)\frac{q^2}{4} + (q-2)\frac{q\sqrt{\delta}}{2} + \|a-I\|_\infty\alpha_1\delta\frac{q^2}{4}, \qquad \alpha_1 > 0.$$

Proof of Claim 2. We have $\phi = (-\Delta u)|w|^{q-2} - |w|^{q-3}w\cdot\nabla|w|$, so

$$\langle -b\cdot w, \phi\rangle = \langle -\Delta u, |w|^{q-2}(-b\cdot w)\rangle - (q-2)\langle w\cdot\nabla|w|, |w|^{q-3}(-b\cdot w)\rangle =: F_1 + F_2.$$

Set

$$B_q := \langle |b|^2|w|^q\rangle.$$

We have

$$F_2 \le (q-2)B_q^{\frac{1}{2}}J_q^{\frac{1}{2}}.$$

Next, we bound F_1. We represent $-\Delta u = \nabla\cdot(a-I)\cdot w - \mu u - b\cdot w + h$, and evaluate: $\nabla\cdot(a-I)\cdot w = \nabla a\cdot w + (a-I)_{i\ell}\nabla_i w_\ell$, so

$$F_1 = \langle \nabla\cdot(a-I)\cdot w, |w|^{q-2}(-b\cdot w)\rangle + \langle(-\mu u - b\cdot w + h), |w|^{q-2}(-b\cdot w)\rangle$$

$$= \langle \nabla a \cdot w, |w|^{q-2}(-b \cdot w)\rangle$$
$$+ \langle (a-I)_{i\ell}\nabla_i w_\ell, |w|^{q-2}(-b \cdot w)\rangle$$
$$+ \langle (-\mu u - b \cdot w + h), |w|^{q-2}(-b \cdot w)\rangle.$$

Set $P_q := \langle |\nabla a|^2 |w|^q\rangle$. We bound F_1 from above by applying consecutively the following estimates:

1) $\langle \nabla a \cdot w, |w|^{q-2}(-b \cdot w)\rangle \leq P_q^{\frac{1}{2}} B_q^{\frac{1}{2}}$.

2) $\langle (a-I)_{i\ell}\nabla_i w_\ell, |w|^{q-2}(-b \cdot w)\rangle \leq \|a-I\|_\infty I_q^{\frac{1}{2}} B_q^{\frac{1}{2}} \leq \|a-I\|_\infty(\alpha_1 B_q + \frac{1}{4\alpha_1} I_q)$.

3) $\langle \mu u, |w|^{q-2} b \cdot w\rangle \leq \frac{\mu}{\mu-\omega_q} B_q^{\frac{1}{2}} \|w\|_q^{\frac{q-2}{2}} \|h\|_q$ for all $\mu > \omega_q := \frac{\lambda\sqrt{\delta}}{2(q-1)}$.

Indeed, $\langle \mu u, |w|^{q-2}(-b \cdot w)\rangle \leq \mu B_q^{\frac{1}{2}} \|w\|_q^{\frac{q-2}{2}} \|u\|_q$ and, by Theorem A.1, $\|u\|_q \leqslant (\mu-\omega_q)^{-1}\|h\|_q$, $\mu > \omega_q$.

4) $\langle b \cdot w, |w|^{q-2} b \cdot w\rangle \leq B_q$.

5) $\langle h, |w|^{q-2}(-b \cdot w)\rangle| \leq B_q^{\frac{1}{2}} \|w\|_q^{\frac{q-2}{2}} \|h\|_q$.

In 3) and 5) we estimate $B_q^{\frac{1}{2}} \|w\|_q^{\frac{q-2}{2}} \|h\|_q \leq \varepsilon_0 B_q + \frac{1}{4\varepsilon_0}\|w\|_q^{q-2}\|h\|_q^2$ ($\varepsilon_0 > 0$).

The above estimates yield:

$$\begin{aligned}\langle -b \cdot w, \phi\rangle &= F_1 + F_2\\ &\leq P_q^{\frac{1}{2}} B_q^{\frac{1}{2}} + \|a-I\|_\infty I_q^{\frac{1}{2}} B_q^{\frac{1}{2}} + B_q + (q-2) B_q^{\frac{1}{2}} J_q^{\frac{1}{2}}\\ &+ \varepsilon_0\Big(\frac{\mu}{\mu-\mu_1} + 1\Big) B_q + C_2'(\varepsilon_0)\|w\|_q^{q-2}\|h\|_q^2.\end{aligned}$$

Selecting $\varepsilon_0 > 0$ sufficiently small, using that the assumption on δ, δ_a are strict inequalities, we can and will ignore below the terms multiplied by ε_0.

Finally, we use in the last estimate: By $b \in \mathbf{F}_\delta$,

$$B_q \leq \frac{q^2}{4}\delta J_q + \lambda\delta\langle |w|^q\rangle$$

(cf. (9)), and by $\nabla a \in \mathbf{F}_{\delta_a}$,

$$P_q \leq \frac{q^2}{4}\delta_a J_q + \lambda\delta_a\|w\|_q^q.$$

This yields Claim 2. □

We estimate the term $\langle h, \phi\rangle$ in (•) as follows.

Claim 3. *For each $\varepsilon_0 > 0$ there exists a constant $C = C(\varepsilon_0) < \infty$ such that*

$$\langle h, \phi\rangle \leq \varepsilon_0 I_q + C\|w\|_q^{q-2}\|h\|_q^2.$$

Proof of Claim 3. We have:

$$\langle h, \phi\rangle = \langle -\Delta u, |w|^{q-2}h\rangle - (q-2)\langle |w|^{q-3} w \cdot \nabla|w|, h\rangle =: F_1 + F_2.$$

Due to $|\Delta u|^2 \leq d|\nabla_r w|^2$ and $\langle |w|^{q-2}h^2\rangle \leq \|w\|_q^{q-2}\|h\|_q^2$,

$$F_1 \leq \sqrt{d} I_q^{\frac{1}{2}} \|w\|_q^{\frac{q-2}{2}} \|h\|_q, \qquad F_2 \leq (q-2) J_q^{\frac{1}{2}} \|w\|_q^{\frac{q-2}{2}} \|h\|_q.$$

Now the standard quadratic estimates yield Claim 3. □

We now apply Claims 1, 2 and 3 in (•). Since the assumption on γ, δ, δ_a in the theorem are strict inequalities, select $\varepsilon_0 > 0$ sufficiently small so that we can ignore the term $\varepsilon_0 I_q$ from Claim 3. We arrive at: There exists $\mu_0 > 0$ such that for all $\mu > \mu_0$

$$(\mu - \mu_0)\|w\|_q^q + (1 - M_1 - M_2)I_q + (q - 2 - N_1 - N_2)J_q \leq C\|h\|_q^q,$$

We select $\alpha = \beta := \frac{1}{q\sqrt{\gamma}}$, $\alpha_1 := \frac{1}{q\sqrt{\delta}}$. By the assumptions of the theorem, the coefficient of I_q

$$1 - M_1 - M_2 = 1 - \frac{q}{4}(\sqrt{\gamma} + \|a - I\|_\infty \sqrt{\delta}) \quad \text{is positive,}$$

so, by $I_q \geq J_q$,

$$(\mu - \mu_0)\|w\|_q^q + \Big[(q-1)(1 - \frac{q\sqrt{\gamma}}{2}) - (\sqrt{\delta}\sqrt{\delta_a} + \delta)\frac{q^2}{4} - (q-2)\frac{q\sqrt{\delta}}{2} - \|a - I\|_\infty \frac{q\sqrt{\delta}}{2}\Big]J_q$$
$$\leq C\|h\|_q^q.$$

By the assumptions of the theorem the coefficient of J_q is positive. Thus, we have

$$(\mu - \mu_0)\|w\|_q^q + cJ_q \leq C\|h\|_q^q, \quad c > 0.$$

In particular, $(\mu - \mu_0)\|w\|_q^q \leq C\|h\|_q^q$, which yields immediately the first estimate in (∗∗). Applying the Sobolev Embedding Theorem in $J_q = \frac{4}{q^2}\langle(\nabla|\nabla u|^{\frac{q}{2}})^2\rangle$, we obtain the second estimate in (∗∗).

