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ON MOD 2 ARITHMETIC DIJKGRAAF–WITTEN INVARIANTS FOR CERTAIN REAL QUADRATIC NUMBER FIELDS

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Abstract

Minhyong Kim introduced arithmetic Chern–Simons invariants for totally imaginary number fields as arithmetic analogues of the Chern–Simons invariants for 3-manifolds. In this paper, we extend Kim's definition to any number field, by using the modified étale cohomology groups and fundamental groups which take real primes into account. We then show explicit formulas of mod 2 arithmetic Dijkgraaf–Witten invariants for real quadratic fields $\mathbb{Q}(\sqrt{p_1 p_2 \cdots p_r})$, where p_i's are distinct prime numbers congruent to 1 mod 4, in terms of the Legendre symbols of p_i's. We also show topological analogues of our formulas for 3-manifolds.

1. Introduction

In recent years, Minhyong Kim ([9], [6]) initiated the study of arithmetic Chern–Simons theory for number fields as an arithmetic analogue of the Dijkgraaf–Witten theory for 3-manifolds [7], based on the analogies between number rings and 3-manifolds, and primes and knots in arithmetic topology [13]. We note that Dijkgraaf–Witten theory may be regarded as a 3-dimensional Chern–Simons gauge theory with finite gauge groups. Kim's theory is concerned with totally imaginary number fields, since it employs some results on étale cohomology groups of the ring of integers of totally imaginary number fields, which no longer hold for number fields with real primes. Therefore, it is desirable to extend Kim's theory for number fields with real primes.

In this paper, we extend Kim's theory for number fields with real primes, by using the modified étale cohomology groups and the modified étale fundamental groups which take real primes into account. We then explicitly calculate the mod 2 arithmetic Dijkgraaf–Witten invariants for real quadratic fields $\mathbb{Q}(\sqrt{p_1 p_2 \cdots p_r})$, where p_i's are distinct prime numbers congruent to 1 mod 4, in terms of the Legendre symbols of p_i's.

Let us outline the construction of arithmetic Chern–Simons invariants and arithmetic Dijkgraaf–Witten invariants in the following. Let n be a positive integer and let K be a finite algebraic number field containing n-th roots of unity. Note that if K has a real prime, then n must be 2. We choose a primitive n-th root of unity ζ_n in K, which induces an isomorphism $\mathbb{Z}/n\mathbb{Z} \cong \mu_n$. Let $\mathcal{O}_K$ denote the ring of integers of K and let $X = \operatorname{Spec} \mathcal{O}_K$ denote the prime spectrum of $\mathcal{O}_K$. Let X_∞ denote the set of infinite primes of K and set $\overline{X} = X \sqcup X_\infty$. Following [3] and [1], we may introduce a Grothendieck topology (site) $\overline{X}_{\text{ét}}$ called the Artin–Verdier site, the topos $\operatorname{Sh}(\overline{X}_{\text{ét}})$ of abelian sheaves on $\overline{X}_{\text{ét}}$, and the modified étale cohomology groups $\mathrm{H}^i(\overline{X}, F)$ for $F \in \operatorname{Sh}(\overline{X}_{\text{ét}})$ and $i \geq 0$. These cohomology groups admit the 3-dimensional

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Artin–Verdier duality and there is an isomorphism $\mathrm{H}^3(\overline{X}, \mathbb{Z}/n\mathbb{Z}) \cong \mathbb{Z}/n\mathbb{Z}$ determined by the choice of ζ_n. In addition, we have the category $\mathrm{FEt}_{\overline{X}}$ of finite étale coverings over $\overline{X}$, which is proven to be a Galois category. We define the modified étale fundamental group $\pi_1(\overline{X})$ as the fundamental group of $\mathrm{FEt}_{\overline{X}}$.

Now, let A be a finite group and let $c \in \mathrm{H}^3(A, \mathbb{Z}/n\mathbb{Z})$. Let $\mathcal{M}(\overline{X}, A) = \mathrm{Hom}_c(\pi_1(\overline{X}), A)/A$ denote the set of conjugacy classes of all continuous homomorphisms $\pi_1(\overline{X}) \to A$. Then, for each $\rho \in \mathcal{M}(\overline{X}, A)$, the *arithmetic Chern–Simons invariant* $CS_c(\rho)$ of ρ associated to c is defined as the image of c under the composition

$$\mathrm{H}^3(A, \mathbb{Z}/n\mathbb{Z}) \xrightarrow{\rho^*} \mathrm{H}^3(\pi_1(\overline{X}), \mathbb{Z}/n\mathbb{Z}) \xrightarrow{j_3} \mathrm{H}^3(\overline{X}, \mathbb{Z}/n\mathbb{Z}) \cong \mathbb{Z}/n\mathbb{Z}$$

of maps, where j_3 is the edge homomorphism in the modified Hochschild–Serre spectral sequence $\mathrm{H}^p(\pi_1(\overline{X}), \mathrm{H}^q(\widetilde{\overline{X}}, \mathbb{Z}/n\mathbb{Z})) \Rightarrow \mathrm{H}^{p+q}(\overline{X}, \mathbb{Z}/n\mathbb{Z})$ (see Section 2 for $\widetilde{\overline{X}}$). The *arithmetic Dijkgraaf–Witten invariant* of $\overline{X}$ associated to c is then defined by

$$Z_c(\overline{X}) = \sum_{\rho \in \mathcal{M}(\overline{X}, A)} \exp\left(\frac{2\pi i}{n} CS_c(\rho)\right).$$

A basic problem is to concretely calculate $CS_c(\rho)$ and $Z_c(\overline{X})$. The papers [6], [5], and [4] are concerned with this problem for the cases where K is totally imaginary and c is some specific cocycle. In this paper, we consider the case where K is the real quadratic field $\mathbb{Q}(\sqrt{p_1 p_2 \cdots p_r})$, each p_i being a prime number congruent to 1 mod 4, $A = \mathbb{Z}/2\mathbb{Z}$, and c is the non-trivial cocycle in $\mathrm{H}^3(A, \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$. For this purpose, we generalize results in [1] and [4], which describe $CS_c(\rho)$ by the Artin symbol in unramified class field theory, for any number fields. Afterward, by using Gauss's genus theory, we explicitly calculate $CS_c(\rho)$ and $Z_c(\overline{X})$, in terms of the Legendre symbols among p_i's.

We remark that for 3-manifolds, the abelian Chern–Simons partition functions and the Dijkgraaf–Witten invariants are given by Gaussian integrals and Gaussian sums (cf. [10, Chapter 3], [15]). Since our formula (Theorem 4.2.2) for number fields is given in a form similar to Gaussian sums, we may expect that the cases with non-abelian gauge groups would be given by a non-abelian generalization of Gaussian sums.

Following the analogies in arithmetic topology, in Section 5, we show a topological counterpart of our main result in the context of Dijkgraaf–Witten theory for 3-manifolds.

Here are the contents of this paper. In Subsection 2.1, notations being as above, we introduce the Artin–Verdier site $\overline{X}_{\text{ét}}$ and the category $\mathrm{FEt}_{\overline{X}}$ of finite étale coverings over $\overline{X}$. We show that $\mathrm{FEt}_{\overline{X}}$ is a Galois category and define the modified étale fundamental group $\pi_1(\overline{X})$ as the automorphism group of the fiber functor of $\mathrm{FEt}_{\overline{X}}$. In Subsection 2.2, we introduce the topos $\mathrm{Sh}(\overline{X}_{\text{ét}})$ of abelian sheaves on $\overline{X}_{\text{ét}}$ and define the modified étale cohomology groups $\mathrm{H}^i(\overline{X}, F)$ for $F \in \mathrm{Sh}(\overline{X}_{\text{ét}})$ and $i \geq 0$. We also show the modified Hochschild–Serre spectral sequence. In Section 3, by using the materials prepared in Subsection 2.1 and 2.2, we extend Kim's definition for all number fields. In Section 4, we firstly extend a result in [1] for $\overline{X}$. Then, we explicitly calculate mod 2 $CS_c(\rho)$ and $Z_c(\overline{X})$ for $K = \mathbb{Q}(\sqrt{p_1 p_2 \cdots p_r})$ with $p_i \equiv 1$ mod 4. In Section 5, as an appendix, we rearrange the theory of Dijkgraaf–Witten invariants for 3-manifolds and show the topological counterparts of our main theorems. This final section aims to clarify the analogy and may be read independently.

The results in this paper were announced by the author at the workshop "Low dimensional topology and number theory XI" held in Osaka University in March 2019. During the preparation of this paper after that, we found the paper [11] which also studies the arithmetic Chern–Simons theory for number fields with real primes. They use compactly supported étale cohomology groups.

Notation. As usual, we denote by $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ the ring of rational integers, the field of rational numbers, the field of real numbers, and the field of complex numbers, respectively. For a commutive ring R, we denote by $R^\times$ the group of units in R. For a number field K, we denote by $\mathcal{O}_K$ the ring of integers of K. We denote by I_K the group of fractional ideals of K, and by $\mathrm{N}\mathfrak{a} \in \mathbb{Z}$ the norm of $\mathfrak{a} \in I_K$. We denote by Cl_K, (resp. Cl_K^+) the ideal class group (resp. the narrow ideal class group) of K.

2. The modified étale cohomology groups for a number ring

In Subsection 2.1, we recall the Artin–Verdier site for a number field. We then define the modified étale fundamental group of the ring of integers, taking the infinite primes into account. In Subsection 2.2, we define the modified étale cohomology groups and show the Hochschild–Serre spectral sequence. We follow the argument of [1] and [3] throughout this section.

2.1. The Artin–Verdier site and the modified étale fundamental group. Let K be a finite algebraic number field and let $X = \mathrm{Spec}\,\mathcal{O}_K$ be the prime spectrum of the ring $\mathcal{O}_K$ of integers of K. Let X_∞ denote the set of infinite primes, namely, real primes and pairs of conjugate complex primes of K, and we set $\overline{X} = X \sqcup X_\infty$. Let Y be a scheme which is étale over X. A real prime of Y is defined by a point $y : \mathrm{Spec}\,\mathbb{C} \to Y$ which factors through $\mathrm{Spec}\,\mathbb{R}$. A complex prime of Y is defined to be a pair of complex conjugate points $y, \overline{y} : \mathrm{Spec}\,\mathbb{C} \to Y$ such that $y \neq \overline{y}$. An infinite prime of Y is a real prime or a complex prime of Y. Let Y_∞ denote the set of infinite primes of Y. Note that an étale morphism $f : Y \to X$ induces $f_\infty : Y_\infty \to X_\infty$. We say that f_∞ is unramified at $y_\infty \in Y_\infty$ if y_∞ is a real prime or if $(y_\infty, f_\infty(y_\infty))$ is a complex prime. Regarding Grothendieck topologies, we refer to [2] and [19].

Definition 2.1.1 ([1, Definition 2.1], [3, Proposition 1.2]). The *Artin–Verdier site* $\overline{X}_{\text{ét}}$ of $\overline{X}$ is the Grothendieck topology consisting of the category $\mathrm{Cat}(\overline{X}_{\text{ét}})$ and a set $\mathrm{Cov}(\overline{X}_{\text{ét}})$ of coverings defined as follows.

- An object in $\mathrm{Cat}(\overline{X}_{\text{ét}})$ is a pair (Y, M), where $f : Y \to X$ is a scheme étale over X and $M \subset Y_\infty$ such that $f_\infty| : M \to X_\infty$ is unramified. A morphism $\varphi : (Y_1, M_1) \to (Y_2, M_2)$ in $\mathrm{Cat}(\overline{X}_{\text{ét}})$ is a morphism of shemes $\varphi : Y_1 \to Y_2$ over X such that the induced map $\varphi_\infty : (Y_1)_\infty \to (Y_2)_\infty$ satisfies $\varphi_\infty(M_1) \subset M_2$.
- A covering in $\mathrm{Cov}(\overline{X}_{\text{ét}})$ is a family of morphisms $\{\varphi_i : (Y_i, M_i) \to (Z, N)\}_{i \in I}$ in $\mathrm{Cat}(\overline{X}_{\text{ét}})$ which satisfies $\bigcup_i \varphi_i(Y_i) = Z$ and $\bigcup_i \varphi_i(M_i) = N$.

