PATHWISE UNIQUENESS OF STOCHASTIC DIFFERENTIAL EQUATIONS DRIVEN BY CAUCHY PROCESSES WITH DRIFT

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Abstract

We consider one-dimensional stochastic differential equations driven by Cauchy processes with drift. This driving process is also known as a strictly 1-stable process. In this paper, we study the pathwise uniqueness of the solution to the stochastic differential equations under a non-Lipschitz condition on the diffusion coefficient.

1. Introduction and main result

Let us consider the one-dimensional stochastic differential equation:

$$(1) \qquad X_t = x + \int_0^t F(X_{s-})dZ_s,$$

where $Z = (Z_t : t \geq 0)$ is a Cauchy process with drift parameter γ characterized by the Lévy–Khintchine representation:

$$\mathbb{E}[e^{iuZ_t}] = \exp\{t(-\pi|u| + i\gamma u)\}.$$

Note that the Lévy process Z is also called a strictly 1-stable process. In this paper, we shall study the pathwise uniqueness of the solution to this stochastic differential equation.

Let us recall some known results on the pathwise uniqueness of the solution to the stochastic differential equation:

$$(2) \qquad X_t = x + \int_0^t F(X_{s-})dZ_s^{(\alpha)},$$

where $Z^{(\alpha)} = (Z_t^{(\alpha)} : t \geq 0)$ is a strictly α-stable process with $0 < \alpha \leq 2$. When $\alpha = 2$, that is, the process $Z^{(2)}$ is a Brownian motion, Yamada and Watanabe [12] have proved the pathwise uniqueness if F is locally $1/2$-Hölder continuous. When $1 < \alpha < 2$, the Lévy measure of the process $Z^{(\alpha)}$ is given by

$$\nu_{r_-,r_+}^\alpha(dz) = |z|^{-\alpha-1}\{r_-\mathbf{1}_{(z<0)} + r_+\mathbf{1}_{(z>0)}\}dz,$$

where r_-, r_+ are non-negative constants such that $r_- + r_+ > 0$, and the drift of the process $Z^{(\alpha)}$ is given by

$$\gamma_{r_-,r_+}^\alpha = -\int_{|z|\geq 1} z\nu_{r_-,r_+}^\alpha(dz).$$

In case of $r_- = r_+$, Komatsu [8] and Bass [2] have proved the pathwise uniqueness if F is locally $1/\alpha$-Hölder continuous. In case of $r_- = 0$, Li and Mytnik [9] have done if F is increasing and locally $(\alpha - 1)/\alpha$-Hölder continuous. In case of $r_- \leq r_+$, Fournier [5] has done if F is $(\alpha - \kappa)/\alpha$-Hölder continuous where

$$\kappa = \frac{1}{\pi} \arccos \left[\frac{(r_-/r_+)^2 \sin^2(\pi\alpha) - \{1 + (r_-/r_+)\cos(\pi\alpha)\}^2}{(r_-/r_+)^2 \sin^2(\pi\alpha) + \{1 + (r_-/r_+)\cos(\pi\alpha)\}^2} \right] \in [\alpha - 1, 1],$$

and satisfies that $\{F(x) - F(y)\}\mathrm{sgn}(y - x) \leq C|x - y|$ where C is a positive constant. Note that κ satisfies $\int_{\mathbb{R}\setminus\{0\}}\{|1 + z|^\kappa - 1 - \kappa z\}\nu^\alpha_{r_-,r_+}(dz) = 0$. In [11], we studied the pathwise uniqueness when the driving process is a more general Lévy process but α-stable processes with $0 < \alpha \leq 1$ are not included. On the other hand, when the process $Z^{(\alpha)}$ is a symmetric α-stable process, Bass, Burdzy and Chen [3] have proved that there exists the function F that is bounded above and below by positive constants and λ-Hölder continuous for $0 < \lambda < 1 \wedge (1/\alpha)$, but under which the pathwise uniqueness fails.

In this paper, we shall consider the stochastic differential equation (1), that is, the equation (2) where $\alpha = 1$ and $r_- = r_+$ but with an additional drift $\gamma > 0$. We may assume that $\gamma > 0$ without loss of generality. If $\gamma < 0$, we shall consider the equation:

$$X_t = x + \int_0^t F(X_{s-})dZ_s = x + \int_0^t \hat{F}(X_{s-})d\hat{Z}_s,$$

where $\hat{F} = -F$ and $\hat{Z} = -Z$ is a Cauchy process with drift $-\gamma$. Now we state our main result.

Theorem 1.1. *Let $\gamma > 0$ and set*

$$\beta = \frac{2}{\pi} \arctan\left(\frac{\gamma}{\pi}\right).$$

Suppose that the coefficient F of the equation satisfies the following conditions:

- *the function F is locally $(1 - \beta)$-Hölder continuous, that is, for each $m \in \mathbb{N}$, there exists a positive constant $C_1(m)$ such that*

(3) $$|F(x) - F(y)| \leq C_1(m)|x - y|^{1-\beta} \quad \text{for } |x|, |y| \leq m;$$

- *for each $m \in \mathbb{N}$, there exists a positive constant $C_2(m)$ such that*

(4) $$\{F(x) - F(y)\}\mathrm{sgn}(x - y) \leq C_2(m)|x - y| \quad \text{for } |x|, |y| \leq m.$$

Then, the solution to the equation (1) is pathwise unique.

Our driving process is a symmetric 1-stable process with non-vanishing drift. If the drift $\gamma = 0$, the result in [3] tells us that the Lipschitz condition on the coefficient F is sharp for the pathwise uniqueness. However, if $\gamma \neq 0$, our result guarantees the pathwise uniqueness under the weaker conditions on F.

