TOPICS IN THE GROTHENDIECK CONJECTURE FOR HYPERBOLIC POLYCURVES OF DIMENSION 2

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Abstract

In this paper, we study the anabelian geometry of hyperbolic polycurves of dimension 2 over sub-p-adic fields. In 1-dimensional case, Mochizuki proved the Hom version of the Grothendieck conjecture for hyperbolic curves over sub-p-adic fields and the pro-p version of this conjecture. In 2-dimensional case, a naive analogue of this conjecture does not hold for hyperbolic polycurves over general sub-p-adic fields. Moreover, the Isom version of the pro-p Grothendieck conjecture does not hold in general. We explain these two phenomena and prove the Hom version of the Grothendieck conjecture for hyperbolic polycurves of dimension 2 under the assumption that the Grothendieck section conjecture holds for some hyperbolic curves.

Contents

0. Introduction

Let K be a field, $\overline{K}$ a separable closure of K, and Y, X normal varieties (cf. Definition 1.3) over K. Write $Y_{\overline{K}}$ (resp. $X_{\overline{K}}$) for the scheme $Y \times_{\mathrm{Spec}\,K} \mathrm{Spec}\,\overline{K}$ (resp. $X \times_{\mathrm{Spec}\,K} \mathrm{Spec}\,\overline{K}$) and G_K for the absolute Galois group $\mathrm{Gal}(\overline{K}/K)$. Take a geometric point $*_Y$ (resp. $*_X$) of $Y_{\overline{K}}$ (resp. $X_{\overline{K}}$). A morphism $f : Y \to X$ over K induces a homomorphism

$$f_* : \pi_1(Y, *_Y) \to \pi_1(X, *_X)$$

over G_K between the étale fundamental groups of Y and X which is uniquely determined up to inner automorphisms induced by elements of $\pi_1(X_{\overline{K}}, *_X)$. Hence, we obtain a natural map

$$\mathrm{Mor}_K(Y, X) \to \mathrm{Hom}_{G_K}(\pi_1(Y, *_Y), \pi_1(X, *_X))/\mathrm{Inn}\,\pi_1(X_{\overline{K}}, *_X),$$

where we write $\mathrm{Mor}_K(Y, X)$ (resp. $\mathrm{Hom}_{G_K}(\pi_1(Y, *_Y), \pi_1(X, *_X))$; $\mathrm{Inn}\,\pi_1(X_{\overline{K}}, *_X)$) for the set of morphisms from Y to X over K (resp. the set of continuous homomorphisms over G_K from $\pi_1(Y, *_Y)$ to $\pi_1(X, *_X)$; the group of inner automorphisms of $\pi_1(X_{\overline{K}}, *_X)$).

In anabelian geometry, the following questions have been studied:

QUESTION 0.1.   1. Write $\mathrm{Isom}_K(Y, X)$ (resp. $\mathrm{Isom}_{G_K}(\pi_1(Y, *_Y), \pi_1(X, *_X))$) for the subset of $\mathrm{Mor}_K(Y, X)$ (resp. $\mathrm{Hom}_{G_K}(\pi_1(Y, *_Y), \pi_1(X, *_X))$) consisting of isomorphisms. Is the map

$$\mathrm{Isom}_K(Y, X) \to \mathrm{Isom}_{G_K}(\pi_1(Y, *_Y), \pi_1(X, *_X))/\mathrm{Inn}\,\pi_1(X_{\overline{K}}, *_X)$$

bijective?

2. Write $\mathrm{Mor}_K^{\mathrm{dom}}(Y, X)$ for the subset of $\mathrm{Mor}_K(Y, X)$ consisting of dominant morphisms and $\mathrm{Hom}_{G_K}^{\mathrm{open}}(\pi_1(Y, *_Y), \pi_1(X, *_X))$ for the subset of $\mathrm{Hom}_{G_K}(\pi_1(Y, *_Y), \pi_1(X, *_X))$ consisting of open homomorphisms. Is the map (cf. [4] Lemma 1.3)

$$\mathrm{Mor}_K^{\mathrm{dom}}(Y, X) \to \mathrm{Hom}_{G_K}^{\mathrm{open}}(\pi_1(Y, *_Y), \pi_1(X, *_X))/\mathrm{Inn}\,\pi_1(X_{\overline{K}}, *_X)$$

bijective?

3. Suppose that $Y = \mathrm{Spec}\,K$. (Hence, we have $\mathrm{Mor}_K(Y, X) = X(K)$). Write $\mathrm{Sect}_{G_K}(\pi_1(X, *_X))$ for the set of sections of the natural surjective homomorphism $\pi_1(X, *_X) \to G_K$. Is the map

$$X(K) \to \mathrm{Sect}_{G_K}(\pi_1(X, *_X))/\mathrm{Inn}\,\pi_1(X_{\overline{K}}, *_X)$$

bijective?

In the case where K is finitely generated over $\mathbb{Q}$ and X is a hyperbolic curve (cf. Definition 1.1.1), Grothendieck conjectured that the maps discussed in Questions 0.1.1, 0.1.2, and a modified version of the map discussed in Question 0.1.3 (see Conjecture 2.1 for this modified version) are bijective [2]. Question 0.1.1 (resp. 0.1.2; 0.1.3) is called the Isom version of the Grothendieck conjecture (resp. the Hom version of the Grothendieck conjecture; the Grothendieck section conjecture).

Suppose that X is a hyperbolic curve. In the case where K is finitely generated over $\mathbb{Q}$, Y is also a hyperbolic curve, and at least one of X and Y is affine, Question 0.1.1 was affirmatively answered by Tamagawa [13]. In the case where K is a sub-p-adic field (i.e., a subfield of a field finitely generated over $\mathbb{Q}_p$ (cf. Definition 1.4)) and Y is a smooth variety, Question 0.1.2 was affirmatively answered by Mochizuki (cf. [7] Theorem A). Also, the injectivity portion of Question 0.1.3 was proved in [7] (cf. Lemma 2.2).

Suppose that X is a hyperbolic polycurve (cf. Definition 1.1), that is, a variety X over K which admits a structure of successive smooth fibrations

$$(1) \qquad\qquad X = X_n \xrightarrow{f_n} X_{n-1} \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_2} X_1 \xrightarrow{f_1} \mathrm{Spec}\,K$$

whose fibers are hyperbolic curves. A hyperbolic polycurve is regarded as a higher dimensional analogue of a hyperbolic curve, and has been studied in anabelian geometry. In the case where K is sub-p-adic and $n \leq 4$, Question 0.1.2 was affirmatively answered by Hoshi under some conditions (cf. [4] Theorem A). Then he solved Question 0.1.1 as a corollary. Moreover, in the case where X is a strongly hyperbolic Artin neighborhood ([12] Definition

6.1) and K is finitely generated over $\mathbb{Q}$, Question 0.1.1 was affirmatively answered by Stix and Schmidt [12].

