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STABILIZATION OF TWO STRONGLY COUPLED HYPERBOLIC EQUATIONS IN EXTERIOR DOMAINS

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Abstract

In this paper we study the behavior of the total energy and the L^2-norm of solutions of two coupled hyperbolic equations by velocities in exterior domains. Only one of the two equations is directly damped by a localized damping term. We show that, when the damping set contains the coupling one and the coupling term is effective at infinity and on captive region, then the total energy decays uniformly and the L^2-norm of smooth solutions is bounded. In the case of two Klein-Gordon equations with equal speeds we deduce an exponential decay of the energy.

1. Introduction and statement of the results

Let Ω be a domain of $\mathbb{R}^d$, $d \geqslant 2$. We denote by Δ the Laplace operator on Ω with Dirichlet boundary condition. We consider the following hyperbolic equation with localized linear damping

$$\begin{cases} \partial_t^2 u - \Delta u + mu + a(x)\partial_t u = 0 & \text{in } \mathbb{R}_+ \times \Omega, \\ u = 0 & \text{on } \mathbb{R}_+ \times \Gamma, \\ (u(0,.), \partial_t u(0,.)) = (u_0, u_1) & \text{in } \Omega, \end{cases} \tag{1.1}$$

where $a \in L^\infty(\Omega)$ is a nonnegative smooth function and $m \in \mathbb{R}_+$. It is easy to verify that the energy given by

$$E_u(t) = \frac{1}{2}\int_\Omega |\partial_t u(t,x)|^2 + |\nabla u(t,x)|^2 + m|u(t,x)|^2\, dx, \tag{1.2}$$

is non-increasing and

$$E_u(0) = \int_0^t \int_\Omega a(x)|\partial_t u(t,x)|^2\, dxdt + E_u(t),\ t > 0.$$

When $m = 0$, the stabilization problem for the linear damped wave equation has been studied by several authors. More precisely, when Ω is bounded, the uniform decay of the total energy is equivalent to the geometric control condition of Bardos et al. [7]. On the other hand, if Ω is not bounded then, in general, the decay rate of the total energy cannot be uniform. Indeed, in the whole space,i.e. $\Omega = \mathbb{R}^d$, Matsumura [19] obtained a precise $L^p - L^q$ type decay estimate for solutions of (1.1), when $a(x) = 1$,

$$E_u(t) \leqslant C(1+t)^{-1-d(\frac{1}{i}-\frac{1}{2})} I_i^2, \tag{1.3}$$

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$$\|u(t,.)\|_{L^2}^2 \leqslant C(1+t)^{-d(\frac{1}{i}-\frac{1}{2})} I_i^2, \tag{1.4}$$

where C is a positive constant, $i \in [1,2]$ and $I_i^2 = \|u_0\|_{H^1}^2 + \|u_1\|_{L^2}^2 + \|u_0\|_{L^i}^2 + \|u_1\|_{L^i}^2$. The proof in [19] is based on a Fourier transform method. In the case of exterior domains and when $a(x) \geqslant a^- > 0$ on Ω, it is easy to show that the weak solution u of the system (1.1) satisfies

$$E_u(t) \leqslant C(1+t)^{-1} I_2^2 \text{ and } \|u(t)\|_{L^2}^2 \leqslant C I_2^2, \text{ for all } t \geqslant 0. \tag{1.5}$$

In [20], Nakao obtained the estimate (1.5) for a damper which is positive near infinity and near a part of the boundary (Lions's condition). Daoulatli in [11] generalized this result by assuming that each trapped ray meets the damping region which is also effective at infinity. Recently, Aloui et al. [6] established the uniform stabilization of the total energy for the system (1.1) when the initial data are compactly supported. They proved that the rate of decay turns out to be the same as those of the heat equation, which shows that the effective damper at space infinity strengthens the parabolic structure in the equation.

In the case $m > 0$, the energy (1.2) contains the L^2 norm. Then, using the semi-group property, the type of decay (1.5) implies the exponential one

$$E_u(t) \leqslant C e^{-\delta t} E_u(0), \text{ for all } t \geqslant 0, \tag{1.6}$$

where C, δ are positive constants. In [23] Zuazua considered the nonlinear Klein-Gordon equations with dissipative term and he proved the exponential decay of energy through the weighted energy method. This result has been generalized by Aloui et al. [5] for more general nonlinearities. We refer the reader to the works of Dehman et al. [9] and Laurent et al. [14] for related results.

In this paper we will study the stabilization problem for a system of two coupled hyperbolic equations in exterior domains. More precisely, let O be a compact domain of $\mathbb{R}^d$ with C^∞ boundary $\Gamma = \partial O$ and $\Omega = \mathbb{R}^d \backslash O$

$$\begin{cases} \partial_t^2 u - \Delta u + m_1 u + b(x)\partial_t v + a(x)\partial_t u = 0 & \text{in } \mathbb{R}_+ \times \Omega, \\ \partial_t^2 v - \gamma^2 \Delta v + m_2 v - b(x)\partial_t u = 0 & \text{in } \mathbb{R}_+ \times \Omega, \\ u = v = 0 & \text{on } \mathbb{R}_+ \times \Gamma, \\ (u(0,.), \partial_t u(0,.)) = (u_0, u_1) & \text{in } \Omega, \\ (v(0,.), \partial_t v(0,.)) = (v_0, v_1) & \text{in } \Omega, \end{cases} \tag{1.7}$$

where $b \in L^\infty(\Omega)$ is a smooth function, $m_1, m_2 \in \mathbb{R}_+$ and γ is a positive constant.
Indirect damping of reversible systems occurs in several applications in engineering and mechanics. In general it is impossible or too expansive to damp all the components of the state, so it is important to study stabilization properties of coupled systems with a reduced number of feedbacks.