REMARK 1.2. The condition (4) can not be removed because the pathwise uniqueness fails for the stochastic differential equation (1) with the initial value $x = 0$ and the coefficient $F(y) = |y|^\theta$ where $\theta \in (0, 1)$:

(5) $$X_t = \int_0^t |X_{s-}|^\theta dZ_s.$$

To show this, we use the similar argument to the proof of Proposition in [8] and Remark 2.3 in [2]. Let $\theta \in (0, 1)$. Set

$$\tau(u) = \int_0^u |Z_s|^{-\theta} ds \quad \text{and} \quad \tau^{-1}(t) = \inf\{u \geq 0 : \tau(u) > t\}.$$

We know that $\mathbb{E}[\tau(u)] < \infty$ and $\tau(u) \to \infty$ as $u \to \infty$. Then, applying Theorem 3 in Kallsen and Shiryaev [7], the process $V = (V_t : t \geq 0)$ defined by

$$V_t = \int_0^{\tau^{-1}(t)} |Z_{s-}|^{-\theta} dZ_s,$$

is a Cauchy process with drift γ. We define the process $U = (U_t : t \geq 0)$ by $U_t = Z_{\tau^{-1}(t)}$, then we obtain

$$U_t = \int_0^t |U_{s-}|^{\theta} dV_s.$$

Hence, (U, V) is a solution to the equation (5) and the process U is not identically zero. However, $(0, V)$ is also a solution to the equation (5). Therefore, the pathwise uniqueness fails for the equation (5).

The organization of this paper is as follows: in Section 2, we prepare some notations and establish the Itô formula for the function given by the convolution of $|x|^{\beta}$ and a mollifier. In Section 3, we compute the remainder term of the Itô formula via the Fourier analysis. In Section 4, we establish the Itô formula for $|x|^{\beta}$ by the limiting argument, and prove our main result.

2. Preliminaries

Let $C_c^{\infty}(\mathbb{R})$ be the elements of $C^{\infty}(\mathbb{R})$ with compact support, and $S(\mathbb{R})$ the Schwartz space of rapidly decreasing functions on $\mathbb{R}$.

Let $Z = (Z_t : t \geq 0)$ be a Cauchy process with drift γ. This process is characterized by the triplet $(\gamma, 0, \nu)$ where ν is the Lévy measure on $\mathbb{R}_0 := \mathbb{R} \setminus \{0\}$ given by

$$\nu(dz) = |z|^{-2} dz,$$

and γ is the drift parameter. In this paper, we are concerned with the case of $\gamma \neq 0$ instead of $\gamma = 0$ which is known as a Cauchy process.

By the Lévy–Khintchine formula, the Lévy symbol η of Z_1 is given by

$$\eta(u) = \mathbb{E}[e^{iuZ_1}] = \exp\{-\pi|u| + i\gamma u\}.$$

By the Lévy–Itô decomposition, it can be represented as

$$Z_t = \gamma t + \int_0^t \int_{|z|<1} z\tilde{N}(ds, dz) + \int_0^t \int_{|z| \geq 1} zN(ds, dz),$$

where $N(ds, dz)$ is a Poisson random measure with the intensity $ds\nu(dz)$, and $\tilde{N}(ds, dz)$ is the compensated Poisson random measure given by $\tilde{N}(ds, dz) = N(ds, dz) - ds\nu(dz)$.

Let $x \in \mathbb{R}$ and $F : \mathbb{R} \to \mathbb{R}$ be continuous. For $i = 1, 2$, we shall consider two càdlàg solutions $X^i = (X_t^i : t \geq 0)$ to the stochastic differential equation:

(6) $X_t^i = x + \int_0^t F(X_{s-}^i)dZ_s$

$$= x + \gamma \int_0^t F(X_s^i)ds + \int_0^t \int_{|z|<1} F(X_{s-}^i)z\tilde{N}(ds,dz) + \int_0^t \int_{|z|\geq 1} F(X_{s-}^i)zN(ds,dz).$$

Throughout this paper, we assume the existence of weak solutions to the stochastic differential equation (6). It is known that the stochastic differential equation has a weak solution under suitable conditions on the coefficient F, see [1, 6, 10, 13]. In this paper, we shall focus on the pathwise uniqueness of the solution to the equation (6) under a non-Lipschitz condition on the coefficient F.

For the sake of simplicity of notations, we write

$$Y_t = X_t^1 - X_t^2, \quad G_t = F(X_t^1) - F(X_t^2).$$

Then, we shall study the difference between the solutions of the equation (6):

$$Y_t = \gamma \int_0^t G_s ds + \int_0^t \int_{|z|<1} G_{s-}z\tilde{N}(ds,dz) + \int_0^t \int_{|z|\geq 1} G_{s-}zN(ds,dz).$$

Let $0 < \beta < 1$ and set $\Phi(x) = |x|^\beta$. Let $\psi : \mathbb{R} \to \mathbb{R}$ be a mollifier, that is, ψ satisfies that $\psi \in C_c^\infty(\mathbb{R})$, $\psi \geq 0$, $\text{Supp}\,\psi \subset [-1, 1]$ and $\int_\mathbb{R} \psi(x)dx = 1$. For $n \in \mathbb{N}$, define $\psi_n : \mathbb{R} \to \mathbb{R}$ by $\psi_n(x) = n\psi(nx)$. Define the convolution of Φ and ψ_n by

$$(\Phi * \psi_n)(x) = \int_\mathbb{R} \Phi(y)\psi_n(x - y)dy,$$

and denote $\Phi * \psi_n$ by Φ_n for the simplicity. For $m \in \mathbb{N}$, define the stopping time T_m by

$$T_m = \inf\{t > 0 : |X_t^1| \wedge |X_t^2| > m\}.$$

We establish the following inequality:

Lemma 2.1. *For each $x, y \in \mathbb{R}$, it holds that*

$$|\Phi_n(x + y) - \Phi_n(x)| \leq |y|^\beta.$$

Proof. This follows from the inequality:

(7) $|x + y|^\beta \leq |x|^\beta + |y|^\beta$

for each $x, y \in \mathbb{R}$. $\square$

Now we can apply the Itô formula for the function Φ_n and obtain that

(8) $\Phi_n(Y_{t \wedge T_m}) = \Phi_n(0) + \gamma \int_0^{t \wedge T_m} \Phi_n'(Y_s)G_s ds$

$$+ \int_0^{t \wedge T_m} \int_{|z|<1} \{\Phi_n(Y_{s-} + G_{s-}z) - \Phi_n(Y_{s-})\}\tilde{N}(ds,dz)$$

$$+ \int_0^{t \wedge T_m} \int_{|z|\geq 1} \{\Phi_n(Y_{s-} + G_{s-}z) - \Phi_n(Y_{s-})\}\tilde{N}(ds,dz)$$

$$+ \int_0^{t \wedge T_m} \int_{\mathbb{R}_0} \{\Phi_n(Y_s + G_s z) - \Phi_n(Y_s) - \Phi_n'(Y_s)G_s z\mathbf{1}_{(|z|<1)}\}\nu(dz)ds.$$

REMARK 2.2. Since Φ'_n is bounded, it follows from the mean value theorem that

$$\mathbb{E}\left[\int_0^{t\wedge T_m}\int_{|z|<1}|\Phi_n(Y_s+G_sz)-\Phi_n(Y_s)|^2\nu(dz)ds\right]$$

$$=\mathbb{E}\left[\int_0^{t\wedge T_m}\int_{|z|<1}\left|\int_0^1\Phi'_n(Y_s+G_s\theta z)G_szd\theta\right|^2\nu(dz)ds\right]$$

$$\leq 4C_3(n)^2C_4(m)^2t\int_{|z|<1}z^2\nu(dz)=8C_3(n)^2C_4(m)^2t,$$

where $C_3(n)=\sup_{x\in\mathbb{R}}|\Phi'_n(x)|$ and $C_4(m)=\sup_{|x|\leq m}|F(x)|$, and hence the 3rd term on the right-hand side of (8) is a square-integrable martingale.

REMARK 2.3. Since F is continuous, it follows from Lemma 2.1 and the inequality (7) that

$$\mathbb{E}\left[\int_0^{t\wedge T_m}\int_{|z|\geq 1}|\Phi_n(Y_s+G_sz)-\Phi_n(Y_s)|\nu(dz)ds\right]$$

$$\leq\mathbb{E}\left[\int_0^{t\wedge T_m}\int_{|z|\geq 1}|G_sz|^\beta\nu(dz)ds\right]\leq 2^\beta C_4(m)^\beta t\int_{|z|\geq 1}|z|^\beta\nu(dz)=\frac{2^{\beta+1}C_4(m)^\beta t}{1-\beta},$$

where $C_4(m)$ is the same constant as in Remark 2.2, and hence the 4th term on the right-hand side of (8) is a martingale.

Moreover, the last term on the right-hand side of (8) can be represented as follows:

Lemma 2.4. *For each $m,n\in\mathbb{N}$ and $t\geq 0$, it holds that*

$$\int_0^{t\wedge T_m}\int_{\mathbb{R}_0}\{\Phi_n(Y_s+G_sz)-\Phi_n(Y_s)-\Phi'_n(Y_s)G_sz\mathbf{1}_{(|z|<1)}\}\nu(dz)ds$$

$$=\int_0^{t\wedge T_m}|G_s|\left(\int_{\mathbb{R}_0}\{\Phi_n(Y_s+z)-\Phi_n(Y_s)-\Phi'_n(Y_s)z\mathbf{1}_{(|z|<1)}\}\nu(dz)\right)\mathbf{1}_{(G_sY_s\neq 0)}ds.$$

Proof. It follows that $G_s=0$ if $Y_s=0$. We then have

$$\Phi_n(Y_s+G_sz)-\Phi_n(Y_s)-\Phi'_n(Y_s)G_sz\mathbf{1}_{(|z|<1)}=0,$$

if $G_sY_s=0$. We consider the integrand on $(G_sY_s\neq 0)$.

Now, we shall show that

(9)
$$\int_0^{t\wedge T_m}\int_{\mathbb{R}_0}\Phi'_n(Y_s)G_sz\mathbf{1}_{\left(|z|<\left|\frac{1}{G_s}\right|,\,|z|\geq 1\right)}\mathbf{1}_{\left(\left|\frac{1}{G_s}\right|>1,\,G_sY_s\neq 0\right)}\nu(dz)ds=0,$$

(10)
$$\int_0^{t\wedge T_m}\int_{\mathbb{R}_0}\Phi'_n(Y_s)G_sz\mathbf{1}_{\left(|z|<1,\,|z|\geq\left|\frac{1}{G_s}\right|\right)}\mathbf{1}_{\left(\left|\frac{1}{G_s}\right|<1,\,G_sY_s\neq 0\right)}\nu(dz)ds=0.$$

From Fubini's theorem and the symmetry of ν, it is enough to show the integrability of integrands on the left-hand side of the above equations. By using the inequality: $|x|^{-1}\log|x|\leq e^{-1}$ for each $|x|>1$, we have

$$\int_0^{t\wedge T_m}\int_{\mathbb{R}_0}|\Phi'_n(Y_s)G_sz|\mathbf{1}_{\left(|z|<\left|\frac{1}{G_s}\right|,\,|z|\geq 1\right)}\mathbf{1}_{\left(\left|\frac{1}{G_s}\right|>1,\,G_sY_s\neq 0\right)}\nu(dz)ds$$

$$= \int_0^{t \wedge T_m} |\Phi_n'(Y_s)G_s| \left(\int_{1 \le |z| < |\frac{1}{G_s}|} \frac{dz}{|z|} \right) \mathbf{1}_{\left(\left| \frac{1}{G_s} \right| > 1, G_s Y_s \ne 0 \right)} ds$$

$$= 2 \int_0^{t \wedge T_m} |\Phi_n'(Y_s)G_s| \left(\log \left| \frac{1}{G_s} \right| \right) \mathbf{1}_{\left(\left| \frac{1}{G_s} \right| > 1, G_s Y_s \ne 0 \right)} ds$$