Remark 2.1.2. In $\mathrm{Cat}(\overline{X}_{\text{ét}})$, the fiber product of morphisms $\varphi_i : (Y_i, M_i) \to (Z, N)$ $(i = 1, 2)$ is defined by $(Y_1 \times_Z Y_2, M_3)$, where $Y_1 \times_Z Y_2$ is the fiber product in the category of schemes and M_3 is the set consisting of points of $(Y_1 \times_Z Y_2)_\infty$ whose images are in M_i under the projections $(Y_1 \underset{Z}{\times} Y_2)_\infty \to Y_{i\infty}$ for $i = 1, 2$. We can check easily that M_3 is isomorphic to $M_1 \underset{N}{\times} M_2$ in the category of sets.

Next, we introduce a Galois category to define the modified étale fundamental group.

We say that $(Y, M) \in \mathrm{Cat}(\overline{X}_{\text{ét}})$ is *finite étale* if $Y \to X$ is a finite étale morphism of schemes over X and $M = Y_\infty$. We denote by $\mathrm{FEt}_{\overline{X}}$ the full subcategory of $\overline{X}_{\text{ét}}$ whose objects are finite étale, and denote by FSets the category of finite sets.

In the following, we often abbreviate (Y, Y_∞) to $\overline{Y}$ for a scheme Y étale over X. Let $\overline{K}$ be an algebraic closure of K and let $\tilde{\eta} : \mathrm{Spec}\,\overline{K} \to X$ be a geometric point. Then we have functors

$$F_{\tilde{\eta}} : \mathrm{FEt}_{\overline{X}} \to \mathrm{FSets};\ \overline{Y} \mapsto \mathrm{Hom}_X(\tilde{\eta}, Y),$$

$$U : \mathrm{FEt}_{\overline{X}} \to \mathrm{FEt}_X;\ \overline{Y} \mapsto Y.$$

We note that the forgetful functor U is fully faithful.

Definition 2.1.3 ([17, V.4]). Let $\mathscr{C}$ be a category and let $F : \mathscr{C} \to \mathrm{FSets}$ be a covariant functor. $\mathscr{C}$ is called *a Galois category* with *a fiber functor F* if $\mathscr{C}$ and F satisfy the following axioms.

(G1) $\mathscr{C}$ has a final object and finite fiber products.

(G2) Finite direct sums exist in $\mathscr{C}$. The quotient of an object by a finite group of automorphisms exist in $\mathscr{C}$.

(G3) Let $u : A_1 \to A_2$ be a morphism in $\mathscr{C}$. Then u factors into a composition $A_1 \xrightarrow{f} A' \xrightarrow{g} A_2$, where f is a strict epimorphism and g is a monomorphism which is an isomorphism on a direct summand of A_2.

(G4) F is a left exact functor.

(G5) F commutes with finite direct sums and the quotient of an object by a finite group of automorphisms. F sends strict epimorphisms to epimorphisms.

(G6) Let $u : A_1 \to A_2$ be a morphism in $\mathscr{C}$ such that $F(u)$ is an isomorphism. Then u is an isomorphism.

Proposition 2.1.4. $\mathrm{FEt}_{\overline{X}}$ *is a Galois category with a fiber functor* $F_{\tilde{\eta}}$.

Proof. We check the six axioms (G1)$\sim$(G6) of Galois categories for $\mathrm{FEt}_{\overline{X}}$ and $F_{\tilde{\eta}}$. It is well-known that the category FEt_X of schemes finite étale over X is a Galois category with a fiber functor $F'_{\tilde{\eta}} : \mathrm{FEt}_X \to \mathrm{FSets}\ Y \mapsto \mathrm{Hom}_X(\tilde{\eta}, Y)$ [17, V.7], so that FEt_X and $F'_{\tilde{\eta}}$ admit the axioms (G1)$\sim$(G6) and $F'_{\tilde{\eta}}$. Let us verify (G1)$\sim$(G6) for $\mathrm{FEt}_{\overline{X}}$.

(G1) $\mathrm{FEt}_{\overline{X}}$ has a final object $(id : X \to X, X_\infty)$. For $\overline{Y_i} \in \mathrm{FEt}_{\overline{X}}$ $(i = 1, 2, \cdots, m)$, we see that $\prod_i Y_i \in \mathrm{FEt}_X$. So we have $\prod_i \overline{Y_i} = \overline{\prod_i Y_i}$ by the universal property of fiber products.

(G2) $\mathrm{FEt}_{\overline{X}}$ has an initial object $(\mathrm{Spec}\,0, (\mathrm{Spec}\,0)_\infty) = (\emptyset, \emptyset)$. In a similar way to (G1), we see that $\mathrm{FEt}_{\overline{X}}$ admits finite direct sums. For $\overline{Y} \in \mathrm{FEt}_{\overline{X}}$ and a finite subgroup $G \subset \mathrm{Aut}_{\overline{X}}(\overline{Y})$, we have $\mathrm{Aut}_{\overline{X}}(\overline{Y}) = \mathrm{Aut}_X(Y)$ by the definition of morphisms of $\mathrm{Cat}(\overline{X}_{\text{ét}})$. So we have the quotient of $Y \to X \in \mathrm{FEt}_X$ by $G \subset \mathrm{Aut}_X(Y)$ and then one can check $\overline{Y}/G = \overline{Y/G}$.

(G3) For any morphism $\overline{Y_1} \to \overline{Y_2}$ in $\mathrm{FEt}_{\overline{X}}$, $Y_1 \to Y_2$ factors as

$Y_1 \xrightarrow{f} Y' \xrightarrow{g} Y' \bigsqcup Y'' \cong Y_2$ in FEt_X, where f is a strict epimorphism and g is a monomorphism. This sequence induces $\overline{Y_1} \xrightarrow{f} \overline{Y'} \xrightarrow{g} \overline{Y'} \bigsqcup \overline{Y''} \cong \overline{Y_2}$.
(G4) and (G5) are obvious since U is fully faithful and $U \circ F'_{\tilde{\eta}} = F_{\tilde{\eta}}$.
(G6) Let $u : \overline{Y_1} \to \overline{Y_2}$ be a morphism in $\mathrm{FEt}_{\overline{X}}$. If $F_{\tilde{\eta}}(u) : F_{\tilde{\eta}}(\overline{Y_1}) \to F_{\tilde{\eta}}(\overline{Y_2})$ is an isomorphism, then $U(u) : Y_1 \to Y_2$ is an isomorphism. Since the forgetful functor U is fully faithful, u is an isomorphism. □

Now we define the modified étale fundamental group.

Definition 2.1.5. The *modified étale fundamental group* $\pi_1(\overline{X}) = \pi_1(\overline{X}, \tilde{\eta})$ with geometric basepoint $\tilde{\eta}$ is defined by the fundamental group of the Galois category $\mathrm{FEt}_{\overline{X}}$ associated to the fiber functor $F_{\tilde{\eta}}$, namely, the group of automorphisms of $F_{\tilde{\eta}}$.

The fundamental theorem of Galois categories is stated as follows.

Theorerm 2.1.6. *There is an equivalence of categories between* $\mathrm{FEt}_{\overline{X}}$ *and the category of finite discrete sets equipped with continuous left actions of* $\pi_1(\overline{X})$.

Next, in order to describe $\pi_1(\overline{X})$ more explicitly, we observe which object is Galois in the Galois category $\mathrm{FEt}_{\overline{X}}$. By the definitions of a connected object and a Galois object in a Galois category, one can see that $\overline{Y} \in \mathrm{FEt}_{\overline{X}}$ is connected in $\mathrm{FEt}_{\overline{X}}$ if and only if $Y \to X$ is connected in FEt_X, and that a connected object $\overline{Y}$ is Galois in $\mathrm{FEt}_{\overline{X}}$ if and only if $\mathrm{Aut}_{\overline{X}}(\overline{Y}) = \mathrm{Aut}_X(Y) \to F'_{\tilde{\eta}}(Y) = F_{\tilde{\eta}}(\overline{Y})$ is bijective, i.e., Y is Galois in FEt_X. Therefore, we have the following Proposition.

Proposition 2.1.7. *Let $\tilde{K}$ (resp. $\tilde{K}^{ab}$) denote the maximal Galois (resp. abelian) extension of K which is unramified over all finite and infinite primes. Then we have the following.*
(1) *There is a natural isomorphisms* $\mathrm{Gal}(\tilde{K}/K) \cong \pi_1(\overline{X})$.
(2) *The abelianization* $\pi_1^{ab}(\overline{X})$ *of* $\pi_1(\overline{X})$ *admits natural isomorphisms*

$$\mathrm{Cl}_K \xrightarrow{\sim} \mathrm{Gal}(\tilde{K}^{ab}/K) \cong \pi_1^{ab}(\overline{X}) \ ; \ [\mathfrak{a}] \mapsto \left(\frac{\tilde{K}^{ab}/K}{\mathfrak{a}} \right)$$

given by the Artin reciprocity law.

2.2. The Artin–Verdier topos and the modified étale cohomology groups. Let $\mathrm{Sh}(\overline{X}_{\text{ét}})$ denote the *Artin–Verdier étale topos*, namely, the category of abelian sheaves on the site $\overline{X}_{\text{ét}}$. Let us recall the decomposition lemma for $\mathrm{Sh}(\overline{X}_{\text{ét}})$ following [1] and [3]. We fix an algebraic closure $\overline{K}$ of K. For each $x \in X_\infty$, we fix an extension $\overline{x}$ of x to $\overline{K}$ and denote by $I_{\overline{x}}$ the inertia group of $\overline{x}$. If x is a real prime, then we have $I_{\overline{x}} \cong \mathbb{Z}/2\mathbb{Z}$; if x is a complex prime, then $I_{\overline{x}}$ is trivial. Let $\eta : \mathrm{Spec}\, K \to X$ denote the generic point. Then, for $F \in \mathrm{Sh}(\overline{X}_{\text{ét}})$, we can regard $\eta^* F = F_\eta$ as a $\mathrm{Gal}(\overline{K}/K)$-module and $I_{\overline{x}} \subset \mathrm{Gal}(\tilde{K}/K)$ acts on $\eta^* F$. We define a site TX_∞ as follows. An object in TX_∞ is a pair (M, m) where M is a finite set and $m : M \to X_\infty$ is a map. A morphism $(M_1, m_1) \to (M_2, m_2)$ in TX_∞ is a map $f : M_1 \to M_2$ such that $m_2 = f \circ m_1$. A covering in TX_∞ is a family of morphisms $\{\varphi_i : (M_i, m_i) \to (M, m)\}_{i \in I}$ in TX_∞ such that m_i is surjective and $M = \bigcup_i \varphi(M_i)$. Hence, each G on TX_∞ is identified with a family of abelian groups $\{G_x\}_{x \in X_\infty}$. We define maps of sites $p : \overline{X}_{\text{ét}} \to TX_\infty$ and $q : \overline{X}_{\text{ét}} \to X_{\text{ét}}$ by the forgetful functors. Then we have functors

$$\mathrm{Sh}(TX_\infty) \underset{p^*}{\overset{p_*}{\leftrightarrows}} \mathrm{Sh}(\overline{X}_{\text{ét}}) \underset{q^*}{\overset{q_*}{\leftrightarrows}} \mathrm{Sh}(X_{\text{ét}}).$$

Next, we define the category $\mathrm{Sh}(\overline{X}_{\text{ét}})'$ as follows. An object in $\mathrm{Sh}(\overline{X}_{\text{ét}})'$ is a triple $(\{G_x\}_{x\in X_\infty}, F, \{\sigma_x : G_x \to (\eta^*F)^{I_x}\}_{x\in X_\infty})$, where $\{G_x\}_{x\in X_\infty} \in \mathrm{Sh}(TX_\infty)$, $F \in \mathrm{Sh}(X_{\text{ét}})$ and $\{\sigma_x : G_x \to (\eta^*F)^{I_x}\}_{x\in X_\infty}$ is a family of homomorphisms of abelian groups. A morphism $(\{G_x\}, F, \{\sigma_x\}) \to (\{G'_x\}, F', \{\sigma'_x\})$ is a pair of morphisms $\{G_x\} \to \{G'_x\}$, and $F \to F'$ such that the induced diagram

$$\begin{array}{ccc} G_x & \xrightarrow{\sigma_x} & (\eta^*F)^{I_x} \\ \downarrow & & \downarrow \\ G'_x & \xrightarrow{\sigma'_x} & (\eta^*F')^{I_x} \end{array}$$

is commutative for each $x \in X_\infty$.