We associate to the system (1.7) the energy functional given by

$$E_{u,v}(t) = \frac{1}{2} \int_\Omega |\nabla u(t,x)|^2 + |\partial_t u(t,x)|^2 + m_1 |u(t,x)|^2 \, dx$$

$$+\frac{1}{2}\int_\Omega \gamma^2|\nabla v(t,x)|^2 + |\partial_t v(t,x)|^2 + m_2|v(t,x)|^2\, dx.$$

Let $\mathcal{H} = \left(H^1_D(\Omega) \times L^2(\Omega)\right)^2$ be the completion of $(C_0^\infty(\Omega))^4$ with respect to the norm

$$\|(w_0, w_1, w_2, w_3)\|_{\mathcal{H}} = \Big(\int_\Omega |\nabla w_0|^2 + \gamma^2|\nabla w_2|^2 + m_1|w_0|^2 + m_2|w_2|^2 + |w_1|^2 + |w_3|^2\, dx\Big)^{\frac{1}{2}}.$$

The linear evolution equation (1.7) can be rewritten under the form

$$\begin{cases} \mathcal{U}_t + \mathcal{A}\mathcal{U} = 0, \\ \mathcal{U}(0) = \mathcal{U}_0 \in \mathcal{H}, \end{cases} \tag{1.8}$$

where

$$\mathcal{U} = \begin{pmatrix} u \\ \partial_t u \\ v \\ \partial_t v \end{pmatrix}, \mathcal{U}_0 = \begin{pmatrix} u_0 \\ u_1 \\ v_0 \\ v_1 \end{pmatrix}$$

and the unbounded operator $\mathcal{A}$ on $\mathcal{H}$ with domain

$$D(\mathcal{A}) = \{\mathcal{U} \in \mathcal{H}, \mathcal{A}\mathcal{U} \in \mathcal{H}\}$$

is defined by

$$\mathcal{A} = \begin{pmatrix} 0 & -Id & 0 & 0 \\ -\Delta + m_1 Id & a & 0 & b \\ 0 & 0 & 0 & -Id \\ 0 & -b & -\gamma^2\Delta + m_2 Id & 0 \end{pmatrix}.$$

From the linear semi-group theory, we can infer that for $\mathcal{U}_0 \in \mathcal{H}$ the problem (1.8) admits a unique solution $\mathcal{U} \in C^0([0, +\infty[, \mathcal{H})$.

In addition, if $\mathcal{U}_0 \in D(\mathcal{A}^n)$, for $n \in \mathbb{N}$, then the solution $\mathcal{U} \in \bigcap_{i=0}^{n} C^{n-i}(\mathbb{R}_+, D(\mathcal{A}^i))$.

It is easy to verify that

$$\frac{d}{dt}E_{u,v}(t) = -\int_\Omega a(x)|\partial_t u(t,x)|^2\, dx. \tag{1.9}$$

Thus $E_{u,v}(t)$ is decreasing with respect to time.

For bounded domain, Kapitonov [13] considered the case of equal speeds ($\gamma = 1$) and proved, under some geometric conditions, the uniform decay

$$E_{u,v}(t) \leqslant Me^{-\beta t}E_{u,v}(0), \text{ for all } t \geqslant 0, \tag{1.10}$$

where $M, \beta > 0$. In [3], Ammar et al studied the indirect stability of system (1.7) in the case of one-dimensional space and when a and b have disjoint supports. More precisely, they established that the "classical" internal damping applied to only one of the equations never gives exponential stability if $\gamma \neq 1$ and for the case $\gamma = 1$ they gave an explicit necessary and sufficient conditions for the stability to occur. In [22], Toufayli generalized this result for different speeds and established, under some geometric conditions, a polynomial stability.

The problem of the indirect stabilization has been also studied for coupled wave equations by displacements (weakly coupled). Indeed Alabau et al [1] considered the following system

$$
(1.11)\qquad \begin{cases} \partial_t^2 u(t,x) - \Delta u(t,x) + b(x)v(t,x) + a(x)\partial_t u(t,x) = 0 & \text{in } \mathbb{R}_+ \times \Omega, \\ \partial_t^2 v(t,x) - \Delta v(t,x) + b(x)u(t,x) = 0 & \text{in } \mathbb{R}_+ \times \Omega, \\ u = v = 0 & \text{on } \mathbb{R}_+ \times \Gamma, \\ (u(0,.), \partial_t u(0,.)) = (u_0, u_1) & \text{in } \Omega, \\ (v(0,.), \partial_t v(0,.)) = (v_0, v_1) & \text{in } \Omega, \end{cases}
$$

where Ω is a bounded domain. They proved that the system (1.11) can not be exponentially stable and when the coupling term is constant they established a polynomial decay. In [2] Alabau et al improved this result by assuming that the regions $\{a > 0\}$ and $\{b > 0\}$ both verify GCC and the coupling term satisfies a smallness assumption. This result has been generalized by Aloui et al [4], for more natural smallness condition on the infinity norm of the coupling term. Recently, Daoulatli [10] showed that the rate of energy decay for solutions to the system on a compact manifold with a boundary is determined from a first order differential equation when the coupling zone and the damping zone verify the GCC.