$$\le 2e^{-1} \int_0^{t \wedge T_m} |\Phi_n'(Y_s)| ds$$

$$\le 2C_3(n)e^{-1}t$$

where $C_3(n)$ is the same constant as in Remark 2.2. Similarly, we have

$$\int_0^{t \wedge T_m} \int_{\mathbb{R}_0} |\Phi_n'(Y_s)G_s z| \mathbf{1}_{\left(|z| < 1, |z| \ge \left| \frac{1}{G_s} \right| \right)} \mathbf{1}_{\left(\left| \frac{1}{G_s} \right| < 1, G_s Y_s \ne 0 \right)} \nu(dz)ds$$

$$= 2 \int_0^{t \wedge T_m} |\Phi_n'(Y_s)G_s| \left| \log \left| \frac{1}{G_s} \right| \right| \mathbf{1}_{\left(\left| \frac{1}{G_s} \right| < 1, G_s Y_s \ne 0 \right)} ds$$

$$= 2 \int_0^{t \wedge T_m} |\Phi_n'(Y_s)G_s| \left(\log |G_s| \right) \mathbf{1}_{(|G_s| > 1, G_s Y_s \ne 0)} ds$$

$$\le 2e^{-1} \int_0^{t \wedge T_m} |\Phi_n'(Y_s)||G_s|^2 ds$$

$$\le 8C_3(n)C_4(m)^2 e^{-1}t,$$

where $C_4(m)$ is the same constant as in Remark 2.2. Hence, we have (9) and (10). From this, we have

$$\int_0^{t \wedge T_m} \int_{\mathbb{R}_0} \{\Phi_n(Y_s + G_s z) - \Phi_n(Y_s) - \Phi_n'(Y_s)G_s z \mathbf{1}_{(|z|<1)}\} \nu(dz)ds$$

$$= \int_0^{t \wedge T_m} \int_{\mathbb{R}_0} \{\Phi_n(Y_s + G_s z) - \Phi_n(Y_s) - \Phi_n'(Y_s)G_s z \mathbf{1}_{\left(|z| < \left| \frac{1}{G_s} \right| \right)}\} \mathbf{1}_{(G_s Y_s \ne 0)} \nu(dz)ds$$

$$= \int_0^{t \wedge T_m} |G_s| \left(\int_{\mathbb{R}_0} \{\Phi_n(Y_s + v) - \Phi_n(Y_s) - \Phi_n'(Y_s)v \mathbf{1}_{(|v|<1)}\} \nu(dv) \right) \mathbf{1}_{(G_s Y_s \ne 0)} ds,$$

by the change of variables $v = G_s z$, and the required result follows. □

Hence, by taking expectations of (8) we have the following:

Proposition 2.5. *For each $m, n \in \mathbb{N}$ and $t \ge 0$, it holds that*

$$\mathbb{E}[\Phi_n(Y_{t \wedge T_m})]$$

$$= \Phi_n(0) + \gamma \mathbb{E} \left[\int_0^{t \wedge T_m} \Phi_n'(Y_s)G_s ds \right]$$

$$+ \mathbb{E} \left[\int_0^{t \wedge T_m} |G_s| \left(\int_{\mathbb{R}_0} \{\Phi_n(Y_s + z) - \Phi_n(Y_s) - \Phi_n'(Y_s)z \mathbf{1}_{(|z|<1)}\} \nu(dz) \right) \mathbf{1}_{(G_s Y_s \ne 0)} ds \right].$$

Proof. This follows from (8), Remarks 2.2, 2.3 and Lemma 2.4. □

3. Computation of the remainder term

In this section, we shall compute the remainder term of the Itô formula via the Fourier analysis. The computation can be found in Engelbert and Kurenok [4] in a more general case. However, we shall compute it a little more simply in our setting for convenience of the reader. Firstly, we set

$$\mathcal{L}f(x) := \int_{\mathbb{R}_0} \{f(x+z) - f(x) - f'(x)z\mathbf{1}_{(|z|<1)}\}\nu(dz),$$

for $f \in C^2(\mathbb{R})$.

REMARK 3.1. For $f \in \mathcal{S}(\mathbb{R})$, the operator $\mathcal{L}$ coincides with the infinitesimal generator of a Cauchy process, see Theorem 3.3.3 (3) in Applebaum [1].

Using the Fourier transform of $f \in L^1(\mathbb{R})$ defined by

$$\mathcal{F}[f](u) := \int_{\mathbb{R}} e^{-iux} f(x)dx \quad \text{for } u \in \mathbb{R},$$

and the inverse Fourier transform defined by

$$\mathcal{F}^{-1}[f](x) := \frac{1}{2\pi} \int_{\mathbb{R}} e^{iux} f(u)du \quad \text{for } x \in \mathbb{R},$$

the operator $\mathcal{L}$ on $\mathcal{S}(\mathbb{R})$ can be represented as follows:

Lemma 3.2. *For each $f \in \mathcal{S}(\mathbb{R})$ and $x \in \mathbb{R}$, it holds that*

$$\mathcal{L}f(x) = -\pi \mathcal{F}^{-1}[|u|\mathcal{F}[f](u)](x).$$

Proof. This follows from the same argument as in the proof of Theorem 3.3.3 (3) in [1]. In fact, by using the identity: $\int_{\mathbb{R}_0} \{e^{iuz} - 1 - iuz\mathbf{1}_{|z|<1}\}\nu(dz) = -\pi|u|$, the required result follows from Fubini's theorem. $\qquad\square$

By using the Fourier transform, we have the following two lemmas:

Lemma 3.3. *For each $x \in \mathbb{R}$, it holds that*

$$\mathcal{L}\Phi_n(x) = C_5(\beta)\mathcal{F}^{-1}\left[|u|^{-\beta}\mathcal{F}[\psi_n](u)\right](x)$$

where

$$C_5(\beta) = -2\pi\Gamma(\beta+1)\cos\left(\frac{\pi(\beta+1)}{2}\right).$$

Proof. It follows from Lemma 2.1 that

$$\int_{|z|\geq 1} |\Phi_n(x+z) - \Phi_n(x)|dz \leq \int_{|z|\geq 1} |z|^\beta \nu(dz) = \frac{2}{1-\beta}.$$