Now we state the decomposition lemma, which was previously proved for $\mathrm{Sh}(\overline{X}_{\text{ét}})$ ([1, Proposition 2.3] and [3, Proposition 1.2]).

Lemma 2.2.1. *There is an equivalence of categories given by the functors*

$$\mathrm{Sh}(\overline{X}_{\text{ét}}) \underset{\Psi}{\overset{\Phi}{\leftrightarrows}} \mathrm{Sh}(\overline{X}_{\text{ét}})'$$

defined by

$$\Phi : S \mapsto (q_*S, p_*S, p_*S \to p_*q^*q_*S),\ \Psi : (\{G_x\}, F, \{\sigma_x\}) \mapsto q^*F \times_{p^*q_*q^*F} p^*\{G_x\}.$$

Proof. We may check the following properties (1)–(4), so that [2, Proposition 2.4] yields the assertion.
(1) q_*(resp. p_*) is left adjoint to q^* (resp. p^*).
(2) q_*, p_* are exact.
(3) p^*, q^* are fully faithful.
(4) For any $S \in \mathrm{Sh}(\overline{X}_{\text{ét}})$, $q_*S = 0$ holds if and only if there exists $G \in \mathrm{Sh}(TX_\infty)$ such that $S = p^*G$.
We refer to [21, Proposition 1.3.3] for (1), (3), and (4). The property (2) follows from the fact that $\overline{X}_{\text{ét}}$, $X_{\text{ét}}$ and TX_∞ have final objects and finite fiber products preserved by p and q. □

REMARK 2.2.2. (1) Via the equivalence of categories in Lemma 2.2.1, we identify p_*, p^*, q_*, q^* with the functors ψ_*, ψ^*, ϕ^*, ϕ_* defined by

$$\phi^*(\{G_x\}, F, \{\sigma_x\}) = F,\ \phi_*F = (\{(\eta^*F)^{I_x}\}, F, \{\mathrm{id}\}),$$
$$\psi^*(\{G_x\}, F, \{\sigma_x\}) = \{G_x\},\ \psi_*\{G_x\} = (\{G_x\}, 0, \{0\}).$$

respectively.
(2) The constant sheaf $\underline{A}_{\overline{X}_{\text{ét}}}$ on $\overline{X}_{\text{ét}}$ associated to an abelian group A satisfies $\underline{A}_{\overline{X}_{\text{ét}}} = \phi_*(\underline{A}_{X_{\text{ét}}})$. In the following, if there is no confusion, we will abbreviate $\underline{A}_{\overline{X}_{\text{ét}}}$ to A.
(3) For $S = (\{G_x\}, F, \{\sigma_x\}) \in \mathrm{Ob}\,\mathrm{Sh}(\overline{X}_{\text{ét}})$, the section of S at $(Y, M) \in \overline{X}_{\text{ét}}$ is given by $F(Y) \times_{\eta^*F} G_{x_1} \times_{\eta^*F} G_{x_2} \times_{\eta^*F} \cdots \times_{\eta^*F} G_{x_r}$, where $\{x_1, x_2, \cdots, x_r\}$ is the image of M by $Y_\infty \to X_\infty$.

DEFINITION 2.2.3. For each $S \in \mathrm{Sh}(\overline{X}_{\text{ét}})$, the cohomology group $\mathrm{H}^i(\overline{X}, S)$ is called the i-th *modified étale cohomology group* of $\overline{X}$ with values in S.

The group $\mathrm{H}^i(\overline{X}, S)$ of the constant sheaf $\mathbb{Z}/n\mathbb{Z}$ is calculated in [3, Proposition 2.13] and [1, Corollary 2.15]. Let us recall the Artin–Verdier duality.

Proposition 2.2.4 (The Artin–Verdier duality [3, Theorem 5.1])**.** *Let F be a constructible sheaf on $X = \operatorname{Spec} \mathcal{O}_K$. We fix an algebraic closure $\overline{K}$ of K. For each $x \in X_\infty$, we fix an extension $\overline{x}$ of x to $\overline{K}$. Let $\eta : \operatorname{Spec} K \to X$ denote the generic point. Let $G_{m,X}$ denote the étale sheaf of units on X. Then we have the following.*

(a) $\mathrm{H}^i(\overline{X}, \phi_* F) = \mathrm{Ext}^i_{\overline{X}}(\phi_* F, \phi_* G_{m,X}) = 0$ *for* $i > 3$.
(b) *The Yoneda pairing*

$$\mathrm{H}^i(\overline{X}, \phi_* F) \times \mathrm{Ext}^{3-i}_{\overline{X}}(\phi_* F, \phi_* G_{m,X}) \to \mathrm{H}^3(\overline{X}, \phi_* G_{m,X}) \cong \mathbb{Q}/\mathbb{Z}$$

is a perfect duality of finite groups for $i \geq 2$.
(c) *If for every $x \in X_\infty$ the inertia group $I_{\overline{x}}$ of $\overline{x}$ acts trivially on the* $\mathrm{Gal}(\overline{K}/K)$*-module $\eta^* F = F_\eta$, then the pairing in* (b) *is perfect for any $i \geq 0$.*

Applying Proposition 2.2.4 to the constant sheaf $F = \mathbb{Z}/n\mathbb{Z}$ on X, we obtain the following Proposition, where we denote by $\mu_n(K)$ the group of n-th roots of unity in K and put $Z_1 = \{(a, \mathfrak{a}) \in K^\times \oplus I_K \mid (a)^{-1} = \mathfrak{a}^n\}$, $B_1 = \{(b^n, (b)^{-1}) \in K^\times \oplus I_K \mid b \in K^\times\}$.

Proposition 2.2.5 ([3, Proposition 2.13], [1, Corollary 2.15])**.** *We have*

$$\mathrm{Ext}^i_{\overline{X}}(\mathbb{Z}/n\mathbb{Z}, \phi_* G_{m,X}) \cong \begin{cases} \mu_n(K) & (i = 0) \\ Z_1/B_1 & (i = 1) \\ \mathrm{Cl}_K/n\mathrm{Cl}_K & (i = 2) \\ \mathbb{Z}/n\mathbb{Z} & (i = 3) \\ 0 & (i > 3), \end{cases}$$

where $G_{m,X}$ is the étale sheaf of units on X. Then we have, by the Artin–Verdier duality,

$$\mathrm{H}^i(\overline{X}, \mathbb{Z}/n\mathbb{Z}) \cong \begin{cases} \mathbb{Z}/n\mathbb{Z} & (i = 0) \\ (\mathrm{Cl}_K/n\mathrm{Cl}_K)^\sim & (i = 1) \\ (Z_1/B_1)^\sim & (i = 2) \\ (\mu_n(K))^\sim & (i = 3) \\ 0 & (i > 3), \end{cases}$$

where $(-)^\sim$ *denotes* $\mathrm{Hom}(-, \mathbb{Q}/\mathbb{Z})$.

REMARK 2.2.6. Assume that K contains primitive n-th roots of unity. For each $v' \in K^\times$, we choose a primitive n-th root $v'^{\frac{1}{n}}$ of v'. By Theorem 2.1.6, for a continuous and surjective homomorphism $\rho : \pi_1(\overline{X}) \to \mathbb{Z}/n\mathbb{Z}$, there is a corresponding Galois object $\overline{Y} \to \overline{X}$ ($Y = \operatorname{Spec} \mathcal{O}_L$) whose Galois group is $\mathbb{Z}/n\mathbb{Z}$. Since L is a cyclic extension of degree n unramified at all finite and infinite primes, there exists $v \in K^\times$ such that $L = K(v^{\frac{1}{n}})$ and there exists $\mathfrak{a} \in I_K$ which satisfies $\mathfrak{a}^n = (v)^{-1}$. By the definition of L and the Galois correspondence, there is an isomorphism $\chi : \mathrm{Gal}(L/K) \xrightarrow{\sim} \mathbb{Z}/n\mathbb{Z}$ such that the following diagram

$$\begin{array}{ccc} \pi_1(\overline{X}) & \xrightarrow{res} & \mathrm{Gal}(L/K) \\ {\scriptstyle\rho}\downarrow & \swarrow{\scriptstyle\chi} & \\ \mathbb{Z}/n\mathbb{Z} & & \end{array}$$

commutes, where $res : \pi_1(\overline{X}) \to \mathrm{Gal}(L/K)$ denotes the restriction map. By Proposition 2.1.7, we also have the following commutative diagram

$$\begin{array}{ccc} \mathrm{Cl}_K & \xrightarrow{\left(\frac{\tilde{K}^{ab}/K}{}\right)} & {\pi_1}^{ab}(\overline{X}) \\ {\scriptstyle\left(\frac{L/K}{}\right)}\downarrow & \swarrow{\scriptstyle res} & \\ \mathrm{Gal}(L/K), & & \end{array}$$

where $\left(\frac{L/K}{}\right) : \mathrm{Cl}_K \to \mathrm{Gal}(L/K)$ denotes the Artin map.

Now we state the extension of Hochschild–Serre spectral sequence.

Theorerm 2.2.7. *Let $\overline{Y} \to \overline{X}$ be a Galois object in $\mathrm{FEt}_{\overline{X}}$. Then for any $S \in \mathrm{Sh}(\overline{X}_{\acute{e}t})$, there is a cohomological spectral sequence*

$$\mathrm{H}^p(\mathrm{Gal}(\overline{Y}/\overline{X}), \mathrm{H}^q(\overline{Y}, S|\overline{Y})) \Rightarrow \mathrm{H}^{p+q}(\overline{X}, S).$$

Proof. Let $\mathrm{Gal}(\overline{Y}/\overline{X})$-mod denote the category of $\mathrm{Gal}(\overline{Y}/\overline{X})$-modules. We consider the functors

$$F_1 : \mathrm{Sh}(\overline{X}_{\acute{e}t}) \to \mathrm{Gal}(\overline{Y}/\overline{X})\text{-mod}, S \mapsto S(\overline{Y})$$
$$F_2 : \mathrm{Gal}(\overline{Y}/\overline{X})\text{-mod} \to \mathrm{Ab}, M \mapsto M^{\mathrm{Gal}(\overline{Y}/\overline{X})},$$

where the action of $G = \mathrm{Gal}(\overline{Y}/\overline{X})$ on $S(\overline{Y})$ is defined by $\sigma.x = S(\sigma)(x)$ for $x \in S(\overline{Y})$ and $\sigma \in G$. In the same manner as in [12, Remark5.4] and [12, Proposition 1.4], we can easily check $(F_2 \circ F_1)(S) = S(\overline{Y})^G = S(\overline{X})$. Let I be an injective object in $\mathrm{Sh}(\overline{X}_{\acute{e}t})$. By replacing Y and X with $\overline{Y}$ and $\overline{X}$ in the argument of [12, Example2.6], one can see that $\mathrm{H}^i(G, I(\overline{Y})) \cong \check{\mathrm{H}}^i(\overline{Y}/\overline{X}, I)$ for any $i \geq 1$. Since I is injective, we have $\check{\mathrm{H}}^i(\overline{Y}/\overline{X}, I) = 0$ by the definition of Čech cohomologies. Therefore, $F_1(I) = I(\overline{Y})$ is a F_2-acyclic object. By the Grothendieck spectral sequence, we obtain the assertion. □

Let $(\overline{Y_i} \to \overline{X}, \overline{Y_i} \to \overline{Y_j})$ denote the inverse system of finite Galois coverings over $\overline{X}$ and put $\widetilde{X} = \varprojlim_i Y_i$, $\widetilde{\overline{X}} = \varprojlim_i \overline{Y_i}$. By $\mathrm{H}^p(\widetilde{X}, \mathbb{Z}/n\mathbb{Z}) = \varprojlim_i \mathrm{H}^p(Y_i, \mathbb{Z}/n\mathbb{Z})$ and the local cohomology sequence [3, Proposition 1.4], we have $\mathrm{H}^p(\widetilde{\overline{X}}, \mathbb{Z}/n\mathbb{Z}) = \varprojlim_i \mathrm{H}^p(\overline{Y_i}, \mathbb{Z}/n\mathbb{Z})$. So on passing to the inverse limit, we obtain the following.