The proof of Lemma 1 is completed. □

5.2. Iteration procedure. The iteration procedure works for an arbitrary uniformly elliptic matrix $a \in (H_u)$.

Define $t[u, v] := \langle \nabla u \cdot a \cdot \nabla \bar{v}\rangle$, $D(t) = W^{1,2}$. There is a unique self-adjoint operator $A \equiv A_2 \geq 0$ on L^2 associated with the form t: $D(A) \subset D(t)$, $\langle Au, v\rangle = t[u, v]$, $u \in D(A)$, $v \in D(t)$. $-A$ is the generator of a positivity preserving L^∞ contraction C_0 semigroup $T_2^t \equiv e^{-tA}$, $t \geq 0$, on L^2. Then $T_r^t := [T^t \upharpoonright L^r \cap L^2]_{L^r \to L^r}$ determines C_0 semigroup on L^r for all $r \in]1, \infty[$. The generator $-A_r$ of $T_r^t (\equiv e^{-tA_r})$ is the desired operator realization of $\nabla \cdot a \cdot \nabla$ in L^r, $r \in]1, \infty[$. (One can furthermore show that T^t extends to a contraction C_0 semigroup on L^1.)

DEFINITION 1. A vector field $b : \mathbb{R}^d \to \mathbb{R}^d$ belongs to $\mathbf{F}_{\delta_1}(A)$, $\delta_1 > 0$, the class of vector fields form-bounded with respect to $A \equiv A_2$, if $b_a^2 := b \cdot a^{-1} \cdot b \in L^1_{\text{loc}}$ and there exists a constant $\lambda = \lambda(\delta_1) > 0$ such that

$$\|b_a(\lambda + A)^{-\frac{1}{2}}\|_{2\to 2} \leq \sqrt{\delta_1}.$$

REMARK 3. It is easily seen that if $a \geq I$ (as in Theorem 1), then $b \in \mathbf{F}_\delta \equiv \mathbf{F}_\delta(-\Delta) \Rightarrow b \in \mathbf{F}_{\delta_1}(A)$ with $\delta_1 = \delta$.

Consider

$$\{a_n\}_{n=1}^\infty \subset C^1(\mathbb{R}^d, \mathbb{R}^d \otimes \mathbb{R}^d) \cap (H_u) \quad \text{with } \sigma \neq \sigma(n), \xi \neq \xi(n),$$

and

$$\{b_n\}_{n=1}^{\infty} \subset C^1(\mathbb{R}^d, \mathbb{R}^d) \cap \bigcap_{m\geq 1} \mathbf{F}_{\delta_1}(A^m), \quad \delta_1 < 1, \quad A^m \equiv A(a_m), \quad \text{with } \lambda \neq \lambda(n, m).$$

(It is easily seen that a_n, b_n in Theorem 1 satisfy these assumptions.)

By Theorem A.1 below, $-\Lambda_r(a_n, b_n) := \nabla \cdot a_n \cdot \nabla - b_n \cdot \nabla$, $D(\Lambda_r(a_n, b_n)) = W^{2,r}$, is the generator of a positivity preserving L^∞ contraction quasi contraction C_0 semigroup on L^r, $r \in]\frac{2}{2-\sqrt{\delta}}, \infty[$, with the resolvent set of $-\Lambda_r(a_n, b_n)$ containing $\mu > \frac{\lambda\delta_1}{2(r-1)}$ for all $n \geq 1$.

Set $u_n := (\mu + \Lambda_r(a_n, b_n))^{-1} f$, $f \in L^1 \cap L^\infty$ and

$$g := u_m - u_n.$$

Lemma 2 (The iteration inequality)**.** *There are positive constants $C = C(d), k = k(\delta_1)$ such that*

$$\|g\|_{rj} \leq (C\sigma^{-1}(\delta_1 + 2\xi\sigma^{-1})(1 + 2\xi)\|\nabla u_m\|_{qj}^2)^{\frac{1}{r}} (r^{2k})^{\frac{1}{r}} \|g\|_{s'(r-2)}^{1-\frac{2}{r}},$$

where $q \in]\frac{2}{2-\sqrt{\delta_1}} \vee (d-2), \frac{2}{\sqrt{\delta_1}}[$, $2s = qj$, $j = \frac{d}{d-2}$, $s' := \frac{s}{s-1}$ and $s'(r-2) > \frac{2}{2-\sqrt{\delta_1}}$, $\mu > \lambda$.

Proof. Write $b_n \in \mathbf{F}_{\delta_1}(A^m)$ equivalently as

$$\langle b_n \cdot a_m^{-1} \cdot b_n, |\varphi|^2\rangle \leq \delta_1 \langle \nabla\varphi \cdot a_m \cdot \nabla\bar{\varphi}\rangle + \lambda\delta_1\langle|\varphi|^2\rangle, \qquad \varphi \in W^{1,2}.$$

Let $\psi = g|g|^{r-2}, v = g|g|^{\frac{r-2}{2}}$. We multiply the equation

$$(\mu + \Lambda_q(a_n, b_n))g = F, \qquad \text{where } F := \nabla \cdot (a_m - a_n) \cdot \nabla u_m + (b_n - b_m) \cdot \nabla u_m,$$

by ψ and integrate to obtain

$$\begin{aligned}
&\mu\|v\|_2^2 + \frac{4}{rr'}\langle \nabla v \cdot a_n \cdot \nabla v\rangle \\
&= -\frac{2}{r}\langle v, b_n \cdot \nabla v\rangle + \langle \nabla \cdot (a_m - a_n) \cdot \nabla u_m, v|v|^{1-\frac{2}{r}}\rangle + \langle (b_n - b_m) \cdot \nabla u_m, v|v|^{1-\frac{2}{r}}\rangle.
\end{aligned}$$

We estimate the terms in the RHS as follows. By the quadratic inequality, using $b_n \in \mathbf{F}_{\delta_1}(A^n)$,

$$\begin{aligned}
|\langle v, b_n \cdot \nabla v\rangle| &\leq \varepsilon\langle b_n \cdot a_n^{-1} \cdot b_n, |v|^2\rangle + (4\varepsilon)^{-1}\langle \nabla v \cdot a_n \cdot \nabla v\rangle \\
&\leq (\varepsilon\delta_1 + (4\varepsilon)^{-1})\langle \nabla v \cdot a_n \cdot \nabla v\rangle + \varepsilon\lambda\delta_1\|v\|_2^2 \\
&= \sqrt{\delta_1}\langle \nabla v \cdot a_n \cdot \nabla v\rangle + (2\sqrt{\delta_1})^{-1}\lambda\delta_1\|v\|_2^2 \qquad (\varepsilon = (2\sqrt{\delta_1})^{-1}).
\end{aligned}$$

$$\begin{aligned}
\langle \nabla \cdot (a_m - a_n) \cdot \nabla u_m, v|v|^{1-\frac{2}{r}}\rangle &= -\left(2 - \frac{2}{r}\right)\langle (a_m - a_n) \cdot \nabla u_m, |v|^{1-\frac{2}{r}}\nabla v\rangle \\
&\leq \left(2 - \frac{2}{r}\right)\xi(\beta\|\nabla v\|_2^2 + \beta^{-1}\| |v|^{1-\frac{2}{r}}|\nabla u_m| \|_2^2) \quad (\beta > 0) \\
&\leq 2\xi(\beta\sigma^{-1}\langle \nabla v \cdot a_n \cdot \nabla v\rangle + \beta^{-1}\| |v|^{1-\frac{2}{r}}|\nabla u_m| \|_2^2).
\end{aligned}$$

$$\begin{aligned}
\langle (b_n - b_m) \cdot \nabla u_m, v|v|^{1-\frac{2}{r}}\rangle &\leq \langle (|b_n| + |b_m|)|v|, |v|^{1-\frac{2}{r}}|\nabla u_m|\rangle \qquad (b_n, b_m \in \mathbf{F}_{\delta_1}(A^n)) \\
&\leq \beta\delta_1\langle \nabla v \cdot a_n \cdot \nabla v\rangle + \beta\lambda\delta_1\|v\|_2^2 + \beta^{-1}\| |v|^{1-\frac{2}{r}}|\nabla u_m| \|_2^2.
\end{aligned}$$

Thus, we obtain

$$\Big[\mu - \Big(\frac{1}{r}\frac{1}{\sqrt{\delta_1}} + \beta\Big)\lambda\delta_1\Big]\|v\|_2^2 + \Big(\frac{4}{rr'} - \frac{2}{r}\sqrt{\delta_1} - \beta(\delta_1 + 2\xi\sigma^{-1})\Big)\langle\nabla v\cdot a_n\cdot\nabla v\rangle$$
$$\leq (1+2\xi)\beta^{-1}\|\,|v|^{1-\frac{2}{r}}|\nabla u_m|\,\|_2^2.$$

