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L^p-ESTIMATES FOR HARMONIC BERGMAN PROJECTION ON A CLASS OF CONVEX DOMAINS OF INFINITE TYPE IN $\mathbb{C}^2$

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Abstract

In this paper, we prove that the Bergman projection extends continuously to a projection from harmonic L^1-functions onto holomorphic L^1-functions and maps continuously L^∞-functions onto the space of Bloch holomorphic functions in a certain class of infinite type, convex domains in $\mathbb{C}^2$.

1. Introduction

Let Ω be a bounded domain in $\mathbb{C}^2$ with smooth boundary $b\Omega$. Let ρ be a defining function for Ω so that $\Omega = \{z \in \mathbb{C}^2 : \rho(z) < 0\}$ and $b\Omega = \{z \in \mathbb{C}^2 : \rho(z) = 0\}$, $\nabla\rho \neq 0$ on $b\Omega$. Let $\mathcal{O}(\Omega)$ be the space of functions that are holomorphic in Ω, with the topology of uniform convergence on compact subsets of Ω. For $1 \leq p \leq \infty$, the Bergman space $A^p(\Omega)$ is the class of all holomorphic $L^p(\Omega)$-functions in Ω with the $\mathbb{R}^4$-Lebesgue measure dV for $\mathbb{C}^2$. The Bergman projection $\mathcal{P}$ is the orthogonal projection of $L^2(\Omega)$ onto the Bergman space $A^2(\Omega)$. One of most important properties of the Bergman projection is that there exists a function $P\colon \Omega\times\Omega \to \mathbb{C}$ such that

$$\mathcal{P}[u](z) = \int_\Omega u(\zeta)P(\zeta, z)\, dV(\zeta), \tag{1.1}$$

for all $u \in L^2(\Omega)$, $z \in \Omega$. Here, $P(\zeta, z)$ is the Bergman kernel on Ω, which is holomorphic with respect to $z \in \Omega$, and anti-holomorphic in ζ, and only depending on Ω. In this paper, we are interested in L^p-boundedness for $\mathcal{P}$ on a class of certain convex domains in several complex variables.

(1) In [25], it is proved that if Ω is a strongly pseudoconvex domain, then $\mathcal{P}$ extends continuously to a bounded operator from $L^p(\Omega)$ to $A^p(\Omega)$, for all $1 < p < \infty$.
(2) Let $\Omega = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^{2/\alpha} < 1\}$ for $\alpha > 1$. In this case, Ω is weakly convex, not strongly pseudoconvex. In [2], the author showed that the Bergman projection on Ω extends to a continuous operator from $L^p(\Omega)$ to $A^p(\Omega)$ for all $p \in (1, \infty)$.
(3) More generally, let $\Omega \subset \mathbb{C}^n$ be a smoothly bounded convex domain of finite line type (see [22] for the definition of this type). McNeal and Stein (in [23]) proved that the

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Bergman projection maps $L^p(\Omega) \to A^p(\Omega)$ boundedly for all $1 < p < \infty$.
(4) In [3] Charpentier and Dupain had showed that the Bergman projection of smoothly bounded pseudoconvex domains whose boundary points are all of finite commutator type (see [24] the definition of this type) in $\mathbb{C}^n$ and with locally diagonalizable Levi form maps $L^p(\Omega)$ continuously into itself, for all $1 < p < \infty$.
(5) Recently, on certain convex domains of infinite type in $\mathbb{C}^2$, for example,

$$\Omega^\infty = \left\{(z_1, z_2) \in \mathbb{C}^2 : \rho(z) = \exp\left(1 + \frac{2}{s}\right) \cdot \exp\left(\frac{-1}{|z_1|^s}\right) + |z_2|^2 - 1 < 0\right\},$$

the L^p-boundedness of the Bergman projection is also investigated by Ha and Khoi (see in [13]), for all $p \in (1, \infty)$.

Although there are also many results related to L^p-boundedness for the Bergman projection on various domains, the power p can not be equal to 1 or ∞. Even when Ω is the unit ball in $\mathbb{C}^n$, for $n \geq 2$, the Bergman projection $\mathcal{P}$ can not be extended continuously from $L^p(\Omega)$ onto $A^p(\Omega)$ when $p = 1$ or $p = \infty$ (for example, see [30, Section 7.1]). This leads us to explore other spaces that are smaller than $L^p(\Omega)$ and the Bergman projection behaves better on these spaces. In particular, for $1 \leq p < \infty$, the harmonic Bergman space $L^p_{\text{Har}}(\Omega)$ is the class of all harmonic functions on Ω which belong to $L^p(\Omega)$. It is clear that $A^p(\Omega) \subset L^p_{\text{Har}}(\Omega) \subset L^p(\Omega)$. Moreover, we also have the following density property which is proved in [21, page 236].

Lemma 1.1. *If $\Omega \subset \mathbb{R}^N$ is a bounded domain with boundary of class C^4 then $L^2_{Har}(\Omega)$ is dense in $L^1_{Har}(\Omega)$.*

For $p = \infty$, we need the following space which is called the Bloch space.

DEFINITION 1.2. A differentiable function u on Ω is said to be a Bloch function if and only if

$$||u||_{\text{Bl}(\Omega)} = \sup_{z\in\Omega} (|\rho(z)| \cdot |u(z)| + |\rho(z)| \cdot |\nabla u(z)|) < \infty.$$

The space of all Bloch functions defined on Ω is denoted by $\text{Bl}(\Omega)$ and by $\text{BlHol}(\Omega) = \text{Bl}(\Omega) \cap \mathcal{O}(\Omega)$ the space of Bloch holomorphic functions on Ω. We also define $||u||_{\text{BlHol}(\Omega)} = ||u||_{\text{Bl}(\Omega)}$ for all $u \in \text{BlHol}(\Omega)$.

The main result in this paper is following.

Main Theorem. *Let Ω be a smoothly bounded convex domain in $\mathbb{C}^2$ admitting an F-type at all boundary points (see* Definition 2.3*). Then the Bergman projection maps continuously*
(1) *$L^1_{Har}(\Omega)$ onto $A^1(\Omega)$.*
(2) *$L^\infty(\Omega)$ onto $BlHol(\Omega)$.*

This result was proved by Ligocka in [21] when Ω is a smoothly bounded, strongly pseudoconvex domain in $\mathbb{C}^n$. In this case, although Ligocka also cannot admit the boundedness from $L^\infty_{\text{Har}}(\Omega)$ onto $L^\infty(\Omega) \cap \mathcal{O}(\Omega)$, she showed that $\mathcal{P}$ maps continuously $L^\infty_{\text{Har}}(\Omega)$ onto the dual to the Hardy space $H^1(b\Omega)$.

