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THE HOPF MONOID AND THE BASIC INVARIANT OF DIRECTED GRAPHS

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Abstract

Aguiar and Ardila defined the Hopf monoid GP of generalized permutahedra and showed that it contains many submonoids that correspond to combinatorial objects. They also give a basic polynomial invariant of generalized permutahedra, which then specializes to submonoids. We define the Hopf monoid of directed graphs and show that it also embeds in GP. The resulting basic invariant coincides with the strict chromatic polynomial of Awan and Bernardi.

1. Introduction

A Hopf monoid is an algebraic structure defined by Aguiar and Mahajan [2]. Hopf monoids may be applied in the study of combinatorial objects, in a spirit similar to earlier work [9, 10, 12, 16]. They provide an useful structure to many combinatorial families by matching the product to merging and the coproduct to splitting operations. On the other hand, Postnikov [11], Stanley [14] and others constructed polyhedral models to study combinatorial objects. For example, generalized permutahedra are equivalent to polymatroids and submodular functions.

In [1], Aguiar and Ardila investigate combinatorial objects by combining these two points of view. They examine generalized permutahedra using a Hopf algebraic structure, which they call the Hopf monoid of generalized permutahedra GP. They also show that GP contains many other Hopf monoids of combinatorial objects, such as graphs and posets. In other words, if we construct a generalized permutahedron (or a submodular function) from a combinatorial object, we investigate our combinatorial object using GP. One application of this idea is the polynomial invariant $\chi_x(n)$. For each element x of a Hopf monoid, this polynomial in n is defined using a so-called character ζ of the Hopf monoid. We call $\chi_x(n)$ the AA polynomial of the character ζ. In many cases, the AA polynomial $\chi_x(n)$ associated to a combinatorial object x is equivalent to some existing invariant. For example, we know that the AA polynomial obtained from the so-called basic character of graphs is the chromatic polynomial. In particular, it satisfies Stanley's reciprocity theorem for graphs [15]. In [1], a reciprocity theorem is established for any AA polynomial. The reciprocity theorem is formulated in terms of the antipode of the Hopf monoid, which is analogous to the inverse in a group.

In this paper, we introduce and investigate the Hopf monoid of directed graphs. We will denote by DG$[I]$ the set of all directed graphs with vertex set I. We define the Hopf monoid

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structure for directed graphs using directed cuts. For technical reasons this has to be done so that the result is a Hopf monoid in vector species, see Section 2, and the notation changes to **DG**.

Next, we define the submodular function z_g on the ground set I obtained from the directed graph g, and the generalized permutahedron $\mathcal{P}(z_g)$ obtained from z_g (see Section 3). We show that $\mathcal{P}(z_g)$ represents the structure of the directed graph g in the following sense.

Theorem 1.1. *For any directed graph g with vertex set I, we have*

$$\mathcal{P}(z_g) = \mathrm{Cone}\{\, e_i - e_j \mid \textit{ the edge } (j,i) \textit{ is in } g\,\} \subset \mathbb{R}I,$$

where Cone *means convex cone and for each $i \in I$, the vector e_i is a standard generator of the vector space $\mathbb{R}I$.*

The cover relation of a partially ordered set gives it a directed graph structure. In that sense, our Hopf monoid **DG** generalizes Aguiar and Mahajan's Hopf monoid **P** of posets [2, Section 13.1.1]. On the other hand, **DG** is essentially a submonoid of the Hopf monoid of preposets [2, Section 13.1.6] because the transitive closure of a directed graph is a preposet[1]. In addition, Theorem 1.1 is a generalization of [1, Proposition 15.1]. We prove it by an application of the max-flow min-cut theorem.

Furthermore, $\mathcal{P}(z_g)$ provides a morphism from the Hopf monoid of directed graphs to the Hopf monoid of generalized permutahedra. From Theorem 1.1, we derive that the AA polynomial obtained from the basic character of directed graphs is equivalent to the strict-chromatic polynomial $\pi_g^{>}(n)$ [3]. In addition, this theorem is a generalization of [1, Proposition 18.5].

Theorem 1.2 (main theorem). *Let ζ be the character on the Hopf monoid of directed graphs* **DG** *defined by*

$$\zeta(g) = \begin{cases} 1 & (\textit{if } g \textit{ has no edges}), \\ 0 & (\textit{otherwise}). \end{cases}$$

Let $\chi_g(n)$ be the AA polynomial obtained from ζ. For any directed graph g, we have

$$\chi_g(n) = \pi_g^{>}(n).$$

The strict-chromatic polynomial is defined by Awan and Bernardi [3] to study properties of directed graphs (see Section 2.6). From the reciprocity theorem of Hopf monoids, it follows that we have $(-1)^{|I|}\pi_g^{>}(-n) = \pi_g^{\geqslant}(n)$, where $\pi_g^{\geqslant}(n)$ is the weak-chromatic polynomial defined in [3] and I is the vertex set of g. This fact is already established in [3], but our proof puts it in a new context. In [3], they also define a 3-variable polynomial invariant $B_g(n,x,y)$ for directed graphs g. We call this the B-polynomial. The strict- and weak-chromatic polynomials are specializations of the B-polynomial. We also find another character involving a parameter q which yields $B_g(n,q,0)$ as its AA polynomial (see Theorem 4.5).

Organization. In section 2, we recall some definitions and properties of Hopf monoids, as well as polynomial invariants of directed graphs. In section 3, we introduce the Hopf monoid of directed graphs and prove Theorem 1.1. In section 4, we introduce two characters for the Hopf monoid of directed graphs and compute the associated AA polynomials. In particular,

[1] We thank the anonymous referee for this comment.

in subsection 4.1, we prove Theorem 1.2.

2. Preliminaries

In this section, we recall some definitions and facts about Hopf monoids that are contained in [1].

2.1. Hopf monoids. First, we will introduce Joyal's notion of set species [5, 10].

DEFINITION 2.1. A set species P satisfies the following conditions.

(1) For each finite set I, a set $\mathrm{P}[I]$ is given.
(2) For each bijection $\sigma : I \to J$, there is an associated map $\mathrm{P}[\sigma] : \mathrm{P}[I] \to \mathrm{P}[J]$. These satisfy $\mathrm{P}[\sigma \circ \tau] = \mathrm{P}[\sigma] \circ \mathrm{P}[\tau]$ and $\mathrm{P}[\mathrm{id}] = \mathrm{id}$.

DEFINITION 2.2. A morphism $f : \mathrm{P} \to \mathrm{Q}$ between set species P and Q is a collection of maps $f_I : \mathrm{P}[I] \to \mathrm{Q}[I]$ which satisfy the following naturality axiom: for each bijection $\sigma : I \to J$, we have $f_J \circ \mathrm{P}[\sigma] = \mathrm{Q}[\sigma] \circ f_I$.

A decomposition is a partition where the parts may be empty and are ordered. We note that the decompositions $I = S \sqcup T$ and $I = T \sqcup S$ are distinct unless $I = S = T = \emptyset$. A composition is a decomposition where no subset is empty.

DEFINITION 2.3. A connected Hopf monoid in set species consists of the following data.