Since Φ_n'' is bounded, it follows from the mean value theorem that

$$\int_{|z|<1} |\Phi_n(x+z) - \Phi_n(x) - \Phi_n'(x)z|\nu(dz) \leq \int_{|z|<1} \int_0^1 |\Phi_n''(x+\theta z)|(1-\theta)z^2 d\theta\nu(dz)$$

$$\leq \frac{C_6(n)}{2} \int_{|z|<1} z^2 \nu(dz) = C_6(n),$$

where $C_6(n) = \sup_{x \in \mathbb{R}} |\Phi_n''(x)|$. Thus, $\mathcal{L}\Phi_n$ is well-defined for $n \in \mathbb{N}$.

Let $0 < \varepsilon < 1$. Set $\Phi_\varepsilon(x) = \Phi(x)e^{-\varepsilon|x|}$ and $\Phi_{\varepsilon,n} = \Phi_\varepsilon * \psi_n$. We then have $\Phi_{\varepsilon,n} \in \mathcal{S}(\mathbb{R})$. Now, we will show that

$$\lim_{\varepsilon \downarrow 0} \mathcal{L}\Phi_{\varepsilon,n}(x) = \mathcal{L}\Phi_n(x).$$

We see that

$$\lim_{\varepsilon \downarrow 0}\{\Phi_{\varepsilon,n}(x+z) - \Phi_{\varepsilon,n}(x) - \Phi'_{\varepsilon,n}(x)z\mathbf{1}_{(|z|<1)}\} = \Phi_n(x+z) - \Phi_n(x) - \Phi'_n(x)z\mathbf{1}_{(|z|<1)}.$$

By Lemma 2.1, we have

$$\Phi_{\varepsilon,n}(x) \leq \Phi_n(x) \leq |x|^\beta + \Phi_n(0) \leq |x|^\beta + 1.$$

Thus, it follows from the inequality (7) that

$$|\Phi_{\varepsilon,n}(x+z) - \Phi_{\varepsilon,n}(x)| \leq |x+z|^\beta + 1 + |x|^\beta + 1 \leq 2|x|^\beta + |z|^\beta + 2,$$

and the right-hand side is integrable with respect to the measure $\nu(dz) = |z|^{-2}dz$ on $(|z| \geq 1)$. By using integration by parts, we have

$$\Phi''_{\varepsilon,n}(x) = \int_{\mathbb{R}} \Phi_\varepsilon(y)\psi_n''(x-y)dy = \int_{\mathbb{R}} \Phi'_\varepsilon(y)\psi_n'(x-y)dy,$$

where the weak derivative Φ'_ε is given by

$$\Phi'_\varepsilon(y) = \beta|y|^{\beta-1}e^{-\varepsilon|y|}\text{sgn}(y) - \varepsilon|y|^\beta e^{-\varepsilon|y|}\text{sgn}(y)$$

for $y \in \mathbb{R}_0$ and $\Phi'_\varepsilon(0) = 0$. By the inequality: $|x|e^{-|x|} \leq e^{-1}$ for $x \in \mathbb{R}$, we have for $y \neq 0$

$$|\Phi'_\varepsilon(y)| \leq \beta|y|^{\beta-1}e^{-\varepsilon|y|} + \varepsilon|y|^\beta e^{-\varepsilon|y|} \leq (\beta + e^{-1})|y|^{\beta-1}.$$

Thus, it follows from the mean value theorem and the inequality (7) that

$$
\begin{aligned}
&|\Phi_{\varepsilon,n}(x+z) - \Phi_{\varepsilon,n}(x) - \Phi'_{\varepsilon,n}(x)z| \\
&= \left|\int_0^1 \Phi''_{\varepsilon,n}(x+\theta z)(1-\theta)z^2 d\theta\right| \\
&\leq \int_0^1 \left(\int_{\mathbb{R}} |\Phi'_\varepsilon(y)\psi_n'(x+\theta z - y)|dy\right)(1-\theta)z^2 d\theta \\
&\leq (\beta + e^{-1})z^2 \int_0^1 \left(\int_{\mathbb{R}} |y|^{\beta-1}|\psi_n'(x+\theta z - y)|dy\right)(1-\theta)d\theta \\
&\leq \frac{C_7(n)(\beta e + 1)}{2e}z^2 \int_{x-|z|-\frac{1}{n}}^{x+|z|+\frac{1}{n}} |y|^{\beta-1}dy \\
&\leq \frac{C_7(n)(\beta e + 1)}{\beta e}z^2 \left(|x|^\beta + |z|^\beta + \left|\frac{1}{n}\right|^\beta\right),
\end{aligned}
$$

where $C_7(n) = \sup_{x \in \mathbb{R}} |\psi_n'(x)|$, since we have $\text{Supp}\,\psi_n' \subset [-n^{-1}, n^{-1}]$. Moreover, the right-hand side of the above inequality is integrable with respect to the measure $\nu(dz) = |z|^{-2}dz$ on $(|z| < 1)$. Hence, it follows from the dominated convergence theorem that

$$\lim_{\varepsilon\downarrow 0}\mathcal{L}\Phi_{\varepsilon,n}(x) = \lim_{\varepsilon\downarrow 0}\int_{\mathbb{R}_0}\{\Phi_{\varepsilon,n}(x+z) - \Phi_{\varepsilon,n}(x) - \Phi'_{\varepsilon,n}(x)z\mathbf{1}_{(|z|<1)}\}\nu(dz)$$

$$= \int_{\mathbb{R}_0}\{\Phi_n(x+z) - \Phi_n(x) - \Phi'_n(x)z\mathbf{1}_{(|z|<1)}\}\nu(dz)$$