The structure of the paper is as follows. Section 2 deals with preliminaries for Cauchy-Fantappiè forms on convex domains admitting the F-type condition. Necessary L^p-estimates for the Cauchy-Riemann equation is provided in Section 3. Section 4 deals with the proof of the Main Theorem.

2. Preliminaries

2.1. The construction of the Cauchy-Fanttapiè form. In this subsection, we recall the construction of Cauchy-Fanttapiè form, which plays an essential role in our representation for the Bergman projection (see [29] for more details).

Let $\Omega \subset \mathbb{C}^2$ be a bounded convex domain with smooth boundary $b\Omega$ and a defining function ρ. By the hypothesis that Ω is convex,

$$\sum_{i,j=1}^{4} \frac{\partial^2 \rho}{\partial x_i \partial x_j}(x) a_i a_j \geq 0,$$

in which $x \in b\Omega$, $z_j = x_{2j-1} + \sqrt{-1} x_{2j}$ and $a \in \mathbb{R}^4$ be a non-zero vector such that $\sum_{j=1}^4 a_j \frac{\partial \rho}{\partial x_j}(x) = 0$ on $b\Omega$. Let us define, for $(\zeta, z) \in b\Omega \times \Omega$:

$$\Phi(\zeta, z) = \sum_{j=1}^{2} \frac{\partial \rho}{\partial \zeta_j}(\zeta)(\zeta_j - z_j). \tag{2.1}$$

The convexity of Ω gives

$$\mathrm{Re}\left(\sum_{j=1}^{2} \frac{\partial \rho}{\partial \zeta_j}(\zeta)(\zeta_j - z_j) \right) \neq 0,$$

so that $\Phi(\zeta, z) \neq 0$ for all $\zeta \in b\Omega, z \in \Omega$. Moreover, the following lemma is a consequence of the definition of $\Phi(\zeta, z)$.

Lemma 2.1. *For any $P \in b\Omega$, there are positive constants δ, c such that for all boundary points $\zeta \in b\Omega \cap B(P, \delta)$, we have*

(1) *$\Phi(\zeta, z)$ is holomorphic in $z \in B(\zeta, \delta)$;*
(2) *$\Phi(\zeta, \zeta) = 0$, and $d_z \Phi|_{z=\zeta} \neq 0$;*
(3) *There exists a constant $A > 0$ such that $|\Phi(\zeta, z)| \geq A$ for all $z \in \Omega$ and $|z - \zeta| \geq c$;*
(4) *$\rho(z) > 0$ for all z with $\Phi(\zeta, z) = 0$ and $0 < |z - \zeta| < c$.*

Here the notation $B(\zeta, r)$ means the Euclidean ball centered at ζ of radius $r > 0$.

Now we set

$$C(\zeta, z) = \frac{1}{2\pi i} \left[\sum_{j=1}^{2} \frac{\partial \rho}{\partial \zeta_j}(\zeta) d\zeta_j \right] \frac{1}{\Phi(\zeta, z)} \quad \text{for } (\zeta, z) \in b\Omega \times \Omega,$$

which is a $(1,0)$-form of ζ-variables. The Cauchy-Leray kernel for the convex domain Ω is

$$\Omega_0 \left(C(\zeta, z) \right) = C(\zeta, z) \wedge (\bar{\partial}_\zeta C(\zeta, z)) \tag{2.2}$$

$$= \sum_{j_0 \in \{1,2\}} \frac{A_{j_0}(\zeta)}{\Phi^2(\zeta, z)} d\zeta_1 \wedge d\zeta_2 \wedge d\bar{\zeta}_{j_0} + \text{non-singular terms}, \tag{2.3}$$

which is a Cauchy-Fantappiè $(2,1)$-form on $b\Omega \times \Omega$, where $A_{j_0}(\zeta)$ is a polynomial involving first and second derivatives in ζ of ρ.

For each $z \in \Omega$ we may (C^1-function) smoothly extend the $(1,0)$-form $C(., z)$ to the interior of Ω as follows

$$\widetilde{C}(\zeta, z) = \frac{1}{2\pi i}\left[\sum_{j=1}^{2} \frac{\partial\rho}{\partial\zeta_j}(\zeta)d\zeta_j\right]\frac{1}{\Phi(\zeta, z) - \rho(\zeta)}.$$

The following Cauchy-Leray Integral Formulas for convex domains are crucial in the Cauchy-Fantappiè theory.

Proposition 2.2. *Suppose that Ω is a bounded, convex domain of class C^∞. The following are true.*

(1) [29, Theorem 3.4] *for any $u \in \mathcal{O}(\Omega) \cap C(b\Omega)$,*

$$u(z) = \int_{b\Omega} u(\zeta)\Omega_0(C(\zeta, z)), \quad z \in \Omega.$$

(2) [18, Proposition 9] *for any $u \in \mathcal{O}(\Omega)$,*

$$u(z) = \int_{\Omega} u(\zeta)\bar{\partial}_\zeta\Omega_0(\widetilde{C}(\zeta, z)), \quad z \in \Omega.$$

2.2. A domain admitting an F-type. In this paper, we focus on studying the class of convex domains admitting the so-called geometric F-type. This type condition were first introduced in [7] to generalize all domains of finite type and many cases of infinite type in the sense of Range in [27, 28].

DEFINITION 2.3. Let $F : [0, \infty) \to [0, \infty)$ be a smooth, strictly increasing function such that

(1) $F(0) = 0$,

(2) $\displaystyle\int_0^\sigma \left|\ln F(r^2)\right| dr < \infty$ for some $\sigma > 0$ which is small enough,

(3) $\dfrac{F(t)}{t}$ is non-decreasing function.

Let $\Omega \subset \mathbb{C}^2$ be a smooth bounded, convex domain. We say that Ω admitting F-type at a point $P \in b\Omega$ if there are positive constants c, c' such that for all $\zeta \in b\Omega \cap B(P, c')$:

$$\rho(z) \gtrsim F(|z - \zeta|^2),$$

for all $z \in B(\zeta, c)$ with $\Phi(\zeta, z) = 0$.

If Ω admits the same F-type at every point on $b\Omega$, we simply call that Ω admitting F-type. In case $F(t) = t^m$, for $m = 1, 2, \ldots$, the F-type notion agrees with the finite type condition in the sense of Range in [27, 28]. Here the notations $\lesssim$ and $\gtrsim$ denote inequalities up to a positive constant, and $\approx$ means the combination of $\lesssim$ and $\gtrsim$.