Corollary 2.2.8. *There is a cohomological spectral sequence*

$$\mathrm{H}^p(\pi_1(\overline{X}), \mathrm{H}^q(\widetilde{\overline{X}}, \mathbb{Z}/n\mathbb{Z})) \Rightarrow \mathrm{H}^{p+q}(\overline{X}, \mathbb{Z}/n\mathbb{Z}).$$

3. Arithmetic Dijkgraaf–Witten invariants for a number ring

In this section, we introduce the notions of arithmetic Chern–Simons invariant and the arithmetic Dijkgraaf–Witten invariant for a number field, by using the modified étale cohomology groups introduced in Section 2. Let $X = \operatorname{Spec} \mathcal{O}_K$ denote the prime spectrum of the ring of integers of a number field K containing primitive n-th roots of unity. We choose a primitive n-th root of unity ζ_n in K, which induces an isomorphism $\mathbb{Z}/n\mathbb{Z} \cong \mu_n$. Let A be a finite group and let $c \in \mathrm{H}^3(A, \mathbb{Z}/n\mathbb{Z})$. Let $\mathcal{M}(\overline{X}, A) = \operatorname{Hom}_c(\pi_1(\overline{X}), A)/A$ denote the set of conjugacy classes of all continuous homomorphisms $\pi_1(\overline{X}) \to A$. Recall that by Proposition 2.2.5 we have the fundamental class isomorphism $\mathrm{H}^3(\overline{X}, \mathbb{Z}/n\mathbb{Z}) \cong \mathbb{Z}/n\mathbb{Z}$ that depends on the choice of ζ_n.

Definition 3.1. For $\rho \in \mathcal{M}(\overline{X}, A)$, the *arithmetic Chern–Simons invariant* $CS_c(\rho)$ associated to c is defined by the image of c under the composition of maps

$$\mathrm{H}^3(A, \mathbb{Z}/n\mathbb{Z}) \xrightarrow{\rho^*} \mathrm{H}^3(\pi_1(\overline{X}), \mathbb{Z}/n\mathbb{Z}) \xrightarrow{j_3} \mathrm{H}^3(\overline{X}, \mathbb{Z}/n\mathbb{Z}) \cong \mathbb{Z}/n\mathbb{Z},$$

where j_3 is the edge homomorphisms in the modified Hochschild–Serre spectral sequence $\mathrm{H}^p(\pi_1(\overline{X}), \mathrm{H}^q(\widetilde{\overline{X}}, \mathbb{Z}/n\mathbb{Z})) \Rightarrow \mathrm{H}^{p+q}(\overline{X}, \mathbb{Z}/n\mathbb{Z})$ of Corollary 2.2.8 We can easily see that $CS_c(\rho)$ is independent of the choice of ρ in its conjugacy class. The map

$$CS_c : \mathcal{M}(\overline{X}, A) \to \mathbb{Z}/n\mathbb{Z}$$

is called *the arithmetc Chern–Simons functional* associated to c. *The arithmetic Dijkgraaf–Witten invariant* of $\overline{X}$ associated to c is then defined by

$$Z_c(\overline{X}) = \sum_{\rho \in \mathcal{M}(\overline{X}, A)} \exp\left(\frac{2\pi i}{n} CS_c(\rho)\right).$$

When $A = \mathbb{Z}/m\mathbb{Z}$, we call $CS_c(\rho)$ and $Z_c(\overline{X})$ *the mod m arithmetic Chern–Simons invariant* and *the mod n arithmetic Dijkgraaf–Witten invariant*, respectively.

Remark 3.2. (1) If K is totally imaginary, so that K has no ramification at infinite primes, then we have $\pi_1(\overline{X}) = \pi_1(X)$ and $\mathrm{H}^i(\overline{X}, \mathbb{Z}/n\mathbb{Z}) = \mathrm{H}^i(X, \mathbb{Z}/n\mathbb{Z})$. Therefore Definition 3.1 is indeed an extension of Kim's definition [9].
(2) When A is abelian, by Proposition 2.1.7, we have

$$\mathcal{M}(\overline{X}, A) = \operatorname{Hom}_c(\pi_1(\overline{X}), A) \cong \operatorname{Hom}(\mathrm{Cl}_K, A).$$

4. Mod 2 arithmetic Dijkgraaf–Witten invariants for the real quadratic number fields $\mathbb{Q}(\sqrt{p_1 \cdots p_r})$ with $p_i \equiv 1 \bmod 4$

In this section, we study explicit formulas for number fields. In Subsection 4.1, we establish an explicit formula (Theorem 4.1.3) relating the Chern–Simons invariant to the Artin Symbol of Kummer extensions over any number field containing primitive n-th roots of unity, extending [1, Proposition 4.2] and [4, Theorem 1.3]. In Subsection 4.2, by using Theorem 4.1.3 and invoking Gauss's genus theory, we explicitly compute the mod 2 arithmetic Dijkgraaf–Witten invariant for a quadratic field $\mathbb{Q}(\sqrt{p_1 \cdots p_r})$ with $p_i \equiv 1 \bmod 4$ (Theorem 4.2.2).

4.1. A formula for Kummer extensions with use of the Artin Symbols. Let us describe the setting in this subsection. We continue to work over any number field K containing primitive n-th roots of unity. Keeping the same notations as in Section 3, we set $A = \mathbb{Z}/n\mathbb{Z}$ and $c = \mathrm{id} \cup \beta(\mathrm{id}) \in \mathrm{H}^3(A, \mathbb{Z}/n\mathbb{Z})$. Here, $id \in \mathrm{H}^1(A, \mathbb{Z}/n\mathbb{Z})$ denotes (the image of) the identity map and

$$\beta : \mathrm{H}^1(A, \mathbb{Z}/n\mathbb{Z}) \to \mathrm{H}^2(A, \mathbb{Z}/n\mathbb{Z})$$

denotes the Bockstein map (connecting homomorphism) induced by the short exact sequence

$$0 \to \mathbb{Z}/n\mathbb{Z} \stackrel{\times n}{\to} \mathbb{Z}/n^2\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \to 0.$$

Let $j_i : \mathrm{H}^i(\pi_1(\overline{X}), \mathbb{Z}/n\mathbb{Z}) \to \mathrm{H}^i(\overline{X}, \mathbb{Z}/n\mathbb{Z})$ $(i = 0, 1, 2, 3, \cdots)$ denote the edge homomorphisms in the modified Hochschild–Serre spectral sequence (Corollary 2.2.7). For each $\rho \in \mathcal{M}(\overline{X}, A) = \mathrm{Hom}_c(\pi_1(\overline{X}), A)$, let ρ_X^* denote the composition

$$\mathrm{H}^1(A, \mathbb{Z}/n\mathbb{Z}) \stackrel{\rho^*}{\longrightarrow} \mathrm{H}^1(\pi_1(\overline{X}), \mathbb{Z}/n\mathbb{Z}) \stackrel{j_1}{\longrightarrow} \mathrm{H}^1(\overline{X}, \mathbb{Z}/n\mathbb{Z})$$

of the natural map j_1 and the induced map ρ^*. Then we have

$$CS_c(\rho) = \rho_X^*(\mathrm{id}) \cup \tilde{\beta}(\rho_X^*(\mathrm{id})) \in \mathrm{H}^3(\overline{X}, \mathbb{Z}/n\mathbb{Z}),$$

where $\cup : \mathrm{H}^1(\overline{X}, \mathbb{Z}/n\mathbb{Z}) \times \mathrm{H}^2(\overline{X}, \mathbb{Z}/n\mathbb{Z}) \to \mathrm{H}^3(\overline{X}, \mathbb{Z}/n\mathbb{Z})$ is the cup product and $\tilde{\beta} : \mathrm{H}^1(\overline{X}, \mathbb{Z}/n\mathbb{Z}) \to \mathrm{H}^2(\overline{X}, \mathbb{Z}/n\mathbb{Z})$ is the Bockstein map.

Remark 4.1.1. For the definition of the cup product in the category of sheaves on any site, we refer to [18, Corollary 3.7].

We recall some calculations of the cohomology of groups.

Lemma 4.1.2. (1) *We have an isomorphism* $\mathrm{H}^i(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) \cong \mathbb{Z}/n\mathbb{Z}$ *for every* $i \geq 0$.
(2) *The cohomology class* $c = \mathrm{id} \cup \beta(\mathrm{id})$ *is represented by a cochain* $\alpha : (\mathbb{Z}/n\mathbb{Z})^3 \to \mathbb{Z}/n\mathbb{Z}$ *defined by*

$$\alpha(g_1, g_2, g_3) = \frac{1}{n}\overline{g_1}(\overline{g_2} + \overline{g_3} - (\overline{g_2 + g_3})) \bmod n,$$

where $\overline{g} \in \{0, 1, \ldots, n-1\}$ *is a representative element of* $g \in \mathbb{Z}/n\mathbb{Z}$.
(3) *The 3rd cohomology group* $\mathrm{H}^3(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/n\mathbb{Z})$ *is generated by* $c = \mathrm{id} \cup \beta(\mathrm{id})$.

Proof. Consider the projective resolution of $\mathbb{Z}[\mathbb{Z}/n\mathbb{Z}]$-modules over $\mathbb{Z}$

$$\cdots \stackrel{\times p}{\to} \mathbb{Z}[\mathbb{Z}/n\mathbb{Z}] \stackrel{\times q}{\to} \mathbb{Z}[\mathbb{Z}/n\mathbb{Z}] \stackrel{\times p}{\to} \mathbb{Z}[\mathbb{Z}/n\mathbb{Z}] \stackrel{\times q}{\to} \mathbb{Z}[\mathbb{Z}/n\mathbb{Z}] \stackrel{\epsilon}{\to} \mathbb{Z},$$

where $p = \sum_{g \in \mathbb{Z}/n\mathbb{Z}} g$, $q = -(0 \bmod n) + (1 \bmod n)$, and $\epsilon(\sum_{g \in \mathbb{Z}/n\mathbb{Z}} a_g g) = \sum_{g \in \mathbb{Z}/n\mathbb{Z}} a_g$, By taking the functor $\mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}/n\mathbb{Z}]}(-, \mathbb{Z}/n\mathbb{Z})$, we obtain the first assertion

$$\mathrm{H}^i(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) \cong \mathbb{Z}/n\mathbb{Z} \ (i \geq 0). \tag{4.1.2.1}$$