Suppose that X is a hyperbolic polycurve of dimension 2. [4] Theorem 3.14, which is a sort of the Hom version of the Grothendieck conjecture, states that every element of the set

$$\operatorname{Hom}_{G_K}^{\mathrm{open}}(\pi_1(Y, *_Y), \pi_1(X, *_X))/\operatorname{Inn} \pi_1(X_{\overline{K}}, *_X)$$

with topologically finitely generated kernel arises from an element of the set $\operatorname{Mor}_K^{\mathrm{dom}}(Y, X)$. (See [6] Theorem B for a generalization of this theorem.) On the other hand, since there exists a K-morphism $f : Y \to X$ which is not dominant and induces an open outer homomorphism between the étale fundamental groups, we cannot expect that Question 0.1.2 is affirmative (cf. [1] XII Corollaire 3.5). However, we can expect that any open outer group homomorphism from $\pi_1(Y, *_Y)$ to $\pi_1(X, *_X)$ over G_K arises from a nonconstant K-morphism from Y to X.

One of the main results of this paper is as follows:

Theorem 0.2 (cf. Theorem 3.4). *Suppose that K is a sub-p-adic field and Y is a normal variety over K. Let $X_2 \to X_1 \to \operatorname{Spec} K$ be a hyperbolic polycurve of dimension* 2 *over K (cf.* Definition 1.1.2*) and suppose that $X = X_2$. Moreover, suppose that the Grothendieck section conjecture (cf.* Question 0.1.3 *and* Conjecture 2.1*) holds for every hyperbolic curve over a field which is finitely generated extension of K with transcendental degree* 1 *(cf.* Remark 3.5*). Then each element of*

$$\operatorname{Hom}_{G_K}^{\mathrm{open}}(\pi_1(Y, *_Y), \pi_1(X_2, *_X))/\operatorname{Inn} \pi_1(X_{2,\overline{K}}, *_X)$$

arises from an element of $\operatorname{Mor}_K^{\mathrm{nonconst}}(Y, X_2)$. *Here,* $\operatorname{Mor}_K^{\mathrm{nonconst}}(Y, X_2)$ *denotes the subset of* $\operatorname{Mor}_K(Y, X_2)$ *consisting of nonconstant morphisms.*

In [7], the Isom and Hom versions of the pro-p Grothendieck conjecture for hyperbolic curves over sub-p-adic fields were studied. Sawada studied the Isom and Hom versions of the pro-p Grothendieck conjecture for hyperbolic polycurves over sub-p-adic fields under some conditions on their fundamental groups [11]. In Section 4, we give examples of hyperbolic polycurves over sub-p-adic fields which show that the Isom and Hom versions of the pro-p Grothendieck conjecture for hyperbolic polycurves over sub-p-adic fields do not hold generally.

The content of each section is as follows:

In Section 1, we give a review of properties of the étale fundamental groups of hyperbolic polycurves. In Section 2, we review the Grothendieck section conjecture for hyperbolic curves over sub-p-adic fields. In Section 3, we give a proof of Theorem 0.2. In Section 4, we give examples of hyperbolic polycurves which show that the anabelianity of hyperbolic polycurves is weaker than that of hyperbolic curves in some sense.

Terminologies for outer homomorphisms of groups: Let G_1 and G_2 be profinite groups. An outer homomorphism $G_1 \to G_2$ is defined to be an equivalence class of continuous homomorphisms $G_1 \to G_2$, where two such homomorphisms are considered equivalent if they differ by composition with an inner automorphism of G_2. Let $\phi : G_1 \to G_2$ be an outer group homomorphism. Note that the kernel of ϕ is uniquely determined and the image of ϕ is determined uniquely up to conjugation. We shall say that ϕ is open (or, alternatively, ϕ is

an outer open homomorphism) if the image of ϕ is open.

1. Notation and basic properties of the étale fundamental groups of hyperbolic curves

In this section, we fix some notations and definitions. We also prove some properties of inertia subgroups of the étale fundamental groups of hyperbolic curves (cf. Proposition 1.5). We start with the definition of hyperbolic curves.

Definition 1.1. Let S be a scheme.

1. We shall say that a scheme X is a hyperbolic curve over S if the following conditions are satisfied:
   - X is a scheme over S.
   - There exists a scheme $\overline{X}$ proper smooth over S with connected 1-dimensional geometric fibers of genus g.
   - There exists an effective Cartier divisor D of $\overline{X}$ which is finite étale over S of rank r.
   - The open subscheme $\overline{X} \setminus D$ of $\overline{X}$ is isomorphic to X over S.
   - $2g + r - 2 > 0$.
2. We shall say that $X_2 \to X_1 \to S$ is a hyperbolic polycurve of relative dimension 2 over S if $X_2 \to X_1$ and $X_1 \to S$ are hyperbolic curves.

Remark 1.2. Let S be a normal scheme and X a hyperbolic curve over S. Then a pair of schemes $(\overline{X}, D)$ which satisfies the conditions in Definition 1.1.1 is uniquely determined by X up to canonical isomorphism from the argument given in the discussion entitled "Curves" in [9] §0. We shall refer to D as the divisor of cusps of the hyperbolic curve $X \to S$.

Definition 1.3. Let K be a field. We shall say that a scheme X over K is a variety if the morphism $X \to \operatorname{Spec} K$ is separated and of finite type with geometrically connected fibers.

Definition 1.4. Let p be a prime number. We shall say that a field K is a sub-p-adic field if there exist a finitely generated extension field L over $\mathbb{Q}_p$ and an injective homomorphism from K to L.

Proposition 1.5. *Let S be a connected locally Noetherian separated normal scheme over $\mathbb{Q}$ and $X \to S$ a hyperbolic curve. Write D for the divisor of cusps of $X \to S$.*

1. *The divisor D is a disjoint union of finitely many normal schemes which are étale over S.*
2. *Let D_0 be an irreducible component of D. Take a geometric point $*$ of X. Choose a decomposition group G_d of D_0 in $\pi_1(X, *)$ and write $\overline{G_d}$ for the image of G_d in $\pi_1(S, *)$. Then we have the following natural commutative diagram of profinite groups with exact horizontal lines and injective vertical arrows:*

(2)
$$
\begin{array}{ccccccccc}
1 & \longrightarrow & \widehat{\mathbb{Z}}(1) & \longrightarrow & G_d & \longrightarrow & \overline{G_d} & \longrightarrow & 1 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \Delta_{X/S} & \longrightarrow & \pi_1(X,*) & \longrightarrow & \pi_1(S,*) & \longrightarrow & 1.
\end{array}
$$

Here, we write $\Delta_{X/S}$ for the kernel of the homomorphism $\pi_1(X,*) \to \pi_1(S,*)$. Moreover, $\overline{G_d}$ is isomorphic to the étale fundamental group of D_0 in a canonical way up to inner automorphism of $\pi_1(S,*)$.