EXAMPLE 2.4. (a) ([29, p. 195]) Let $\Omega \subset \mathbb{C}^2$ be a strictly convex or strongly pseudoconvex domain with its smooth, strictly plurisubharmonic defining function ρ. Then

$$\operatorname{Re}\Phi(\zeta, z) \geq \rho(\zeta) - \rho(z) + \lambda_0|\zeta - z|^2,$$

for $|\zeta - z|$ and $|\rho(\zeta)|$ small, and $\lambda_0 > 0$.

Hence, when $\zeta \in \overline{\Omega} \cap \{|\zeta - z| < c\}$, and $\Phi(\zeta, z) = 0$, we have

$$\rho(z) - \rho(\zeta) \gtrsim F(|z - \zeta|^2),$$

with $F(t) = t$. So Ω is of F-type.

(b) ([31, Theorem 3.1]) The complex ellipsoid is

$$\Omega = \{(z_1, z_2) \in \mathbb{C}^2 : \rho(z) = |z_1|^{2m_1} + |z_2|^{2m_2} - 1 < 0\} \quad (m_1, m_2 \in \mathbb{N}).$$

Then there exist constants $c, C > 0$ such that

$$\operatorname{Re}\Phi(\zeta, z) \geq -\rho(z) + \rho(\zeta) + C|\zeta - z|^{2m},$$

for $\zeta \in \overline{\Omega}$, $z \in \Omega$ with $|\zeta - z| < c$, and $m = \max\{m_1, m_2\}$. Thus Ω is a convex domain admitting an F-type, with $F(t) = t^m$.

(c) ([26, Proposition 1]) Let $\Omega \subset \mathbb{C}^2$ be a convex domain with real analytic boundary, i.e., ρ is a real analytic function. Then, there exist constants $c, C > 0$ and a positive integer m such that

$$\operatorname{Re}\Phi(\zeta, z) \geq -\rho(z) + \rho(\zeta) + C|\zeta - z|^{2m},$$

for $\zeta \in \overline{\Omega}$, $z \in \Omega$ with $|\zeta - z| < c$. That means Ω is a domain admitting an F-type, with $F(t) = t^m$.

(d) ([32, Lemma 3]) Let

$$\Omega^\infty = \left\{(z_1, z_2) \in \mathbb{C}^2 : \rho(z) = \exp\left(1 + \frac{2}{s}\right). \exp\left(\frac{-1}{|z_1|^s}\right) + |z_2|^2 - 1 < 0\right\}.$$

Then, there exists a constant $c > 0$ such that for all $\zeta, z \in \overline{\Omega}$ with $|\zeta - z| < c$

$$\operatorname{Re}\Phi(\zeta, z) \gtrsim \rho(\zeta) - \rho(z) + \exp\left(1 + \frac{2}{s}\right)\exp\left\{\frac{-1}{32|\zeta - z|^{2s}}\right\},$$

for $0 < s < 1/2$. Hence Ω^∞ is a convex domain admitting an F-type, with $F(t) = \exp\left(\frac{-1}{32t^s}\right)$.

The following lemma provides an important lower estimate for the Cauchy-Fantappiè form. Its proof is rather similar to the proof of [7, Lemma 3.3] with a negligible modification, so we omit it here.

Lemma 2.5. *Let Ω be a smoothly bounded, convex domain in $\mathbb{C}^2$ admitting an F-type at $P \in b\Omega$. Then there is a positive constant c such that the support function $\Phi(\zeta, z)$ satisfies the following estimate*

$$|\Phi(\zeta, z) - \rho(\zeta)| \gtrsim |\rho(\zeta)| + |\rho(z)| + |\operatorname{Im}\Phi(\zeta, z)| + F(|z - \zeta|^2), \tag{2.4}$$

for every $\zeta \in \overline{\Omega} \cap B(P, c)$, and $z \in \Omega$, $|z - \zeta| < c$.

3. L^p-estimates for the $\bar{\partial}$-equation

This section deals with L^p-estimates for solutions of the $\bar{\partial}$-equation on the convex domain admitting F-type. Our method is inspired by the same technique by Khanh in [16] and Khanh et. al. in [11]. Firstly, it is necessary to recall the formula of the $\bar{\partial}$-solution constructed by Henkin-Ramirez, see [26] or [5] for more details.

Lemma 3.1. *Let $\Omega \Subset \mathbb{C}^2$ be a smoothly bounded, convex domain. Let $\varphi = \sum_{j=1}^{2} \varphi_j(z) d\bar{z}_j$ be a bounded, C^1, $\bar{\partial}$-closed $(0,1)$-form on $\overline{\Omega}$. Then*

$$\bar{\partial}\mathcal{T}[\varphi](z) = \varphi(z)$$

on Ω, where

$$\mathcal{T}[\varphi](z) = \frac{1}{2\pi^2} \int_{\zeta \in b\Omega} \frac{\frac{\partial\rho(\zeta)}{\partial\zeta_1}(\bar{\zeta}_2 - \bar{z}_2) - \frac{\partial\rho(\zeta)}{\partial\zeta_2}(\bar{\zeta}_1 - \bar{z}_1)}{\Phi(\zeta, z)|\zeta - z|^2} \varphi(\zeta) \wedge d\zeta_1 \wedge d\zeta_2 + \frac{1}{4\pi^2} \int_{\Omega} \frac{\varphi_1(\zeta)(\bar{\zeta}_1 - \bar{z}_1) - \varphi_2(\zeta)(\bar{\zeta}_2 - \bar{z}_2)}{|\zeta - z|^4} d\bar{\zeta}_1 \wedge d\bar{\zeta}_2 \wedge d\zeta_1 \wedge d\zeta_2. \tag{3.1}$$

In this section, we prove that

Theorem 3.2. *Let Ω be a smoothly bounded convex domain in $\mathbb{C}^2$ admitting a type F at all boundary points. If $\varphi \in L^p_{(0,1)}(\Omega)$, then $\mathcal{T}[\varphi] \in L^p(\Omega)$ and*

$$\|\mathcal{T}[\varphi]\|_{L^p(\Omega)} \le C_p \|\varphi\|_{L^p_{(0,1)}(\Omega)}$$

for all $1 \le p \le \infty$.

Although this theorem is proved in [10] with $\Omega \subset \mathbb{C}^n$ $(n \ge 2)$, we recall its proof for convenience. In order to prove this theorem, we recall the following well-known result from harmonic analysis (see Theorem B.7, Appendix B in [4] for more details).