Since $r > \frac{2}{2-\sqrt{\delta_1}} \Leftrightarrow \frac{2}{r'} - \sqrt{\delta_1} > 0$, we can fix $k > 1$ sufficiently large so that $\frac{4}{rr'} - \frac{2}{r}\sqrt{\delta_1} = \frac{2}{r}(\frac{2}{r'} - \sqrt{\delta_1}) > 2r^{-k}$. Fix β by $\beta(\delta_1 + 2\xi\sigma^{-1}) = \frac{4}{rr'} - \frac{2}{r}\sqrt{\delta_1} - r^{-k}$ $(\geq r^{-k})$. Thus

$$\Big[\mu - \Big(\frac{1}{r}\frac{1}{\sqrt{\delta_1}} + \beta\Big)\lambda\delta_1\Big]\|v\|_2^2 + r^{-k}\langle\nabla v\cdot a_n\cdot\nabla v\rangle$$
$$\leq (\delta_1 + 2\xi\sigma^{-1})(1+2\xi)r^k\|\,|v|^{1-\frac{2}{r}}|\nabla u_m|\,\|_2^2.$$

The choice of μ $(\mu > \lambda)$ ensures that the expression in the square brackets is strictly positive. Indeed, $\mu - (\frac{1}{r}\frac{1}{\sqrt{\delta_1}} + \beta)\lambda\delta_1 > \mu - (\frac{1}{r}\frac{1}{\sqrt{\delta_1}} + (\frac{4}{rr'} - \frac{2}{r}\sqrt{\delta_1})\frac{1}{\delta_1 + 2\xi\sigma^{-1}})\lambda\delta_1 > \mu - \lambda$. Thus

$$\langle\nabla v\cdot a_n\cdot\nabla v\rangle \leq (\delta_1 + 2\xi\sigma^{-1})(1+2\xi)r^{2k}\|\nabla u_m\|_{2s}^2\|v\|_{2s'(1-\frac{2}{r})}^{2(1-\frac{2}{r})},$$

and so

$$\|\nabla v\|_2^2 \leq \sigma^{-1}(\delta_1 + 2\xi\sigma^{-1})(1+2\xi)r^{2k}\|\nabla u_m\|_{2s}^2\|v\|_{2s'(1-\frac{2}{r})}^{2(1-\frac{2}{r})}.$$

By the Sobolev Embedding Theorem, $c_d\|v\|_{2j}^2 \leq \|\nabla v\|_2^2$.

The proof of Lemma 2 is completed. □

Lemma 3. *In the notation of* Lemma 2, *assume that* $\sup_m \|\nabla u_m\|_{qj}^2 < \infty$, $\mu > \mu_0$. *Then for any* $r_0 > \frac{2}{2-\sqrt{\delta_1}}$

$$\|g\|_\infty \leq B\|g\|_{r_0}^\gamma, \quad \mu \geq 1 + \mu_0 \vee \lambda\delta_1,$$

where $\gamma = (1 - \frac{s'}{j})(1 - \frac{s'}{j} + \frac{2s'}{r_0})^{-1} > 0$, *and* $B = B(d, \delta_1) < \infty$.

Proof. Let $D := C\sigma^{-1}(\delta_1 + 2\xi\sigma^{-1})(1+2\xi)\sup_m \|\nabla u_m\|_{qj}^2 < \infty$. We iterate the inequality of Lemma 2,

$$(\star) \qquad \|g\|_{rj} \leq D^{\frac{1}{r}}(r^{\frac{1}{r}})^{2k}\|g\|_{s'(r-2)}^{1-\frac{2}{r}}$$

as follows. Successively setting $s'(r_1 - 2) = r_0$, $s'(r_2 - 2) = jr_1$, $s'(r_3 - 2) = jr_2, \dots$ so that $r_n = (t-1)^{-1}(t^n(\frac{r_0}{s'} + 2) - t^{n-1}\frac{r_0}{s'} - 2)$, where $t = \frac{j}{s'} > 1$, we get from $(\star)$

$$\|g\|_{r_n j} \leq D^{\alpha_n}\Gamma_n\|g\|_{r_0}^{\gamma_n},$$

where

$$\alpha_n = \frac{1}{r_1}\Big(1 - \frac{2}{r_2}\Big)\Big(1 - \frac{2}{r_3}\Big)\dots\Big(1 - \frac{2}{r_n}\Big) + \frac{1}{r_2}\Big(1 - \frac{2}{r_3}\Big)\Big(1 - \frac{2}{r_4}\Big)\dots\Big(1 - \frac{2}{r_n}\Big)$$
$$+ \dots + \frac{1}{r_{n-1}}\Big(1 - \frac{2}{r_n}\Big) + \frac{1}{r_n};$$
$$\gamma_n = \Big(1 - \frac{2}{r_1}\Big)\Big(1 - \frac{2}{r_2}\Big)\dots\Big(1 - \frac{2}{r_n}\Big);$$
$$\Gamma_n = \Big[r_n^{r_n^{-1}}\, r_{n-1}^{r_{n-1}^{-1}(1-2r_n^{-1})}\, r_{n-2}^{r_{n-2}^{-1}(1-2r_{n-1}^{-1})(1-2r_n^{-1})} \dots r_1^{r_1^{-1}(1-2r_2^{-1})\dots(1-2r_n^{-1})}\Big]^{2k}.$$

Since $\alpha_n = t^n - r_n^{-1}(t-1)^{-1}$ and $\gamma_n = r_0 t^{n-1}(s' r_n)^{-1}$,

$$\alpha_n \le \alpha \equiv \left(\frac{r_0}{s'} + 2 - \frac{r_0}{j}\right)^{-1},$$

and

$$\inf_n \gamma_n > \gamma = \left(1 - \frac{s'}{j}\right)\left(1 - \frac{s'}{j} + \frac{2s'}{r_0}\right)^{-1} > 0, \qquad \sup_n \gamma_n < 1.$$

Note that $\|g\|_{r_0} \to 0$ as $n, m \to \infty$ (Theorem A.1), and so $\|g\|_{r_0}^{\gamma_n} \le \|g\|_{r_0}^{\gamma}$ for all large enough n, m.

Finally, since

$$\Gamma_n^{\frac{1}{2k}} = r_n^{r_n^{-1}} r_{n-1}^{t r_n^{-1}} r_{n-2}^{t^2 r_n^{-1}} \dots r_1^{t^{n-1} r_n^{-1}} \quad \text{and} \quad b t^n \le r_n \le a t^n,$$

where $a = r_1(t-1)^{-1},\ b = r_1 t^{-1}$, we have

$$\begin{aligned}\Gamma_n^{\frac{1}{2k}} &\le (at^n)^{(bt^n)^{-1}} (at^{n-1})^{(bt^{n-1})^{-1}} \dots (at)^{(bt)^{-1}} \\ &= \left[a^{t^n - t^{-n}(t-1)^{-1}} t^{\sum_{i=1}^n i t^{-i}}\right]^{\frac{1}{b}} \le \left[a^{(t-1)^{-1}} b^{t(t-1)^2}\right]^{\frac{1}{b}}.\end{aligned}$$

The proof of Lemma 3 is completed. □

Remark. That γ is strictly greater than 0 is the main concern of the iteration procedure.

5.3. Proof of Proposition 1. Let a be as in the assumptions of the theorem. By Theorem A.1, $u_n := (\mu + \Lambda_{r_0}(a_n, b_n))^{-1} f$, $f \in L^1 \cap L^\infty$, are well defined for all $\mu > \frac{\lambda\delta}{2(r_0-1)}$ (we use that $b \in \mathbf{F}_\delta \Rightarrow b \in \mathbf{F}_{\delta_1}(A)$ with $\delta_1 = \delta$ since $a \ge I$).

The second inequality in $(**)$ verifies the assumptions of Lemma 3, which in turn yields

$$\|u_n - u_m\|_\infty \le B\|u_n - u_m\|_{r_0}^{\gamma}, \quad r_0 > 2, \quad \gamma > 0.$$

Since by Theorem A.1 the sequence $\{u_n\}$ is fundamental in L^{r_0}, we obtain that $\{u_n\}$ is fundamental in C_∞.

We will also need

Lemma 4. *Let* $U_n := (\mu + \Lambda_{r_0}(a_n, b_n))^{-1} F,\ \mu > \lambda,\ F := (-\nabla \cdot (a_n - I) \cdot \nabla + b_n \cdot \nabla)(\mu - \Delta)^{-1} f,\ f \in C_c^1$. *There are constants* $0 < \tilde{\gamma} \le 1$, $\tilde{B}$ *and* $\hat{B}$ *independent of n such that*

$$\|U_n\|_\infty \le \tilde{B}\|U_n\|_{r_0}^{\tilde{\gamma}},$$

$$\|\mu U_n\|_\infty \le \hat{B}\|\mu U_n\|_{r_0}^{\tilde{\gamma}}$$

whenever $r_0 > \frac{2}{2-\sqrt{\delta_1}}$.