In the sequel, we fix a constant $R_0 > 0$ such that

$$
O \subset B_{R_0} = \{x \in \mathbb{R}^d, |x| < R_0\}.
$$

Suppose that there exist two positive constants a^- and b^- such that the damping set $\omega_a := \{a(x) > a^- > 0\}$ and the coupling set $\omega_b := \{b(x) > b^- > 0\}$ are non-empty open subsets of Ω. As usual for damped wave (resp. Klein-Gordon) equations, we have to make some geometric assumptions on the sets ω_a and ω_b so that the energy of a single wave decays sufficiently rapidly at infinity. Here, we shall use the Geometric control condition.

DEFINITION 1.1. (see [7, 15]) We say that a set ω of Ω satisfies the geometric control condition **GCC** if there exists $T > 0$ such that from every point in Ω the generalized geodesic meets the set ω in a time $t < T$.

If ω satisfies **GCC**, we set

$$
T_\omega = \inf\{T > 0, (\omega, T) \text{ satisfies } \textbf{GCC}\}.
$$

We need also the following assumptions

($\mathcal{A}_1$) There exists $C > 0$ such that $0 \leqslant b(x) \leqslant Ca(x)$, $\forall x \in \Omega$.

($\mathcal{A}_2$) There exists $R_1 > R_0$ such that
- $B^c_{R_1} \subset \omega_a \cap \omega_b$, if $(m_1, m_2) \in \mathbb{R}_+ \times \mathbb{R}^*_+$,
- $B^c_{R_1} \subset \omega_b$ and $a(x) = \beta b(x)$, $|x| \geqslant R_1$, for some $\beta > 0$, if $m_1 = m_2 = 0$.

For $\gamma \in \mathbb{R}^*_+$, we set

$$
I_\gamma^2 = E_{u,v}(0) + (1 - \frac{1}{\gamma^2})^2 E_{\partial_t u, \partial_t v}(0) + \|(u,v)(0)\|^2_{L^2 * L^2}
$$

and

$$\mathcal{H}_\gamma = \begin{cases} \mathcal{H} \cap (L^2(\Omega))^4, \text{ if } \gamma = 1, \\ D(\mathcal{A}) \cap (L^2(\Omega))^4, \text{ if } \gamma \neq 1. \end{cases}$$

With this notation, we can state the stability result for the system (1.7).

Theorem 1.1. *Let $\gamma \in \mathbb{R}_+^*$ and $(m_1, m_2) \in \{(0,0)\} \cup \mathbb{R}_+ \times \mathbb{R}_+^*$. We assume that ω_b satisfies the **GCC** and that the assumptions $(\mathcal{A}_1)$ and $(\mathcal{A}_2)$ hold. Then for any solution (u, v) of the system* (1.7) *with initial data $(u_0, u_1, v_0, v_1) \in \mathcal{H}_\gamma$, we have*

$$E_{u,v}(t) \leqslant C(1+t)^{-1} I_\gamma^2 \text{ and } \quad \|(u,v)(t)\|^2_{L^2 * L^2} \leqslant C I_\gamma^2, \text{ for all } t \geqslant 0, \tag{1.12}$$

where C is positive constant. In addition for $(u_0, u_1, v_0, v_1) \in \mathcal{H}$, $E_{u,v}(t)$ converges to zero as t goes to infinity.

In the case of Klein-Gordon-type systems we obtain the following uniform decay.

Corollary 1. *Let $m_1, m_2, \gamma \in \mathbb{R}_+^*$. Assume that ω_b satisfies the **GCC** and the assumptions $(\mathcal{A}_1)$ and $(\mathcal{A}_2)$ hold.*

▹ *If $\gamma = 1$, then there exist positive constants C and α such that*

$$E_{u,v}(t) \leqslant C e^{-\alpha t} E_{u,v}(0), \text{ for all } t \geqslant 0, \tag{1.13}$$

for all solution (u, v) of the system (1.7) *with initial data $(u_0, u_1, v_0, v_1) \in \mathcal{H}_1$.*

▹ *If $\gamma \neq 1$, then there exists a positive constant C such that*

$$E_{u,v}(t) \leqslant \frac{C}{t^n} \sum_{k=0}^{n} E_{\partial_t^k u, \partial_t^k v}(0), \text{ for all } t \geqslant 0, \tag{1.14}$$

for all solution (u, v) of the system (1.7) *with initial data $(u_0, u_1, v_0, v_1) \in D(\mathcal{A}^n)$.*

Remark 1.