Theorem 3.3 (Marcinkiewicz Interpolation)**.** *Let $(S_1, \mu_1)$and (S_2, μ_2) be two measure spaces and p_0, p_1, q_0, q_1 be numbers such that $1 \le p_j \le q_j \le \infty$, for $j = 0, 1$ and $q_0 \ne q_1$. If $T : L^{p_j}(S_1, \mu_1) \to L^{q_j}(S_2, \mu_2)$ is bounded, for $j = 0, 1$, then $T : L^{p_t}(S_1, \mu_1) \to L^{q_t}(S_2, \mu_2)$ is bounded, for each (p_t, q_t), provided that*

$$\frac{1}{p_t} = \frac{1-t}{p_0} + \frac{1}{p_1} \quad \text{and} \quad \frac{1}{q_t} = \frac{1-t}{q_0} + \frac{t}{q_1}$$

with $0 < t < 1$.

Proof of Theorem 3.2. By the Marcinkiewicz Interpolation Theorem, it suffices to show that

$$\|T[\varphi]\|_{L^1(\Omega)} \le C_1 \|\varphi\|_{L^1_{(0,1)}(\Omega)} \tag{3.2}$$

and

$$\|T[\varphi]\|_{L^\infty(\Omega)} \le C_\infty \|\varphi\|_{L^\infty_{(0,1)}(\Omega)}. \tag{3.3}$$

Since the second integral in the representation of $\mathcal{T}$ in (3.1) is the Bochner-Martinelli operator over Ω, it is bounded from $L^p_{(0,1)}(\Omega) \to L^p(\Omega)$ for all $1 \le p \le \infty$, so it is not significant in our analysis. The problematic subject is the boundary integral in (3.1).

Firstly, we prove the estimate (3.3) since its proof is shorter than the second estimate's one. Let $\varphi \in L^\infty_{(0,1)}(\Omega)$, we have

$$\left| \frac{1}{2\pi^2} \int_{\zeta \in b\Omega} \frac{\frac{\partial\rho(\zeta)}{\partial\zeta_1}(\bar{\zeta}_2 - \bar{z}_2) - \frac{\partial\rho(\zeta)}{\partial\zeta_2}(\bar{\zeta}_1 - \bar{z}_1)}{\Phi(\zeta, z)|\zeta - z|^2} \varphi(\zeta) \wedge d\zeta_1 \wedge d\zeta_2 \right|$$

$$\lesssim \|\varphi\|_{L^\infty_{(0,1)}(\Omega)} \int_{b\Omega} \frac{dS(\zeta)}{|\Phi(\zeta,z)||\zeta - z|},$$

where dS is the surface element on $b\Omega$.

By the same argument in [8], for a small $0 < \sigma < c/12$ (c is the constant in Lemma 2.5), we choose a cutoff function $\psi \in C^\infty(\mathbb{C}^2 \times \mathbb{C}^2)$ such that $\psi(\zeta, z) = 1$ on the set $\{(\zeta, z) \in \mathbb{C}^2 \times \mathbb{C}^2 : |\rho(z)| + |\operatorname{Im}(\Phi(\zeta, z))| + F(|\zeta - z|^2) < \sigma/2\}$ and $\psi(\zeta, z) = 0$ on the set $\{(\zeta, z) \in \mathbb{C}^2 \times \mathbb{C}^2 : |\rho(z)| + |\operatorname{Im}(\Phi(\zeta, z))| + F(|\zeta - z|^2) > \sigma\}$. Hence,

$$\int_{b\Omega} \frac{dS(\zeta)}{|\Phi(\zeta,z)||\zeta - z|} = \int_{b\Omega} (1 - \psi(\zeta, z)) \frac{dS(\zeta)}{|\Phi(\zeta,z)||\zeta - z|} + \int_{b\Omega} \psi(\zeta, z) \frac{dS(\zeta)}{|\Phi(\zeta,z)||\zeta - z|}.$$

By the construction of ψ, the first component in above sum is a non-singular integral, then

$$\int_{b\Omega} \frac{dS(\zeta)}{|\Phi(\zeta,z)||\zeta - z|} \lesssim 1 + \int_{b\Omega} \psi(\zeta, z) \frac{dS(\zeta)}{|\Phi(\zeta,z)||\zeta - z|}.$$

Since $b\Omega$ is compact, there exists a finite family of points $\{p_j\}_{j=1,\dots,k}$ such that $b\Omega$ is covered by $\{B(p_j, \sigma)\}_{j=1,\dots,k}$. We also change the coordinate of each ball with the linear map T_{p_j} which makes p_j to 0. So we only consider the boundary integral on each $b\Omega \cap B(p_j, \sigma)$. For convenience, we also still use p for p_j.

Since $b\Omega$ is a 3-dimensional regularly imbedded submanifold in $\mathbb{R}^4$, dS can be identified with the unique positive 3-dimensional Lebesgue measure on $b\Omega$, see [17, Appendix II] for more details. By applying Lemma 2.5 and changing to the new variables $t_1 + it_2 = \zeta_1 - z_1$, $t_3 = \operatorname{Im} \Phi(\zeta, z)$ and , we get

$$\begin{aligned}
\int_{b\Omega \cap B(p,\sigma)} \frac{dS(\zeta)}{|\Phi(\zeta,z)||\zeta - z|} &\lesssim \int_{b\Omega \cap B(p,\sigma)} \frac{dS(\zeta)}{(|\rho(z)| + |\operatorname{Im} \Phi(\zeta, z)| + F(|\zeta_1 - z_1|^2))\, |\zeta_1 - z_1|} \\
&\lesssim \int_{|(t_1,t_2,t_3)| \le \sigma} \frac{dt_1 dt_2 dt_3}{\left(|\rho(z)| + |t_3| + F(t_1^2 + t_2^2)\right) |(t_1, t_2)|} \\
&\lesssim \int_{|(r,t_3)| \le \sigma} \frac{r dr dt_3}{(|\rho(z)| + |t_3| + F(r^2))\, r} \\
&\lesssim \int_0^\sigma |\ln F(r^2)| dr < \infty.
\end{aligned}$$

Notice that, in the above second estimate, introduction of (t_1, t_2, t_3)-coordinates in the integral involves a bounded Jacobian factor. Then we obtain the desired $L^\infty(\Omega)$-boundedness.