(1) A set species H such that the set $\mathrm{H}[\emptyset]$ is a singleton.
(2) For each finite set I and each decomposition $I = S \sqcup T$, product and coproduct maps

$$\mu_{S,T} : \mathrm{H}[S] \times \mathrm{H}[T] \to \mathrm{H}[I] \text{ and } \Delta_{S,T} : \mathrm{H}[I] \to \mathrm{H}[S] \times \mathrm{H}[T]$$

satisfying the naturality, unitality, and compatibility axioms given in [1, Section 2.2].

Fix a decomposition $I = S \sqcup T$. For $x \in \mathrm{H}[S]$, $y \in \mathrm{H}[T]$, and $z \in \mathrm{H}[I]$, we write

$$\mu_{S,T} : (x, y) \mapsto x \cdot y \text{ and } \Delta_{S,T} : z \mapsto (z|_S, z/_S).$$

We call $x \cdot y \in \mathrm{H}[I]$ the product of x and y. We call $z|_S \in \mathrm{H}[S]$ the restriction of z to S and $z/_S \in \mathrm{H}[T]$ the contraction of S from z.

DEFINITION 2.4. A morphism $f : \mathrm{H} \to \mathrm{K}$ between Hopf monoids H and K is a morphism of species which respects products, restrictions, and contractions; that is, we have

$$\begin{array}{ll} f_J(\mathrm{H}[\sigma](x)) = \mathrm{K}[\sigma](f_I(x)) & \text{for all bijections } \sigma : I \to J \text{ and all } x \in \mathrm{H}[I], \\ f_I(x \cdot y) = f_S(x) \cdot f_T(y) & \text{for all } I = S \sqcup T \text{ and all } x \in \mathrm{H}[S], y \in \mathrm{H}[T], \\ f_S(z|_S) = f_I(z)|_S,\ f_T(z/_S) = f_I(z)/_S & \text{for all } I = S \sqcup T \text{ and all } z \in \mathrm{H}[I]. \end{array}$$

Next, we introduce Hopf monoids in vector species. All vector spaces and tensor products below are over a fixed field $\Bbbk$.

DEFINITION 2.5. A vector species $\mathbf{P}$ satisfies the following conditions.

(1) For each finite set I, a vector space $\mathbf{P}[I]$ is given.

(2) To each bijection $\sigma : I \to J$, there is an associated map $\mathbf{P}[\sigma] : \mathbf{P}[I] \to \mathbf{P}[J]$.

These satisfy the same axioms as in Definition 2.1. A morphism of vector species $f : \mathbf{P} \to \mathbf{Q}$ is a collection of linear maps $f_I : \mathbf{P}[I] \to \mathbf{Q}[I]$ satisfying the naturality axiom of Definition 2.2.

DEFINITION 2.6. A connected Hopf monoid in vector species is a vector species $\mathbf{H}$ with $\mathbf{H}[\emptyset] = \Bbbk$ that is equipped with linear maps

$$\mu_{S,T} : \mathbf{H}[S] \otimes \mathbf{H}[T] \to \mathbf{H}[I] \text{ and } \Delta_{S,T} : \mathbf{H}[I] \to \mathbf{H}[S] \otimes \mathbf{H}[T]$$

for each decomposition $I = S \sqcup T$, subject to the same axioms as in Definition 2.3. We use similar notations as for Hopf monoids in set species;

$$\mu_{S,T}(x \otimes y) = x \cdot y \text{ and } \Delta_{S,T}(z) = \sum z|_S \otimes z/_S.$$

The following is a consequence of the associativity axiom. For any decomposition $I = S_1 \sqcup \cdots \sqcup S_k$ with $k \geq 2$, there are unique maps

$$\mu_{S_1,\dots,S_k} : \mathbf{H}[S_1] \otimes \cdots \otimes \mathbf{H}[S_k] \to \mathbf{H}[I]$$

$$\Delta_{S_1,\dots,S_k} : \mathbf{H}[I] \to \mathbf{H}[S_1] \otimes \cdots \otimes \mathbf{H}[S_k]$$

obtained by respectively iterating the product maps μ or the coproduct maps Δ in any meaningful way. These maps are well-defined and we refer to them as the higher products and coproducts of H.

Consider the linearization functor Set $\to$ **Vec** which sends a set to the vector space with basis the given set. Applying the linearization functor to a set species P gives a vector species, which we denote with **P**. If H is a Hopf monoid in set species, then its linearization **H** is a Hopf monoid in vector species. When **H** is the linearization of a Hopf monoid H in set species, then higher products and higher coproducts take the form

$$\mu_{S_1,\dots,S_k}(x_1 \otimes \cdots \otimes x_k) = x_1 \cdot \cdots \cdot x_k,$$
$$\Delta_{S_1,\dots,S_k}(z) = z_1 \otimes \cdots \otimes z_k$$

whenever $x_i \in \mathrm{H}[S_i]$ for $i = 1, \dots, k$ and $z \in \mathrm{H}[I]$, respectively. We refer to $z_i \in \mathrm{H}[S_i]$ as the i-th minor of z corresponding to the decomposition $I = S_1 \sqcup \cdots \sqcup S_k$.

2.2. The Hopf monoid of generalized permutahedra. In this section, we will introduce generalized permutahedra, following Postnikov [11]. We remark that they are equivalent to polymatroids, which were defined earlier by Edmonds [7]. In this paper, we use the same notation as in [1, Section 4].

DEFINITION 2.7. A generalized permutahedron $\mathfrak{p} \subseteq \mathbb{R}I$ is a polyhedron whose normal fan $\mathcal{N}_\mathfrak{p}$ is a coarsening of the braid arrangement $\mathcal{B}_I = \mathcal{N}_{\pi_I}$ in $\mathbb{R}^I$, where $\pi_I \subset \mathbb{R}I$ is the standard permutahedron.

DEFINITION 2.8. An extended generalized permutahedron $\mathfrak{p} \subseteq \mathbb{R}I$ is a polyhedron whose normal fan $\mathcal{N}_\mathfrak{p}$ is a coarsening of a subfan of the braid arrangement $\mathcal{B}_I = \mathcal{N}_{\pi_I}$ in $\mathbb{R}^I$.

Next, we give generalized permutahedra the structure of a Hopf monoid in vector species. In order to do this, we recall two propositions.

Proposition 2.9. [1, Proposition 5.1] *Let $I = S \sqcup T$ be a decomposition. If $\mathfrak{p} \subseteq \mathbb{R}I$ and $\mathfrak{q} \subseteq \mathbb{R}T$ are bounded generalized permutahedra, then $\mathfrak{p} \times \mathfrak{q} \subseteq \mathbb{R}I$ is a bounded generalized permutahedron.*

Proposition 2.10. [8, Theorem 3.15] *Let $\mathfrak{p} \subset \mathbb{R}I$ be a generalized permutahedron and $I = S \sqcup T$ be a decomposition. By definition, the linear function $\mathbf{1}_S = \sum_{i \in S} \mathbf{1}_i$ is maximized at the face $\mathfrak{p}_{S,T}$ of $\mathfrak{p}$. Then there exist generalized permutahedra $\mathfrak{p}|_S \subset \mathbb{R}S$ and $\mathfrak{p}/_S \subset \mathbb{R}T$ such that*

$$\mathfrak{p}_{S,T} = \mathfrak{p}|_S \times \mathfrak{p}/_S.$$

We call $\mathfrak{p}|_S$ and $\mathfrak{p}/_S$ the restriction and contraction of $\mathfrak{p}$ with respect to S, respectively.