3. *Let S' be another connected locally Noetherian separated normal scheme and $S' \to S$ a dominant morphism. Suppose that $* \to X$ factors through $* \to X \times_S S' \to X$. Write D'_0 for the irreducible component of the divisor of cusps of $X \times_S S' \to S'$ over D_0 determined by G_d and G'_d for the decomposition group of D'_0 in $\pi_1(X \times_S S', *)$ over G_d. Then we have a natural isomorphism $G'_d \cong G_d \times_{\pi_1(S,*)} \pi_1(S', *)$.*

Proof. Since the morphism $D \to S$ is étale, the assertion 1 holds. Next, we show the assertion 2. We may assume that $*$ is a geometric generic point. Let $K(S)$ be the function field of S and $G_{K(S)}$ the absolute Galois group of $K(S)$ determined by $*$. Write $X_{K(S)}$ for the scheme $X \times_S \operatorname{Spec} K(S)$. Then $D_0 \times_S \operatorname{Spec} K(S)$ is an irreducible component of the divisor of cusps of the hyperbolic curve $X_{K(S)} \to \operatorname{Spec} K(S)$. Choose a decomposition group $G_d^{K(S)}$ of $D_0 \times_S \operatorname{Spec} K(S)$ in $\pi_1(X_{K(S)}, *)$ over G_d and write $\overline{G_d}^{K(S)}$ for the image of $G_d^{K(S)}$ in $G_{K(S)}$. We obtain the following diagram of profinite groups with exact horizontal lines by [13] Lemma (2.2) and [4] Proposition 2.4 (i)(ii):

$$
\begin{array}{ccccccccc}
1 & \longrightarrow & \widehat{\mathbb{Z}}(1) & \longrightarrow & G_d^{K(S)} & \longrightarrow & \overline{G_d}^{K(S)} & \longrightarrow & 1 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \Delta_{X/S} & \longrightarrow & \pi_1(X_{K(S)}, *) & \longrightarrow & G_{K(S)} & \longrightarrow & 1 \\
& & \| & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \Delta_{X/S} & \longrightarrow & \pi_1(X,*) & \longrightarrow & \pi_1(S,*) & \longrightarrow & 1.
\end{array}
$$

Note that $D_0 \times_S \operatorname{Spec} K(S)$ is the spectrum of a finite separable extension field of $K(S)$ and, by [13] Lemma (2.2), $\overline{G_d}^{K(S)}$ is isomorphic to the absolute Galois group of this field. Since the homomorphism $G_d^{K(S)} \to G_d$ is surjective and D_0 is finite étale over S, the assertion 2 holds. The assertion 3 follows from the assertion 2 and [4] Proposition 2.4 (i)(ii).   □

2. The Grothendieck section conjecture

In this section, we recall the Grothendieck section conjecture for hyperbolic curves over sub-p-adic fields.

Let K be a field of characteristic 0, $\overline{K}$ an algebraic closure of K, G_K the absolute Galois group $\operatorname{Gal}(\overline{K}/K)$, X a hyperbolic curve over K, and D the divisor of cusps of the hyperbolic curve X. Write $X_{\overline{K}}$ for the scheme $X \times_{\operatorname{Spec} K} \operatorname{Spec} \overline{K}$. Take a geometric point $*$ of $X_{\overline{K}}$. Write $\operatorname{Sect}_{G_K}(\pi_1(X,*))$ for the set of continuous sections of the homomorphism $\pi_1(X,*) \to G_K$.

First, we state "the Grothendieck section conjecture" in a general setting.

Conjecture 2.1 (cf. Question 0.1.3). **   1.** *Suppose that X is a proper hyperbolic curve over K. Then the natural map*

(3)
$$X(K) \to \mathrm{Sect}_{G_K}(\pi_1(X, *))/\mathrm{Inn}\, \pi_1(X_{\overline{K}}, *)$$

is bijective.

2. *Write $\mathrm{Sect}_{G_K}^{CD}(\pi_1(X, *))$ for the subset of $\mathrm{Sect}_{G_K}(\pi_1(X, *))$ consisting of sections whose images are contained in a decomposition group of some closed point of D. Then the map* (3) *induces a map*

(4)
$$X(K) \to (\mathrm{Sect}_{G_K}(\pi_1(X, *)) \setminus \mathrm{Sect}_{G_K}^{CD}(\pi_1(X, *)))/\mathrm{Inn}\, \pi_1(X_{\overline{K}}, *)$$

and this map is bijective (cf. Example 2.4.1).

Lemma 2.2. *Suppose that K is a sub-p-adic field. The map* (4) *is well-defined and injective.*

Proof. The well-definedness portion follows from the proof of [10] Theorem 1.3 (iv) and [7] Theorem C. The injectivity portion follows from [7] Theorem C.   □

REMARK 2.3.   1. Suppose that K is a generalized sub-p-adic (not necessarily sub-p-adic) field. In this case, as written in [3] Introduction, the injectivity portion of Lemma 2.2 also holds (cf. the proof of [7] Theorem C and [8] Theorem 4.12 and Remark following Theorem 4.12). Moreover, the well-definedness portion of Lemma 2.2 also holds by its proof.

2. There exist (generalized) sub-p-adic fields such that the Grothendieck section conjecture does not hold for hyperbolic curves over them. Let p be a prime number and suppose that K is the field of fractions of a henselization of $\mathbb{Z}_{(p)}$. Write $\widehat{K}$ for the completion of the field K. Let $\overline{\widehat{K}}$ be an algebraic closure of $\widehat{K}$ and fix an embedding $\overline{K} \hookrightarrow \overline{\widehat{K}}$ over K. Then we have $\mathrm{Gal}(\overline{K}/K) \cong \mathrm{Gal}(\overline{\widehat{K}}/\widehat{K})$. Suppose that $X(\widehat{K})$ has uncountably infinitely many $\widehat{K}$-rational points. (For example, suppose that X has a K-rational point x and a finite morphism $X \to \mathbb{P}^1_K$ étale at x. Then, by the theory of locally analytic manifolds and the implicit function theorem, X has uncountably infinitely many $\widehat{K}$-rational points.) Since the cardinality of the set $X(K)$ is at most countable, the induced map $X(K) \to X(\widehat{K})$ is not surjective. Therefore, the Grothendieck section conjecture for X does not hold.

EXAMPLE 2.4. Suppose that D has a K-rational point x.