By applying the snake lemma to the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C^1(\mathbb{Z}/n\mathbb{Z},\mathbb{Z}/n\mathbb{Z}) & \xrightarrow{\times n} & C^1(\mathbb{Z}/n\mathbb{Z},\mathbb{Z}/n^2\mathbb{Z}) & \longrightarrow & C^1(\mathbb{Z}/n\mathbb{Z},\mathbb{Z}/n\mathbb{Z}) & \longrightarrow & 0 \\ & & \big\downarrow d & & \big\downarrow d & & \big\downarrow d & & \\ 0 & \longrightarrow & C^2(\mathbb{Z}/n\mathbb{Z},\mathbb{Z}/n\mathbb{Z}) & \xrightarrow{\times n} & C^2(\mathbb{Z}/n\mathbb{Z},\mathbb{Z}/n^2\mathbb{Z}) & \longrightarrow & C^2(\mathbb{Z}/n\mathbb{Z},\mathbb{Z}/n\mathbb{Z}) & \longrightarrow & 0, \end{array}$$

we obtain the second assertion. For the third assertion, by (4.1.2.1), it suffices to show that for each $n' = 1, 2, \ldots, n-1$, the cohomology class $n'c$ is not zero in $\mathrm{H}^3(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/n\mathbb{Z})$. Assume that there is a cochain $b \in C^2(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/n\mathbb{Z})$ such that $db = n'\alpha$. Then, for each $(g_1, g_2, g_3) \in (\mathbb{Z}/n\mathbb{Z})^3$, we have

$$(n'\alpha)(g_1, g_2, g_3) = b(g_2, g_3) - b(g_1 + g_2, g_3) + b(g_1, g_2 + g_3) - b(g_1, g_2).$$

So we obtain

$$\sum_{i=0}^{n-1}(n'\alpha)(1 \bmod n, i \bmod n, 1 \bmod n) = 0. \tag{4.1.2.2}$$

By the definition of α, we also have

$$(n'\alpha)(g_1, g_2, g_3) = \frac{n'}{n}\overline{g_1}(\overline{g_2} + \overline{g_3} - (\overline{g_2 + g_3})) \bmod n.$$

So we obtain

$$\sum_{i=0}^{n-1}(n'\alpha)(1 \bmod n, i \bmod n, 1 \bmod n) = n' \bmod n.$$

This contradicts the equation (4.1.2.2). □

Now we show the main assertion of this subsection. We keep the same notations as in Remark 2.2.6.

Theorerm 4.1.3. *Let $X = \operatorname{Spec} \mathcal{O}_K$ denote the prime spectrum of the ring of integers of a number field K containing primitive n-th roots of unity. Let $\rho : \pi_1(\overline{X}) \to \mathbb{Z}/n\mathbb{Z}$ be a continuous and surjective homomorphism. Set $A = \mathbb{Z}/n\mathbb{Z}$ and $c = \mathrm{id} \cup \beta(\mathrm{id}) \in \mathrm{H}^3(A, \mathbb{Z}/n\mathbb{Z})$. Let $L = K(v^{\frac{1}{n}})$ denote the Kummer extension corresponding to Ker ρ as in* Remark 2.2.6*, so that L/K is unramified at all finite and infinite primes and there exist some $\mathfrak{a} \in I_K$ and $v \in K^\times$ with $\mathfrak{a}^n = (v)^{-1}$. Let $\chi : \mathrm{Gal}(L/K) \xrightarrow{\sim} \mathbb{Z}/n\mathbb{Z}$ denote the natural isomorphism induced by ρ. Then we have*

$$CS_c(\rho) = \chi\left(\left(\frac{L/K}{\mathfrak{a}}\right)\right).$$

Proof. When K is totally imaginary, the assertion holds by [4, Theorem 1.3]. So we consider the case K has real primes and $n = 2$. By direct calculation, we see that $\rho^*(\mathrm{id}) \in \mathrm{H}^1(\pi_1(\overline{X}), \mathbb{Z}/2\mathbb{Z})$ corresponds to $\rho \in \mathrm{Hom}_c(\pi_1(\overline{X}), \mathbb{Z}/2\mathbb{Z})$ via the natural isomorphism

$$\mathrm{H}^1(\pi_1(\overline{X}), \mathbb{Z}/2\mathbb{Z}) \cong \mathrm{Hom}_c(\pi_1(\overline{X}), \mathbb{Z}/2\mathbb{Z}).$$

Then, by Proposition 2.2.5 and Remark 2.2.6 , $\rho_{\overline{X}}^*(\mathrm{id}) = j_1 \circ \rho^*(\mathrm{id}) \in \mathrm{H}^1(\overline{X}, \mathbb{Z}/2\mathbb{Z})$ corresponds to the composition $\chi \circ \left(\frac{L/K}{\,\cdot\,}\right) \in \mathrm{Hom}_c(\mathrm{Cl}_K, \mathbb{Z}/2\mathbb{Z})$ via the natural isomorphism

$$\mathrm{H}^1(\overline{X}, \mathbb{Z}/2\mathbb{Z}) \cong \mathrm{Hom}_c(\mathrm{Cl}_K, \mathbb{Z}/2\mathbb{Z}).$$

We regard $\tilde{\beta}(\rho_X^*(\mathrm{id}))$ as an element in $\mathrm{Ext}^1_{\overline{X}}(\mathbb{Z}/2\mathbb{Z}, \phi_* G_{m,X})^{\sim} = (Z_1/B_1)^{\sim}$ through Artin–Verdier duality. Then by [1, Corollary 3.13], we have

$$\rho_X^*(\mathrm{id}) \cup \tilde{\beta}(\rho_X^*(\mathrm{id})) = \tilde{\beta}(\rho_X^*(\mathrm{id}))([(v, \mathfrak{a})]) = \rho_X^*(\mathrm{id})(\tilde{\beta}'([(v, \mathfrak{a})])),$$

where $\tilde{\beta}' : \mathrm{Ext}^1_{\overline{X}}(\mathbb{Z}/2\mathbb{Z}, \phi_* G_{m,X}) \to \mathrm{Ext}^2_{\overline{X}}(\mathbb{Z}/2\mathbb{Z}, \phi_* G_{m,X})$ is the connecting homomorphism induced by the short exact sequence

$$0 \to \mathbb{Z}/2\mathbb{Z} \overset{\times 2}{\to} \mathbb{Z}/2^2\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0.$$

By replacing X with $\overline{X}$ in the proof of [1, Lemma 4.1], one can see $\tilde{\beta}'([(v, \mathfrak{a})]) = [\mathfrak{a}]$. Hence we see that $CS_c(\rho) = 0$ holds if and only if $\left(\frac{L/K}{\mathfrak{a}}\right) \in \mathrm{Gal}(L/K)$ is trivial. Therefore, we obtain the assertion. □

4.2. Explicit formulas of the mod 2 arithmetic Dijkgraaf–Witten invariants for real quadratic number fields $\mathbb{Q}(\sqrt{p_1 p_2 \cdots p_r})$ with $p_i \equiv 1$ mod 4. In the following, we consider the case $K = \mathbb{Q}(\sqrt{p_1 p_2 \cdots p_r})$, where each p_i is a prime number such that $p_i \equiv 1 \bmod 4$. We keep the notation as in the previous subsection and suppose that $n = 2$, $A = \mathbb{Z}/2\mathbb{Z}$, and $c = \mathrm{id} \cup \beta(\mathrm{id}) \in \mathrm{H}^3(A, \mathbb{Z}/2\mathbb{Z})$. Assume that the norm of the fundamental unit in $\mathcal{O}_K^\times$ is -1. Then the narrow ideal class group Cl_K^+ is the same as Cl_K.

REMARK 4.2.1. The fundamental unit of $\mathbb{Q}(\sqrt{5 \cdot 13 \cdot 61})$ is $\epsilon = \dfrac{63 + \sqrt{5 \cdot 13 \cdot 61}}{2}$ with $\mathrm{Nr}\,\epsilon = 1$. We eliminate such cases to use Gauss's genus theory.

By $p_i \equiv 1 \bmod 4$, the discriminant of K is $p_1 p_2 \cdots p_r$. We define the abelian multiplicative 2-group $T_\times$ by

$$T_\times = \{(x_1, x_2, \cdots, x_r) \in \{\pm 1\}^r \mid \prod_{i=1}^r x_i = 1\},$$

and put $e_{ij}^\times = (1, \cdots, 1, \overset{i\text{-th}}{-1}, 1, \cdots, 1, \overset{j\text{-th}}{-1}, 1, \cdots, 1) \in T_\times$ for each (i, j) with $1 \leqq i < j \leqq r$. In addition, we define an additive 2-group T_+ by

$$T_+ = \{(x_1, x_2, \cdots, x_r) \in (\mathbb{Z}/2\mathbb{Z})^r \mid \sum_{i=1}^r x_i = 0\}.$$

and put $e_{ij}^+ \overset{def}{=} (0, \cdots, 0, \overset{i\text{-th}}{1}, 0, \cdots, 0, \overset{j\text{-th}}{1}, 0, \cdots, 0) \in T_+$ for each (i, j) with $1 \leqq i < j \leqq r$. Then we have a standard isomorphism $T_+ \to T_\times; (x_i)_i \mapsto ((-1)^{x_i})_i$. By Gauss's genus theory [16, §4.7], there is an isomorphism

$$\mathrm{Cl}_K^+/2\mathrm{Cl}_K^+ \overset{\sim}{\longrightarrow} T_\times,$$

given by

$$[\mathfrak{a}] \mapsto \left(\left(\frac{\mathrm{N}\mathfrak{a}}{p_1}\right), \left(\frac{\mathrm{N}\mathfrak{a}}{p_2}\right), \cdots, \left(\frac{\mathrm{N}\mathfrak{a}}{p_r}\right)\right),$$

where $\left(\frac{-}{p_i}\right)$ denotes the Legendre symbol. Therefore, by Proposition 2.1.7, we obtain the following isomorphisms

$$\begin{aligned}\mathrm{Hom}_c(\pi_1(\overline{X}), \mathbb{Z}/2\mathbb{Z}) &\cong \mathrm{Hom}(\mathrm{Cl}_K^+/2\mathrm{Cl}_K^+, \mathbb{Z}/2\mathbb{Z})\\ &\cong \mathrm{Hom}(T_\times, \{\pm 1\}) \cong \mathrm{Hom}(T_+, \mathbb{Z}/2\mathbb{Z}).\end{aligned}$$

We denote the corresponding elements in those groups by the same letters.

Now we prove the following formula.

Theorerm 4.2.2. *Notations being as above, for each nontrivial* $\rho \in \mathrm{Hom}(T_\times, \{\pm 1\})$, *the arithmetic Chern–Simons invariant satisfies*

$$(-1)^{CS_c(\rho)} = \prod_{\substack{i<j \\ \rho(e_{ij}^\times)=-1}} \left(\frac{p_j}{p_i}\right).$$

Proof. Define elements $b_1, b_2, \cdots, b_{r-1} \in T_\times$ by

$$b_1 = (-1, 1, 1, \cdots, -1), b_2 = (1, -1, 1, 1, \cdots, -1), \cdots, b_{r-1} = (1, 1, \cdots, 1, -1, -1)$$

so that the tuple $(b_1, b_2, \cdots, b_{r-1})$ is a basis of $T_\times$. Let $J = \{j_1, j_2 \cdots, j_m\} \subset \{1, 2, \cdots, r-1\}$ with $j_1 < j_2 \cdots < j_m$ and suppose that $\rho(b_i) = -1$ if and only if $i \in J$. Note that $\rho(e_{ij}^\times) = -1$ holds if and only if the intersection $\{i, j\} \cap J$ consists of one element. Let L denote the abelian unramified extension of K corresponding to $2\mathrm{Cl}_K$ via the class field theory (Proposition 2.1.7), namely, we put

$$L = \mathbb{Q}(\sqrt{p_1}, \sqrt{p_2}, \cdots, \sqrt{p_r}).$$

Let L_ρ denote the unramified Kummer extension of K corresponding to $\mathrm{Ker}\rho \subset \pi_1(\overline{X})$, so that we have

$$L_\rho = K(\sqrt{v}), \quad \mathfrak{a}_v^2 = (v)^{-1} \qquad (4.2.2.1)$$

for some $v \in K^\times$ and $\mathfrak{a}_v \in I_K$. In order to apply Theorem 4.1.3, let us explicitly find such v. Let $\mathbf{a} = (a_1, a_2, \cdots, a_r) \in T_\times$ and let $\mathfrak{a} \in I_K$ whose image $[\mathfrak{a}]$ corresponds to $\mathbf{a}$ via the isomorphism $\mathrm{Cl}_K^+/2\mathrm{Cl}_K^+ \xrightarrow{\sim} T_\times$ of Gauss's genus theory. Then the Artin symbol $\left(\frac{L/K}{\mathfrak{a}}\right) \in \mathrm{Gal}(L/K)$ is characterized by

$$\left(\frac{L/K}{\mathfrak{a}}\right)(\sqrt{p_i}) = a_i\sqrt{p_i} \ \ (i = 1, 2, \cdots, r).$$