Theorem 2.11. [1, Theorem 5.6] *Let $\mathbf{GP}_+[I]$ be the vector space spanned by the extended generalized permutahedra on I. Define a product and a coproduct as follows.*

For extended generalized permutahedra $\mathfrak{p} \in \mathbf{GP}[S]$ and $\mathfrak{q} \in \mathbf{GP}[T]$, their product is given by

$$\mathfrak{p} \cdot \mathfrak{q} := \mathfrak{p} \times \mathfrak{q} \in \mathbf{GP}_+[I].$$

For an extended generalized permutahedron $\mathfrak{p} \in \mathrm{GP}_+[I]$, its coproduct with respect to $S \sqcup T$ is given by

$$\Delta_{S,T}(\mathfrak{p}) = \begin{cases} \mathfrak{p}|_S \otimes \mathfrak{p}/_S & (\text{ if } \mathfrak{p} \text{ is bounded in the direction of } \mathbf{1}_S \text{ }), \\ 0 & (\text{ otherwise }), \end{cases}$$

where the restriction $\mathfrak{p}|_S$ and contraction $\mathfrak{p}/_S$ are defined in Proposition 2.10.

These operations turn the vector species $\mathbf{GP}_+$ into a Hopf monoid.

2.3. Submodular functions and generalized permutahedra. Let 2^I be the power set of a finite set I. A Boolean function on I is a function $z : 2^I \to \mathbb{R}$ such that $z(\emptyset) = 0$. Let $\mathbf{BF}[I]$ be the vector space generated freely by Boolean functions on I. If we define a product and a coproduct as in [1, Section 12.1] (cf. the definition in Theorem 2.13 below), then we obtain the Hopf monoid $\mathbf{BF}$ in vector species. A Boolean function z on I is submodular if $z(A \cup B) + z(A \cap B) \leq z(A) + z(B)$ for every $A, B \subseteq I$. Let $\mathbf{SF}[I]$ be the vector space spanned freely by submodular functions on I. Then $\mathbf{SF}$ is a Hopf submonoid of $\mathbf{BF}$. Next, we introduce extended submodular functions, which are related to extended generalized permutahedra.

DEFINITION 2.12. Let $z : 2^I \to \mathbb{R} \cup \{\infty\}$ be an extended Boolean function with $z(\emptyset) = 0$ and $z(I) \neq \infty$. We say that z is submodular if

$$z(A \cup B) + z(A \cap B) \leq z(A) + z(B)$$

for all $A, B \subseteq I$.

Theorem 2.13. [1, Section 12.4] *Let $\mathbf{SF}_+[I]$ be the vector space spanned freely by extended submodular functions on I. Fix a decomposition $I = S \sqcup T$. Define a product and a coproduct as follows.*

If $u \in \mathbf{SF}_+[S]$ *and* $v \in \mathbf{SF}_+[T]$*, define their product* $\mu_{S,T}(u \otimes v) \in \mathbf{SF}_+[I]$ *to be*

$$\mu_{S,T}(u \otimes v)(E) = u(E \cap S) + v(E \cap T) \text{ for } E \subseteq I.$$

If $z \in \mathbf{SF}_+[I]$*, define its coproduct* $\Delta_{S,T}(z) \in \mathbf{SF}_+[S] \otimes \mathbf{SF}_+[T]$ *to be*

$$\Delta_{S,T}(z) = \begin{cases} z|_S \otimes z/_S & (\text{ if } z(S) \neq \infty\,), \\ 0 & (\text{ if } z(S) \neq \infty\,), \end{cases}$$

where the restriction $z|_S$ *and contraction* $z/_S$ *are defined by*

$$z|_S(E) := z(E) \text{ for } E \subseteq S$$

and

$$z/_S(E) := z(E \cup S) - z(S) \text{ for } E \subseteq T.$$

These operations turn $\mathbf{SF}_+$ *into a Hopf monoid in vector species.*

Next, we will introduce the isomorphism between the Hopf monoid of extended submodular functions $\mathbf{SF}_+$ and the Hopf monoid of generalized permutahedra $\mathbf{GP}_+$. For $x \in \mathbb{R}I$ and $A \subseteq I$, we denote

$$x(A) = \sum_{i \in A} x_i.$$

Definition 2.14. The base polyhedron of a given extended Boolean function $z : 2^I \to \mathbb{R} \cup \{\infty\}$ is the set

$$\mathcal{P}(z) := \left\{ x \in \mathbb{R}I \;\middle|\; \sum_{i \in I} x_i = z(I) \text{ and } \sum_{i \in A} x_i \leq z(A) \text{ for all } A \subseteq I \right\}.$$

Theorem 2.15. *For a polyhedron* $\mathfrak{p}$ *in* $\mathbb{R}I$*, the following are equivalent.*

(1) *The polyhedron* $\mathfrak{p}$ *is an extended generalized permutahedron.*
(2) *There exists an extended submodular function* $z : 2^I \to \mathbb{R} \cup \{\infty\}$ *such that* $\mathfrak{p} = \mathcal{P}(z)$*.*

This theorem is compiled from results in [8, 11, 13] by Aguiar and Ardila [1, Theorem 12.3].

Theorem 2.16. [1, Theorem 12.7] *The collection of maps*

$$\mathbf{SF}_+[I] \to \mathbf{GP}_+[I], \;\; z \mapsto \mathcal{P}(z)$$

is an isomorphism of Hopf monoids in vector species $\mathbf{SF}_+ \cong \mathbf{GP}_+$*.*

2.4. The AA polynomial invariant of a character. Next, we will introduce a polynomial obtained from a character of the Hopf monoid, which we call the AA polynomial.

Definition 2.17. Let $\mathbf{H}$ be a connected Hopf monoid in vector species. A character ζ on $\mathbf{H}$ is a collection of linear maps $\zeta_I : \mathbf{H}[I] \to \Bbbk$ for each finite set I satisfying the following axioms.

(1) Naturality. For each bijection $\sigma : I \to J$ and $x \in \mathbf{H}[I]$, we have $\zeta_J(\mathbf{H}[\sigma](x)) = \zeta_I(x)$.
(2) Multiplicativity. For each $I = S \sqcup T$, $x \in \mathbf{H}[S]$, and $y \in \mathbf{H}[T]$, we have $\zeta_I(x \cdot y) = \zeta_S(x)\zeta_T(y)$.

(3) Unitality. The map $\zeta_\emptyset : \mathbf{H}[\emptyset] \to \Bbbk$ sends $1 \in \Bbbk = \mathbf{H}[\emptyset]$ to $1 \in \Bbbk$.