For the $L^1(\Omega)$-boundedness, we must convert the integral from the boundary $b\Omega$ to the interior Ω. By the Stoke's Theorem and the assumption that $\bar\partial\varphi = 0$ (in the distribution sense), the boundary integral in (3.1) equals

$$\mathcal{H}[\varphi](z) = \int_\Omega H(\zeta, z)\varphi(\zeta) \wedge d\zeta_1 \wedge d\zeta_2,$$

where

$$H(\zeta, z) = \frac{1}{2\pi^2} \bar\partial_\zeta \left(\frac{\frac{\partial\rho(\zeta)}{\partial\zeta_1}\left(\bar\zeta_2 - \bar z_2\right) - \frac{\partial\rho(\zeta)}{\partial\zeta_2}\left(\bar\zeta_1 - \bar z_1\right)}{(\Phi(\zeta, z) - \rho(\zeta))|\zeta - z|^2} \right).$$

Recalling the covering $\{B(p_j, \sigma)\}_{j=1,\dots,k}$ of $b\Omega$, we need to show that

$$\int_{\Omega\cap B(p,\sigma)} |\mathcal{H}\varphi(z)| dV(z) \lesssim \|\varphi\|_{L^1_{(0,1)}(\Omega)}.$$

Let $\varphi \in L^1_{(0,1)}(\Omega)$, we have

$$\begin{aligned}
\int_{\Omega\cap B(p,\sigma)} |\mathcal{H}\varphi(z)| dV(z) &= \iint_{(\zeta,z)\in\Omega\times(\Omega\cap B(p,\sigma))} |H(\zeta,z)\varphi(\zeta)|\, dV(\zeta,z) \\
&\lesssim \iint_{(\zeta,z)\in\Omega\times(\Omega\cap B(p,\sigma))} \frac{O(|\zeta-z|)}{|\Phi(\zeta,z)-\rho(\zeta)|^2|\zeta-z|^2}\cdot|\varphi(\zeta)| dV(\zeta,z) \\
&\lesssim \iint_{(\zeta,z)\in\Omega\times(\Omega\cap B(p,\sigma)),|\zeta-z|<c} \frac{|\varphi(\zeta)|}{|\Phi(\zeta,z)-\rho(\zeta)|^2|\zeta-z|} dV(\zeta,z) \\
&+ \iint_{(\zeta,z)\in\Omega\times(\Omega\cap B(p,\sigma)),|\zeta-z|\geq c} \frac{|\varphi(\zeta)|}{|\Phi(\zeta,z)-\rho(\zeta)|^2|\zeta-z|} dV(\zeta,z) \\
&= (I) + (II),
\end{aligned}$$

where $0 < \sigma < c$ is small enough. Since (II) is a non-singular integral, we only need to estimate the term (I). Let us consider the changing of variables $(\alpha,\omega) = (\alpha_1,\alpha_2,\omega_1,\omega_2) = (\zeta_1,\zeta_2,z_1-\zeta_1,\rho(z)+i\,\mathrm{Im}(\Phi(\zeta,z)))$ and let J be its Jacobian, then

$$\det(J) = \frac{\partial\,\mathrm{Im}(\Phi(\zeta,z))}{\partial\,\mathrm{Im}\, z_2}\frac{\partial\rho(z)}{\partial\,\mathrm{Re}\, z_2} - \frac{\partial\,\mathrm{Im}(\Phi(\zeta,z))}{\partial\,\mathrm{Re}\, z_2}\frac{\partial\rho(z)}{\partial\,\mathrm{Im}\, z_2}.$$

By a possible rotation and dilation of Ω, we can assume that $\nabla\rho(0) = (0,0,0,-1)$. A direct calculation then establishes that if σ is chosen sufficiently small so that $\det(J) \neq 0$. Applying the estimate in Lemma 2.5, we get

$$\begin{aligned}
(I) &\lesssim \int_{(\omega,\alpha)\in(\Omega\cap B(0,\sigma))\times B(0,\sigma)} \frac{|\varphi(\alpha)|}{\left(|\omega_2| + F(|\omega_1^2|)\right)^2 |\omega_1|} dV(\alpha,\omega) \\
&\lesssim \|\varphi\|_{L^1(\Omega)} \int_{B(0,\sigma)} \frac{dV(\omega)}{\left(|\omega_2| + F(|\omega_1|^2)\right)^2 |\omega_1|} \\
&\lesssim \|\varphi\|_{L^1(\Omega)} \int_{|(t_1,t_2,t_3,t_4)|<\sigma} \frac{dt_1dt_2dt_3dt_4}{(|t_3|+|t_4|+F(t_1^2+t_2^2))^2\sqrt{t_1^2+t_2^2}} \\
&\qquad\qquad (\text{where } \omega_1 = t_1+it_2, \omega_2 = t_3+it_4) \\
&\lesssim \|\varphi\|_{L^1(\Omega)} \int_{|(t_1,t_2)|<\sigma} \frac{|\ln(F(t_1^2+t_2^2))|dt_1dt_2}{\sqrt{t_1^2+t_2^2}} \\
&\lesssim \|\varphi\|_{L^1(\Omega)} \int_0^\sigma |\ln(F(r^2))|dr \quad (\text{using the polar coordinates } r = |(t_1,t_2)|) \\
&\lesssim \|\varphi\|_{L^1(\Omega)} \quad (\text{by the second condition on } F).
\end{aligned}$$

Hence the proof of Theorem 3.2 is complete due to the Fubini - Tonelli Theorem from the measure theory. □

4. Proof of the main result

The main idea to prove Main Theorem is based on techniques of Ligocka in [21] on strongly pseudoconvex domains. In our proof, we extend her setup on convex domains admitting F-type.

For $u \in C^1(\overline{\Omega}) \cap \mathcal{O}(\Omega)$ and u is holomorphic on Ω, by the Stoke Theorem, we get

$$u(z) = \int_\Omega u(\zeta)\bar{\partial}_\zeta \Omega_0(\widetilde{C}(\zeta, z)), \quad z \in \Omega.$$

By the smoothness of each component in $\Omega_0((\widetilde{C}(\zeta, z))$ then the form $\bar{\partial}_\zeta\Omega_0((\widetilde{C}(\zeta, z))$ also is a smooth form on $\overline{\Omega} \times \Omega$.

For $0 < c < \delta$ (c is the constant in Lemma 2.5), let us define $\Omega_\delta = \{z \in \mathbb{C}^2 : \rho(z) < \delta\}$ and let P_z be the Hörmander solution operator to the $\bar{\partial}$-equation in the variables $z \in \Omega_\delta$ (the existence of P_z can be found in [12]).