1. We show that, in the case where X is affine, the map (3) is not surjective in general. The decomposition group of x in the fundamental group $\pi_1(X, *)$ is isomorphic to the absolute Galois group $G_{K((T))}$ of the field of Laurent series over K by [13] Lemma (2.2). Since the characteristic of K is 0, there exists a continuous section of the homomorphism $G_{K((T))} \to G_K$. (Indeed, we can construct such a section by considering a compatible system $(T^{1/n})_{n \geq 1}$.) Therefore, we obtain a section $G_K \to \pi_1(X, *)$ which is not defined by a rational point of X by Lemma 2.2.

2. Here, we give an example of outer homomorphism over G_K between the étale fundamental group of hyperbolic curves over K. We do not fix geometric points and do

not write base points of étale fundamental groups. The morphism $\operatorname{Spec} K((T)) \to \operatorname{Spec} K[T, 1/T]$ induces an outer isomorphism

$$(G_{K((T))} =)\,\pi_1(\operatorname{Spec} K((T))) \to \pi_1(\operatorname{Spec} K[T, \frac{1}{T}])$$

between their fundamental groups. By composing the surjective outer homomorphism $\pi_1(\mathbb{P}^1_K \setminus \{0, 1, \infty\}) \to \pi_1(\operatorname{Spec} K[T, \frac{1}{T}])$ induced by the open immersion $\mathbb{P}^1_K \setminus \{0, 1, \infty\} \to \operatorname{Spec} K[T, \frac{1}{T}]$, the inverse of the above outer isomorphism, and an outer isomorphism from $G_{K((T))}$ to a decomposition group of x in $\pi_1(X)$, we obtain an outer homomorphism $\phi : \pi_1(\mathbb{P}^1_K \setminus \{0, 1, \infty\}) \to \pi_1(X)$ whose image is a decomposition group of x. Therefore, $\operatorname{Im} \phi$ neither is open in $\pi_1(X)$ nor determines a section of the homomorphism $\pi_1(X) \to G_K$.

3. Sections for hyperbolic polycurves of dimension 2

In this section, we prove the Hom version of the Grothendieck conjecture for morphisms from regular varieties to hyperbolic polycurves of dimension 2 over sub-p-adic fields under the assumption that the Grothendieck section conjecture for hyperbolic curves holds.

Let K be a field of characteristic 0, $X_2 \to X_1 \to \operatorname{Spec} K$ a hyperbolic polycurve of dimension 2 over K, K_1 the function field of X_1, $\overline{K}_1$ an algebraic closure of K_1, and $\overline{K}$ the algebraic closure of K in K_1. Write G_K (resp. G_{K_1}) for the absolute Galois group $\operatorname{Gal}(\overline{K}_1/K_1)$ (resp. $\operatorname{Gal}(\overline{K}/K)$) and X_{2,K_1} for the scheme $X_2 \times_{X_1} \operatorname{Spec} K_1$. In this section, for any normal variety W over K or K_1, we consider a geometric point of $W \times_{\operatorname{Spec} K} \operatorname{Spec} \overline{K}$ or $W \times_{\operatorname{Spec} K_1} \operatorname{Spec} \overline{K}_1$ and write Π_W (resp. Δ_W) for the étale fundamental group of W (resp. $W \times_{\operatorname{Spec} K} \operatorname{Spec} \overline{K}$ or $W \times_{\operatorname{Spec} K_1} \operatorname{Spec} \overline{K}_1$). We omit base points of étale fundamental groups in this notation, because we only consider outer homomorphisms unless otherwise noted. Write $\Delta_{2,1}$ for the kernel of the homomorphism $\Pi_{X_2} \to \Pi_{X_1}$ induced by the structure morphism $X_2 \to X_1$. Since the profinite group $\Pi_{X_{2,K_1}}$ is isomorphic to the profinite group $\Pi_{X_2} \times_{\Pi_{X_1}} G_{K_1}$ by [4] Proposition 2.4 (ii), we have the following commutative diagram of profinite groups with exact horizontal lines:

$$
\begin{array}{ccccccccc}
1 & \longrightarrow & \Delta_{2,1} & \longrightarrow & \Pi_{X_{2,K_1}} & \longrightarrow & G_{K_1} & \longrightarrow & 1 \\
& & \| & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \Delta_{2,1} & \longrightarrow & \Pi_{X_2} & \longrightarrow & \Pi_{X_1} & \longrightarrow & 1.
\end{array}
$$

We write $\operatorname{Sect}_{\Pi_{X_1}}(\Pi_{X_2})$ for the set of continuous sections of the homomorphism $\Pi_{X_2} \to \Pi_{X_1}$. Let $(\overline{X}_2, D)$ be the smooth compactification of the hyperbolic curve $X_2 \to X_1$ (cf. Remark 1.2). Since X_1 is normal, we have a decomposition $D = \coprod_{1 \le i \le n} D_i$ by Proposition 1.5.1, where each D_i is a normal scheme. Write θ_i for the generic point of D_i. We shall write $\operatorname{Sect}_{\Pi_{X_1}}^{CD}(\Pi_{X_2})$ for the set of continuous sections of the homomorphism $\Pi_{X_2} \to \Pi_{X_1}$ whose images are contained in a decomposition group of some θ_i in Π_{X_2}.

Lemma 3.1. *There exists a natural injective map*

$$\operatorname{Sect}_{\Pi_{X_1}}(\Pi_{X_2})/\operatorname{Inn}(\Delta_{2,1}) \to \operatorname{Sect}_{G_{K_1}}(\Pi_{X_{2,K_1}})/\operatorname{Inn}(\Delta_{2,1})$$

which induces a map

$$\mathrm{Sect}^{CD}_{\Pi_{X_1}}(\Pi_{X_2})/\mathrm{Inn}\,(\Delta_{2,1}) \to \mathrm{Sect}^{CD}_{G_{K_1}}(\Pi_{X_{2,K_1}})/\mathrm{Inn}\,(\Delta_{2,1})$$

and a map

$$(\mathrm{Sect}_{\Pi_{X_1}}(\Pi_{X_2}) \setminus \mathrm{Sect}^{CD}_{\Pi_{X_1}}(\Pi_{X_2}))/\mathrm{Inn}\,(\Delta_{2,1})$$
$$\to (\mathrm{Sect}_{G_{K_1}}(\Pi_{X_{2,K_1}}) \setminus \mathrm{Sect}^{CD}_{G_{K_1}}(\Pi_{X_{2,K_1}}))/\mathrm{Inn}\,(\Delta_{2,1}).$$

Proof. Since the group $\Pi_{X_{2,K_1}}$ is isomorphic to the group $\Pi_{X_2} \times_{\Pi_{X_1}} G_{K_1}$ by [4] Proposition 2.4 (ii), we obtain a natural map

$$\mathrm{Sect}_{\Pi_{X_1}}(\Pi_{X_2})/\mathrm{Inn}\,(\Delta_{2,1}) \to \mathrm{Sect}_{G_{K_1}}(\Pi_{X_{2,K_1}})/\mathrm{Inn}\,(\Delta_{2,1}).$$

The injectivity of this map follows from the surjectivity of the homomorphism $G_{K_1} \to \Pi_{X_1}$.