Proof. Arguing exactly as in the proof of Lemma 2, we obtain the inequalities

$$\|U_n\|_{rj} \le (C\delta_1 \|\nabla(\mu - \Delta)^{-1} f\|_{qj}^2)^{\frac{1}{r}} (r^{2k})^{\frac{1}{r}} \|U_n\|_{x'(r-2)}^{1-\frac{2}{r}},$$

$$\|\mu U_n\|_{rj} \le (C\delta_1 \|\nabla f\|_{qj}^2)^{\frac{1}{r}} (r^{2k})^{\frac{1}{r}} \|\mu U_n\|_{x'(r-2)}^{1-\frac{2}{r}};$$

their iteration provides the required result. □

Lemma 5. *In the notation of* Lemma 4*, we have*

$$\|\mu U_n\|_r \leq \Big(\frac{8}{r}\Big(\frac{2}{r'} - \sqrt{\delta_1}\Big)\Big)^{-\frac{1}{2}} \Big(\mu - \frac{\lambda\delta_1}{\delta_1}\Big)^{-\frac{1}{2}} \|\nabla f\|_r$$

whenever $r > \frac{2}{2-\sqrt{\delta_1}}$.

Proof. Arguing again as in the proof of Lemma 2, we obtain the inequality ($\beta > 0$)

$$\Big[\mu - \Big(\frac{1}{r}\frac{1}{\sqrt{\delta_1}} + \beta\Big)\lambda\delta_1\Big]\|v\|_2^2 + \Big(\frac{4}{rr'} - \frac{2}{r}\sqrt{\delta_1} - \beta\delta_1\Big)\|\nabla v\|_2^2 \leq (4\beta)^{-1}\| \, |v|^{1-\frac{2}{r}}|f_\mu| \, \|_2^2,$$

where $v := U_n|U|_n^{\frac{r-2}{2}}$ and $f_\mu := \nabla(\mu - \Delta)^{-1} f$. Putting here $\beta\delta_1 = \frac{4}{rr'} - \frac{2}{r}\sqrt{\delta_1}$ and noticing that

$$\delta_1\Big[\mu - \Big(\frac{1}{r}\frac{1}{\sqrt{\delta_1}} + \beta\Big)\lambda\delta_1\Big] = \mu - \lambda\Big(\frac{4}{rr'} - \frac{\sqrt{\delta_1}}{r}\Big) \geq \mu - \lambda,$$

we have

$$\frac{8}{r}\Big(\frac{2}{r'} - \sqrt{\delta_1}\Big)(\mu - \lambda)\|v\|_2^2 \leq \delta_1 \|v\|_2^{2(1-\frac{2}{r})}\|f_\mu\|_r^2.$$

It remains to note that $\|f_\mu\|_r \leq \mu^{-1}\|\nabla f\|_r$. □

Lemma 6. *s-C_∞-* $\lim_{\mu\uparrow\infty} \mu(\mu + \Lambda_{C_\infty}(a_n, b_n))^{-1} = 1$ *uniformly in n.*

Proof. Clearly it suffices to show that

$$\lim_{\mu\uparrow\infty} \sup_n \|\mu[(\mu + \Lambda_r(a_n, b_n))^{-1} - (\mu - \Delta)^{-1}]f\|_\infty = 0 \qquad \text{for all } f \in C_c^1.$$

Since

$$\begin{aligned} &- [(\mu + \Lambda_r(a_n, b_n))^{-1} - (\mu - \Delta)^{-1}]f \\ &= (\mu + \Lambda_r(a_n, b_n))^{-1}(-\nabla \cdot (a_n - I) \cdot \nabla + b_n \cdot \nabla)(\mu - \Delta)^{-1} f = U_n, \end{aligned}$$

we obtain by Lemma 4 and Lemma 5 that

$$\|\mu U_n\|_\infty \leq \hat{B}\|\mu U_n\|_{r_0}^{\tilde{\gamma}} \leq \dot{B}(\mu - \lambda)^{-\frac{\tilde{\gamma}}{2}}\|\nabla f\|_{r_0}^{\tilde{\gamma}},$$

which yields the required. □

We are in position to complete the proof of Proposition 1.

The fact that $\{u_n\}$ is fundamental in C_∞, and Lemma 6, verify conditions of the Trotter Approximation Theorem [10, Ch. IX, sect. 2]: the limit

$$s\text{-}C_\infty\text{-}\lim e^{-t\Lambda_{C_\infty}(a_n,b_n)} \quad (\text{loc. uniformly in } t \geq 0)$$

exists and determines a contraction C_0 semigroup on C_∞, $e^{-t\Lambda_{C_\infty}(a,b)}$. The latter is positivity preserving since $e^{-t\Lambda_{C_\infty}(a_n,b_n)}$ are. Thus, $e^{-t\Lambda_{C_\infty}(a,b)}$ is a Feller semigroup.

By the construction of $e^{-t\Lambda_{C_\infty}(a,b)}$, we have the following consistency property:

$$(10) \qquad (\mu + \Lambda_{C_\infty}(a, b))^{-1} = ((\mu + \Lambda_q(a, b))^{-1} \upharpoonright L^q \cap C_\infty)^{\text{clos}}_{C_\infty \to C_\infty}, \quad \mu > \mu_0.$$

The last assertion of Proposition 1 now follows (10), $(**)$ and the Sobolev Embedding Theorem.

The proof of Proposition 1 is completed.

The proof of assertion (i) of Theorem 1 is completed.

6. Proof of Theorem 1(ii),(iii): The martingale problem

The Feller semigroup $e^{-t\Lambda_{C_\infty}(a,\nabla a+b)}$ and the next two estimates will allow us to use the approach of [15] where we considered the case $a = I$.

Lemma 7. *In the assumptions of* Theorem 1*, there exist constants* $\mu_0 > 0$ *and* $C_i = C_i(\delta, \delta_a, \gamma, q, \mu)$*,* $i = 1, 2$*, such that, for all* $h \in C_c$ *and* $\mu > \mu_0$*, we have:*

$$\left\|(\mu + \Lambda_{C_\infty}(a, \nabla a + b))^{-1}|b_m|h\right\|_\infty \leq C_1\||b_m|^{\frac{2}{q}}h\|_q, \tag{11}$$

$$\|(\mu + \Lambda_{C_\infty}(a, \nabla a + b))^{-1}|b_m - b_n|h\|_\infty \leq C_2\left\||b_m - b_n|^{\frac{2}{q}}h\right\|_q. \tag{12}$$

We will also need a weighted variant of Lemma 7 . Define the weight

$$\rho(y) \equiv \rho_l(y) := (1 + l|y|^2)^{-\nu}, \quad \nu > \frac{d}{2q} + 1, \quad l > 0, \quad y \in \mathbb{R}^d.$$

Clearly,

$$|\nabla\rho| \leq \nu\sqrt{l}\rho, \quad |\Delta\rho| \leq 2\nu(2\nu + d + 2)l\rho. \tag{13}$$

Lemma 8. *In the assumptions of* Theorem 1*, there exist constants* $\mu_0 > 0$ *and* $K_1 = K_1(\delta, \gamma, \delta_a, q)$ *and* $K_2 = K_2(\delta, \gamma, \delta_a, q, \mu)$ *such that, for all* $h \in C_c(\mathbb{R}^d)$*,* $\mu > \mu_0$ *and sufficiently small* $l = l(\delta, \gamma, \delta_a, q) > 0$*, we have:*

$$\left\|\rho(\mu + \Lambda_{C_\infty}(a_n, \nabla a_n + b_n))^{-1}h\right\|_\infty \leq K_1\|\rho h\|_q, \tag{E_1}$$

$$\left\|\rho(\mu + \Lambda_{C_\infty}(a_n, \nabla a_n + b_n))^{-1}|b_m|h\right\|_\infty \leq K_2\||b_m|^{\frac{2}{q}}\rho h\|_q. \tag{E_2}$$

We need the weight ρ in order to control the behaviour of the semigroups $e^{-t\Lambda_{C_\infty}(a_n,\nabla a_n+b_n)}$ at infinity in absence of a uniform (in n) Gaussian upper bound on their integral kernels.

We prove Lemma 7 and Lemma 8 in the next section.

We return to the proof of assertions (ii) and (iii) of Theorem 1. By construction,

$$e^{-t\Lambda_{C_\infty}(a,\nabla a+b)} = s\text{-}C_\infty\text{-}\lim e^{-t\Lambda_{C_\infty}(a_n,\nabla a_n+b_n)} \quad (\text{loc. uniformly in } t \geq 0), \tag{14}$$

where $\Lambda_{C_\infty}(a_n, \nabla a_n + b_n) := -a_n \cdot \nabla^2 + b_n \cdot \nabla$, $D(\Lambda_{C_\infty}(a_n, \nabla a_n + b_n)) := (1 - \Delta)^{-1}C_\infty$.