Let $u : K^\times \to K^\times/(K^\times)^2$ denote the natural projection. By Remark 2.2.6, the class $u(v) \in K^\times/(K^\times)^2$ is characterized by

$$\left(\frac{L_\rho/K}{\mathfrak{a}}\right)(\sqrt{v})/\sqrt{v} = \rho(\mathbf{a}).$$

Since $\left(\frac{L_\rho/K}{\mathfrak{a}}\right)$ is the restriction of $\left(\frac{L/K}{\mathfrak{a}}\right)$ to L_ρ, we may put

$$v = p_{j_1}p_{j_2}\cdots p_{j_m}/p_1p_2\cdots p_r.$$

Since the minimal polynomial of $(1+\sqrt{p_1p_2\cdots p_r})/2$ over $\mathbb{Q}$ is congruent to $(2X-1)^2$ mod p_i, we have

$$(p_i) = \mathfrak{p}_i{}^2,$$

where $\mathfrak{p}_i = (p_i, \sqrt{p_1p_2\cdots p_r})$ is the prime ideal of $\mathcal{O}_K$. Hence we have

$$\mathfrak{a}_v = \mathfrak{p}_1\mathfrak{p}_2\cdots\mathfrak{p}_r/\mathfrak{p}_{j_1}\mathfrak{p}_{j_2}\cdots\mathfrak{p}_{j_m}.$$

We see that the composite map

$$\chi' : \mathrm{Gal}(L/K) \overset{\sim}{\to} \mathrm{Cl}_K/2\mathrm{Cl}_K \overset{\sim}{\to} T_\times \overset{\rho}{\twoheadrightarrow} \{\pm 1\}$$

sends $\left(\frac{L/K}{\mathfrak{a}_v}\right) \in \mathrm{Gal}(L/K)$ to $\prod_{i=1}^{r}\left(\frac{\mathrm{N}\mathfrak{a}_v}{p_i}\right) \in \{\pm 1\}$. By the quadratic residue and the assumption that $p_l \equiv 1 \bmod 4$ $(l = 1, \ldots, r)$, we have $\left(\frac{p_j}{p_i}\right) = \left(\frac{p_i}{p_j}\right)$ for any distinct $i, j \in \{1, 2, \ldots, r\}$. So we obtain

$$\begin{aligned}\prod_{i=1}^{r}\left(\frac{\mathrm{N}\mathfrak{a}_v}{p_i}\right) &= \prod_{i=1}^{r}\prod_{\substack{1\le j\le r\\ j\notin J}}\left(\frac{p_j}{p_i}\right)\\ &= \prod_{\substack{1\le i\le r\\ i\in J}}\prod_{\substack{1\le j\le r\\ j\notin J}}\left(\frac{p_i}{p_j}\right)\\ &= \prod_{\substack{i<j\\ \rho(e_{ij}^{\times})=-1}}\left(\frac{p_j}{p_i}\right).\end{aligned}$$

The last equation follows from the fact that $\rho(e_{ij}^{\times}) = -1$ holds if and only if the intersection $\{i, j\} \cap J$ consists of one element. On the other hand, since L_ρ is the unramified Kummer extension of K corresponding to $\mathrm{Ker}\rho \subset \pi_1(\overline{X})$, the composite map $\chi' : \mathrm{Gal}(L/K) \overset{\sim}{\to} \mathrm{Cl}_K/2\mathrm{Cl}_K \overset{\sim}{\to} T_\times \overset{\rho}{\twoheadrightarrow} \{\pm 1\}$ induces the natural isomorphism $\chi'' : \mathrm{Gal}(L_\rho/K) = \mathrm{Gal}(L/K)/(\mathrm{Ker}\rho) \overset{\sim}{\to} \{\pm 1\}$. Let $\chi : \mathrm{Gal}(L_\rho/K) \overset{\sim}{\to} \mathbb{Z}/2\mathbb{Z}$ denote the natural isomorphism induced by $\rho : \pi_1(\overline{X}) \to \mathbb{Z}/2\mathbb{Z}$. We see that χ is equal to the composite map $\mathrm{Gal}(L_\rho/K) \overset{\chi''}{\to} \{\pm 1\} \overset{\sim}{\to} \mathbb{Z}/2\mathbb{Z}$. Therefore, by Theorem 4.1.3, we have

$$(-1)^{CS_c(\rho)} = \chi''\left(\left(\frac{L_\rho/K}{\mathfrak{a}_v}\right)\right) = \prod_{\substack{i<j\\ \rho(e_{ij}^{\times})=-1}}\left(\frac{p_j}{p_i}\right).$$

□

Since the invariant $CS_c(0)$ of the trivial representation $0 \in \mathrm{Hom}(T_+, \mathbb{Z}/2\mathbb{Z})$ is zero, we have the following

Corollary 4.2.3. *For* $\rho \in \mathrm{Hom}(T_+, \mathbb{Z}/2\mathbb{Z})$, *we have*

$$CS_c(\rho) = \sum_{i<j}\rho(e_{ij}^{+})\mathrm{lk}_2(p_i, p_j),$$

where $\mathrm{lk}_2(p_i, p_j)$ *denotes the modulo* 2 *linking number of* p_i *and* p_j *defined by* $(-1)^{\mathrm{lk}_2(p_i,p_j)} = \left(\frac{p_i}{p_j}\right)$.

By Definition 3.1, the mod 2 arithmetic Dijkgraaf–Witten invariant is given by

$$Z_c(\overline{X}) = \sum_{\rho \in \mathrm{Hom}(T_+, \mathbb{Z}/2\mathbb{Z})} (-1)^{CS_c(\rho)}.$$

Hence we obtain the following.

Corollary 4.2.4. *The mod* 2 *arithmetic Dijkgraaf–Witten invariant is given by*

$$Z_c(\overline{X}) = \sum_{\rho \in \mathrm{Hom}(T_+, \mathbb{Z}/2\mathbb{Z})} \left(\prod_{i<j} \left(\frac{p_i}{p_j} \right)^{\rho(e_{ij}^+)} \right).$$

Example 4.2.5. Here are some numerical examples of $CS_c(\rho)$ and $Z_c(\overline{X})$ for the case $r = 3$. We define ρ_0, ρ_1, ρ_2 and ρ_3 in $\mathrm{Hom}(T_+, \mathbb{Z}/2\mathbb{Z})$ by

$$\rho_0(1,1,0) = 0,\ \rho_0(0,1,1) = 0,\ \rho_0(1,0,1) = 0,$$

$$\rho_1(1,1,0) = 1,\ \rho_1(0,1,1) = 0,\ \rho_1(1,0,1) = 1,$$

$$\rho_2(1,1,0) = 0,\ \rho_2(0,1,1) = 1,\ \rho_2(1,0,1) = 1,$$

$$\rho_3(1,1,0) = 1,\ \rho_3(0,1,1) = 1,\ \rho_3(1,0,1) = 0,$$

so that $\mathrm{Hom}(T_+, \mathbb{Z}/2\mathbb{Z}) = \{\rho_0, \rho_1, \rho_2, \rho_3\}$.
(1) $K = \mathbb{Q}(\sqrt{5 \cdot 29 \cdot 37})$:

$$\mathrm{lk}_2(5, 29) = 0,\ \mathrm{lk}_2(29, 37) = 1,\ \mathrm{lk}_2(37, 5) = 1,$$

$$CS_c(\rho_0) = 0,\ CS_c(\rho_1) = 1,\ CS_c(\rho_2) = 0,\ CS_c(\rho_3) = 1,$$

$$Z_c(\overline{X}) = 0.$$

(2) $K = \mathbb{Q}(\sqrt{5 \cdot 13 \cdot 73})$:

$$\mathrm{lk}_2(5, 13) = \mathrm{lk}_2(13, 73) = \mathrm{lk}_2(73, 5) = 1,$$

$$CS_c(\rho_0) = CS_c(\rho_1) = CS_c(\rho_2) = CS_c(\rho_3) = 0,$$

$$Z_c(\overline{X}) = 4.$$

5. Appendix on mod 2 Dijkgraaf–Witten invariants for double branched covers of the 3-sphere

In this section, we present topological analogues of Theorem 4.2.2, Corollary 4.2.3 and Corollary 4.2.4 in the content of Dijkgraaf–Witten theory for 3-manifolds, in the spirit of arithmetic topology. For this purpose, we firstly display $\mathrm{M}^2\mathrm{K}^2\mathrm{R}$-dictionary, due to Mazur, Kapranov&Reznikov, Morishita, and Kim, concerning the analogies between 3-dimensional topology and number theory (cf. [13]):

Based on this dictionary, in Subsection 5.1, we introduce the Dijkgraaf–Witten invariants for 3-manifolds in a slightly different manner from the original one. In Subsections 5.2 and 5.3, we present topological analogues of Subsections 4.1 and 4.2 respectively.

3-dimensional topology	number theory
connected, oriented, and closed 3-manifold M	compactified spectrum of a number ring $\overline{X} = \overline{\mathrm{Spec}\ \mathcal{O}_K}$
knot $\mathcal{K} : S^1 \to M$	maximal ideal $\mathfrak{p} : \mathrm{Spec}\ (\mathcal{O}_K/\mathfrak{p}) \to \mathrm{Spec}\ (\mathcal{O}_K)$
link $L = \mathcal{K}_1 \cup \mathcal{K}_2 \cup \cdots \cup \mathcal{K}_r$	finite set of maximal ideals $S = \{\mathfrak{p}_1, \mathfrak{p}_2, \cdots, \mathfrak{p}_r\}$
fundamental group $\pi_1(M)$	modified étale fundamental group $\pi_1(\overline{X})$
1-cycle group $Z_1(M)$	ideal group I_K
mod 2 linking number $\mathrm{lk}(\mathcal{K}_1, \mathcal{K}_2) \bmod 2$	Legendre symbol $\left(\frac{p_1}{p_2}\right)$
1-boundary group $B_1(M)$ $\partial : C_2(M) \to Z_1(M); S \mapsto \partial S$	principal ideal group P_K $\partial : K^\times \to I_K; a \mapsto (a)$
1st integral homology group $\mathrm{H}_1(M) = Z_1(M)/B_1(M)$	ideal class group $\mathrm{Cl}_K = I_K/P_K$
Hurewicz isomorphism $\pi_1(M)^{ab} \cong \mathrm{Gal}(M^{ab}/M) \cong \mathrm{H}_1(M)$	Artin reciprocity $\pi_1(\overline{X})^{ab} \cong \mathrm{Gal}(\tilde{K}^{ab}/K) \cong \mathrm{Cl}_K$
Poincaré duality $\mathrm{H}^i(M, \mathbb{Z}/n\mathbb{Z}) \cong \mathrm{H}_{3-i}(M, \mathbb{Z}/n\mathbb{Z})$	Artin–Verdier duality $\mathrm{H}^i(\overline{X}, \mathbb{Z}/n\mathbb{Z}) \cong \mathrm{Ext}^{3-i}_{\overline{X}}(\mathbb{Z}/n\mathbb{Z}, \phi_* G_{m,X})^{\sim}$

5.1. Dijkgraaf–Witten invariants for 3-manifolds. In this subsection, we introduce the Dijkgraaf–Witten invariants in a manner slightly different from the original one [7] to clarify the analogy between the Dijkgraaf–Witten invariant for a 3-manifold and that for a number ring. In order to define the invariant, we show the following proposition, which is a topological analogue of Corollary 2.2.8.