Let $s_X : \Pi_{X_1} \to \Pi_{X_2}$ be a section of the homomorphism $\Pi_{X_2} \to \Pi_{X_1}$ and θ an element of $\{\theta_i \mid 1 \leq i \leq n\}$. Write G_d^{XK} for a decomposition group of θ in $\Pi_{X_{2,K_1}}$ and G_d^X for the image of G_d^{XK} in Π_{X_2}. Note that G_d^X coincides with a decomposition group of θ in Π_{X_2}. Write s_{XK} for the section of the homomorphism $\Pi_{X_{2,K_1}} \to G_{K_1}$ determined by s_X. It suffices to show that the image of the homomorphism s_{XK} is contained in G_d^{XK} if and only if the image of the homomorphism s_X is contained in G_d^X. This follows from Proposition 1.5.3. Hence, we finish the proof of Lemma 3.1. $\square$

Theorem 3.2. *Let $X_2 \to X_1 \to \mathrm{Spec}\,K$ be a hyperbolic polycurve of dimension 2 over K. Suppose that the Grothendieck section conjecture holds for the hyperbolic curve $X_{2,K_1} \to \mathrm{Spec}\,K_1$. Then the natural map*

$$(5) \qquad\qquad \mathrm{Sect}_{X_1}(X_2) \to (\mathrm{Sect}_{\Pi_{X_1}}(\Pi_{X_2}))/\mathrm{Inn}\,(\Delta_{2,1})$$

factors through

$$(6) \qquad\qquad \mathrm{Sect}_{X_1}(X_2) \to (\mathrm{Sect}_{\Pi_{X_1}}(\Pi_{X_2}) \setminus \mathrm{Sect}^{CD}_{\Pi_{X_1}}(\Pi_{X_2}))/\mathrm{Inn}\,(\Delta_{2,1})$$

and the homomorphism (6) *is bijective.*

Proof. Consider the following diagram:

$$
\begin{array}{ccccc}
\mathrm{Sect}_{X_1}(X_2) & \dashrightarrow & S(\Pi \setminus CD) & \hookrightarrow & S(\Pi) \\
\downarrow & & \downarrow & & \downarrow \\
X_{2,K_1}(K_1) & \longrightarrow & S(G \setminus CD) & \hookrightarrow & S(G),
\end{array}
$$

where we write $S(\Pi \setminus CD)$ (resp. $S(\Pi)$; $S(G \setminus CD)$; $S(G)$) for the set $(\mathrm{Sect}_{\Pi_{X_1}}(\Pi_{X_2}) \setminus \mathrm{Sect}^{CD}_{\Pi_{X_1}}(\Pi_{X_2}))/\mathrm{Inn}\,(\Delta_{2,1})$ (resp. $(\mathrm{Sect}_{\Pi_{X_1}}(\Pi_{X_2}))/\mathrm{Inn}\,(\Delta_{2,1})$; $(\mathrm{Sect}_{G_{K_1}}(\Pi_{X_{2,K_1}}) \setminus \mathrm{Sect}^{CD}_{G_{K_1}}(\Pi_{X_{2,K_1}}))/\mathrm{Inn}\,(\Delta_{2,1})$; $(\mathrm{Sect}_{G_{K_1}}(\Pi_{X_{2,K_1}}))/\mathrm{Inn}\,(\Delta_{2,1})$). The right rectangle is discussed in Lemma 3.1. The first vertical arrow is induced by base change, and hence injective. The curved arrow in the first horizontal line is (5) and the biggest rectangle of the diagram is commutative. The left homomorphism of the second horizontal line is bijective by the assumption of Theorem 3.2. By using these discussion and Lemma 3.1, (6) is induced and injective. Moreover, each element of

$$(\text{Sect}_{\Pi_{X_1}}(\Pi_{X_2}) \setminus \text{Sect}_{\Pi_{X_1}}^{CD}(\Pi_{X_2}))/\text{Inn}(\Delta_{2,1})$$

is defined by a section of the morphism $X_2 \to X_1$ by [4] Lemma 2.10 and the surjectivity of the first homomorphism of the second horizontal line. Hence, we finish the proof of Theorem 3.2. $\qquad\square$

Corollary 3.3. *Suppose that the morphism $X_2 \to X_1$ is proper and the Grothendieck section conjecture holds for the hyperbolic curve $X_{2,K_1} \to \text{Spec } K_1$. Then the map $\text{Sect}_{X_1}(X_2) \to \text{Sect}_{\Pi_{X_1}}(\Pi_{X_2})/\text{Inn}(\Delta_{2,1})$ is bijective.*

Proof. Since the morphism $X_2 \to X_1$ is proper, we have $\text{Sect}_{\Pi_{X_1}}^{CD}(\Pi_{X_2}) = \emptyset$. Therefore, Corollary 3.3 follows from Theorem 3.2. $\qquad\square$

Theorem 3.4. *Suppose that K is a sub-p-adic field. Let Y be a normal variety over K. Suppose that the Grothendieck section conjecture holds for every hyperbolic curve over a field which is finitely generated over K of transcendental degree 1 (cf. Remark 3.5). Then, for any outer open homomorphism $\phi \in \text{Hom}_{G_K}^{\text{open}}(\Pi_Y, \Pi_{X_2})/\text{Inn}(\Delta_X)$, there exists a nonconstant morphism $Y \to X$ inducing ϕ.*

Proof. Write ϕ_1 for the composite outer homomorphism

$$\Pi_Y \xrightarrow{\phi} \Pi_{X_2} \to \Pi_{X_1}.$$

Then the outer homomorphism ϕ_1 is induced by a unique dominant K-morphism $f_1 : Y \to X_1$ by [4] Theorem 3.3. Write K_1' for the normalization of K_1 in the function field of Y, η for the scheme $\text{Spec } K_1'$, X_1' for the open subscheme of the normalization of X_1 in K_1' determined by the image of Y, Y_η for the scheme $Y \times_{X_1'} \eta$, and $G_{K_1'}$ for the étale fundamental group of η (, which is isomorphic to the absolute Galois group of K_1'). Then we have the following commutative diagram of profinite groups:

If the image of the induced outer homomorphism

$$(7) \qquad\qquad \text{Ker}(\Pi_{Y_\eta} \to G_{K_1'}) \to \Delta_{2,1}$$

is nontrivial, ϕ arises from a morphism $Y \to X_2$ over K by [4] Lemma 3.4 (iv). Suppose that the outer homomorphism (7) is trivial. Note that we have natural isomorphisms $\Pi_{X_2 \times_{X_1} \eta} \cong \Pi_{X_2} \times_{\Pi_{X_1}} G_{K_1'}$ and $\text{Ker}(\Pi_{X_2 \times_{X_1} \eta} \to G_{K_1'}) \simeq \Delta_{2,1}$ by [2] Proposition 2.4 (ii) and the outer homomorphism $\Pi_{Y_\eta} \to G_{K_1'}$ is surjective. Hence, the image of the induced outer homomorphism $\Pi_{Y_\eta} \to \Pi_{X_2 \times_{X_1} \eta}$ defines a section s of the outer homomorphism $\Pi_{X_2 \times_{X_1} \eta} \to G_{K_1'}$. Suppose that

$$s \in \mathrm{Sect}_{G_{K_1'}}^{CD}(\Pi_{X_2 \times_{X_1} \eta})/\mathrm{Inn}(\Delta_{2,1}).$$

Then the group $\mathrm{Im}(\Pi_{Y_\eta} \to \Pi_{X_2})$ is not open in Π_{X_2} by Proposition 1.5.2 and 3. Since the outer homomorphism $\Pi_{Y_\eta} \to \Pi_Y$ is surjective, the image of ϕ coincides with $\mathrm{Im}(\Pi_{Y_\eta} \to \Pi_{X_2})$. Therefore, the image of ϕ is not open, which contradicts the assumption on ϕ. By the Grothendieck section conjecture for the hyperbolic curve $X_2 \times_{X_1} \eta \to \eta$, we have a K_1'-morphism $Y_\eta \to X_2 \times_{X_1} \eta$ inducing the outer homomorphism $\Pi_{Y_\eta} \to \Pi_{X_2 \times_{X_1} \eta}$. Then by [4] Lemma 2.10, there exists a K-morphism $Y \to X_2$ inducing ϕ such that the composite morphism $Y_\eta \to Y \to X_2$ coincides with the composite morphism $Y_\eta \to X_2 \times_{X_1} \eta \to X_2$. □

REMARK 3.5. Let Y be as in Theorem 3.4. Suppose that the Grothendieck section conjecture holds for every hyperbolic curve over a field which is finitely generated over K of transcendental degree $\dim Y$ and the morphism $X_2 \to X_1$ is proper. Write η (resp. G_η) for the spectrum (resp. the absolute Galois group) of the function field of Y. Then we have a diagram of profinite groups
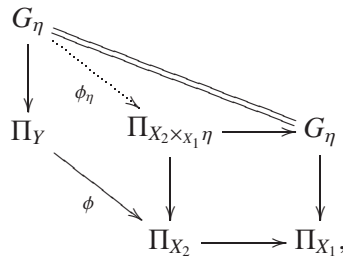
where ϕ_η is the outer homomorphism induced by using the isomorphism

$$\Pi_{X_2 \times_{X_1} \eta} \cong \Pi_{X_2} \times_{\Pi_{X_1}} G_\eta.$$

By Grothendieck section conjecture and [4] Lemma 2.10, we can prove that ϕ is induced by a K-morphism $Y \to X_2$. Then we can show Theorem 3.4 without using the assumption that ϕ is open.

4. Examples of hyperbolic polycurves

In this section, we give examples of hyperbolic polycurves which show that the anabelianity of hyperbolic polycurves is weaker than that of hyperbolic curves in some sense.

As we write in Section 0, Mochizuki proved the Hom version of the pro-p Grothendieck conjecture for hyperbolic curves over sub-p-adic fields (cf. [7]). Moreover, Sawada proved a pro-p analogue of [4] Theorem A under a certain assumption on the étale fundamental groups of hyperbolic polycurves (cf. [11]). We construct examples which show that the Isom version of the pro-p Grothendieck conjecture for hyperbolic polucurves over sub-p-adic fields does not hold in general in this section.

Let K be a field of characteristic 0, $\overline{K}$ an algebraic closure of K, and p a prime number.

NOTATION-DEFINITION 4.1. 1. Let G be a profinite group. We write G^p for the maximal pro-p quotient of G (i.e., the inverse limit of the inverse system consisting of the

quotient groups of G by open normal subgroups such that the orders of the quotient groups are powers of p).

2. For any variety X over K, we write Π_X (resp. Δ_X; $\Pi_X^{(p)}$) for the étale fundamental group of X (resp. the étale fundamental group of $X \times_{\mathrm{Spec}\,K} \mathrm{Spec}\,\overline{K}$; the quotient group $\Pi_X/\mathrm{Ker}(\Delta_X \to \Delta_X^p)$) in this section.

First, we prove an elementary lemma.

Lemma 4.2. *Let*

$$1 \to N \to G \to H \to 1$$

be an exact sequence of profinite groups.

1. *We have an exact sequence*

$$(N/[N, \mathrm{Ker}(G \to G^p)])^p \to G^p \to H^p \to 1.$$

   Here, "$[-,-]$" denotes the topological closure of the commutator subgroup.

2. *Suppose that we have a section $s : H \to G$ of the homomorphism $G \to H$ and write $N_{\mathrm{Ker}(H \to H^p)}$ for the maximal quotient group of N on which $\mathrm{Ker}(H \to H^p)$ acts trivially. Then we have an exact sequence*

$$(N_{\mathrm{Ker}(H \to H^p)})^p \to G^p \to H^p \to 1.$$

Proof.   1. Since the image of $[N, \mathrm{Ker}(G \to G^p)]$ in G^p is trivial, we obtain an exact sequence

$$N/[N, \mathrm{Ker}(G \to G^p)] \to G^p \to H^p \to 1$$

and hence also an exact sequence

$$(N/[N, \mathrm{Ker}(G \to G^p)])^p \to G^p \to H^p \to 1.$$

2. Since we have $s(\mathrm{Ker}(H \to H^p)) \subset \mathrm{Ker}(G \to G^p)$, the assertion follows from 1.

$\square$

We show a lemma for Example 4.4.

Lemma 4.3. *Suppose that $p \neq 2$. Let H be a hyperelliptic curve over K and ι the hyperelliptic involution of H. Suppose that there exist K-rational points h, h' of H which are fixed by the action of ι. By considering a geometric point over the fixed point h, we obtain actions of ι on $\Delta_{H\setminus\{h'\}}$ and Δ_H. Then we have $(\Delta_H)_{\langle\iota\rangle}^p = \{1\}$ and $(\Delta_{H\setminus\{h'\}})_{\langle\iota\rangle}^p = \{1\}$.*

Proof. Since the profinite groups $\Delta_{H\setminus\{h'\}}$ and Δ_H are topologically finitely generated, it suffices to show that $(\Delta_H)_{\langle\iota\rangle}^{p,\mathrm{ab}} = \{1\}$ and $(\Delta_{H\setminus\{h'\}})_{\langle\iota\rangle}^{p,\mathrm{ab}} = \{1\}$. By [5] Lemma 1.11, the action of ι on the abelian profinite group $(\Delta_{H\setminus\{h'\}}^{p,\mathrm{ab}}) \cong (\Delta_H^{p,\mathrm{ab}})$ is same as the multiplication by -1. Therefore,

$$(\Delta_H^{p,\mathrm{ab}})_{\langle\iota\rangle} = \Delta_H^{p,\mathrm{ab}}/2\Delta_H^{p,\mathrm{ab}} = \{1\}.$$