Let $\mathbb{P}_x^n$ be the probability measures associated with $e^{-t\Lambda_{C_\infty}(a_n,\nabla a_n+b_n)}$, $n = 1, 2, \dots$

Set $\mathbb{E}_x := \mathbb{E}_{\mathbb{P}_x}$, and $\mathbb{E}_x^n := \mathbb{E}_{\mathbb{P}_x^n}$.

Lemma 9. *For every* $x \in \mathbb{R}^d$ *and* $t > 0$*,* $\mathbb{P}_x[X(t) = \infty] = 0$*.*

The proof repeats the proof of [15, Lemma 2], where we use crucially estimate (E_1) of Lemma 8. Lemma 9 yields Theorem 1(ii).

Set $\Omega_D := D([0, \infty[, \mathbb{R}^d)$, the subspace of $\bar{\Omega}_D$ $(:= D([0, \infty[, \bar{\mathbb{R}}^d))$ consisting of the trajectories $X(t) \neq \infty$, $0 \leq t < \infty$. Let $\mathcal{F}'_t := \sigma(X(s) \mid 0 \leq s \leq t, X \in \Omega_D)$, $\mathcal{F}'_\infty := \sigma(X(s) \mid 0 \leq$

$s < \infty, X \in \Omega_D$). By Lemma 9, $(\Omega_D, \mathcal{F}'_\infty)$ has full $\mathbb{P}_x$-measure in $(\bar{\Omega}_D, \mathcal{F}_\infty)$. We denote the restriction of $\mathbb{P}_x$ from $(\bar{\Omega}_D, \mathcal{F}_\infty)$ to $(\Omega_D, \mathcal{F}'_\infty)$ again by $\mathbb{P}_x$.

Lemma 10. *For every $x \in \mathbb{R}^d$ and $g \in C_c^\infty(\mathbb{R}^d)$,*

$$g(X(t)) - g(x) + \int_0^t (-a \cdot \nabla^2 g + b \cdot \nabla g)(X(s))ds,$$

is a martingale relative to $(\Omega_D, \mathcal{F}'_t, \mathbb{P}_x)$.

The proof follows the proof of [15, Lemma 3] where we apply estimates (11) and (12) of Lemma 7.

Armed with Lemma 10, we show, repeating the proof of [15, Lemma 4], that for each $x \in \mathbb{R}^d$, Ω has full $\mathbb{P}_x$-measure in Ω_D. We denote the restriction of $\mathbb{P}_x$ from $(\Omega_D, \mathcal{F}'_\infty)$ to $(\Omega, \mathcal{G}_\infty)$ again by $\mathbb{P}_x$. In view of Lemma 10, we obtain

Lemma 11. *For every $x \in \mathbb{R}^d$ and $g \in C_c^\infty(\mathbb{R}^d)$,*

$$g(X(t)) - g(x) + \int_0^t (-a \cdot \nabla^2 g + b \cdot \nabla g)(X(s))ds, \quad X \in \Omega,$$

is a continuous martingale relative to $(\Omega, \mathcal{G}_t, \mathbb{P}_x)$.

Using Lemma 11 and estimate (E_2) of Lemma 8, we follow the proof of [15, Lemma 5] to obtain

Lemma 12. *For every $x \in \mathbb{R}^d$ and $t > 0$, $\mathbb{E}_x \int_0^t |b(X(s))|ds < \infty$, and, for $f(y) = y_i$ or $f(y) = y_i y_j$, $1 \le i, j \le d$,*

$$f(X(t)) - f(x) + \int_0^t (-a \cdot \nabla^2 f + b \cdot \nabla f)(X(s))ds, \quad X \in \Omega,$$

is a continuous martingale relative to $(\Omega, \mathcal{G}_t, \mathbb{P}_x)$.

Lemma 12 yields Theorem 1(iii).

The proof of Theorem 1(ii)(iii) is completed.

7. Proofs of Lemmas 7 and 8

The proof of Lemma 7 is obtained via a simple modification of the proof of Lemma 8. We will attend to it in the end of this section.

Proof of Lemma 8 . We follow the proof of Lemma 1. Since a_n, b_n are C^∞ smooth, we have, in view of Theorem A.1, for all $\mu > \frac{\lambda\delta}{2(q-1)}$

$$(\mu + \Lambda_{C_\infty}(a_n, \nabla a_n + b_n))^{-1} \upharpoonright C_\infty \cap L^q = (\mu + \Lambda_q(a_n, \nabla a_n + b_n))^{-1} \upharpoonright C_\infty \cap L^q,$$

(note that, by our assumptions, $\nabla a_n + b_n \in \mathbf{F}_{\delta_a + \delta}$ with $\delta_a + \delta < 1$).

Thus, it suffices to prove estimates (E_1), (E_2) in Lemma 8 for $(\mu + \Lambda_q(a_n, \nabla a_n + b_n))^{-1}$.

Set $A_q^n := -\nabla \cdot a_n \cdot \nabla$, $D(A_q^n) := W^{2,q}$. To shorten the proof, we introduce notation

$$\hat{b}_n := \nabla a_n + b_n \in \mathbf{F}_{\delta_0}, \quad \delta_0 := \delta_a + \delta.$$

Put

$$0 \leq u_n := (\mu + \Lambda_q(a_n, \hat{b}_n))^{-1} h, \quad 0 \leq h \in C_c^1,$$

where $\Lambda_q(a_n, \hat{b}_n) = A_q^n + \hat{b}_n \cdot \nabla$ $(= -a_n \cdot \nabla^2 + b_n \cdot \nabla)$, $D(\Lambda_q(a_n, \hat{b}_n)) = W^{2,q}$, $n \geq 1$. Clearly, since $a_n, b_n \in C^\infty$, we have $u_n \in W^{3,q}$.

For brevity, we omit index n everywhere below, and write $u \equiv u_n$, $a \equiv a_n$, $\hat{b} \equiv \hat{b}_n$, $A_q \equiv A_q^n$. Set

$$\eta := \rho^q, \quad w := \nabla u, \quad I_q := \sum_{r=1}^d \langle (\nabla_r w)^2 |w|^{q-2} \eta \rangle, \quad J_q := \langle (\nabla |w|)^2 |w|^{q-2} \eta \rangle,$$

$$I_q^a := \sum_{r=1}^d \langle (\nabla_r w \cdot a \cdot \nabla_r w) |w|^{q-2} \eta \rangle, \quad J_q^a := \langle (\nabla |w| \cdot a \cdot \nabla |w|) |w|^{q-2} \eta \rangle.$$

By (13),

$$|\nabla \eta| \leq c_1 \sqrt{l} \eta, \quad |\Delta \eta| \leq c_2 l \eta. \tag{15}$$

Recall $[F, G] := FG - GF$.

Proof of (E_1). We multiply the equation $\mu u + \Lambda_q(a, \hat{b}) u = h$ by the test function

$$\phi := -\nabla \cdot (\eta w |w|^{q-2}) \equiv -\sum_{r=1}^d \nabla_r (\eta w_r |w|^{q-2})$$

and integrate:

$$\mu \langle \eta |w|^q \rangle + \langle A_q w, \eta w |w|^{q-2} \rangle + \langle [\nabla, A_q] u, \eta w |w|^{q-2} \rangle = \langle -\hat{b} \cdot \nabla u, \phi \rangle + \langle h, \phi \rangle,$$

$$\mu \langle \eta |w|^q \rangle + I_q^a + (q-2) J_q^a + R_q^1 + \langle [\nabla, A_q] u, \eta w |w|^{q-2} \rangle = \langle -\hat{b} \cdot \nabla u, \phi \rangle + \langle h, \phi \rangle,$$

where $R_q^1 := \langle \nabla \eta \cdot a \cdot \nabla |w|, |w|^{q-1} \rangle$.

We will get rid of the terms containing $\nabla \eta$, which we denote by R_q^k, towards the end of the proof, by selecting the constant l in the definition of $\eta = \rho^q$ to be sufficiently small, and then appealing to the estimates (15).

By our assumption $a \geq I$, and so $I_q^a \geq I_q$, $J_q^a \geq J_q$. We obtain the *principal inequality*

$$\mu \langle \eta |w|^q \rangle + I_q + (q-2) J_q \leq -\langle [\nabla, A_q] u, \eta w |w|^{q-2} \rangle + \langle -\hat{b} \cdot \nabla u, \phi \rangle + \langle h, \phi \rangle - R_q^1. \tag{$\bullet\bullet$}$$

We estimate $\langle [\nabla, A_q] u, \eta w |w|^{q-2} \rangle := \sum_{r=1}^d \langle [\nabla_r, A_q] u, \eta w_r |w|^{q-2} \rangle$ first. Below we omit the summation sign in repeated indices.