Proposition 5.1.1. *Let M be a connected compact 3-manifold. Then, for $n \geq 2$, there is a cohomological spectral sequence*

$$\mathrm{H}^p(\pi_1(M), \mathrm{H}^q(\widetilde{M}, \mathbb{Z}/n\mathbb{Z})) \Rightarrow \mathrm{H}^{p+q}(M, \mathbb{Z}/n\mathbb{Z}),$$

where $\widetilde{M}$ denotes the universal covering of M.

Proof. Although this may be well known, we give a proof for the sake of readers. Since M is compact, the singular cohomology $\mathrm{H}^i(M, \mathbb{Z}/n\mathbb{Z})$ can be identified with the cohomology of the constant sheaf $\mathbb{Z}/n\mathbb{Z}$ on M. So we show the assertion for the cohomology of the constant sheaf. We denote by $\mathrm{Gal}(\widetilde{M}/M)$-mod the category of $\mathrm{Gal}(\widetilde{M}/M)$-modules. We consider the functors

$$F_1 : \mathrm{Sh}(M) \to \mathrm{Gal}(\widetilde{M}/M)\text{-mod}, S \mapsto S(\widetilde{M})$$
$$F_2 : \mathrm{Gal}(\widetilde{M}/M)\text{-mod} \to \mathrm{Ab}, R \mapsto R^{\mathrm{Gal}(\widetilde{M}/M)},$$

where the action of $G = \mathrm{Gal}(\widetilde{M}/M)$ on $S(\widetilde{M})$ is defined by $\sigma.x = S(\sigma)(x)$ for $x \in S(\widetilde{M})$ and $\sigma \in G$. In a similar way to Proposition 2.2.7, we can check $(F_2 \circ F_1)(S) = S(\widetilde{M})^G = S(M)$ and that F_1 sends any injective object I to a F_2-acyclic object. Therefore we have the expected spectral sequence by the Grothendieck spectral sequence and $\pi_1(M) \cong \mathrm{Gal}(\widetilde{M}/M)$. $\square$

Now we define the Dijkgraaf–Witten invariant for a 3-manifold.

Definition 5.1.2. Let M be a connected oriented closed 3-manifold and let $c \in \mathrm{H}^3(A, \mathbb{Z}/n\mathbb{Z})$ for a finite group A and $n \geq 2$. Let $\mathcal{M}(M, A) = \mathrm{Hom}(\pi_1(M), A)/A$ denote the set of conjugacy classes of all homomorphisms $\pi_1(M) \to A$. Note that the fundamental class $[M]$ generates $\mathrm{H}_3(M, \mathbb{Z}/n\mathbb{Z}) \cong \mathbb{Z}/n\mathbb{Z}$. For each $\rho \in \mathcal{M}(M, A)$, the *Chern–Simons invariant* $CS_c(\rho)$ of ρ associated to c is defined by the image of c under the composition of the maps

$$\mathrm{H}^3(A, \mathbb{Z}/n\mathbb{Z}) \xrightarrow{\rho^*} \mathrm{H}^3(\pi_1(M), \mathbb{Z}/n\mathbb{Z}) \xrightarrow{j_3} \mathrm{H}^3(M, \mathbb{Z}/n\mathbb{Z}) \xrightarrow{<\ ,[M]>} \mathbb{Z}/n\mathbb{Z},$$

where j_3 denotes the edge homomorphisms in the spectral sequence

$$\mathrm{H}^p(\pi_1(M), \mathrm{H}^q(\widetilde{M}, \mathbb{Z}/n\mathbb{Z})) \Rightarrow \mathrm{H}^{p+q}(M, \mathbb{Z}/n\mathbb{Z})$$

of Proposition 5.1.1. The *Dijkgraaf–Witten invariant* of M associated to c is then defined by

$$Z_c(M) = \sum_{\rho \in \mathcal{M}(M,A)} \exp\left(\frac{2\pi i}{n} CS_c(\rho)\right).$$

When $A = \mathbb{Z}/m\mathbb{Z}$, we call $CS_c(\rho)$ and $Z_c(M)$ the *mod m Chern–Simons invariant* and the *mod m Dijkgraaf–Witten invariant* respectively.

Remark 5.1.3. The Dijkgraaf–Witten invariant was originally defined as follows [7]. Let M and A be as in Definition 5.1.2. Let $\mathrm{B}A$ denotes a classifying space for A. Consider $U(1) = \{z \in \mathbb{C} \mid |z| = 1\}$ and let $c \in \mathrm{H}^3(A, U(1))$. Then the Dijkgraaf–Witten invariant $DW_c(M)$, is defined by

$$DW_c(M) = \sum_{\rho \in \mathrm{Hom}(\pi_1(M),A)} < {f_\rho}^* c, [M] >,$$

where $f_\rho : M \to \mathrm{B}A$ denotes the classifying map with respect to ρ and $<\ ,\ > : \mathrm{H}^3(M, U(1)) \times \mathrm{H}_3(M, \mathbb{Z}) \to U(1)$ denotes the natural pairing.

The relation between this definition and Definition 5.1.2 is given as follows. Suppose that $A = \mathbb{Z}/n\mathbb{Z}$, so that there is an isomorphism $\mathrm{H}^3(A, U(1)) \cong \mu_n \subset U(1)$ sending c to an n-th root of unity $\zeta_{n,c}$ in $U(1)$. Then, we may verify that for any $\rho \in \mathrm{Hom}(\pi_1(M), A)$, the equality

$$\zeta_{n,c}^{CS_{id \cup \beta(id)}(\rho)} =< {f_\rho}^* c, [M] >$$

holds, where $id \cup \beta(id)$ is a natural generator of $\mathrm{H}^3(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/n\mathbb{Z})$ (see Lemma 4.1.2). In particular, when $\zeta_{n,c} = \exp(\frac{2\pi i}{n})$, we have

$$DW_c(M) = Z_{\mathrm{id} \cup \beta(\mathrm{id})}(M).$$

5.2. A formula with use of the Hurewicz isomorphism. In this subsection, we show a topological analogue of Theorem 4.1.3. Let us describe the setting in this subsection. Keeping the same notations as in Subsection 5.1, we set $A = \mathbb{Z}/n\mathbb{Z}$ and $c = \mathrm{id} \cup \beta(\mathrm{id}) \in \mathrm{H}^3(A, \mathbb{Z}/n\mathbb{Z})$, where $id \in \mathrm{H}^1(A, \mathbb{Z}/n\mathbb{Z})$ is the identity map and

$$\beta^i : \mathrm{H}^i(A, \mathbb{Z}/n\mathbb{Z}) \to \mathrm{H}^{i+1}(A, \mathbb{Z}/n\mathbb{Z}) \ (i = 0, 1, 2, \cdots)$$

is the Bockstein map induced by the short exact sequence

$$(*) \qquad 0 \to \mathbb{Z}/n\mathbb{Z} \xrightarrow{\times n} \mathbb{Z}/n^2\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \to 0.$$

In addition, for $i = 1, 2, \cdots$, let $\beta^i : \mathrm{H}^i(M, \mathbb{Z}/n\mathbb{Z}) \to \mathrm{H}^{i+1}(M, \mathbb{Z}/n\mathbb{Z})$ and $\beta_i : \mathrm{H}_i(M, \mathbb{Z}/n\mathbb{Z}) \to \mathrm{H}_{i-1}(M, \mathbb{Z}/n\mathbb{Z})$ denote the Bockstein maps of the singular homology and cohomology induced by $(*)$. Furthermore, for $i = 1, 2, \cdots$, let $\tilde{\beta}_i : \mathrm{H}_i(M, \mathbb{Z}/n\mathbb{Z}) \to \mathrm{H}_{i-1}(M, \mathbb{Z})$ denote the Bockstein map of the singular homology induced by the short exact sequence

$$0 \to \mathbb{Z} \xrightarrow{\times n} \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \to 0.$$

Let $j_i : \mathrm{H}^i(\pi_1(M), \mathbb{Z}/n\mathbb{Z}) \to \mathrm{H}^i(M, \mathbb{Z}/n\mathbb{Z})$ $(i = 0, 1, 2, 3, \cdots)$ denote the edge homomorphisms in the spectral sequence of Proposition 5.1.1. We will abbreviate $j_i \circ \rho^*$ to ρ_M^* for $\rho \in \mathcal{M}(M, A) = \mathrm{Hom}(\pi_1(M), A)/A$. We denote by $\Phi^i : \mathrm{H}^i(M, \mathbb{Z}/n\mathbb{Z}) \xrightarrow{\sim} \mathrm{H}_{3-i}(M, \mathbb{Z}/n\mathbb{Z})$ $(i = 0, 1, 2, 3)$ the isomorphism of the Poincaré duality defined by $u \mapsto u \cap [M]$, where

$$\cap : \mathrm{H}^i(M, \mathbb{Z}/n\mathbb{Z}) \times \mathrm{H}_3(M, \mathbb{Z}/n\mathbb{Z}) \to \mathrm{H}_{3-i}(M, \mathbb{Z}/n\mathbb{Z})$$

denotes the cap product. Note that, by the universal coefficient theorems, we have

$$\mathrm{H}_1(M, \mathbb{Z}/n\mathbb{Z}) \cong \mathrm{H}_1(M) \otimes \mathbb{Z}/n\mathbb{Z} \cong \mathrm{H}_1(M)/n\mathrm{H}_1(M).$$

Together with the Hurewicz isomorphism, we obtain the isomorphisms

$$\begin{aligned} \mathrm{Hom}(\pi_1(M), \mathbb{Z}/n\mathbb{Z}) &\cong \mathrm{Hom}(\mathrm{H}_1(M), \mathbb{Z}/n\mathbb{Z}) \cong \mathrm{Hom}(\mathrm{H}_1(M, \mathbb{Z}/n\mathbb{Z}), \mathbb{Z}/n\mathbb{Z}) \\ &\cong \mathrm{H}^1(M, \mathbb{Z}/n\mathbb{Z}). \end{aligned}$$

We see that each $\rho \in \mathrm{Hom}(\pi_1(M), \mathbb{Z}/n\mathbb{Z})$ corresponds to $\rho_M^*(\mathrm{id}) \in \mathrm{H}^1(M, \mathbb{Z}/n\mathbb{Z})$ via these isomorphisms. We denote by $\tilde{\rho} \in \mathrm{Hom}(\mathrm{H}_1(M, \mathbb{Z}/n\mathbb{Z}), \mathbb{Z}/n\mathbb{Z})$ the homomorphism corresponding to ρ and $\rho_M^*(\mathrm{id})$. Now we show the main assertion in this subsection.