$$= \mathcal{L}\Phi_n(x).$$

By Lemma 3.2, we have

$$\mathcal{L}\Phi_{\varepsilon,n}(x) = -\pi\mathcal{F}^{-1}\left[|u|\mathcal{F}[\Phi_{\varepsilon,n}](u)\right](x) = -\pi\mathcal{F}^{-1}\left[|u|\mathcal{F}[\Phi_{\varepsilon}](u)\mathcal{F}[\psi_n](u)\right](x),$$

since $\Phi_{\varepsilon,n} \in \mathcal{S}(\mathbb{R})$. Thus, it is sufficient to show that

$$\lim_{\varepsilon\downarrow 0}\left\{-\pi\mathcal{F}^{-1}\left[|u|\mathcal{F}[\Phi_{\varepsilon}](u)\mathcal{F}[\psi_n](u)\right](x)\right\} = C_5(\beta)\mathcal{F}^{-1}\left[|u|^{-\beta}\mathcal{F}[\psi_n](u)\right](x).$$

By using the identity:

$$(11) \qquad \int_0^\infty x^{\xi-1}e^{-wx}dx = \Gamma(\xi)w^{-\xi}$$

for each $\xi > 0$ and $\mathrm{Re}(w) > 0$, we have

$$\mathcal{F}[\Phi_{\varepsilon}](u) = \int_{\mathbb{R}}|x|^\beta e^{-\varepsilon|x|-iux}dx$$

$$= \int_0^\infty x^\beta e^{-(\varepsilon+iu)x}dx + \int_0^\infty x^\beta e^{-(\varepsilon-iu)x}dx$$

$$= \Gamma(\beta+1)\left\{(\varepsilon+iu)^{-\beta-1} + (\varepsilon-iu)^{-\beta-1}\right\},$$

since $\beta + 1 > 0$ and $\mathrm{Re}(\varepsilon \pm iu) = \varepsilon > 0$. We then have for $u \neq 0$,

$$\lim_{\varepsilon\downarrow 0}|u|\mathcal{F}[\Phi_{\varepsilon}](u) = 2\Gamma(\beta+1)|u|^{-\beta}\cos\left(\frac{\pi(\beta+1)}{2}\right).$$

Moreover, we have

$$\||u|\mathcal{F}[\Phi_{\varepsilon}](u)\mathcal{F}[\psi_n](u)| \leq 2\Gamma(\beta+1)|u|^{-\beta}\,|\mathcal{F}[\psi_n](u)|,$$

and the right-hand side is integrable on $\mathbb{R}$, since $-\beta > -1$ and $\mathcal{F}[\psi_n] \in \mathcal{S}(\mathbb{R})$. Hence, it follows from the dominated convergence theorem that

$$\lim_{\varepsilon\downarrow 0}\left\{-\pi\mathcal{F}^{-1}\left[|u|\mathcal{F}[\Phi_{\varepsilon}](u)\mathcal{F}[\psi_n](u)\right](x)\right\}$$

$$= -2\pi\Gamma(\beta+1)\cos\left(\frac{\pi(\beta+1)}{2}\right)\mathcal{F}^{-1}\left[|u|^{-\beta}\mathcal{F}[\psi_n](u)\right](x),$$

and the required result follows. $\qquad\square$

Lemma 3.4. *For each $x \in \mathbb{R}$, it holds that*

$$\mathcal{F}^{-1}\left[|u|^{-\beta}\mathcal{F}[\psi_n](u)\right](x) = C_8(\beta)\int_{\mathbb{R}}|x-y|^{\beta-1}\psi_n(y)dy$$

where

$$C_8(\beta) = \frac{\Gamma(1-\beta)}{\pi}\cos\left(\frac{\pi(\beta-1)}{2}\right).$$

Proof. Let $\varepsilon > 0$. We have

$$\left| |u|^{-\beta} e^{-\varepsilon|u|} \mathcal{F}[\psi_n](u) \right| \leq |u|^{-\beta} \left| \mathcal{F}[\psi_n](u) \right|,$$

and the right-hand side is integrable on $\mathbb{R}$ since $-\beta > -1$ and $\mathcal{F}[\psi_n] \in \mathcal{S}(\mathbb{R})$. Hence, it follows from the dominated convergence theorem that

$$\lim_{\varepsilon \downarrow 0} \mathcal{F}^{-1} \left[|u|^{-\beta} e^{-\varepsilon|u|} \mathcal{F}[\psi_n](u) \right](x) = \mathcal{F}^{-1} \left[|u|^{-\beta} \mathcal{F}[\psi_n](u) \right](x).$$

By Fubini's theorem, we have

$$\mathcal{F}^{-1} \left[|u|^{-\beta} e^{-\varepsilon|u|} \mathcal{F}[\psi_n](u) \right](x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{iux} |u|^{-\beta} e^{-\varepsilon|u|} \left(\int_{\mathbb{R}} \psi_n(y) e^{-iuy} dy \right) du$$

$$= \frac{1}{2\pi} \int_{\mathbb{R}} \left(\int_{\mathbb{R}} e^{-iu(y-x)} |u|^{-\beta} e^{-\varepsilon|u|} du \right) \psi_n(y) dy$$

$$= \frac{1}{2\pi} \int_{\mathbb{R}} \mathcal{F} \left[|u|^{-\beta} e^{-\varepsilon|u|} \right](y - x) \psi_n(y) dy$$

$$= \frac{1}{2\pi} \int_{\mathbb{R}} \mathcal{F} \left[|u|^{-\beta} e^{-\varepsilon|u|} \right](y) \psi_n(x + y) dy.$$

By the identity (11), we have

$$\mathcal{F} \left[|u|^{-\beta} e^{-\varepsilon|u|} \right](y) = \Gamma(1 - \beta) \left\{ (\varepsilon + iy)^{\beta-1} + (\varepsilon - iy)^{\beta-1} \right\},$$

since $1 - \beta > 0$. We then have for $y \neq 0$,

$$\lim_{\varepsilon \downarrow 0} \mathcal{F} \left[|u|^{-\beta} e^{-\varepsilon|u|} \right](y) = 2\Gamma(1 - \beta) |y|^{\beta-1} \cos \left(\frac{\pi(\beta - 1)}{2} \right).$$