- The typical model of damping and coupling terms that satisfy the hypothesis of Theorem 1.1 is $a = \chi_{\omega_1 \cup B^c_{R_1}}$ and $b = \chi_{\omega_2 \cup B^c_{R_2}}$, where $R_2 > R_1$, $\omega_2 \subset \omega_1$ and ω_2 controls the captive rays.
- To our best knowledge, our result is new for the indirect stabilization problem in exterior domains (even in the free case).
- Note that in the case of negative coupling term, our results remain true under the same hypothesis with the change of b by $-b$.
- Remark that, when $\gamma = 1$, the energy of the system (1.7) decays as fast as that of the corresponding scalar damped equation. So the coupling through velocities, in this case, allows a full transmission of the damping effects, quite different from the coupling through the displacements.
- To prove our main result we study the energy first at infinity (Section 2) and then in bounded regions (Section 3). Keeping, only the second step, we can obtain the exponential energy decay for the system (1.7) in bounded domains with Dirichlet boundary condition.
- Due to technical difficulties we did not cover the Klein-Gordon-Wave case ($m_1 > 0$, $m_2 = 0$); we will be interested in the forthcoming work.

We conclude this introduction with an outline of the rest of this paper. In Section 2 we estimate the total energy at infinity by multiplier arguments. Section 3 is devoted to the study of the energy in bounded domain. The proof of this result is based on observability estimate for scalar wave equation. In order to control the compact terms, we prove in section 4 a weak observability estimate that is based on a unique continuation result. Finally, in Section 5 we combine the results of the previous sections to established our main results.

We denote by $\Omega_R := \Omega \cap B_R$, $C_{R,R'} = \Omega \cap (B_{R'} \backslash B_R)$, when $0 < R < R'$,

$$E^R(u, v, t) = \frac{1}{2} \int_{|x|>R} |\partial_t u(t, x)|^2 + |\nabla u(t, x)|^2 + m_1 |u(t, x)|^2 \, dx + \frac{1}{2} \int_{|x|>R} |\partial_t v(t, x)|^2 + \gamma^2 |\nabla v(t, x)|^2 + m_2 |v(t, x)|^2 \, dx,$$

$$E_R(u, v, t) = \frac{1}{2} \int_{\Omega_R} |\partial_t u(t, x)|^2 + |\nabla u(t, x)|^2 + m_1 |u(t, x)|^2 \, dx + \frac{1}{2} \int_{\Omega_R} |\partial_t v(t, x)|^2 + \gamma^2 |\nabla v(t, x)|^2 + m_2 |v(t, x)|^2 \, dx,$$

and $A \lesssim B$ means $A \leqslant CB$ for some positive constant C.

2. Estimate of energy near infinity

The main result of this section is as follows.

Proposition 2.1. *Let $\gamma \in \mathbb{R}_+^*$ and $(m_1, m_2) \in \{(0, 0)\} \cup \mathbb{R}_+ \times \mathbb{R}_+^*$. Let $R_1 > 0$ be such that $(\mathcal{A}_2)$ is satisfied and $R_2 > R_1$. Then for every $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that for all solution (u, v) of* (1.7) *with initial data $(u_0, u_1, v_0, v_1) \in \mathcal{H}_\gamma$, we have*

$$\|(u, v)(t)\|^2_{L^2 * L^2} + \int_0^t E^{R_2}(u, v, s) ds \leqslant C_\varepsilon (E_{u,v}(0) + (1 - \frac{1}{\gamma^2})^2 E_{\partial_t u, \partial_t v}(0)) + \varepsilon \int_0^t E_{u,v}(s) \, ds + C_\varepsilon \Big(\int_0^t \int_{\Omega_{R_2}} |u|^2 + |v|^2 \, dx ds + \|(u, v)(0)\|^2_{L^2 * L^2} \Big), \tag{2.1}$$

for all $t > 0$.

Let $\varphi \in C^\infty(\mathbb{R}^d)$ be a function satisfying $0 \leqslant \varphi \leqslant 1$ and

$$\varphi(x) = \begin{cases} 1 & \text{for } |x| \geqslant R_2 \\ 0 & \text{for } |x| \leqslant R_1. \end{cases}$$

To prove Proposition 2.1, we need the following Lemma.

Lemma 2.1. *We assume the hypothesis of* Proposition 2.1 *and we consider φ as above. Then for every $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that for all solution (u, v) of* (1.7) *with initial data $(u_0, u_1, v_0, v_1) \in \mathcal{H}_\gamma$, we have*

$$\int_0^t \int_\Omega b(x) \varphi |\partial_t v|^2 \, dx ds \leqslant C_\varepsilon (E_{u,v}(0) + (1 - \frac{1}{\gamma^2})^2 E_{\partial_t u, \partial_t v}(0)) + C_\varepsilon \int_0^t \int_{\Omega_{R_2}} |v|^2 \, dx ds + \varepsilon \int_0^t E_{u,v}(s) \, ds, \tag{2.2}$$

for all $t > 0$.