Definition 4.1. For $(\zeta, z) \in \overline{\Omega} \times \overline{\Omega}_\delta$, let us define

$$\begin{aligned} Q(\zeta, z) &= -P_z\left(\bar{\partial}_z\bar{\partial}_\zeta\Omega_0((\widetilde{C}(\zeta, z))\right), \\ G(\zeta, z) &= Q(\zeta, z) + \bar{\partial}_\zeta\Omega_0((\widetilde{C}(\zeta, z)), \end{aligned}$$

where $G(\zeta, z)$ is holomorphic in z.

The fact $Q(\zeta, z) \in C^\infty(\overline{\Omega}) \times C^1(\overline{\Omega})$ implies that

$$G(\zeta, z) = \frac{1}{\pi^2}\frac{1}{(\Phi(\zeta, z) - \rho(\zeta)^3}\left[\det\begin{pmatrix} \rho(\zeta) & \dfrac{\partial\rho}{\partial\zeta_1}(\zeta) & \dfrac{\partial\rho}{\partial\zeta_2}(\zeta) \\ \dfrac{\partial\rho}{\partial\bar{\zeta}_1}(\zeta) & \dfrac{\partial^2\rho}{\partial\zeta_1\partial\bar{\zeta}_1}(\zeta) & \dfrac{\partial^2\rho}{\partial\zeta_2\partial\bar{\zeta}_1}(\zeta) \\ \dfrac{\partial\rho}{\partial\bar{\zeta}_2}(\zeta) & \dfrac{\partial^2\rho}{\partial\zeta_1\partial\bar{\zeta}_2}(\zeta) & \dfrac{\partial^2\rho}{\partial\zeta_2\partial\bar{\zeta}_2}(\zeta) \end{pmatrix}\right]$$
$$d\zeta_1 \wedge d\bar{\zeta}_1 \wedge d\zeta_2 \wedge d\bar{\zeta}_2 + \text{non-singular terms.}$$

Let u be a holomorphic function defined on Ω_δ, since

$$\begin{aligned} \int_\Omega u(\zeta)P_z(\bar{\partial}_z\bar{\partial}_\zeta\Omega_0((\widetilde{C}(\zeta, z))) &= \int_\Omega P_z(u(\zeta)\bar{\partial}_z\bar{\partial}_\zeta\Omega_0((\widetilde{C}(\zeta, z))) \\ &= P_z(\int_\Omega u(\zeta)\bar{\partial}_z\bar{\partial}_\zeta\Omega_0((\widetilde{C}(\zeta, z))) \\ &= P_z(\int_\Omega u(\zeta)\bar{\partial}_\zeta\bar{\partial}_z\Omega_0((\widetilde{C}(\zeta, z))) \\ &= P_z(\int_{b\Omega} u(\zeta)\bar{\partial}_z\Omega_0((\widetilde{C}(\zeta, z))) \\ &= 0 \quad \text{(see [15, 1.4.2])}, \end{aligned}$$

we have the reproductive property of $G(\zeta, z)$ that $u(z) = \displaystyle\int_\Omega u(\zeta)G(\zeta, z)$ for all $z \in \Omega$. More generally, let $u \in L^2(\Omega)$, we define

$$\mathcal{G}[u](z) = \int_\Omega u(\zeta)G(\zeta, z)$$

and its dual

$$\mathcal{G}^*[u](z) = \int_\Omega u(\zeta)\overline{G(\zeta, z)}.$$

Then $\mathcal{G} : L^2(\Omega) \to A^2(\Omega)$ is a well-defined, continuous operator. Moreover, we also have:

Theorem 4.2. [8, Theorem 3.4]*[Ligocka's decomposition]. Let $\Omega \subset \mathbb{C}^2$ be a smoothly bounded, convex domain. Assume that Ω admits a F-type at all boundary points for some function F. Then $\mathcal{P}[u](z) = \mathcal{G}(I - \mathcal{B})^{-1}[u](z) = (I + \mathcal{B})^{-1}\mathcal{G}^*[u](z)$, where*

$$\mathcal{B}[u](z) = \mathcal{G}^*[u](z) - \mathcal{G}[u](z).$$

Moreover, $\mathcal{B}$ maps continuously $L^\infty(\Omega)$ into $\Lambda^f(\Omega)$, where

$$f(d^{-1}) = \left(\int_0^d \frac{\sqrt{F^*(t)}}{t} dt\right)^{-1}$$

and F^* is the inversion function of F.

The proof of Main Theorem is based on the following two propositions:

Proposition 4.3. *Let Ω be a smoothly bounded, convex domain in $\mathbb{C}^2$ admitting an F-type at all boundary points. Then $\mathcal{P}$ maps continuously $L^\infty(\Omega)$ onto $BlHol(\Omega)$.*

This proposition is exactly the second statement of the Main Theorem.

Proposition 4.4. *Let Ω be a smoothly bounded, convex domain in $\mathbb{C}^2$ admitting an F-type at all boundary points. Then $BlHol(\Omega)$ represents the dual space $\left(A^1(\Omega)\right)^*$ via the following scalar product:*

$$\langle u, v\rangle_1 = \langle u, \mathcal{L}^1 v\rangle_0 = \int_\Omega u(z)\overline{\mathcal{L}^1 v(z)} dV(z), \qquad u \in A^1(\Omega), v \in BlHol(\Omega),$$

where $\mathcal{L}^1 : C^\infty(\overline{\Omega}) \to C^\infty(\overline{\Omega})$ is the Bell's extension operator of first order (see [1]*) which is defined by*

$$\mathcal{L}^1 u = u - \Delta\left(\frac{u\phi\rho^2}{2|\nabla\rho|^2}\right),$$

where ϕ is an arbitrarily smooth function equal to 1 in a neighborhood of $b\Omega$ and equal to 0 in a neighborhood of the set $\{z \in \mathbb{C}^2 : \nabla\rho(z) = 0\}$.

The operator $\mathcal{L}^1$ plays an important role in studying various duality relations between spaces of holomorphic and harmonic functions (see References in [1]). Some fundamental properties of $\mathcal{L}^1$ which can be also found in [1] are:

(1) $\mathcal{L}^1 u$ vanishes on $b\Omega$.
(2) $(u - \mathcal{L}^1 u)$ is orthogonal to $L^2_{\mathrm{Har}}(\Omega)$.
(3) if $u \in \mathrm{BlHol}(\Omega)$ then $\mathcal{L}^1 u \in L^\infty(\Omega)$.

In all proofs below we need the following inequality of the Bell's operator which is contained in [21, p. 231].