Claim 4.

$$|\langle [\nabla_r, A_q] u, \eta w_r |w|^{q-2} \rangle| \leq \alpha \gamma \frac{q^2}{4} J_q + \frac{1}{4\alpha} I_q + (q-2) \Big[\beta \gamma \frac{q^2}{4} + \frac{1}{4\beta} \Big] J_q \qquad \gamma = \sum_{r,\ell} \gamma_{r\ell}$$
$$+ R_q^2 + (\alpha + (q-2)\beta) R_q^3 + (\alpha + (q-2)\beta) \lambda \gamma \langle \eta |w|^q \rangle, \qquad (\alpha, \beta > 0)$$

where $R_q^2 := \langle (\nabla_r a_{i\ell}) w_\ell, w_r |w|^{q-2} \nabla_i \eta \rangle$, $R_q^3 := \frac{q}{2} \langle \nabla |w|, |w|^{q-1} \nabla \eta \rangle + \frac{1}{4} \langle |w|^q \frac{(\nabla \eta)^2}{\eta} \rangle$.

The proof repeats the proof of Claim 1.

Claim 5. *There exist constants C_i ($i = 0, 1, 2$) such that*

$$\langle -\hat{b}\cdot w, \phi\rangle \le \Big[(\sqrt{\delta_0}\sqrt{\delta_a}+\delta_0)\frac{q^2}{4}+(q-2)\frac{q\sqrt{\delta_0}}{2}\Big]J_q$$
$$+\|a-I\|_\infty\Big[\alpha_1\delta_0\frac{q^2}{4}J_q+\frac{1}{4\alpha_1}I_q\Big]+C_0\|\eta^{\frac{1}{q}}w\|_q^q+C_1\|\eta^{\frac{1}{q}}w\|_q^{q-2}\|\eta^{\frac{1}{q}}h\|_q^2+C_2R_q^3+R_q^4,$$

where $R_q^4 := -\langle\nabla\eta, w|w|^{q-2}(-\hat{b}\cdot w)\rangle$, and $\alpha_1 > 0$.

Proof. The proof repeats the proof of Claim 2, with 3) replaced by

3°) $\langle\mu u, \eta|w|^{q-2}\hat{b}\cdot w\rangle \le \frac{\mu}{\mu-\mu_1}B_q^{\frac{1}{2}}\|\eta^{\frac{1}{q}}w\|_q^{\frac{q-2}{2}}\|\eta^{\frac{1}{q}}h\|_q$ for some $\mu_1 > 0$, for all $\mu > \mu_1$.

Indeed, $\langle\mu u, \eta|w|^{q-2}(-\hat{b}\cdot w)\rangle \le \mu B_q^{\frac{1}{2}}\|\eta^{\frac{1}{q}}w\|_q^{\frac{q-2}{2}}\|\eta^{\frac{1}{q}}u\|_q$ and $\|\eta^{\frac{1}{q}}u\|_q \leqslant (\mu-\mu_1)^{-1}\|\eta^{\frac{1}{q}}h\|_q$, $\mu > \mu_1$, for appropriate $\mu_1 > 0$. To prove the last estimate, we multiply $(\mu + \Lambda_q(a, \hat{b}))u = h$ by ηu^{q-1} to obtain

$$\mu\langle u, \eta u^{q-1}\rangle - \langle\nabla\cdot a\cdot w, \eta u^{q-1}\rangle = \langle -\hat{b}\cdot w, \eta u^{q-1}\rangle + \langle h, \eta u^{q-1}\rangle,$$

$$\mu\|\eta^{\frac{1}{q}}u\|_q^q + \frac{4(q-1)}{q^2}\langle\eta\nabla u^{\frac{q}{2}}\cdot a\cdot\nabla u^{\frac{q}{2}}\rangle + R_q^5 = \langle -\hat{b}\cdot w, \eta u^{q-1}\rangle + \langle h, \eta u^{q-1}\rangle,$$

where $R_q^5 := \frac{2}{q}\langle a\cdot\nabla u^{\frac{q}{2}}, (\nabla\eta)u^{\frac{q}{2}}\rangle$. In the RHS we apply the quadratic inequality to $\langle -\hat{b}\cdot\nabla u, \eta u^{q-1}\rangle$ to obtain:

$$\mu\|\eta^{\frac{1}{q}}u\|_q^q + \frac{4(q-1)}{q^2}\langle\eta\nabla u^{\frac{q}{2}}\cdot a\cdot\nabla u^{\frac{q}{2}}\rangle + R_q^5$$
$$\le \kappa\frac{2}{q}\langle\eta(\nabla u^{\frac{q}{2}})^2\rangle + \frac{1}{2\kappa q}\langle\eta\hat{b}^2u^q\rangle + \langle h, \eta u^{q-1}\rangle \qquad (\kappa > 0),$$

$$\mu\|\eta^{\frac{1}{q}}u\|_q^q + \frac{4(q-1)}{q^2}\langle\eta\nabla u^{\frac{q}{2}}\cdot a\cdot\nabla u^{\frac{q}{2}}\rangle + R_q^5$$
$$\le \kappa\frac{2}{q}\langle\eta(\nabla u^{\frac{q}{2}})^2\rangle + \frac{1}{2\kappa q}\langle\eta\hat{b}^2u^q\rangle + \|\eta^{\frac{1}{q}}h\|_q\|\eta^{\frac{1}{q}}u\|_q^{q-1}.$$

Since $a \ge I$, we can replace in the LHS $\langle\eta\nabla u^{\frac{q}{2}}\cdot a\cdot\nabla u^{\frac{q}{2}}\rangle$ by $\langle\eta(\nabla u^{\frac{q}{2}})^2\rangle$. By $\hat{b} \in \mathbf{F}_{\delta_0}$, $\langle\eta\hat{b}^2u^q\rangle \le \delta_0\langle\eta(\nabla u^{\frac{q}{2}})^2\rangle + 2\langle\nabla u^{\frac{q}{2}}, \nabla\eta\rangle + \langle(\nabla\eta)^2u^q\rangle + \lambda\delta_0\langle\eta u^q\rangle$, and thus we arrive at

$$(\mu-\mu_1)\|\eta^{\frac{1}{q}}u\|_q^q + \Big[\frac{4(q-1)}{q^2} - \kappa\frac{2}{q} - \frac{1}{2\kappa q}\delta_0\Big]\langle\eta(\nabla u^{\frac{q}{2}})^2\rangle \le -R_q^5 + R_q^6 + \|\eta^{\frac{1}{q}}h\|_q\|\eta^{\frac{1}{q}}u\|_q^{q-1},$$

where $\mu_1 := \lambda\delta_0$, $R_q^6 := \frac{1}{2\kappa q}(2\langle\nabla u^{\frac{q}{2}}, \nabla\eta\rangle + \langle(\nabla\eta)^2u^q\rangle)$. We select $\kappa := \frac{\sqrt{\delta_0}}{2}$. Since by the assumptions of the lemma $q > \frac{2}{2-\sqrt{\delta_0}}$, the coefficient of $\langle\eta(\nabla u^{\frac{q}{2}})^2\rangle$ is positive. In turn, by (15),

$$-R_q^5 \le c_2\sqrt{l}\|a\|_\infty\langle|\nabla u^{\frac{q}{2}}|, \eta u^{\frac{q}{2}}\rangle \le \frac{c_2}{2}\sqrt{l}\|a\|_\infty(\langle\eta(\nabla u^{\frac{q}{2}})^2\rangle + \langle\eta u^q\rangle).$$

We estimate R_q^6 similarly. The required estimate $(\mu-\mu_1)\|\eta^{\frac{1}{q}}u\|_q \le \|\eta^{\frac{1}{q}}h\|_q$ now follows upon selecting l in the definition of η $(= \rho^q)$ sufficiently small at expense of increasing μ_1 slightly. This completes the proof of 3°). □

Claim 6. *For each $\varepsilon_0 > 0$ there exists a constant $C = C(\varepsilon_0) < \infty$ such that*

$$\langle h, \phi\rangle \le \varepsilon_0 I_q + C\|\eta^{\frac{1}{q}} w\|_q^{q-2}\|\eta^{\frac{1}{q}} h\|_q^2 + R_q^7,$$

where $R_q^7 := -\langle \nabla\eta \cdot w|w|^{q-2}, h\rangle$.

The proof repeats the proof of Claim 3.