Proof of Lemma 2.1. Multiplying the first and the second equation of (1.7) respectively by $\varphi\partial_t v$ and $\frac{1}{\gamma^2}\varphi\partial_t u$ and integrating the sum of these results on $[0,t]\times\Omega$, we obtain

$$\Big[\int_\Omega \frac{1}{\gamma^2}\varphi\partial_t u\partial_t v + m_1\varphi uv\,dx\Big]_0^t + \int_0^t\int_\Omega b(x)\varphi|\partial_t v|^2\,dxds$$
$$= \int_0^t\int_\Omega \frac{1}{\gamma^2}b(x)\varphi|\partial_t u|^2 - \varphi a(x)\partial_t u\partial_t v + \varphi\Delta u\partial_t v$$
$$+ (m_1 - \frac{m_2}{\gamma^2})\varphi v\partial_t u + \varphi\Delta v\partial_t u - (1-\frac{1}{\gamma^2})\varphi\partial_t v\partial_t^2 u\,dxds.$$

Note that

$$\int_0^t\int_\Omega \varphi\Delta u\partial_t v\,dxds = \Big[\int_\Omega \varphi\Delta uv\,dx\Big]_0^t - \int_0^t\int_\Omega \varphi\Delta\partial_t uv\,dxds \tag{2.3}$$
$$= -\Big[\int_\Omega \nabla u(\nabla\varphi v + \varphi\nabla v)\,dx\Big]_0^t - \int_0^t\int_\Omega \Delta(\varphi v)\partial_t u\,dxds$$
$$= -\int_0^t\int_\Omega (\Delta\varphi v + \Delta v\varphi + 2\nabla v\nabla\varphi)\partial_t u\,dxds$$
$$-\Big[\int_\Omega \nabla u(\nabla\varphi v + \varphi\nabla v)\,dx\Big]_0^t.$$

Then using Young's inequality, we get

$$\Big[F_\gamma\Big]_0^t + \int_0^t\int_\Omega b(x)\varphi|\partial_t v|^2\,dxds \lesssim \int_0^t\int_\Omega ((\frac{1}{\gamma^2}a(x)+2)\varphi + C_\varepsilon|\nabla\varphi|^2)|\partial_t u|^2$$
$$+ C_\varepsilon\varphi(1-\frac{1}{\gamma^2})^2|\partial_t^2 u|^2 + |\Delta\varphi|^2|v|^2\,dxds$$
$$+\varepsilon\int_0^t\int_\Omega |\nabla v|^2 + (m_1 - \frac{m_2}{\gamma^2})^2\|\varphi\|_\infty|v|^2$$
$$+ |\partial_t u|^2 + \|\varphi\|_\infty|\partial_t v|^2\,dxds,$$

where

$$F_\gamma = \int_\Omega \varphi(\frac{1}{\gamma^2}\partial_t u\partial_t v + m_1 uv) + \nabla u(\nabla\varphi v + \varphi\nabla v)\,dx.$$

By hypothesis

$$supp(\varphi) \subset \{x\in\Omega, a(x) > a^-\}, \tag{2.4}$$

so, we deduce that

$$\Big[F_\gamma\Big]_0^t + \int_0^t\int_\Omega b(x)\varphi|\partial_t v|^2\,dxds \lesssim C_\varepsilon\int_0^t\int_\Omega a(x)(|\partial_t u|^2 \tag{2.5}$$
$$+ (1-\frac{1}{\gamma^2})^2|\partial_t^2 u|^2)\,dxds + \int_0^t\int_{\Omega_{R_2}} |v|^2\,dxds + \varepsilon\int_0^t E_{u,v}(s)\,ds.$$

Using the energy decay (1.9) and the fact that $(m_1, m_2)\in\{(0,0)\}\cup\mathbb{R}_+\times\mathbb{R}_+^*$, we can see that

$$\Big|F_\gamma(s)\Big| \lesssim E_{u,v}(s) \lesssim E_{u,v}(0), \quad \forall\, s \geqslant 0. \tag{2.6}$$

Combining (1.9), (2.5) and (2.6), we obtain (2.2). □

Lemma 2.2. *Let $\gamma \in \mathbb{R}_+^*$ and $(m_1, m_2) = (0,0)$. Let $R_1 > 0$ be such that $(\mathcal{A}_2)$ is satisfied and $R_2 > R_1$. Then for every $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that for all solution (u,v) of (1.7) with initial data $(u_0, u_1, v_0, v_1) \in \mathcal{H}_\gamma$, we have*

$$\begin{aligned}(2.7)\qquad \|(u,v)(t)\|^2_{L^2*L^2} + \int_0^t E^{R_2}(v,s)ds &\leqslant C_\varepsilon(E_{u,v}(0) + (1-\frac{1}{\gamma^2})^2 E_{\partial_t u, \partial_t v}(0)) \\ &+ \varepsilon \int_0^t E_{u,v}(s)\, ds + C_\varepsilon(\int_0^t \int_{\Omega_{R_2}} |u|^2 + |v|^2\, dxds + \|(u,v)(0)\|^2_{L^2*L^2}),\end{aligned}$$

for all $t > 0$. Where $E^{R_2}(v,t) = \frac{1}{2}\int_{|x|>R_2} |\partial_t v(t,x)|^2 + \gamma^2 |\nabla v(t,x)|^2\, dx$.