Lemma 4.5. *If $u \in BlHol(\Omega)$ then $\mathcal{L}^1 u \in L^\infty(\Omega)$ and*

$$\|\mathcal{L}^1 u\| \lesssim \|u\|_{BlHol(\Omega)}.$$

Proof of Proposition 4.3. Since the continuity of $\mathcal{B}$ in Theorem 4.2 and the fact that $\mathrm{Ker}[I-\mathcal{B}]=\{0\}$, $I-\mathcal{B}$ is a Fredholm isomorphism of $L^\infty(\Omega)$. Thus, it is sufficient to prove that $\mathcal{G}$ maps continuously $L^\infty(\Omega)$ into $\mathrm{BlHol}(\Omega)$.

Let $u\in L^\infty(\Omega)$, we must show that

$$(|\rho(z)|\cdot|\mathcal{G}u(z)|+|\rho(z)|\cdot|\nabla\mathcal{G}u(z)|)\lesssim\|u\|_\infty \tag{4.1}$$

for all $z\in\Omega$. We consider the first term in (4.1)

$$\begin{aligned}|\rho(z)|\cdot|\mathcal{G}u(z)|&=|\rho(z)|\cdot\left|\int_\Omega u(\zeta)G(\zeta,z)dV(\zeta)\right|\\&\le|\rho(z)|\cdot\|u\|_\infty\int_\Omega|G(\zeta,z)|\,dV(\zeta)\\&\le\|u\|_\infty\left(|\rho(z)|\int_\Omega|Q(\zeta,z)|\,dV(\zeta)+|\rho(z)|\int_\Omega\left|\bar\partial_\zeta\Omega_0((\widetilde{C}(\zeta,z))\right|dV(\zeta)\right)\\&\lesssim\|u\|_\infty\left(1+|\rho(z)|\int_\Omega\left|\bar\partial_\zeta\Omega_0((\widetilde{C}(\zeta,z))\right|dV(\zeta)\right).\end{aligned}$$

For $0<c<\sigma$ (c is the constant in Lemma 2.5), let $h\in C^\infty(\mathbb{C}^2)$ be a cut-off function such that $h=1$ on $\{\zeta\in\mathbb{C}^2:|\rho(\zeta)|+|\rho(z)|+|\operatorname{Im}(\Phi(\zeta,z))|+F(|\zeta-z|^2)<\sigma/2\}$ and $h=0$ on $\{\zeta\in\mathbb{C}^2:|\rho(\zeta)|+|\rho(z)|+|\operatorname{Im}(\Phi(\zeta,z))|+F(|\zeta-z|^2)>\sigma\}$. Then,

$$\begin{aligned}\int_\Omega\left|\bar\partial_\zeta\Omega_0((\widetilde{C}(\zeta,z))\right|dV(\zeta)&=\int_\Omega(1-h(\zeta))\left|\bar\partial_\zeta\Omega_0((\widetilde{C}(\zeta,z))\right|dV(\zeta)\\&\quad+\int_\Omega h(\zeta)\left|\bar\partial_\zeta\Omega_0((\widetilde{C}(\zeta,z))\right|dV(\zeta)\\&\lesssim1+\int_\Omega h(\zeta)\left|\bar\partial_\zeta\Omega_0((\widetilde{C}(\zeta,z))\right|dV(\zeta)\\&\lesssim1+\int_{|\rho(\zeta)|+|\rho(z)|+|\operatorname{Im}(\Phi(\zeta,z))|+F(|\zeta-z|^2)<\sigma}\left|\bar\partial_\zeta\Omega_0((\widetilde{C}(\zeta,z))\right|dV(\zeta)\\&\lesssim\int_{|\rho(\zeta)|+|\rho(z)|+|\operatorname{Im}(\Phi(\zeta,z))|+F(|\zeta-z|^2)<\sigma}\left|\bar\partial_\zeta\Omega_0((\widetilde{C}(\zeta,z))\right|dV(\zeta).\end{aligned}$$

Since $|\bar\partial_\zeta\Omega_0((\widetilde{C}(\zeta,z))|$ is dominated by $\dfrac{1}{|\Phi(\zeta,z)-\rho(\zeta)|^3}$ when ζ near to z, we obtain

$$\begin{aligned}&\int_{|\rho(\zeta)|+|\rho(z)|+|\operatorname{Im}(\Phi(\zeta,z))|+F(|\zeta-z|^2)<\sigma}\left|\bar\partial_\zeta\Omega_0((\widetilde{C}(\zeta,z))\right|dV(\zeta)\\&\qquad\lesssim\int_{|\rho(\zeta)|+|\rho(z)|+|\operatorname{Im}(\Phi(\zeta,z))|+F(|\zeta-z|^2)<\sigma}\frac{1}{|\Phi(\zeta,z)-\rho(\zeta)|^3}dV(\zeta).\end{aligned}$$

To estimate the last integral in the above inequality, we use the following Henkin coordinates on Ω (see [29, Lemma V3.4]). These coordinates do exist since $\nabla\rho(\zeta)|_{\zeta=z}$ and $\nabla\operatorname{Im}\Phi(\zeta,z)|_{\zeta=z}$ are nonzero and are not proportial.

Lemma 4.6 (Henkin's coordinates)**.** *There exist positive constants M, a and $\eta\le c$, and, for each z with $dist(z,b\Omega)\le a$, there is a smooth local coordinate system $(t_1,t_2,t_3,t_4)=t=t(\zeta,z)$ on the ball $B(z,c)$ such that we have*

$$\begin{cases} t(z,z) = 0, \\ t_1(\zeta) = \rho(\zeta) - \rho(z), \\ t_2(\zeta) = \operatorname{Im}(\Phi(\zeta,z)), \\ |t| < \delta \quad for\ \zeta \in B(z,c), \\ |J_{\mathbb{R}}(t)| \le M \quad and \quad |det J_{\mathbb{R}}(t)| \ge \dfrac{1}{M}, \end{cases}$$

where $J_{\mathbb{R}}(t)$ is the Jacobian of the transformation t.