Claims 4, 5 and 6 applied in (•) yield (assuming $\varepsilon_0 > 0$ is chosen sufficiently small): There exists $\mu_0 > \mu_1$ such that

$$\begin{aligned}&(\mu-\mu_0)\|\eta^{\frac{1}{q}} w\|_q^q + I_q + (q-2)J_q - \alpha\gamma\tfrac{q^2}{4}J_q - \tfrac{1}{4\alpha}I_q - (q-2)\Big[\beta\gamma\tfrac{q^2}{4} + \tfrac{1}{4\beta}\Big]J_q \\ &-\Big((\sqrt{\delta_0}\sqrt{\delta_a}+\delta_0)\tfrac{q^2}{4} + (q-2)\tfrac{q\sqrt{\delta_0}}{2}\Big)J_q - \|a-I\|_\infty\Big(\alpha_1\delta_0\tfrac{q^2}{4}J_q + \tfrac{1}{4\alpha_1}I_q\Big) \\ &\le C\|\eta^{\frac{1}{q}} h\|_q^q - R_q^1 + R_q^2 + CR_q^3 + R_q^4 + R_q^7, \qquad \alpha=\beta := \tfrac{1}{q\sqrt{\gamma}}, \quad \alpha_1 := \tfrac{1}{q\sqrt{\delta_0}}.\end{aligned}$$

By the assumptions of the theorem, the coefficient of I_q, $1 - \frac{q}{4}(\sqrt{\gamma} + \|a-I\|_\infty\sqrt{\delta_0}) - \varepsilon_0$ is positive, so by $I_q \ge J_q$ we have

$$\begin{aligned}&(\mu-\mu_0)\|\eta^{\frac{1}{q}} w\|_q^q + \Big[(q-1)(1-\frac{q\sqrt{\gamma}}{2}) - (\sqrt{\delta_0}\sqrt{\delta_a}+\delta_0)\frac{q^2}{4} - (q-2)\frac{q\sqrt{\delta_0}}{2} - \|a-I\|_\infty\frac{q\sqrt{\delta_0}}{2}\Big]J_q \\ &\le C\|\eta^{\frac{1}{q}} h\|_q^q - R_q^1 + R_q^2 + CR_q^3 + R_q^4 + R_q^7.\end{aligned}$$

By the assumptions of the theorem the coefficient of J_q is positive. Selecting l in the definition of η sufficiently small, we eliminate the terms R_q^k ($k = 1, 2, 3, 4, 7$) using the estimates (15) as in the proof of 3°), at expense of increasing μ_0 and decreasing the coefficient of J_q slightly, arriving at

$$(\mu-\mu_0)\|\eta^{\frac{1}{q}} w\|_q^q + cJ_q \le C\|\eta^{\frac{1}{q}} h\|_q^q, \quad c > 0.$$

In $J_q \equiv \frac{4}{q^2}\langle \eta(\nabla|\nabla u|^{\frac{q}{2}})^2\rangle$, we commute η and ∇ using (15), arriving at

$$\langle (\nabla|\nabla(\eta^{\frac{1}{q}} u)|^{\frac{q}{2}})^2\rangle \le C'\|\eta^{\frac{1}{q}} h\|_q^q.$$

Applying the Sobolev Embedding Theorem twice, we obtain (E_1).

Proof of (E_2). We modify the proof of Lemma 1 by including the weight η in the same way as in the proof of (E_1) above, and replacing h by $|b_m|h$. Now $u = (\mu + \Lambda_q(a, \hat{b}))^{-1}|b_m|h$, where $0 \leqslant h \in C_c$.

We obtain an obvious analogue of Claim 2 as follows:

We replace 3) in its proof with

3′) $\langle \hat{b}\cdot w, \eta|w|^{q-2}\mu u_n\rangle \leqslant \mu C(\mu)B_q^{\frac{1}{2}}\|\eta^{\frac{1}{q}} w\|_q^{\frac{q-2}{2}}\|\eta^{\frac{1}{q}}|b_m|^{\frac{2}{q}} h\|_q$,

where we have used $\|\eta^{\frac{1}{q}} u_n\|_q \leqslant C(\mu)\|\eta^{\frac{1}{q}}|b_m|^{\frac{2}{q}} h\|_q$. The proof of the last estimate follows the proof of the analogous estimate in 3°), but now we estimate $\langle h, \eta u^{q-1}\rangle$ by Young's inequality:

$$\begin{aligned}\langle |b_m|h, \eta u^{q-1}\rangle &\le \frac{q-1}{q}\sigma^{\frac{q}{q-1}}\langle \eta|b_m|^{\frac{q-2}{q-1}}u^q\rangle + \frac{\sigma^{-q}}{q}\langle \eta|b_m|^2h^q\rangle \qquad (\sigma > 0) \\ &\le \frac{q-1}{q}\sigma^{\frac{q}{q-1}}\langle \eta(1+|b_m|^2)u^q\rangle + \frac{\sigma^{-q}}{q}\langle \eta|b_m|^2h^q\rangle.\end{aligned}$$

It remains to apply $b_m \in \mathbf{F}_\delta$ in order to estimate $\langle \eta(1+|b_m|^2)u^q\rangle$ in terms of $\langle \eta(\nabla u^{\frac{q}{2}})^2\rangle$, $\|\eta^{\frac{1}{q}} u\|_q^q$

and the terms containing $\nabla\eta$ which can be discarded at expense on increasing μ_0. Selecting $\sigma > 0$ sufficiently small, we obtain the required estimate.

Next, we replace 5) by

5′) $\langle |b_m|h, \eta|w|^{q-2}(-\hat{b}\cdot w)\rangle| \leq B_q^{\frac{1}{2}}\langle \eta(|b_m|h)^2|w|^{q-2}\rangle^{\frac{1}{2}}$, where, in turn,

$$\langle \eta(|b_m|h)^2|w|^{q-2}\rangle \leq \frac{q-2}{q}\epsilon^{\frac{q}{q-2}}\langle \eta|b_m|^2|w|^q\rangle + \frac{2}{q}\epsilon^{-\frac{2}{q}}\langle \eta|b_m|^2h^q\rangle$$

(use $b_m \in \mathbf{F}_\delta$)

$$\leq \frac{q-2}{q}\epsilon^{\frac{q}{q-2}}\Big[\frac{q^2}{4}\delta J_q + R_q^3 + \lambda\delta\langle |w|^q\eta\rangle\Big] + \frac{2}{q}\epsilon^{-\frac{2}{q}}\langle \eta|b_m|^2h^q\rangle \tag{16}$$

where $\epsilon > 0$ is to be chosen sufficiently small.

We obtain an analogue of Claim 3 by replacing the estimate $\langle |w|^{q-2}h^2\rangle \leq \|w\|_q^{q-2}\|h\|_q^2$ in its proof by (16). The analogue of R_q^7 is $-\langle \nabla\eta\cdot w|w|^{q-2}, |b_m|h\rangle$, which we eliminate as follows. By (15),

$$-\langle \nabla\eta\cdot w|w|^{q-2}, |b_m|h\rangle \leqslant c_1^2 l\langle \eta(|b_m|h)^2|w|^{q-2}\rangle^{\frac{1}{2}}\|\eta^{\frac{1}{q}}w\|_q^{\frac{q}{2}},$$

so applying (16) to the first multiple in the RHS we estimate in the LHS in terms of the quantities that we can control, by multiplied by the constant l that we can choose arbitrarily small.

The rest of the proof repeats the proof of (E_1). □

Proof of Lemma 7 . The proof of (11) repeats the proof of (E_2) with η taken to be $\equiv 1$ (and so all terms R_q^k disappear). The proof of (12) also repeats the proof of (E_2) with $\eta \equiv 1$, where we take into account that $b_m - b_n \in \mathbf{F}_\delta$. □

Appendix A

We use notation introduced in the beginning of Section 5.2.

Theorem A.1 ([12, Theorem 4.6]). *Let $a, a_n \in (H_u)$, $b, b_n : \mathbb{R}^d \to \mathbb{R}^d$, $A \equiv A(a)$, $A^n \equiv A(a_n)$, $n = 1, 2, \ldots$ Assume that*

(i) $$b \in \mathbf{F}_{\delta_1}(A),\ b_n \in \mathbf{F}_{\delta_1}(A^n),\quad \delta_1 < 1,\quad \textit{with } \lambda \neq \lambda(n),$$

(ii) $$a_n \to a \textit{ strongly in } L^2_{\rm loc}(\mathbb{R}^d, \mathbb{R}^d\otimes\mathbb{R}^d),\quad b_n \to b \textit{ strongly in } L^2_{\rm loc}(\mathbb{R}^d, \mathbb{R}^d).$$

Then:

(j) *The limit*

$$e^{-t\Lambda_r(a,b)} := s\text{-}L^r\text{-}\lim_{n\to\infty} e^{-t\Lambda_r(a_n,b_n)}\quad (\textit{loc. uniformly in } t \geq 0),\qquad r \in \Big]\frac{2}{2-\sqrt{\delta_1}}, \infty\Big[,$$

where $\Lambda_r(a_n, b_n) := -\nabla\cdot a_n\cdot\nabla + b_n\cdot\nabla$, $D(\Lambda_r(a_n, b_n)) = W^{2,r}$, exists and determines a quasi contraction C_0 semigroup on L^r.