Definition 2.18. Let $\mathbf{H}$ be a connected Hopf monoid and $\zeta : \mathbf{H} \to \Bbbk$ be a character of $\mathbf{H}$. Define, for each element $x \in \mathbf{H}[I]$ and each natural number $n \in \mathbb{N}$,

$$\chi_x(n) := \sum_{I=S_1\sqcup\cdots\sqcup S_n} (\zeta_{S_1} \otimes \cdots \otimes \zeta_{S_n}) \circ \Delta_{S_1,\cdots,S_n}(x),$$

summing over all decompositions of I into n disjoint subsets which are allowed to be empty.

Remark 2.19. For a set I and an element $x \in \mathbf{H}[I]$, the function χ_x is defined on $\mathbb{N}$ and takes values in $\Bbbk$. If we take $n = 0$, we have

$$\chi_x(0) = \begin{cases} \zeta_\emptyset(x) & (\text{ if } I = \emptyset\,), \\ 0 & (\text{ otherwise }). \end{cases}$$

Furthermore we note that $\chi_x(1) = \zeta_I(x)$.

Recall that a composition of a finite set $I \neq \emptyset$ is a decomposition $I = S_1 \sqcup \cdots \sqcup S_k$ in which each subset S_i is nonempty. We write compositions as $I \vDash (S_1, \ldots, S_k)$.

Proposition 2.20. [1, Proposition 16.1] *Let* $\mathbf{H}$ *be a connected Hopf monoid in vector species,* $\zeta : \mathbf{H} \to \Bbbk$ *be a character, and let* χ *be defined by* Definition 2.18. *Fix a finite set* I *and an element* $x \in \mathbf{H}[I]$.

For each $n \in \mathbb{N}$*, it holds that*

$$\chi_x(n) = \sum_{k=0}^{|I|} \chi_x^{(k)} \binom{n}{k}$$

where, for each $k = 0, \ldots, |I|$*, we have*

$$\chi_x^{(k)} = \sum_{I \vDash (T_1,\ldots,T_k)} (\zeta_{T_1} \otimes \cdots \otimes \zeta_{T_k}) \circ \Delta_{T_1,\ldots,T_k}(x) \in \Bbbk,$$

summing over all compositions $(T_1, \ldots, T_k)$ *of* I*. Therefore* χ_x *is a polynomial function of* n *of degree at most* $|I|$.

Let $\sigma : I \to J$ *be a bijection,* $x \in \mathbf{H}[I]$ *and* $y := \mathbf{H}[\sigma](x) \in \mathbf{H}[J]$*. Then* $\chi_x = \chi_y$.

Proposition 2.21. [1, Proposition 16.3] *Let* $\mathbf{H}$ *and* $\mathbf{K}$ *be two Hopf monoids in vector species. Suppose* $\zeta^{\mathbf{H}}$ *is a character on* $\mathbf{H}$ *and* $\zeta^{\mathbf{K}}$ *is a character on* $\mathbf{K}$*. We will denote by* $f : \mathbf{H} \to \mathbf{K}$ *a morphism of Hopf monoids such that*

$$\zeta^{\mathbf{K}}(f(x)) = \zeta^{\mathbf{H}}(x)$$

for any I *and* $x \in \mathbf{H}[I]$*. Let* $\chi^{\mathbf{H}}$ *and* $\chi^{\mathbf{K}}$ *be the polynomial invariants corresponding to* $\zeta^{\mathbf{H}}$ *and* $\zeta^{\mathbf{K}}$*, respectively. Then*

$$\chi^{\mathbf{K}}_{f(x)} = \chi^{\mathbf{H}}_x$$

for any I *and* $x \in \mathbf{H}[I]$.

Definition 2.22. Let $\mathbf{H}$ be a connected Hopf monoid in vector species. The antipode of $\mathbf{H}$ is the collection of maps $\mathrm{s}_I : \mathbf{H}[I] \to \mathbf{H}[I]$ given by $\mathrm{s}_\emptyset = \mathrm{id}$ and, for each finite set $I \neq \emptyset$,

$$\mathrm{s}_I = \sum_{\substack{I=(S_1,\dots,S_k)\\ k\geq 1}} (-1)^k \mu_{S_1,\dots,S_k} \circ \Delta_{S_1,\dots,S_k}.$$

We call this equation Takeuchi's formula.

Proposition 2.23 (Reciprocity for polynomial invariants)**.** [1, Proposition 16.5] *Let* **H** *be a connected Hopf monoid,* $\zeta : \mathbf{H} \to \Bbbk$ *be a character, and* χ *be the AA polynomial invariant obtained from* ζ*. Let* s *be the antipode of* **H**. *Then*

$$\chi_x(-1) = \zeta_I(\mathrm{s}_I(x)).$$

More generally, for every $n \in \mathbb{N}$*, we have*

$$\chi_x(-n) = \chi_{\mathrm{s}_I(x)}(n).$$

2.5. The basic character and the basic invariant of GP. We introduce the basic character β and its associated AA invariant χ on the Hopf monoid of generalized permutahedra **GP**. We call χ the basic invariant. The property of **GP** which we introduce in this section, also holds for extended generalized permutahedra $\mathbf{GP}_+$.

DEFINITION 2.24. The basic character β of **GP** is defined by

$$\beta(\mathfrak{p}) = \begin{cases} 1 & (\text{ if } \mathfrak{p} \text{ is a point }),\\ 0 & (\text{ otherwise }),\end{cases}$$

for a generalized permutahedron $\mathfrak{p} \subset \mathbb{R}I$. The basic invariant χ of **GP** is the AA polynomial obtained from the basic character β.

Given a generalized permutahedron $\mathfrak{p} \subset \mathbb{R}I$ and a linear functional $y \in \mathbb{R}^I$, the generalized permutahedron $\mathfrak{p}$ is called directionally generic in the direction of y if the y-maximal face $\mathfrak{p}_y$ is a point. In this case, we will also say that y is $\mathfrak{p}$-generic and that $\mathfrak{p}$ is y-generic.

Proposition 2.25. [1, Proposition 17.3] *At a natural number n, the basic invariant* χ *of a generalized permutahedron* $\mathfrak{p} \subset \mathbb{R}I$ *is given by*

$$\chi_{\mathfrak{p}}(n) = (\textit{number of } \mathfrak{p}\textit{-generic functions } y : I \to [n]).$$

Here, we call $y : I \to [n]$ $\mathfrak{p}$-generic if its linear extension to an element of $\mathbb{R}^I$ is $\mathfrak{p}$-generic.

Proposition 2.26. [1, Proposition 17.4] *At a negative integer* $-n$*, the basic invariant* χ *of a generalized permutahedron* $\mathfrak{p} \subset \mathbb{R}I$ *is given by*

$$(-1)^{|I|}\chi_{\mathfrak{p}}(-n) = \sum_{y:I\to[n]} (\textit{number of vertices of } \mathfrak{p}_y).$$

Propositions 2.25 and 2.26 were first proved in [6], using Stanley's combinatorial reciprocity theorem. Aguiar and Ardila give Hopf theoretic proofs of these results in [1].