Proof of Lemma 2.2. We write the system (1.7) in the form

$$(2.8)\qquad \begin{cases} \partial_t^2 u - \Delta u + \dfrac{a(x)}{b(x)}\partial_t^2 v - \dfrac{a(x)}{b(x)}\gamma^2 \Delta v + b(x)\partial_t v = 0 & \text{in } \mathbb{R}_+ \times \Omega_{R_1^c}, \\ -\partial_t^2 v + \gamma^2 \Delta v + b(x)\partial_t u = 0 & \text{in } \mathbb{R}_+ \times \Omega_{R_1^c}. \end{cases}$$

Multiplying the first equation of (2.8) by φv and the second one by $\frac{1}{\gamma^2}\varphi u$ and integrating the sum of these results on $[0,t] \times \Omega$, we obtain

$$\begin{aligned}\int_\Omega \frac{\varphi b(x)}{2}(\frac{1}{\gamma^2}|u(t)|^2 + |v(t)|^2)\, dx + \beta \int_0^t \int_\Omega \varphi(|\partial_t v|^2 + \gamma^2 |\nabla v|^2)\, dxds \\ = \int_0^t \int_\Omega 2\varphi\beta|\partial_t v|^2 + \frac{\gamma^2 \beta \Delta\varphi}{2}|v|^2 - \nabla u(\nabla\varphi v + \varphi \nabla v) \\ + \nabla v(\nabla \varphi u + \varphi \nabla u) + (1 - \frac{1}{\gamma^2})\varphi \partial_t u \partial_t v\, dxds \\ + \int_\Omega \frac{\varphi b(x)}{2}(\frac{1}{\gamma^2}|u(0)|^2 + |v(0)|^2)\, dx - \Big[G_\gamma\Big]_0^t,\end{aligned}$$

where

$$G_\gamma = \int_\Omega \varphi(\partial_t u v + \beta \partial_t v v - \frac{1}{\gamma^2}\partial_t v u)\, dx.$$

According to Lemma 2.1, hypothesis $(\mathcal{A}_2)$ and using Young's inequality, we deduce that

$$\begin{aligned}(2.9)\qquad \int_\Omega \varphi(|u(t)|^2 + |v(t)|^2)\, dx + \int_0^t \int_\Omega \varphi(|\partial_t v|^2 + \gamma^2 |\nabla v|^2)\, dxds \\ \lesssim C_\varepsilon E_{u,v}(0) + (1 - \frac{1}{\gamma^2})^2 E_{\partial_t u, \partial_t v}(0) + \|(u,v)(0)\|^2_{L^2*L^2} \\ + \int_0^t \int_{\Omega_{R_2}} |v|^2 + |u|^2\, dxds + \varepsilon \int_0^t E_{u,v}(s)\, ds + \left|\Big[G_\gamma\Big]_0^t\right|.\end{aligned}$$

But we have

$$\begin{aligned}\Big|G_\gamma(t)\Big| &\lesssim C_{\varepsilon_1} E_{u,v}(t) + \varepsilon_1 \int_\Omega \varphi(|u(t)|^2 + |v(t)|^2)\, dx \\ &\lesssim C_{\varepsilon_1} E_{u,v}(0) + \varepsilon_1 \int_\Omega \varphi(|u(t)|^2 + |v(t)|^2)\, dx.\end{aligned}$$

So, for ε_1 small enough we get

$$\int_\Omega \varphi(|u(t)|^2+|v(t)|^2)\,dx+\int_0^t\int_\Omega \varphi(|\partial_t v|^2+\gamma^2|\nabla v|^2)\,dx\,ds \tag{2.10}$$
$$\lesssim C_\varepsilon E_{u,v}(0)+(1-\frac{1}{\gamma^2})^2E_{\partial_t u,\partial_t v}(0)+\|(u,v)(0)\|^2_{L^2*L^2}$$
$$+\int_0^t\int_{\Omega_{R_2}}|v|^2+|u|^2\,dxds+\varepsilon\int_0^t E_{u,v}(s)\,ds.$$

Since

$$\varphi\equiv 1\ for\ |x|\geqslant R_2, \tag{2.11}$$

we deduce that

$$\int_{|x|>R_2}|u(t)|^2+|v(t)|^2\,dx+\int_0^t E^{R_2}(v,s)\,ds$$
$$\leqslant\int_{\Omega_{R_1^c}}\varphi(|u(t)|^2+|v(t)|^2)\,dx+\int_0^t\int_{\Omega_{R_1^c}}\varphi(|\partial_t v|^2+\gamma^2|\nabla v|^2)\,dxds.$$

Combining this estimate with (2.10), we conclude (2.7). This finishes the proof of Lemma 2.2. □

Now we give the proof of Proposition 2.1.

Proof of Proposition 2.1. We distinguish the case $m_1=m_2=0$ and the case where $m_1\in\mathbb{R}_+$ and $m_2\in\mathbb{R}_+^*$.