Therefore, for some $0 < \sigma' < \sigma$ small enough,

$$\begin{aligned} &\int_{|\rho(\zeta)|+|\rho(z)|+|\operatorname{Im}(\Phi(\zeta,z))|+F(|\zeta-z|^2)<\sigma} \frac{|\rho(z)|}{|\Phi(\zeta,z)-\rho(\zeta)|^3} dV(\zeta) \\ &\qquad \le \int_{|\rho(\zeta)|+|\rho(z)|+|\operatorname{Im}(\Phi(\zeta,z))|+F(|\zeta-z|^2)<\sigma} \frac{1}{|\Phi(\zeta,z)-\rho(\zeta)|^2} dV(\zeta) \\ &\qquad \lesssim \int_{|(t_1,\dots,t_4)|\le\sigma} \frac{1}{(|t_1|+|t_2|+F(|(t_3,t_4)|^2))^2} dt_1 \dots dt_4 \\ &\qquad \lesssim \iint_{(r_1,r_2)\in(0,\sigma')^2} \frac{r_1 r_2}{r_1^2+F^2(r_2^2)} dr_1 dr_2 \\ &\qquad \text{(using the polar coordinates } r_1 = |(t_1,t_2)| \text{ and } r_2 = |(t_3,t_4)|) \\ &\qquad \lesssim \int_0^{\sigma'} |\ln F(r^2)| dr < \infty. \end{aligned}$$

Next, for the second term in (4.1), we have the note that $\left|\dfrac{\partial}{\partial z_j}\bar{\partial}_\zeta \Omega_0(\widetilde{C}(\zeta,z))\right|$ is dominated by $\dfrac{|\zeta - z|}{|\Phi(\zeta,z)-\rho(\zeta)|^4}$. Thus, for all $z \in \Omega$, using the Henkin coordinates again, we have

$$\begin{aligned} |\rho(z)|\cdot|\nabla\mathcal{G}u(z)| &\lesssim |\rho(z)|\cdot\|u\|_\infty \int_\Omega \frac{dV(\zeta)}{|\Phi(\zeta,z)-\rho(\zeta)|^4} \\ &\le \|u\|_\infty \left(\int_\Omega (1-h(\zeta)) \frac{dV(\zeta)}{|\Phi(\zeta,z)-\rho(\zeta)|^3} + \int_\Omega h(\zeta) \frac{dV(\zeta)}{|\Phi(\zeta,z)-\rho(\zeta)|^3} \right) \\ &\lesssim \|u\|_\infty \left(1 + \int_\Omega h(\zeta) \frac{dV(\zeta)}{|\Phi(\zeta,z)-\rho(\zeta)|^3} \right) \\ &\lesssim \|u\|_\infty \left(1 + \int_{|(t_1,\dots,t_4)|\le\sigma'} \frac{1}{(|t_1|+|t_2|+F(|(t_3,t_4)|^2))^2 |(t_3,t_4)|} dt_1 \dots dt_4 \right) \\ &\lesssim \|u\|_\infty \left(1 + \iint_{(t_3,t_4)\in(0,\sigma')^2} \frac{1}{(|(t_3,t_4)|} \ln F(|(t_3,t_4)|^2) dt_3 dt_4 \right) \\ &\lesssim \|u\|_\infty \left(1 + \int_0^{\sigma'} |\ln F(r^2)| dr \right) < \infty. \end{aligned}$$

Therefore we conclude that for all $u \in L^\infty(\Omega)$, $\mathcal{G}[u] \in \operatorname{BlHol}(\Omega)$. So $\mathcal{G}$ is continuous from $L^\infty(\Omega)$ to $\operatorname{BlHol}(\Omega)$.

Finally, we show that $\mathcal{P}$ is onto $\operatorname{BlHol}(\Omega)$. Let any $u \in \operatorname{BlHol}(\Omega)$ then $v = \mathcal{L}^1 u \in L^\infty(\Omega)$. Thus we get

$$\mathcal{P}[v](z) - u(z) = \mathcal{P}[v-u](z)$$

$$= \int_\Omega K_\Omega(\zeta, z) \cdot (v(\zeta) - u(\zeta)) dV(\zeta)$$
$$= 0$$

since $(u - \mathcal{L}^1 u)$ is orthogonal to $L^2_{\mathrm{Har}}(\Omega)$ and $K_\Omega(\zeta, z)$ is L^2-(pluri)harmonic in ζ-variables. Therefore $\mathcal{P}[v] = u$. □

To prove the Proposition 4.4, we shall use the following fact.

Lemma 4.7. *Let Ω be a smoothly bounded, convex domain in $\mathbb{C}^2$ admitting an F-type at all boundary points. Then $A^2(\Omega)$ is dense in $A^1(\Omega)$.*

Proof. In the Theorem 3.2, we have shown that the Henkin operator $\mathcal{T}$ solving the $\bar\partial$-equation is continuous from $L^1_{(0,1)}(\Omega)$ into $L^1(\Omega)$ when Ω is any smoothly bounded, convex domain in $\mathbb{C}^2$ admitting an F-type at all boundary points. Therefore, the proof now follows from exactly the same lines in [21, page 235] (for strongly pseudoconvex domains) or in [9, (i), Theorem 1.3] (for the convex domain of infinite type Ω^∞). □

Proof of Proposition 4.4. Proposition 4.3 and Lemma 4.7 imply Proposition 4.4 by the same argument with [21, p.236]. □

Finally the first part of Main Theorem follows from Proposition 4.4 and Lemma 4.5 by the same argument with [21, p. 237].

To end this section, we provide a corollary of the Main Theorem. Let $\bar\partial^*$ be the Hilbert adjoint of $\bar\partial$ in $L^2(\Omega)$ and $\mathcal{N}$ be the operator solving the $\bar\partial$-Neumann problem $\Box\alpha = \varphi$, where φ is a $(0,1)$-form. If u is an arbitrary square integrable function in Ω, then we have

$$\mathcal{P}[u] = u - \bar\partial^* \mathcal{N}[\bar\partial u].$$

Let $\varphi \in L^p_{(0,1)}(\Omega)$ be a $\bar\partial$-closed form (in the distribution sense) and $u = \mathcal{T}[\varphi]$ be the Henkin solution solving the $\bar\partial$-equation. Then we can rewrite

$$\bar\partial^* \mathcal{N}[\varphi] = \mathcal{T}[\varphi] - \mathcal{P}[\mathcal{T}[\varphi]].$$

Combining the Main Theorem, Theorem 3.2 and the L^p-boundedness of $\mathcal{P}$ in [13], we have:

Corollary 4.8. *Let Ω be a smoothly bounded convex domain in $\mathbb{C}^2$ admitting a type F at all boundary points. Let φ be a $\bar\partial$-closed* $(0,1)$ *form. Then*

(1) *If $\varphi \in L^p_{(0,1)}(\Omega)$, for $1 < p < \infty$, $\bar\partial^* \mathcal{N}[\varphi] \in L^p(\Omega)$.*
(2) *If $\varphi \in L^1_{(0,1)}(\Omega)$ with harmonic coefficients, $\bar\partial^* \mathcal{N}[\varphi] \in L^1(\Omega)$.*
(3) *If $\varphi \in L^\infty_{(0,1)}(\Omega)$, $\bar\partial^* \mathcal{N}[\varphi] \in BlHol(\Omega)$.*

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