Moreover, we have

$$\left| \mathcal{F}[|u|^{-\beta} e^{-\varepsilon|u|}](y) \psi_n(x + y) \right| \leq 2\Gamma(1 - \beta) |y|^{\beta-1} \psi_n(x + y),$$

and the right-hand side is integrable with respect to the Lebesgue measure dy on $\mathbb{R}$, since $\beta - 1 > -1$ and $\psi_n \in C_c^\infty(\mathbb{R})$. Hence, it follows from the dominated convergence theorem that

$$\lim_{\varepsilon \downarrow 0} \frac{1}{2\pi} \int_{\mathbb{R}} \mathcal{F} \left[|u|^{-\beta} e^{-\varepsilon|u|} \right](y) \psi_n(x + y) dy = \frac{\Gamma(1 - \beta)}{\pi} \cos \left(\frac{\pi(\beta - 1)}{2} \right) \int_{\mathbb{R}} |y|^{\beta-1} \psi_n(x + y) dy$$

$$= \frac{\Gamma(1 - \beta)}{\pi} \cos \left(\frac{\pi(\beta - 1)}{2} \right) \int_{\mathbb{R}} |x - y|^{\beta-1} \psi_n(y) dy,$$

and the required result follows. $\qquad\square$

Hence, we have the following:

Proposition 3.5. *For each $x \in \mathbb{R}$, it holds that*

$$\mathcal{L}\Phi_n(x) = C_9(\beta) \int_{\mathbb{R}} |x - y|^{\beta-1} \psi_n(y) dy$$

where

$$C_9(\beta) = \pi\beta \tan \left(\frac{\pi\beta}{2} \right).$$

Proof. By using Euler's reflection formula: $\Gamma(\xi)\Gamma(1 - \xi) = \pi/\sin(\pi\xi)$ for $0 < \xi < 1$, we have

$$C_5(\beta)C_8(\beta) = 2\beta\Gamma(\beta)\Gamma(1 - \beta)\sin^2\left(\frac{\pi\beta}{2}\right) = \frac{2\pi\beta}{\sin(\pi\beta)}\sin^2\left(\frac{\pi\beta}{2}\right) = \pi\beta\tan\left(\frac{\pi\beta}{2}\right),$$

and the required result follows from Lemmas 3.3 and 3.4. $\qquad\square$

4. Pathwise uniqueness

In this section, we shall study the pathwise uniqueness of the solution to the stochastic differential equation (1). To prove our main result, we shall take the expectations of the Itô formula for $|Y_t|^\beta$.

Firstly, by Proposition 3.5, we can rewrite Proposition 2.5 as follows:

Proposition 4.1. *For each $m, n \in \mathbb{N}$ and $t \geq 0$, it holds that*

$$(12) \qquad \mathbb{E}[\Phi_n(Y_{t\wedge T_m})] = \Phi_n(0) + \gamma\mathbb{E}\left[\int_0^{t\wedge T_m} \Phi_n'(Y_s)G_s ds\right]$$

$$+ C_9(\beta)\mathbb{E}\left[\int_0^{t\wedge T_m} |G_s|\left(\int_{\mathbb{R}} |Y_s - y|^{\beta-1}\psi_n(y)dy\right)ds\right],$$

where $C_9(\beta)$ is the same constant as in Proposition 3.5.

Proof. This follows from Propositions 2.5 and 3.5. $\qquad\square$

Now we shall take the limits of (12) as $n \to \infty$. Since $\Phi_n(Y_{t\wedge T_m}) \to |Y_{t\wedge T_m}|^\beta$ as $n \to \infty$ and by Lemma 2.1

$$\Phi_n(Y_{t\wedge T_m}) \leq |Y_{t\wedge T_m}|^\beta + \Phi_n(0) \leq |Y_{t\wedge T_m}|^\beta + 1 \leq (2m)^\beta + 1,$$

it follows from the dominated convergence theorem that

$$\lim_{n\to\infty} \mathbb{E}[\Phi_n(Y_{t\wedge T_m})] = \mathbb{E}[|Y_{t\wedge T_m}|^\beta].$$

Before we take the limit of the right-hand side, we establish the following inequality:

Lemma 4.2. *For each $n \in \mathbb{N}$ and $x \in \mathbb{R}_0$, it holds that*

$$\int_{\mathbb{R}} |x - y|^{\beta-1}\psi_n(y)dy \leq C_{10}(\beta)|x|^{\beta-1},$$

where the positive constant $C_{10}(\beta)$ does not depend on n.

Proof. Firstly, we have for $|x| > 2/n$

$$\int_{\mathbb{R}} |x - y|^{\beta-1}\psi_n(y)dy = \int_{|y|<\frac{1}{n}} |x - y|^{\beta-1}\psi_n(y)dy \leq \left(|x| - \frac{1}{n}\right)^{\beta-1} \leq \left|\frac{x}{2}\right|^{\beta-1}.$$

Next, we have for $0 < |x| \leq 2/n$

$$\int_{\mathbb{R}} |x - y|^{\beta-1}\psi_n(y)dy = \int_{\mathbb{R}} |y|^{\beta-1}\psi_n(x - y)dy$$

$$= n\int_{\mathbb{R}} |y|^{\beta-1}\psi(nx - ny)dy$$

$$= \int_{\mathbb{R}} \left| \frac{y}{n} \right|^{\beta-1} \psi(nx - y)dy$$

$$\leq C_{11}n^{1-\beta} \int_{nx-1}^{nx+1} |y|^{\beta-1}dy$$

$$\leq C_{11}2^{1-\beta}|x|^{\beta-1} \int_{|y|\leq 3} |y|^{\beta-1}dy = \frac{C_{11}2^{2-\beta}3^{\beta}}{\beta}|x|^{\beta-1},$$

where $C_{11} = \sup_{x\in\mathbb{R}} \psi(x)$. Note that $C_{11} \geq 1/2$ since ψ satisfies that $\text{Supp}\,\psi \subset [-1, 1]$ and $\int_{\mathbb{R}} \psi(y)dy = 1$.