First case $m_1=m_2=0$. Multiplying the first equation of (1.7) by φu and integrating on $[0,t]\times\Omega$, we obtain

$$\Big[\int_\Omega\varphi(\partial_t uu+\frac{a(x)|u|^2}{2}+b(x)uv)\,dx\Big]_0^t+\int_0^t\int_\Omega\varphi(|\nabla u|^2+|\partial_t u|^2)\,dxds \tag{2.12}$$
$$=\int_0^t\int_\Omega 2\varphi|\partial_t u|^2+\frac{\Delta\varphi}{2}|u|^2+\varphi b(x)v\partial_t u\,dxds.$$

Note that we have

$$\int_0^t\int_\Omega\varphi b(x)v\partial_t u\,dxds=\int_0^t\int_\Omega\varphi v(\partial_t^2 v-\gamma^2\Delta v)\,dxds \tag{2.13}$$
$$=\Big[\int_\Omega\varphi\partial_t vv\,dx\Big]_0^t+\int_0^t\int_\Omega\varphi(\gamma^2|\nabla v|^2-|\partial_t v|^2)-\gamma^2\frac{\Delta\varphi}{2}|v|^2\,dxds.$$

So, combining this identity with (2.12) and using (2.4), we get

$$\int_0^t\int_\Omega\varphi(|\partial_t u|^2+|\nabla u|^2)\,dxds\lesssim\int_0^t\int_\Omega a(x)|\partial_t u|^2+\int_0^t\int_\Omega\varphi(|\partial_t v|^2 \tag{2.14}$$
$$+\gamma^2|\nabla v|^2)\,dxds+\int_0^t\int_{\Omega_{R_2}}|u|^2+|v|^2\,dxds$$
$$+\left|\Big[\int_\Omega\varphi(\partial_t uu+b(x)uv+\frac{a(x)|u|^2}{2}-\partial_t vv)\,dx\Big]_0^t\right|.$$

Using that,

$$\Big|\int_\Omega \varphi(\partial_t uu + b(x)uv + \frac{a(x)|u|^2}{2} - \partial_t vv)(t)\,dx\Big| \lesssim E_{u,v}(0) + \int_\Omega \varphi(|u(t)|^2 + |v(t)|^2)\,dx$$

and

$$\Big|\int_\Omega \varphi(\partial_t uu + b(x)uv + \frac{a(x)|u|^2}{2} - \partial_t vv)(0)\,dx\Big| \lesssim E_{u,v}(0) + \|(u,v)(0)\|^2_{L^2 * L^2},$$

we obtain

$$\begin{aligned}(2.15)\qquad \int_0^t\int_\Omega \varphi(|\partial_t u|^2 + |\nabla u|^2)\,dxds &\lesssim E_{u,v}(0) + \int_\Omega \varphi(|u(t)|^2 + |v(t)|^2)\,dx\\ &+ \int_0^t\int_\Omega \varphi(|\partial_t v|^2 + \gamma^2|\nabla v|^2)\,dx\,ds + \int_0^t\int_{\Omega_{R_2}} |u|^2 + |v|^2\,dxds + \|(u,v)(0)\|^2_{L^2 * L^2}.\end{aligned}$$

According to (2.7), we find

$$\begin{aligned}(2.16)\qquad \int_0^t E^{R_2}(u,s)ds &\lesssim C_\varepsilon E_{u,v}(0) + (1-\frac{1}{\gamma^2})^2 E_{\partial_t u,\partial_t v}(0) + \varepsilon\int_0^t E_{u,v}(s)\,ds\\ &+ \int_0^t\int_{\Omega_{R_2}} |u|^2 + |v|^2\,dxds + \|(u,v)(0)\|^2_{L^2 * L^2},\end{aligned}$$

where $E^{R_2}(u,t) = \frac{1}{2}\int_{|x|>R_2} |\partial_t u(t,x)|^2 + |\nabla u(t,x)|^2\,dx$.
Combining (2.7) and (2.16), we conclude (2.1).
Second case $m_1 \in \mathbb{R}_+$ **and** $m_2 \in \mathbb{R}_+^*$. Multiplying the first and the second equation of (1.7) respectively by φu and φv and integrating the sum of these results on $[0,t]\times\Omega$, we obtain

$$\begin{aligned}(2.17)\qquad \int_\Omega \varphi\frac{a(x)|u(t)|^2}{2}\,dx &+ \int_0^t\int_\Omega \varphi(|\partial_t u|^2 + |\nabla u|^2 + m_1|u|^2 + |\partial_t v|^2\\ &+ \gamma^2|\nabla v|^2 + m_2|v|^2)\,dxds = \int_0^t\int_\Omega 2\varphi(|\partial_t u|^2 + |\partial_t v|^2)\,dxds\\ &+ \int_0^t\int_\Omega \frac{\Delta\varphi}{2}(|u|^2 + \gamma^2|v|^2) + 2\varphi b(x)v\partial_t u\,dxds\\ &- \Big[\int_\Omega \varphi(\partial_t uu + \partial_t vv + b(x)uv)\,dx\Big]_0^t + \int_\Omega \varphi\frac{a(x)|u(0)|^2}{2}\,dx\\ &\lesssim \int_0^t\int_\Omega a(x)|\partial_t u|^2 + \varphi|\partial_t v|^2 + \varepsilon\|\varphi\|_\infty|v|^2\,dxds\\ &- \Big[\int_\Omega \varphi(\partial_t uu + \partial_t vv + b(x)uv)\,dx\Big]_0^t + \int_\Omega \varphi\frac{a(x)|u(0)|^2}{2}\,dx\\ &+ \int_0^t\int_{\Omega_{R_2}} |u|^2 + |v|^2\,dxds.\end{aligned}$$