2.6. Awan–Bernardi's polynomial invariant for directed graphs. Awan and Bernardi investigate polynomial invariants for directed graphs [3]. In particular, they define the chromatic polynomial of a directed graph. A directed graph with vertex set I consists of directed edges. Let us denote by (i, j) the directed edge from $i \in I$ to $j \in I$. From here on, we assume that our directed graphs do not contain parallel edges. The presence of such would not change any of our constructions. We will denote by $g = (I, E)$ the directed graph g with vertex set I and directed edge set E.

DEFINITION 2.27. [3, Definition 5.1] Let $g = (I, E) \in \mathrm{DG}[I]$ be a directed graph where E is the directed edge set of g. The strict-chromatic polynomial $\pi_g^{>}(n)$ of g is defined by

$$\pi_g^{>}(n) = |\{f : I \to [n] \mid f(u) < f(v) \text{ for any } (u, v) \in E\}|.$$

We call such functions $f : I \to [n]$ the order-preserving maps.

The weak-chromatic polynomial $\pi_g^{\geqslant}(n)$ of g is defined by

$$\pi_g^{\geqslant}(n) = |\{f : I \to [n] \mid f(u) \leq f(v) \text{ for any } (u, v) \in E\}|.$$

Awan and Bernardi use Ehrhart theory to study these polynomials. In particular, they show that these polynomials are given by counting lattice points in integer multiple a certain polytope [3, Section 6].

Next, we will define a three variable polynomial invariant for directed graphs.

DEFINITION 2.28. Let $g = (I, E)$ be a directed graph with vertex set I. The B-polynomial of g is defined by

$$B_g(n, y, z) = \sum_{f:V\to[n]} y^{|\{(u,v)\in E | f(v)>f(u)\}|} z^{|\{(u,v)\in E | f(v)<f(u)\}|}.$$

The strict- and weak-chromatic polynomials are obtained from this B-polynomial.

Proposition 2.29. *Let $g = (I, E)$ be a directed graph with vertex set I. Then we have*

$$\pi_g^{>}(n) = \text{the coefficient of } y^{|E|} \text{ in } B_g(n, y, 1),$$

and

$$\pi_g^{\geqslant}(n) = B_g(n, 1, 0).$$

EXAMPLE 2.30. Let $I = \{0, 1, 2\}$ and let g be the directed graph with vertex set I in Figure 1. We get

$$B_g(n, y, z) = \binom{n}{1} + (2y^2 + 2z^2 + 2yz)\binom{n}{2} + (y^3 + z^3 + 2yz(y + z))\binom{n}{3}.$$

Moreover, we have

$$\pi_g^{>}(n) = \binom{n}{3} \text{ and } \pi_g^{\geqslant}(n) = \binom{n}{1} + 2\binom{n}{2} + \binom{n}{3}.$$

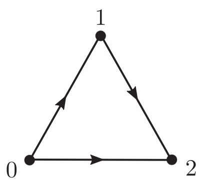


Fig. 1. The directed graph g

3. The Hopf monoid of directed graphs

Let $\mathbf{DG}[I]$ denote the vector space spanned by directed graphs with vertex set I. We use a bijection $\sigma : I \to J$ to relabel the vertices of a directed graph $g \in \mathbf{DG}[I]$ and turn it into a graph $\mathbf{DG}[\sigma](g) \in \mathbf{DG}[J]$. Thus $\mathbf{DG}$ is a vector species.

We claim that $\mathbf{DG}$ is a Hopf monoid in vector species with the following operations. Let $I = S \sqcup T$ be a decomposition. The product $\mu_{S,T} : \mathbf{DG}[S] \otimes \mathbf{DG}[T] \to \mathbf{DG}[I]$ is given by

$$\mu_{S,T}(g_1 \otimes g_2) = g_1 \cdot g_2,$$

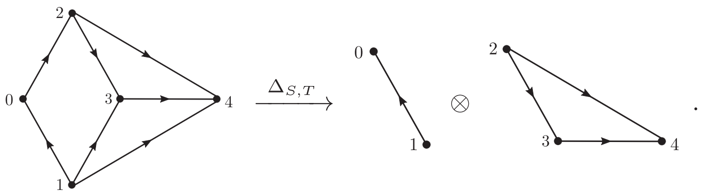
where the graph $g_1 \cdot g_2$ is the disjoint union of g_1 and g_2. So an edge of $g_1 \cdot g_2$ is an edge of g_1 or g_2. The restriction $g|_S \in \mathbf{DG}[S]$ is the induced subgraph on S, which consists of the edges whose ends are in S.

We say S is a lower half of the directed graph g if every directed edge which connects S and T is oriented from S. The coproduct $\Delta_{S,T} : \mathbf{DG}[I] \to \mathbf{DG}[S] \otimes \mathbf{DG}[T]$ is given by

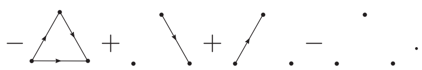
$$\Delta_{S,T}(g) = \begin{cases} g|_S \otimes g|_T & (\text{ if } S \text{ is a lower half of } g\text{ }), \\ 0 & (\text{ otherwise }). \end{cases}$$

We may easily check that the Hopf monoid axioms hold.

EXAMPLE 3.1. For $I = \{0, 1, 2, 3, 4\}$, let $S = \{0, 1\}$ and $T = \{2, 3, 4\}$. With this decomposition $I = S \sqcup T$, we have, for example,



EXAMPLE 3.2. We consider the antipode of the Hopf monoid of directed graphs. We let $I = \{0, 1, 2\}$ and we define $g \in \mathbf{DG}[I]$ as in Figure 1. The lower halves in this directed graph are $\{0, 1, 2\}$, $\{0\}$, and $\{0, 1\}$. So we get the antipode $\mathrm{s}_I(g) \in \mathbf{DG}[I]$ of g from Takeuchi's formula in Definition 2.22 as follows.

$\mathrm{s}_I(g) =$ 

As we noted in the introduction, directed graphs (via their transitive closures) are special cases of so-called preposets. Indeed, $\mathbf{DG}$ embeds into the Hopf monoid of preposets [2, Section 13.1.6]

For any set $A \subset I$ and any directed graph $g \in \mathbf{DG}[I]$, let us define the function z_g by

$$z_g(A) = \begin{cases} 0 & (\text{ } A \text{ is a lower half of } g\text{ }), \\ \infty & (\text{ otherwise }). \end{cases} \tag{3.1}$$

We note that I is always a lower half of g and hence $z_g(I) = 0$.

Lemma 3.3. *For any $g \in \mathbf{DG}[I]$, the extended Boolean function z_g is submodular.*

Proof. If $A, B \subseteq I$ are lower halves of g, then $A \cup B$ and $A \cap B$ are also lower halves of g. From the definition of z_g, if $z_g(A) = z_g(B) = 0$, then the subsets A, B are lower halves of g. Therefore if $z_g(A) = z_g(B) = 0$, then we also have $z_g(A \cap B) = z_g(A \cup B) = 0$. Hence z_g is a submodular function. □

Before the proof of Theorem 1.1, we recall the max-flow min-cut theorem. For this, we need the notion of a flow of a directed graph. For any vertex v of the directed graph g, we let $E_+(v)$ (resp. $E_-(v)$) be the set of incoming (resp. outgoing) edges from (resp. to) v in g. We call the vertex a sink (resp. source) if the vertex has only incoming (resp. outgoing) edges.