$\square$

EXAMPLE 4.4. Suppose that $p \neq 2$ and K is a finite extension field of $\mathbb{Q}_p$. We construct a proper hyperbolic polycurve Z over a field K, such that the natural map

$$\mathrm{Isom}_K(Z,Z) \to \mathrm{Isom}_{G_K}(\Pi_Z^{(p)}, \Pi_Z^{(p)})/\mathrm{Inn}(\Delta_Z^p)$$

is not injective.   Here, $\mathrm{Isom}_K(Z,Z)$ is the set of automorphisms of Z over K, and $\mathrm{Isom}_{G_K}(\Pi_Z^{(p)}, \Pi_Z^{(p)})$ is the set of automorphisms of $\Pi_Z^{(p)}$ over G_K. This shows that it is impossible to detect an automorphism of a hyperbolic polycurve from the corresponding G_K-outer automorphism of its pro-p fundamental group. In particular, the Isom version of the pro-p Grothendieck conjecture, which is true for hyperbolic curves ([7]) or hyperbolic polycurves with suitable conditions up to dimension 4 ([11]), cannot be true for general hyperbolic polycurves.

Let X_1 be a proper hyperbolic curve over K, and assume that there exists a homomorphism $\Pi_{X_1} \to \mathbb{Z}/2\mathbb{Z}$ which induces a surjection $\Delta_{X_1} \to \mathbb{Z}/2\mathbb{Z}$. We write $X_1' \to X_1$ for the étale covering space of X_1 corresponding to $\mathrm{Ker}\,(\Pi_{X_1} \to \mathbb{Z}/2\mathbb{Z})$ and ι_1 for a generator of $\mathrm{Aut}(X_1'/X_1)$. Let X_2 be a hyperbolic curve over K whose automorphism group over K has a subgroup isomorphic to $\mathbb{Z}/2\mathbb{Z} = \langle \iota_2 \rangle$ such that X_2 has a fixed point x_2 under the action of $\mathbb{Z}/2\mathbb{Z}(= \langle \iota_2 \rangle)$. Moreover, assume that the maximal quotient group $(\Delta_{X_2})_{\mathbb{Z}/2\mathbb{Z}}^p$ of $(\Delta_{X_2})^p$ on which $\mathbb{Z}/2\mathbb{Z}$ acts trivially via a geometric point over x_2 is trivial (cf. Lemma 4.3).

Consider the action of $\mathbb{Z}/2\mathbb{Z}$ on $X_2 \times_{\mathrm{Spec}\,K} X_1'$ induced by (ι_2, ι_1). Write Z for the quotient scheme of $X_2 \times_{\mathrm{Spec}\,K} X_1'$ by this $\mathbb{Z}/2\mathbb{Z}$-action. By construction, we have a Cartesian diagram

$$
\begin{array}{ccc}
X_2 \times_{\mathrm{Spec}\,K} X_1' & \longrightarrow & X_1' \\
\downarrow & & \downarrow \\
Z & \longrightarrow & X_1.
\end{array}
$$

Since the morphism $X_1' \to X_1$ is finite etale, $Z \to X_1$ is a hyperbolic curve whose geometric generic fiber coincides with that of $X_2 \times_{\mathrm{Spec}\,K} X_1' \to X_1'$. Hence, we obtain exact sequences of profinite groups

$$1 \to \Delta_{X_2} \to \Pi_Z \to \Pi_{X_1} \to 1$$

and

$$1 \to \Delta_{X_2} \to \Delta_Z \to \Delta_{X_1} \to 1$$

by [4] Proposition 2.4 (i). Since the section $X_1' \to X_2 \times_{\mathrm{Spec}\,K} X_1'$ of the morphism $X_2 \times_{\mathrm{Spec}\,K} X_1' \to X_1'$ determined by the point x_2 is compatible with the actions of $\mathbb{Z}/2\mathbb{Z}$, we have a section $X_1 \to Z$ of the morphism $Z \to X_1$ by taking the quotient schemes by $\mathbb{Z}/2\mathbb{Z}$. Therefore, the homomorphism $\Pi_Z \to \Pi_{X_1}$ has a section which also determines a section of the homomorphism $\Delta_Z \to \Delta_{X_1}$. Consider the following restriction (to Δ_{X_1}) of the action of Π_{X_1} on Δ_{X_2} induced by the section:

(8) $\Delta_{X_1}(\subset \Pi_{X_1}) \to \mathrm{Aut}(\Delta_{X_2})$.

By the construction of Z, the action (8) coincides with the composite homomorphism

$$\Delta_{X_1} \to \Delta_{X_1}/\Delta_{X_1'} \simeq \Pi_{X_1}/\Pi_{X_1'}$$
$$= \langle \iota_1 \rangle \cong \mathbb{Z}/2\mathbb{Z}$$
$$\cong \langle \iota_2 \rangle \subset \{f \in \mathrm{Aut}(X_2/\mathrm{Spec}\, K) \mid f(x_2) = x_2\} \to \mathrm{Aut}(\Delta_{X_2}).$$

Since the image of the restriction of the action (8) to the subgroup $\mathrm{Ker}(\Delta_{X_1} \to \Delta_{X_1}^p)$, i.e., the image of the composite homomorphism

$$\mathrm{Ker}(\Delta_{X_1} \to \Delta_{X_1}^p) \subset \Delta_{X_1}(\subset \Pi_{X_1}) \to \mathrm{Aut}(\Delta_{X_2}),$$

is $\langle \iota_2 \rangle$ by the assumption $2 \neq p$, the group $\mathrm{Ker}\,(\Delta_Z^p \to \Delta_{X_1}^p)$ is a quotient group of $(\Delta_{X_2})_{\langle \iota_2 \rangle}^p$ by Lemma 4.2.2. Thus, we have

$$\Delta_Z^p \cong \Delta_{X_1}^p$$

by the assumption that $(\Delta_{X_2})_{\langle \iota_2 \rangle}^p$ is trivial. Hence, we have

$$\Pi_Z^{(p)} \cong \Pi_{X_1}^{(p)}.$$

It suffices to show that the scheme Z has a nontrivial automorphism over X_1, since such an automorphism induces the trivial outer automorphism of $\Pi_Z^{(p)}(\cong \Pi_{X_1}^{(p)})$ (over G_K). Since the automorphism $(\iota_2, \mathrm{id}_{X_1'})$ of $X_2 \times_{\mathrm{Spec}\, K} X_1'$ over X_1' is compatible with the diagonal action of $\mathbb{Z}/2\mathbb{Z}$, this automorphism defines a nontrivial automorphism of Z over X_1.