(jj) *The resolvent set of $-\Lambda_r(a, b)$, $-\Lambda_r(a_n, b_n)$ contains $\Big\{\zeta\in\mathbb{C} \;\Big|\; \mathrm{Re}\zeta > \frac{\lambda\sqrt{\delta_1}}{2(r-1)}\Big\}$.*

Remark 4. We note that $v(t,\cdot) = e^{-t\Lambda_r(a,b)}f(\cdot)$, $f \in L^r$, is a unique weak solution in L^r to Cauchy problem for parabolic equation $(\partial_t - \nabla\cdot a\cdot\nabla + b\cdot\nabla)v(t,\cdot) = 0$, $v(0+,\cdot) = f(\cdot)$. See [29, Theorem 1.1] for details.

Appendix B

The solution to the martingale problem of Theorem 1(iii) is unique in the following sense. In the assumptions of Theorem 1, let $\|a - I\|_\infty + \delta < 1$. If $\{\mathbb{Q}_x\}_{x\in\mathbb{R}^d}$ is another solution to the martingale problem (iv) such that

$$\mathbb{Q}_x = w\text{-}\lim_n \mathbb{P}_x(\tilde{a}_n, \tilde{b}_n) \quad \text{for every } x \in \mathbb{R}^d,$$

where smooth $\tilde{b}_n$, $\tilde{a}_n$ satisfy $(\mathbf{C}_1)$, $(\mathbf{C}_2)$ with form-bounds $\tilde{\delta}$, $\tilde{\gamma}_{rk}$, $\tilde{\gamma}_a$ (provided that $\lambda \neq \lambda(n)$) satisfying (2), then $\{\mathbb{Q}_x\}_{x\in\mathbb{R}^d} = \{\mathbb{P}_x\}_{x\in\mathbb{R}^d}$.

Proof. The ideas in the following argument are rather standard, cf. [28, Ch.3, sect. 2].

For $f \in C_c^\infty$, $x \in \mathbb{R}^d$, denote

$$R^n_\mu f(x) := \mathbb{E}_{\mathbb{P}^n_x}\int_0^\infty e^{-\mu s} f(X(s))ds \quad \Big(= (\mu + \Lambda_{C_\infty}(\tilde{a}_n, \nabla\tilde{a}_n + \tilde{b}_n))^{-1} f(x)\Big),$$

$$R^Q_\mu f(x) := \mathbb{E}_{\mathbb{Q}_x}\int_0^\infty e^{-\mu s} f(X(s))ds, \quad \mu > 0.$$

Let us show that $(\mu + \Lambda_{C_\infty}(a, \nabla a + b))^{-1} f(x) = R^Q_\mu f(x)$ for all $\mu > 0$ sufficiently large; this would imply that $\{\mathbb{Q}_x\}_{x\in\mathbb{R}^d} = \{\mathbb{P}_x\}_{x\in\mathbb{R}^d}$.

We have:

1) $R^n_\mu f(x) \to R^Q_\mu f(x)$ (the assumption).

2) $\|R^Q_\mu f\|_2 \leqslant (\mu - \omega_2)^{-1}\|f\|_2$, $\mu > \omega_2$.

Indeed, $R^n_\mu f = (\mu + \Lambda_2(\tilde{a}_n, \nabla\tilde{a}_n + \tilde{b}_n))^{-1} f$, $f \in C_c^\infty$. Since $e^{-t\Lambda_2(\tilde{a}_n, \nabla\tilde{a}_n + \tilde{b}_n)}$ is a quasi contraction on L^2, $\|(\mu + \Lambda_2(\tilde{a}_n, \nabla\tilde{a}_n + \tilde{b}_n))^{-1}\|_{2\to 2} \leqslant (\mu - \omega_2)^{-1}$, $\mu > \omega_2$, $0 < \omega_2 \neq \omega_2(n)$. Thus, $\|R^n_\mu f\|_2 \leqslant (\mu - \omega_2)^{-1}\|f\|_2$ for all n. Now 2) follows from 1) by a weak compactness argument in L^2.

By 2), R^Q_μ admits extension by continuity to L^2, which we denote by $R^Q_{\mu,2}$.

3) $\|(-(a - I)\cdot\nabla^2 + b\cdot\nabla)(\mu - \Delta)^{-1}\|_{2\to 2} \leqslant \|a - I\|_\infty + \delta$ (we use $b \in \mathbf{F}_\delta$).

4) $(\mu + \Lambda_2(a, \nabla a + b))^{-1} f = (\mu - \Delta)^{-1}\big(1 + (-(a - I)\cdot\nabla^2 + b\cdot\nabla)(\mu - \Delta)^{-1}\big)^{-1} f$.

Indeed, by our assumptions $\|a - I\|_\infty + \delta < 1$, so in view of 3) the RHS is well defined. Clearly, 4) holds for $a = a_n$, $b = b_n$. We pass to the limit $n \to \infty$ using Theorem A.1.

5) $(\mu + \Lambda_{C_\infty}(a, \nabla a + b))^{-1} f = R^Q_\mu f$ a.e. on $\mathbb{R}^d$.

Indeed, since $\{\mathbb{Q}_x\}$ is a weak solution of (SDE_I), we have by Itô's formula

$$(\mu - \Delta)^{-1}h = R^Q_\mu[(1 + (-(a - I)\cdot\nabla^2 + b\cdot\nabla)(\mu - \Delta)^{-1})h], \quad h \in C_c^\infty.$$

Since $\|(1 + (-(a - I)\cdot\nabla^2 + b\cdot\nabla)(\mu - \Delta)^{-1})\|_{2\to 2} < \infty$ (by 3)), we have, in view of 2),

$$(\mu - \Delta)^{-1}g = R^Q_{\mu,2}[(1 + (-(a - I)\cdot\nabla^2 + b\cdot\nabla)(\mu - \Delta)^{-1})g], \quad g \in L^2.$$

Take $g = (1 + (-(a - I)\cdot\nabla^2 + b\cdot\nabla)(\mu - \Delta)^{-1})^{-1} f$, $f \in C_c^\infty$. Then by 4) $(\mu + \Lambda_2(a, \nabla a + b))^{-1} f =$

$R^{Q}_{\mu,2}f$. By the consistency property $(\mu+\Lambda_{C_\infty}(a,\nabla a+b))^{-1}|_{C_c^\infty\cap L^2} = (\mu+\Lambda_2(a,\nabla a+b))^{-1}|_{C_c^\infty\cap L^2}$, and the result follows.

6) Fix a $q > 2 \vee (d-2)$ satisfying the assumptions of Theorem 1. Since $R^n_\mu f = (\mu + \Lambda_q(\tilde{a}_n, \nabla\tilde{a}_n + \tilde{b}_n))^{-1}f$, we obtain by $(\star\star\star)$ that for all $\mu > \mu_0$

$$\|\nabla R^n_\mu f\|_{qj} \leqslant K\|f\|_q, \quad j = \frac{d}{d-2}, \quad \mu > \mu_0.$$

By a weak compactness argument in L^{qj}, in view of 1), we have $|\nabla R^Q_\mu f| \in L^{qj}$, and there is a subsequence of $\{R^n_\mu f\}$ (without loss of generality, it is $\{R^n_\mu f\}$ itself) such that

$$\nabla R^n_\mu f \xrightarrow{w} \nabla R^Q_\mu f \quad \text{in } L^{qj}(\mathbb{R}^d, \mathbb{R}^d).$$

By Mazur's Lemma, there is a sequence of convex combinations of the elements of $\{\nabla R^n_\mu f\}_{n=1}^\infty$ that converges to $\nabla R^Q_\mu f$ strongly in $L^{qj}(\mathbb{R}^d, \mathbb{R}^d)$, i.e.

$$\sum_\alpha c_\alpha \nabla R^{n_\alpha}_\mu f \xrightarrow{s} \nabla R^Q_\mu f \quad \text{in } L^{qj}(\mathbb{R}^d, \mathbb{R}^d).$$

Now, in view of 1), the latter and the Sobolev Embedding Theorem yield $\sum_\alpha c_\alpha R^{n_\alpha}_\mu f \xrightarrow{s} R^Q_\mu f$ in C_∞. Therefore, by 5), $(\mu + \Lambda_{C_\infty}(a, \nabla a + b))^{-1}f(x) = R^Q_\mu f(x)$ for all $x \in \mathbb{R}^d$, $f \in C_c^\infty$, as needed. □

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