DEFINITION 3.4. Let $g = (I, E)$ be a directed graph with vertex set I, which has a source vertex α and a sink vertex ω. We fix an arbitrary function $c : E \to \mathbb{R}_{\geq 0}$ and call it the capacity function of the directed graph g. The function $f : E \to R_{\geq 0}$ is a flow if f satisfies the following conditions.

(1) For any edge $e \in E$, we have $f(e) \leq c(e)$.
(2) For any vertex $i \in I$ except α and ω, we have

$$\sum_{e \in E_+(i)} f(e) = \sum_{e \in E_-(i)} f(e).$$

For any flow f of g, the value $[f]$ is defined by

$$[f] = \sum_{e \in E_+(\omega)} f(e) = \sum_{e \in E_-(\alpha)} f(e).$$

The decomposition $I = S \sqcup T$ is a cut of g if the source α of g is in S and the sink ω of g is in T. We define the capacity of the cut $I = S \sqcup T$ by

$$c(S, T) = \sum_{i \in S} \sum_{j \in T} c((i, j)),$$

where (i, j) is a directed edge in g.

Theorem 3.5 (max-flow min-cut theorem)**.** *Let $g \in \mathrm{DG}[I]$ be a directed graph with vertex set I, which has source vertex and sink vertex. Let $c : E \to \mathbb{R}_+$ be a capacity function of g. The maximal value of a flow is equal to the minimal capacity of a cut. That is, we have*

$$\max_{f:flow} [f] = \min_{I = S \sqcup T:cut} c(S, T).$$

Now we are in a position to prove Theorem 1.1.

Proof of Theorem 1.1. Let us write $C(g) = \mathrm{Cone}\{ e_i - e_j \mid \text{ the edge } (j, i) \text{ is in } g \}$ for any directed graph $g = (I, E)$. We will prove that, for any directed graph g, we have $\mathcal{P}(z_g) = C(g)$.

First, we will prove that $\mathcal{P}(z_g) \supset C(g)$. It suffices to prove that the generators $e_i - e_j$ satisfy the conditions of Definition 2.14 for $z = z_g$. For any lower half A of g, the a-th coordinates ($a \in A$) of $e_i - e_j$ are equal to 0 or -1. We have $(e_i - e_j)(A) \leq 0 = z_g(A)$. We may easily check that $(e_i - e_j)(I) = 0$. So we have $(e_i - e_j) \in \mathcal{P}(z_g)$. Therefore we have $\mathcal{P}(z_g) \supset C(g)$.

Conversely, we will show $\mathcal{P}(z_g) \subset C(g)$. We take $x \in \mathcal{P}(z_g)$. That is, for any lower half

S of g, we have $x(S) \leq 0$ and we have $x(I) = 0$. For any $g \in \mathrm{DG}[I]$, we will construct a new directed graph g' from g. The vertex set of g' is defined to be $I \cup \{\alpha, \omega\}$. For any vertex $i \in I$, we add at most one new edge to form the edge set E' of g'. If the i-th coordinate of x is positive, we add a new edge from i to ω. If the i-th coordinate of x is negative, we add a new edge from α to i. See Figure 2. For any edge e of g', we will define the capacity $c(e)$. If e is an old edge (i.e., $e \in E$), the capacity of the edge is defined to be $c(e) = \infty$. For a new edge e incident to the vertex $i \in I$, if the i-th coordinate of x is positive, the capacity of the edge is defined to be $c(e) = x(i)$. If the i-th coordinate of x is negative, the capacity of the edge is defined to be $c(e) = -x(i)$. Our graph g' has two trivial directed cuts, formed by separating either α or ω from the rest of the vertices. Since $x(I) = 0$, both these cuts have the same capacity

$$\sum_{i \in I\,:\, x(i)>0} x(i) = \sum_{i \in I\,:\, x(i)<0} (-x(i)).$$

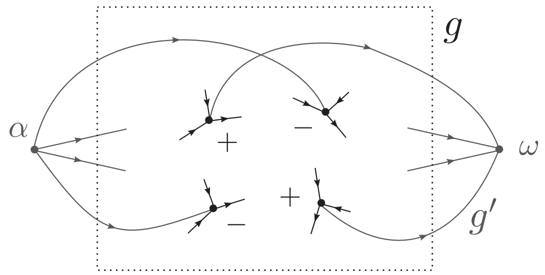


Fig. 2. The black graph is the old graph g. We construct a new graph g' from g by connecting each old vertex to at most one of the new vertices α and ω.

Next, we show that the two cuts just mentioned are minimal in the directed graph g' with the capacity function $c : E' \to \mathbb{R}_{\geq 0}$. For any cut $I \cup \{\alpha, \omega\} = A \sqcup B$ (where $\alpha \in A$ and $\omega \in B$), the sum of the capacities of the edges from A to B is finite if and only if there are no edges which go from $A \setminus \{\alpha\}$ to $B \setminus \{\omega\}$ in g. That means that $I = A \setminus \{\alpha\} \sqcup B \setminus \{\omega\}$ is a directed cut in g so that $B \setminus \{\omega\}$ is its lower half. Let us denote the set of edges from A to ω by A^+ and the set of edges from α to A by A^-. We define the sets of edges B^+, B^- in a similar way, see Figure 3. Then, the capacity of the cut $I \cup \{\alpha, \omega\} = A \sqcup B$ equals

$$\sum_{e \in A^+} c(e) + \sum_{e \in B^-} c(e).$$

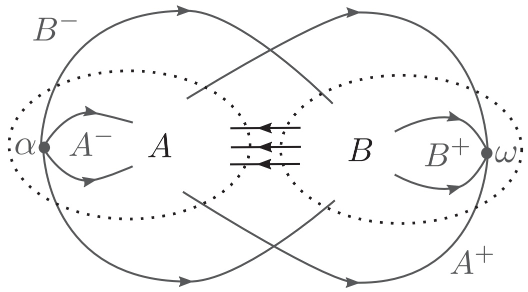


Fig. 3. The black regions represent the old graph g and the red edges are the new edges. The left subset $B \setminus \{\omega\}$ is a lower half of g.

Since $B \setminus \{\omega\}$ is a lower half of g, we have $x(B \setminus \{\omega\}) \leq 0$. That implies

$$\sum_{e \in B^+} c(e) - \sum_{e \in B^-} c(e) \leq 0.$$

Therefore, we get

$$\sum_{e \in A^+} c(e) + \sum_{e \in B^-} c(e) \geq \sum_{e \in A^+} c(e) + \sum_{e \in B^+} c(e).$$

The right hand side of this inequality is the capacity of the cut $(I \cup \{\alpha\}) \cup \{\omega\}$. Hence, we see that this value M is indeed minimal.