Theorerm 5.2.1. *Notations being as above, let $u \in Z_2(M, \mathbb{Z}/n\mathbb{Z})$ be a 2-cycle that represents $\Phi^1(\rho_M^*(\mathrm{id})) \in \mathrm{H}_2(M, \mathbb{Z}/n\mathbb{Z})$. Then there is a 2-chain $D \in C_2(M, \mathbb{Z})$ such that $D \bmod n = u$ and there is a 1-cycle $\mathfrak{a} \in Z_1(M, \mathbb{Z})$ satisfying $\partial D = n\mathfrak{a}$. Let $[\mathfrak{a}]$ denote the homology class in $\mathrm{H}_1(M, \mathbb{Z}/n\mathbb{Z})$ defined by $\mathfrak{a}$. Then we have*

$$CS_c(\rho) = \tilde{\rho}([\mathfrak{a}]).$$

Proof. We consider the following commutative diagram,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C_2(M, \mathbb{Z}) & \xrightarrow{\times n} & C_2(M, \mathbb{Z}) & \xrightarrow{\bmod n} & C_2(M, \mathbb{Z}/n\mathbb{Z}) & \longrightarrow & 0 \\ & & \downarrow \partial & & \downarrow \partial & & \downarrow \partial & & \\ 0 & \longrightarrow & C_1(M, \mathbb{Z}) & \xrightarrow{\times n} & C_1(M, \mathbb{Z}) & \xrightarrow{\bmod n} & C_1(M, \mathbb{Z}/n\mathbb{Z}) & \longrightarrow & 0. \end{array}$$

By the upper short exact sequence, there is a 2-chain $D \in C_2(M, \mathbb{Z})$ such that $D \bmod n = u$. Hence, we have

$$(\partial D) \bmod n = \partial(D \bmod n) = \partial u = 0.$$

Therefore, by the lower short exact sequence, there is a 1-cycle $\mathfrak{a} \in Z_1(M, \mathbb{Z})$ such that $\partial D = n\mathfrak{a}$. For the latter assertion, by direct calculation, we can check $\Phi^2 \circ \beta^1 = \beta_2 \circ \Phi^1$. Then, by Definition 5.1.2, we have,

$$\begin{aligned} CS_c(\rho) &= < \rho_M^*(\mathrm{id}) \cup \beta^1(\rho_M^*(\mathrm{id})), [M] > \\ &= < \rho_M^*(\mathrm{id}), \Phi^2(\beta^1(\rho_M^*(\mathrm{id}))) > \\ &= \tilde{\rho}(\beta_2(\Phi^1(\rho_M^*(\mathrm{id})))). \end{aligned}$$

Next, we consider the following commutative diagram,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{\times n} & \mathbb{Z} & \longrightarrow & \mathbb{Z}/n\mathbb{Z} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle p_2} & & \downarrow{\scriptstyle id} & & \\ 0 & \longrightarrow & \mathbb{Z}/n\mathbb{Z} & \xrightarrow{\times n} & \mathbb{Z}/n^2\mathbb{Z} & \longrightarrow & \mathbb{Z}/n\mathbb{Z} & \longrightarrow & 0, \end{array}$$

where p_1 and p_2 are natural projections, and id is the identity map. By considering the connecting homomorphism with respect to the singular homologies for each row, we see that $\beta_2 = p_{1*} \circ \tilde{\beta}_2$. Then the required statement immediately follows by the definition of $\tilde{\beta}_2$. □

5.3. A topological analogue of the explicit formulas of the mod 2 Dijkgraaf–Witten invariants for double branched covers of the 3-sphere. In this subsection, we prove topological analogues of Theorem 4.2.2, Corollary 4.2.3, and Corollary 4.2.4. Keeping the notations as in Subsections 4.2, 5.1, and 5.2, we consider the case $A = \mathbb{Z}/2\mathbb{Z}$ and $c = \mathrm{id} \cup \beta(\mathrm{id}) \in \mathrm{H}^3(A, \mathbb{Z}/2\mathbb{Z})$ in Definition 5.1.2. A tame knot $\mathcal{K}$ is the image of a continuous embedding $S^1 \to S^3$ which extends to an embedding of a solid torus. Let $\mathcal{L} = \mathcal{K}_1 \cup \mathcal{K}_2 \cup \cdots \cup \mathcal{K}_r$ be a tame link in the 3-sphere S^3. Let $h : M \to S^3$ denote the double covering ramified along $\mathcal{L}$, that is, h is obtained by the Fox completion [8] of the unramified covering $Y \to X := S^3 \backslash \mathcal{L}$ corresponding to the kernel of the surjective homomorphism $\mathrm{H}_1(X) \to \mathbb{Z}/2\mathbb{Z}$ that maps any meridian of $\mathcal{K}_i$ to $1 \in \mathbb{Z}/2\mathbb{Z}$. Recall that T_+ denotes the abelian group defined by

$$T_+ = \{(x_1, x_2, \cdots, x_r) \in (\mathbb{Z}/2\mathbb{Z})^r \mid \sum_{i=1}^{r} x_i = 0\}$$

and we put

$$e_{ij}^+ = (0, \cdots, 0, \overset{i\text{-th}}{1}, 0, \cdots, 0, \overset{j\text{-th}}{1}, 0, \cdots, 0) \in T_+$$

for each (i, j) with $1 \leqq i < j \leqq r$. By the topological analogue of Gauss's genus theory [14, Corollary], there is an isomorphism

$$g : \mathrm{H}_1(M)/2\mathrm{H}_1(M) \xrightarrow{\sim} T_+ \tag{5.3.1}$$

given by

$$[\mathfrak{a}] \mapsto (\mathrm{lk}(h_*(\mathfrak{a}), \mathcal{K}_i) \bmod 2),$$

where lk(,) denotes the linking number. Hence we obtain the isomorphisms

$$\begin{aligned}\mathrm{Hom}(\pi_1(M),\mathbb{Z}/2\mathbb{Z}) &\cong \mathrm{Hom}(\mathrm{H}_1(M),\mathbb{Z}/2\mathbb{Z}) \cong \mathrm{Hom}(\mathrm{H}_1(M,\mathbb{Z}/2\mathbb{Z}),\mathbb{Z}/2\mathbb{Z})\\ &\cong \mathrm{Hom}(T_+,\mathbb{Z}/2\mathbb{Z})\\ &\cong \mathrm{H}^1(M,\mathbb{Z}/2\mathbb{Z}).\end{aligned}$$

by Subsection 5.2.

Now we prove a topological analogue of Corollary 4.2.3.

Theorerm 5.3.1. *Notations being as above, for* $\rho \in \mathrm{Hom}(T_+,\mathbb{Z}/2\mathbb{Z})$*, we have*

$$CS_c(\rho) = \sum_{i<j} \rho(e_{ij}^+)\mathrm{lk}(\mathcal{K}_i,\mathcal{K}_j) \bmod 2.$$

Proof. Define elements $b_1, b_2, \cdots, b_{r-1} \in T_+$ by

$$b_1 = (1,0,0,\cdots,1), b_2 = (0,1,0,0,\cdots,1), \cdots, b_{r-1} = (0,0,\cdots,0,1,1)$$

so that the tuple $(b_1, b_2, \cdots, b_{r-1})$ is a basis of T_+. Let $J = \{j_1, j_2 \cdots, j_m\} \subset \{1, 2, \cdots, r-1\}$ with $j_1 < j_2 \cdots < j_m$ and suppose that $\rho(b_i) = 1$ if and only if $i \in J$. For each $i = 1, 2, \cdots, r$, let $\mathcal{S}_i$ be a Seifert surface of $\mathcal{K}_i$ in S^3, and put $\tilde{\mathcal{K}}_i := h^{-1}(\mathcal{K}_i)$ and $\tilde{\mathcal{S}}_i := h^{-1}(\mathcal{S}_i)$. Let $u \in Z_2(M,\mathbb{Z}/n\mathbb{Z})$ be a 2-cycle that represents $\Phi^1(\rho_M^*(\mathrm{id})) \in \mathrm{H}_2(M,\mathbb{Z}/n\mathbb{Z})$. There is a 2-chain $D \in C_2(M,\mathbb{Z})$ such that $D \bmod 2 = u$ and a 1-cycle $\mathfrak{a}_\rho \in Z_1(M,\mathbb{Z})$ satisfying $\partial D = 2\mathfrak{a}_\rho$. In order to apply Theorem 5.2.1, let us explicitly find such a D. Let $\mathbf{a} = (a_1, a_2, \cdots, a_r) \in T_+$ and let $\mathfrak{a} \in Z_1(M,\mathbb{Z})$ whose image $[\mathfrak{a}]$ corresponds to $\mathbf{a}$ via the isomorphism $\mathrm{H}_1(M)/2\mathrm{H}_1(M) \xrightarrow{\sim} T_+$ of the topological analogue of Gauss's genus theory. We note that the mod 2 linking number $(\mathrm{lk}(h_*(\mathfrak{a}), h_*(\partial D)) \bmod 2)$ is equal to the mod 2 intersection number of $\mathfrak{a}$ and D. Therefore, by the Poincaré duality, a 2-chain $D \in C_2(M,\mathbb{Z})$ satisfies $u = D \bmod 2 \in Z_2(M,\mathbb{Z}/n\mathbb{Z})$ for some u with $[u] = \Phi^1(\rho_M^*(\mathrm{id}))$ if and only if

$$\mathrm{lk}(h_*(\mathfrak{a}), h_*(\partial D)) \bmod 2 = \rho(\mathbf{a}).$$

Therefore, we may put

$$D = \sum_{i=1}^{r} \tilde{\mathcal{S}}_i - \sum_{i\in\{j_1,j_2,\cdots,j_m\}} \tilde{\mathcal{S}}_i.$$

In this case, the 1-cycle

$$\mathfrak{a}_\rho = \sum_{i=1}^{r} \tilde{\mathcal{K}}_i - \sum_{i\in\{j_1,j_2,\cdots,j_m\}} \tilde{\mathcal{K}}_i$$

satisfies $\partial D = 2\mathfrak{a}_\rho$. By Theorem 5.2.1, we obtain

$$\begin{aligned}CS_c(\rho) &= \tilde{\rho}([\mathfrak{a}_\rho])\\ &= \rho(g([\mathfrak{a}_\rho]))\\ &= \rho((\mathrm{lk}(h_*(\mathfrak{a}_\rho),\mathcal{K}_i) \bmod 2)\\ &= \sum_{l=1}^{m} \mathrm{lk}(h_*(\mathfrak{a}_\rho),\mathcal{K}_{j_l}) \bmod 2\\ &= \sum_{l=1}^{m} \sum_{i\notin\{j_1,j_2,\cdots,j_m\}} \mathrm{lk}(\mathcal{K}_i,\mathcal{K}_{j_l}) \bmod 2\end{aligned}$$

$$= \sum_{i<j} \rho(e_{ij}^{+})\mathrm{lk}(\mathcal{K}_i, \mathcal{K}_j) \bmod 2.$$

□

By Definition 5.1.2, the mod 2 Dijkgraaf–Witten invariant is given by

$$Z_c(M) = \sum_{\rho\in\mathrm{Hom}(T_+,\mathbb{Z}/2\mathbb{Z})} \exp\left(\pi i\, CS_c(\rho)\right).$$

Hence we obtain the following.

Corollary 5.3.2. *Notations being as above, we have*

$$Z_c(M) = \sum_{\rho\in\mathrm{Hom}(T_+,\mathbb{Z}/2\mathbb{Z})} \prod_{i<j} (-1)^{\rho(e_{ij}^{+})\mathrm{lk}(\mathcal{K}_i,\mathcal{K}_j)}.$$

Example 5.3.3. Let L be a two-bridge link $B(a,b)$ ($0 < a < b$, b: even, $(a,b) = 1$). So we have $r = 2$ and $\mathrm{Hom}(T_+, \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$. Then, the double branched cover M is the lens space $L(a,b)$. By Proposition 5.3.1 and [20, p.540 and p.543], for each $0 \neq \rho \in \mathrm{Hom}(T_+, \mathbb{Z}/2\mathbb{Z})$, we have

$$CS_c(\rho) = \sum_{k=1}^{b/2} (-1)^{\lfloor (2k-1)a/b \rfloor} \bmod 2,$$

where $\lfloor\ \rfloor$ denotes the floor function. Therefore, we also have

$$Z_c(M) = \begin{cases} 2, & \text{if } \displaystyle\sum_{k=1}^{b/2} (-1)^{\lfloor (2k-1)a/b \rfloor} \text{ is even,} \\ 0, & \text{if otherwise.} \end{cases}$$

Remark 5.3.4. In the context of quantum topology, Murakami, Ohtsuki and Okada calculated the mod n Dijkgraaf–Witten invariant for the 3-manifold obtained by a Dehn surgery on S^3 along a framed link and expressed the mod n Dijkgraaf–Witten invariant in terms of Gaussian sums and the linking matrix of the framed link [15, Proposition 9.1]. Since our formula (Theorem 4.2.2) for number fields is given in a form similar to Gaussian sums, we may expect that the cases with non-abelian gauge groups would be given by a non-abelian generalization of Gaussian sums.

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