Even if we change X_2 to another hyperbolic curve satisfying the above condition for X_2, the geometrically pro-p étale fundamental group ($\Pi^{(p)} = \Pi/\mathrm{Ker}(\Delta \to \Delta^p)$) of the resulting polycurve is isomorphic to $\Pi_Z^{(p)}$ over G_K, since we have the isomorphism $\Pi_Z^{(p)} \cong \Pi_{X_1}^{(p)}$. Therefore this example gives a counterexample to the Isom version of the pro-p Grothendieck conjecture for hyperbolic polycurves. Since we have the isomorphism $\Delta_Z^p \cong \Delta_{X_1}^p$, we cannot even determine the dimension of a hyperbolic polycurve X over $\overline{K}$ from its pro-p étale fundamental group Δ_X^p.

EXAMPLE 4.5. We give another example of non-isomorphic hyperbolic polycurves over a mixed characteristic local field K with residual field of characteristic p and of order q, whose geometrically pro-p étale fundamental groups are isomorphic over G_K. This gives another counterexample to the Isom version of the pro-p Grothendieck conjecture for hyperbolic polycurves.

Let l be a prime number such that $l|q-1$. Let X_2 be the hyperbolic curve $\mathbb{P}_K^1 \setminus (\{\infty\} \cup \mu_l)$ over K. Fix a primitive l-th root of unity $\zeta \in \mu_l$. Let $\iota : \mathbb{P}_K^1 \to \mathbb{P}_K^1$ be the automorphism $z \mapsto z\zeta$. The morphism ι induces a $\mathbb{Z}/l\mathbb{Z}$-action on X_2 over K which fixes $0 \in X_2$. Let X_1 be a hyperbolic curve over K, and assume that there exists a homomorphism $\Pi_{X_1} \to \mathbb{Z}/l\mathbb{Z}$ which induces a surjection $\Delta_{X_1} \to \mathbb{Z}/l\mathbb{Z}$. We can obtain a scheme Z via the construction same as that in Example 4.4 by replacing $\mathbb{Z}/2\mathbb{Z}$ by $\mathbb{Z}/l\mathbb{Z}$. Then the fixed point $0 \in X_2$ defines a section $X_1 \to Z$, which determines sections $\Delta_{X_1} \to \Delta_Z$ and $\Pi_{X_1} \to \Pi_Z$. Since $p \neq l$, we obtain an exact sequence

$$(\Delta_{X_2})_{\langle \iota \rangle}^p \to \Delta_Z^p \to \Delta_{X_1}^p \to 1$$

by using the same argument as that in Example 4.4. The group $(\Delta_{X_2})_{\langle \iota \rangle}^{p,\mathrm{ab}}$ is generated by 1 element, which shows that the group $(\Delta_{X_2})_{\langle \iota \rangle}^p$ is an abelian group. Therefore, the kernel

of the homomorphism $\Delta_Z^p \to \Delta_{X_1}^p$ is a quotient group of $(\Delta_{X_2})_{\langle \iota \rangle}^{\mathrm{ab}}$. Since we have $(\Delta_{X_2})_{\langle \iota \rangle}^{\mathrm{ab}} = (\Delta_{X_2})_{\Delta_{X_1}}^{\mathrm{ab}} = (\Delta_{X_2}/[\Delta_{X_2}, \Delta_{X_2}])_{\Delta_{X_1}} = \Delta_{X_2}/[\Delta_{X_2}, \Delta_Z]$, we obtain the commutative diagram with exact horizontal lines

$$
\begin{array}{ccccccccc}
1 & \longrightarrow & \Delta_{X_2} & \longrightarrow & \Delta_Z & \longrightarrow & \Delta_{X_1} & \longrightarrow & 1 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & (\Delta_{X_2})_{\langle \iota \rangle}^{\mathrm{ab}} & \longrightarrow & \Delta_Z/[\Delta_{X_2}, \Delta_Z] & \longrightarrow & \Delta_{X_1} & \longrightarrow & 1.
\end{array}
$$

The second line of this diagram also splits, and thus we have the decomposition

$$
\Delta_Z/[\Delta_{X_2}, \Delta_Z] = (\Delta_{X_2})_{\langle \iota \rangle}^{\mathrm{ab}} \times \Delta_{X_1},
$$

and hence the decomposition $(\Delta_Z/[\Delta_{X_2}, \Delta_Z])^p \cong (\Delta_{X_2})_{\langle \iota \rangle}^{p,\mathrm{ab}} \times \Delta_{X_1}^p$. Since

$$
(\Delta_{X_2})_{\langle \iota \rangle}^p \cong (\Delta_{X_2})_{\langle \iota \rangle}^{p,\mathrm{ab}},
$$

we have the isomorphism $\Delta_Z^p \cong (\Delta_Z/[\Delta_{X_2}, \Delta_Z])^p$, and therefore we obtain the decomposition $\Delta_Z^p = (\Delta_{X_2})_{\langle \iota \rangle}^{p,\mathrm{ab}} \times \Delta_{X_1}^p$. Note that $\Delta_{X_2}^{\mathrm{ab}}$ is isomorphic to $\widehat{\mathbb{Z}}(1) \otimes_{\mathbb{Z}} (\underset{z \in \mu_l}{\oplus} \mathbb{Z}e_z)$ as a Π_{X_1}-module. This shows that $\Pi_Z^{(p)}(= \Pi_Z/\mathrm{Ker}(\Delta_Z \to \Delta_Z^p))$ is isomorphic to $\mathbb{Z}_p(1) \rtimes \Pi_{X_1}^{(p)}$, which is defined by the action

$$
\Pi_{X_1}^{(p)}(= \Pi_{X_1}/\mathrm{Ker}(\Delta_{X_1} \to \Delta_{X_1}^p)) \to G_K \to \mathrm{Aut}(\mathbb{Z}_p(1)).
$$

Therefore, $\Pi_Z^{(p)}$ does not depend on l. Moreover, if we consider the étale covering space of Z corresponding to $p^n \mathbb{Z}_p(1) \rtimes \Pi_{X_1}^{(p)} \subset \mathbb{Z}_p(1) \rtimes \Pi_{X_1}^{(p)}$, its geometrically pro-$p$ étale fundamental group is isomorphic to $\mathbb{Z}_p(1) \rtimes \Pi_{X_1}^{(p)}$ over G_K. However, the Euler characteristic of the étale covering space is larger than that of Z and therefore it is not isomorphic to Z.

Note that the order of the group $\mathrm{Aut}(\Pi_Z^{(p)})/\mathrm{Inn}(\Delta_Z^p)$ is infinite since it contains $\mathbb{Z}_p^{\times}$. Also, note that the group Δ_Z^p is not center-free.

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