Using Theorem 3.5, we may take a flow f on g' such that

$$[f] = \sum_{e \in E_-(\alpha)} f(e) = \sum_{e \in E_+(\omega)} f(e) = M.$$

Notice that, for $[f]$ to reach the capacity of the trivial cuts, the value of f has to be exactly the capacity on all of our new edges in $E' \setminus E$. So, for any vertex $i \in I$ with incoming and outgoing edges $E_\pm(i)$, we have

$$\begin{aligned} \sum_{e \in E_+(i)} f(e) - \sum_{e \in E_-(i)} f(e) &= \begin{cases} f((i,\omega)) & (\text{ if } x(i) > 0\,), \\ -f((\alpha,i)) & (\text{ if } x(i) < 0\,), \end{cases} \\ &= \begin{cases} c((i,\omega)) & (\text{ if } x(i) > 0\,), \\ -c((\alpha,i)) & (\text{ if } x(i) < 0\,), \end{cases} \\ &= x(i) \end{aligned}$$

That is, if we take $\lambda_{ij} = f((j,i)) \geq 0$ for any edge (i, j), we have $x = \sum \lambda_{ij}(e_i - e_j)$. Hence, we have $x \in \mathcal{C}(g)$, i.e., we have $\mathcal{P}(z_g) \subset \mathcal{C}(g)$.

Therefore, we have $\mathcal{P}(z_g) = \mathcal{C}(g)$, which proves the theorem. $\square$

We will see in the next section how the proof of the main theorem relies on Theorem 1.1. But before that, we need another essential ingredient which is an extension of [1, Proposition 15.6]. It in fact follows from [1, Proposition 15.11] but we give a short proof for completeness.

Recall that the Hopf monoid $\mathbf{DG}[I]$ in vector species of directed graphs is the vector space spanned freely by directed graphs with vertex set I. For a directed graph $g \in \mathbf{DG}[I]$, let us rename the extended submodular function z_g of (3.1) as low_g. That is, for a directed graph g, $\mathrm{low}_g : 2^I \to \mathbb{R} \cup \{\infty\}$ is defined by

$$\mathrm{low}_g(S) = \begin{cases} 0 & (\text{ if } S \text{ is a lower half of } g\ ), \\ \infty & (\text{ otherwise }). \end{cases}$$

Proposition 3.6. *The map* low : $\mathbf{DG} \to \mathbf{SF}_+$ *is a morphism of Hopf monoids in vector species.*

Proof. First, we examine products. Let $I = S \sqcup T$ be a decomposition. Let $g_1 \in \mathbf{DG}[S]$, $g_2 \in \mathbf{DG}[T]$ be directed graphs. We denote by $g_1 \cdot g_2$ the disjoint union of g_1 and g_2. For any subset $J \subseteq I$, the subset J is a lower half of $g_1 \cdot g_2$ if and only if $J \cap S$ is a lower half of g_1 and $J \cap T$ is a lower half of g_2. Therefore we have

$$\begin{aligned} \mathrm{low}_{g_1 \cdot g_2}(J) &= \mathrm{low}_{g_1}(J \cap S) + \mathrm{low}_{g_2}(J \cap T) \\ &= (\mathrm{low}_{g_1} \cdot \mathrm{low}_{g_2})(J). \end{aligned}$$

Next, we look at coproduct in the Hopf monoid. For a decomposition $I = S \sqcup T$ and a directed graph $g \in \mathbf{DG}[I]$, we will prove that $\Delta_{S,T}(\mathrm{low}_g) = \mathrm{low}_{\Delta_{S,T}(g)}$. I.e., we have $(\mathrm{low}_g)|_S = (\mathrm{low}_{g|_S})$ and $(\mathrm{low}_g)/_S = (\mathrm{low}_{g/_S})$.

Suppose S is not a lower half of g. Then we have $\Delta_{S,T}(g) = 0$ and thus $\mathrm{low}_{\Delta_{S,T}(g)} = 0$. On the other hand, we have $\mathrm{low}_g(S) = \infty$. From the definition of the coproduct of submodular functions, we obtain $\Delta_{S,T}(\mathrm{low}_g) = 0$. Therefore we get $\Delta_{S,T}(\mathrm{low}_g) = \mathrm{low}_{\Delta_{S,T}(g)}$.

Suppose S is a lower half of g. Then we have $\mathrm{low}_g(S) = 0$. To see that low is compatible with restriction, we note that, for any $R \subseteq S$,

$$\mathrm{low}_{g|_S}(R) = \begin{cases} 0 & (\text{ if } R \text{ is a lower half of } g|_S\ ), \\ \infty & (\text{ otherwise }), \end{cases}$$

and we have

$$\begin{aligned} (\mathrm{low}_g)|_S(R) &= \mathrm{low}_g(R) \\ &= \begin{cases} 0 & (\text{ if } R \text{ is a lower half of } g\ ), \\ \infty & (\text{ otherwise }). \end{cases} \end{aligned}$$

Since R is a lower half of $g|_S$ if and only if R is a lower half of g, we have $\mathrm{low}_{g|_S} = (\mathrm{low}_g)|_S$.

To see that low is compatible with contraction, we note that, for any $R \subseteq T$,

$$\begin{aligned} \mathrm{low}_{g/_S}(R) &= \mathrm{low}_{g|_T}(R) \\ &= \mathrm{low}_g(R) \\ &= \begin{cases} 0 & (\text{ if } R \text{ is a lower half of } g|_T\ ), \\ \infty & (\text{ otherwise }), \end{cases} \end{aligned}$$

and we have

$$\begin{aligned} (\mathrm{low}_g)/_S(R) &= \mathrm{low}_g(R \sqcup S) - \mathrm{low}_g(S) \\ &= \mathrm{low}_g(R \sqcup S) \end{aligned}$$

$$= \begin{cases} 0 & (\text{ if } R \sqcup S \text{ is a lower half of } g\ ), \\ \infty & (\text{ otherwise }). \end{cases}$$

Since R is a lower half of $g|_T$ if and only if $R \sqcup S$ is a lower half of g, we have $\mathrm{low}_{g/S} = (\mathrm{low}_g)/_S$.

Therefore, we conclude that low is a morphism of Hopf monoids. □

4. Polynomial invariants of directed graphs from characters

In this section, we introduce two characters and their associated AA polynomials χ. Moreover we obtain combinatorial formulae for $\chi(n)$ and $\chi(-n)$ for $n \in \mathbb{N}$.

4.1. Basic invariant. First, we introduce the basic character β of the Hopf monoid **DG** and its associated AA polynomial $\chi(x)$, called basic invariant.

DEFINITION 4.1. The basic character ζ of **DG** is given by

$$\zeta_I(g) = \begin{cases} 1 & (\text{ if } g \text{ has no edges }), \\ 0 & (\text{ otherwise }). \end{cases}$$

for a directed graph $g \in \mathbf{DG}[I]$. The basic invariant χ of **DG** is the AA polynomial obtained from ζ.

Now, we are in a position to prove our main theorem.

Proof of Theorem 1.2. First, we get a morphism $\mathbf{DG} \to \mathbf{SF}_+ \to \mathbf{GP}_+$ of Hopf monoids using Theorem 2.16 and Proposition 3.6. Let g be a directed graph on the vertex set I.