Hence, we have for each $x \neq 0$

$$\int_{\mathbb{R}} |x - y|^{\beta-1}\psi_n(y)dy \leq \left(2^{1-\beta} \vee \frac{C_{11}2^{2-\beta}3^{\beta}}{\beta} \right)|x|^{\beta-1} = \frac{C_{11}2^{2-\beta}3^{\beta}}{\beta}|x|^{\beta-1}.$$

The proof is complete.   □

Now, we will take the limit of the right-hand side of (12) as $n \to \infty$.

Lemma 4.3. *Under the condition* (3)*, it holds that*

$$\lim_{n\to\infty} \mathbb{E}\left[\int_0^{t\wedge T_m} \Phi'_n(Y_s)G_s ds \right] = \beta\mathbb{E}\left[\int_0^{t\wedge T_m} G_s|Y_s|^{\beta-1}\text{sgn}(Y_s)\mathbf{1}_{(G_sY_s\neq 0)}ds \right].$$

Proof. It follows that $G_s = 0$ if $Y_s = 0$. We then have $\Phi'_n(Y_s)G_s = 0$ if $G_sY_s = 0$. Now we consider the integrand on $(G_sY_s \neq 0)$. By using integration by parts, we have

$$\Phi'_n(x) = \beta \int_{\mathbb{R}} |x - y|^{\beta-1}\text{sgn}(x - y)\psi_n(y)dy.$$

Since $\lim_{n\to\infty} \Phi'_n(x) = \beta|x|^{\beta-1}\text{sgn}(x)$ for $x \neq 0$, we have

$$\lim_{n\to\infty} \Phi'_n(Y_s)G_s = \beta G_s|Y_s|^{\beta-1}\text{sgn}(Y_s).$$

By Lemma 4.2 and the condition (3), we have

$$|\Phi'_n(Y_s)G_s| \leq C_{10}(\beta)\beta|G_s||Y_s|^{\beta-1} \leq C_1(m)C_{10}(\beta)\beta.$$

Hence, the required result follows from the dominated convergence theorem.   □

Lemma 4.4. *Under the condition* (3)*, it holds that*

$$\lim_{n\to\infty} \mathbb{E}\left[\int_0^{t\wedge T_m} |G_s|\left(\int_{\mathbb{R}} |Y_s - y|^{\beta-1}\psi_n(y)dy \right)ds \right] = \mathbb{E}\left[\int_0^{t\wedge T_m} |G_s||Y_s|^{\beta-1}\mathbf{1}_{(G_sY_s\neq 0)}ds \right].$$

Proof. It follows that $G_s = 0$ if $Y_s = 0$. We then have

$$|G_s|\left(\int_{\mathbb{R}} |Y_s - y|^{\beta-1}\psi_n(y)dy \right) = 0,$$

if $G_sY_s = 0$. Now we consider the integrand on $(G_sY_s \neq 0)$. We have

$$\lim_{n\to\infty} |G_s|\left(\int_{\mathbb{R}} |Y_s - y|^{\beta-1}\psi_n(y)dy \right) = |G_s||Y_s|^{\beta-1}.$$

By Lemma 4.2 and the condition (3), we have

$$\left| G_s \right| \left| \int_{\mathbb{R}} |Y_s - y|^{\beta-1} \psi_n(y) dy \right| \leq C_{10}(\beta)|G_s||Y_s|^{\beta-1} \leq C_1(m)C_{10}(\beta).$$

Hence, the required result follows from the dominated convergence theorem. $\qquad\square$

We then have the following:

Proposition 4.5. *Under the condition* (3)*, it holds that*

$$\mathbb{E}[|Y_{t \wedge T_m}|^\beta] = \beta\gamma\mathbb{E}\left[\int_0^{t \wedge T_m} G_s |Y_s|^{\beta-1} \mathrm{sgn}(Y_s) \mathbf{1}_{(G_s Y_s \neq 0)} ds \right]$$
$$+ C_9(\beta)\mathbb{E}\left[\int_0^{t \wedge T_m} |G_s||Y_s|^{\beta-1} \mathbf{1}_{(G_s Y_s \neq 0)} ds \right],$$

where $C_9(\beta)$ *is the same constant as in* Proposition 3.5.

Proof. This follows from Proposition 4.1, and Lemmas 4.3 and 4.4 $\qquad\square$

Now, we shall prove our main result.
Proof of Theorem 1.1. Let $\gamma > 0$ and set

$$\beta = \frac{2}{\pi} \arctan\left(\frac{\gamma}{\pi}\right).$$

Then, by $0 < \beta < 1$, the constant $C_9(\beta)$ in Proposition 3.5 is given by $C_9(\beta) = \beta\gamma$. By Proposition 4.5 and the condition (4), we have

$$\mathbb{E}[|Y_{t \wedge T_m}|^\beta] = 2\beta\gamma\mathbb{E}\left[\int_0^{t \wedge T_m} |G_s||Y_s|^{\beta-1} \mathbf{1}_{(G_s Y_s > 0)} ds \right] \leq 2C_2(m)\beta\gamma\mathbb{E}\left[\int_0^{t \wedge T_m} |Y_s|^\beta ds \right].$$

Hence, by Gronwall's inequality, we have $\mathbb{E}[|Y_{t \wedge T_m}|^\beta] = 0$. Since $T_m \uparrow \infty$ as $m \to \infty$, it follows from the monotone convergence theorem that

$$0 = \lim_{m \to \infty} \mathbb{E}[|Y_{t \wedge T_m}|^\beta] = \mathbb{E}[|Y_t|^\beta],$$

and the required result follows. $\qquad\square$

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