The directed graph cone $C(g)$ is a point if and only if g has no edges. Therefore, when we restrict the basic character β of $\mathbf{GP}_+$ to directed graph cones, we obtain the basic character ζ of directed graphs. From Proposition 2.21, we have $\chi_g(n) = \chi_{C(g)}(n)$, where $\chi_{C(g)}(n)$ is the AA polynomial of the directed graph cone $C(g) \in \mathbf{GP}_+[I]$ obtained from the basic character β of the Hopf monoid of extended generalized permutahedra. Using Proposition 2.25, it follows that $\chi_g(n)$ is the number of $C(g)$-generic functions $y : I \to [n]$. Now, thanks to Theorem 1.1, the normal fan to $C(g)$ is a single cone cut out by inequalities $y(i) \leq y(j)$ for the vertices $i, j \in I$ so that g has a directed edge from i to j. So the $C(g)$-generic functions are the strictly order-reversing maps in g. We remark that there is a natural bijection between strictly order-reversing maps $I \to [n]$ and strictly order-preserving maps $I \to [n]$, and the proof is complete. □

Furthermore, this polynomial satisfies a reciprocity rule.

Theorem 4.2. *Let g be an acyclic directed graph with vertex set I and $n \in \mathbb{N}$. If the polynomial $\chi_g(n)$ is the basic invariant for the Hopf monoid* $\mathbf{DG}[I]$*, then we have*

$$(-1)^{|I|}\chi_g(-n) = \pi_g^{\geqslant}(n),$$

where $\pi_g^{\geqslant}(n)$ is the weak-chromatic polynomial of g.

Proof. We will show the theorem using Proposition 2.26. The directed graph cone $C(g)$ is pointed if and only if the directed graph g has no directed cycles. If $y : I \to [n]$ is order-reversing, then there is a y-maximum face $C(g)_y$, and it contains that single vertex. If y is not order reversing, then $C(g)$ is unbounded from above in the direction of y. So the left hand side, by Proposition 2.26, is the number of order-reversing maps. These are bijective with order-preserving maps, whose number is the right hand side. □

Corollary 4.3. *For any acyclic directed graph g with vertex set I and for any $n \in \mathbb{N}$, we have*

$$(-1)^{|I|}\pi_g^{>}(-n) = \pi_g^{\geqslant}(n).$$

This corollary is equivalent to the acyclic version of [3, Lemma 6.5]. Here we get the proof without invoking Ehrhart reciprocity.

EXAMPLE 4.4. Let g be the directed graph of Figure 1. The basic invariant $\chi_g(n)$ of g is

$$\chi_g(n) = \pi_g^{\geqslant}(n) = \binom{n}{3}.$$

4.2. Edge character. Next, we introduce another character and its associated AA polynomial. This AA polynomial turns out to be a specialization of Awan–Bernardi's B-polynomial.

Theorem 4.5. *For a directed graph $g = (I, E) \in \mathbf{DG}[I]$, let the character η be defined by $\eta(g) = q^{|E|}$. Let $\psi_g(n)$ be the AA polynomial obtained from η. For any directed graph $g = (I, E)$, we have*

$$\psi_g(n) = q^{|E|}B_g(n, 1/q, 0).$$

Proof. First, we may easily check that η is a character of the Hopf monoid of directed graphs.

Let $I = S_1 \sqcup \cdots \sqcup S_n$ be a decomposition. The coloring $f : I \to [n]$ is defined by $f(i) = k$ for $i \in S_k$. In this coloring, if $\Delta_{S_1\sqcup\cdots\sqcup S_n}(g) = 0$, then there is an edge (i, j) in g such that $f(j) < f(i)$. If $\Delta_{S_1\sqcup\cdots\sqcup S_n} \neq 0$, any edge of the directed graph $\mu(\Delta_{S_1\sqcup\cdots\sqcup S_n}(g))$ is an edge (i, j) such that $f(i) = f(j)$. Furthermore, the edges of g which did not remain in $\mu(\Delta_{S_1\sqcup\cdots\sqcup S_n}(g))$ are the edges (i, j) such that $f(i) < f(j)$.

From these observations, we have

$$\begin{aligned}
\psi_g(n) &= \sum_{\substack{f:I\to[n] \\ \nexists (i,j)\in E \text{ s.t. } f(j)<f(i)}} q^{|\{(i,j)\in E \mid f(i)=f(j)\}|} \\
&= q^{|E|} \sum_{\substack{f:I\to[n] \\ \nexists (i,j)\in E \text{ s.t. } f(j)<f(i)}} \left(\frac{1}{q}\right)^{|\{(i,j)\in E \mid f(i)>f(j)\}|} \\
&= q^{|E|}B_g(n, 1/q, 0).
\end{aligned}$$

□

EXAMPLE 4.6. Let g be the directed graph of Figure 1. We compute $\psi_g(n)$, which is the AA polynomial obtained from the edge character η, from Theorem 4.5 and Example 2.30 as follows:

$$\psi_g(n) = q^3\binom{n}{1} + 2q\binom{n}{2} + \binom{n}{3}.$$

Finally, we get a reciprocity theorem from Theorem 2.23. Let $E(g)$ be the set of edges of the directed graph g. For any directed graph $g = (I, E)$ with vertex set I, we call a partition $I = S_1 \sqcup \cdots \sqcup S_k$ order preserving if any edge between S_i and S_j $(i < j)$ is oriented from S_i to S_j.

Corollary 4.7. *Let $B_g(n, x, y)$ be Awan–Bernardi's B-polynomial of the directed graph $g = (I, E)$ with vertex set I. Then we have*

$$q^{|E|}B_g(-1, 1/q, 0) = \sum_{\substack{I=S_1\sqcup\ldots\sqcup S_k \\ \text{order preserving}}} (-1)^k q^{|E(g|_{S_1})|+\cdots+|E(g|_{S_k})|}.$$

More generally, for every $n \in \mathbb{N}$, we have

$$q^{|E|}B_g(-n, 1/q, 0) = \sum_{\substack{I=S_1\sqcup\ldots\sqcup S_k \\ \text{order preserving}}} (-1)^k \prod_{i=1}^{n} q^{|E(g|_{S_i})|} B_{g|_{S_i}}(n, 1/q, 0).$$

Proof. We will show the second formula. Using Theorem 2.23, we have $\psi_g(-n) = \psi_{\mathrm{s}_I(g)}(n)$. From Theorem 4.5, we have

$$\begin{aligned} q^{|E|}B_g(-n, 1/q, 0) &= \psi_g(-n) \\ &= \psi_{\mathrm{s}_I(g)}(n) \\ &= \sum_{\substack{I\vDash(S_1,\ldots,S_k) \\ k\geq 1}} (-1)^k \psi_{\mu_{S_1,\ldots,S_k}\circ\Delta_{S_1,\ldots,S_k}(g)}(n) \end{aligned}$$

For a composition $I\vDash(S_1, \ldots, S_k)$, the partition $I = S_1\sqcup\cdots\sqcup S_n$ is order preserving if and only if $\mu_{S_1,\ldots,S_k} \circ \Delta_{S_1,\ldots,S_k}(g) \neq 0$. Moreover, for a disjoint union $g_1 \cup g_2$, we have $B_{g_1\cup g_2} = B_{g_1}B_{g_2}$. The statement follows. $\square$

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