Using the following estimates for ε_2 small enough

$$\Big|\int_\Omega \varphi((\partial_t uu + \partial_t vv + b(x)uv)(t))\,dx\Big| \lesssim C_{\varepsilon_2}E_{u,v}(0) + \varepsilon_2\int_\Omega \varphi|u(t)|^2\,dx,$$

$$\Big|\int_\Omega \varphi((\partial_t uu + \partial_t vv + b(x)uv)(0))\,dx\Big| \lesssim E_{u,v}(0) + \|u(0)\|^2_{L^2},$$

and according to Lemma 2.1, we infer (2.1). The proof of Proposition 2.1 is now completed. □

3. Estimate of energy in bounded region

In this section, we will study the energy in bounded domain. For this aim, we consider a function $\psi \in C_0^\infty(\mathbb{R}^d)$ such that $0 \leqslant \psi \leqslant 1$ and

$$\psi(x) = \begin{cases} 1 & \text{for } |x| \leqslant R_3, \\ 0 & \text{for } |x| \geqslant R_4, \end{cases}$$

where $R_4 > R_3 > R_1$ and $R_1 > 0$ be such that $(\mathcal{A}_2)$ is satisfied.

It is easy to verify that $(u^i, v^i) = (\psi u, \psi v)$ satisfies the following system

$$\begin{cases} \partial_t^2 u^i - \Delta u^i + m_1 u^i + b(x)\partial_t v^i + a(x)\partial_t u^i = -2\nabla\psi\nabla u - u\Delta\psi & \text{in } \mathbb{R}_+ \times \Omega_{R_4}, \\ \partial_t^2 v^i - \gamma^2\Delta v^i + m_2 v^i - b(x)\partial_t u^i = -2\gamma^2\nabla\psi\nabla v - \gamma^2 v\Delta\psi & \text{in } \mathbb{R}_+ \times \Omega_{R_4}, \\ u^i = v^i = 0, & \text{on } \mathbb{R}_+ \times \partial\Omega_{R_4}, \\ (u_0^i, u_1^i, v_0^i, v_1^i) = (\psi u_0, \psi u_1, \psi v_0, \psi v_1). \end{cases} \tag{3.1}$$

Proposition 3.1. *Let $\gamma \in \mathbb{R}_+^*$, $(m_1, m_2) \in \{(0,0)\} \cup \mathbb{R}_+ \times \mathbb{R}_+^*$ and ψ be as above. Assume that the assumption $(\mathcal{A}_1)$ holds and that (ω_b, T) geometrically controls Ω for some $T > 0$. Then for every $\varepsilon > 0$, there exist $C_\varepsilon > 0$ such that for all solution (u, v) of* (1.7) *with initial data $(u_0, u_1, v_0, v_1) \in \mathcal{H}_\gamma$, we have*

$$\int_t^{t+T} E_{R_3}(u, v, s)ds \leqslant C_\varepsilon \int_t^{t+T}\int_\Omega a(x)(|\partial_t u|^2 + (1 - \frac{1}{\gamma^2})^2|\partial_t^2 u|^2)\,dxds \tag{3.2}$$
$$+ \varepsilon \int_t^{t+T} E_{u,v}(s)\,ds + C_\varepsilon \int_t^{t+T}\int_{\Omega_{R_4}} |u|^2 + |v|^2\,dxds + C_\varepsilon \int_t^{t+T} E^{R_3}(u, v, s)ds - \Big[\mathcal{K}_\gamma\Big]_t^{t+T}$$

for all $t > 0$. Where

$$\mathcal{K}_\gamma = -\int_\Omega \frac{b(x)}{\gamma^2}\partial_t u^i \partial_t v^i + \nabla u^i \nabla((b(x)v^i) + m_1 b(x) u^i v^i\,dx.$$

In order to prove Proposition 3.1 we need the following result.

Lemma 3.1. *Assume that the hypothesis of* Proposition 3.1 *hold. Then for every $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that for all solution (u, v) of* (1.7) *with initial data $(u_0, u_1, v_0, v_1) \in \mathcal{H}_\gamma$, we have*

$$\int_t^{t+T}\int_\Omega b(x)^2|\partial_t v^i|^2\,dxds \leqslant C_\varepsilon \int_t^{t+T}\int_\Omega a(x)(|\partial_t u|^2 + (1 - \frac{1}{\gamma^2})^2|\partial_t^2 u|^2)\,dxds \tag{3.3}$$
$$+ \varepsilon \int_t^{t+T} E_{u,v}(s)\,ds + C_\varepsilon \int_t^{t+T}\int_{\Omega_{R_4}} |v|^2 + |u|^2\,dxds$$
$$+ C_\varepsilon \int_t^{t+T}\int_{C_{R_3,R_4}} |\nabla u|^2 + |\nabla v|^2\,dxds - \Big[\mathcal{K}_\gamma\Big]_t^{t+T},$$

for all $t > 0$.

Proof of Lemma 3.1 . We multiply the first and the second equation of (3.1) respectively by $b(x)\partial_t v^i$ and $\frac{b(x)}{\gamma^2}\partial_t u^i$ and we integrate the sum of these results on $[t, t+T]\times\Omega$, we get

$$\begin{aligned}\Big[\mathcal{K}_\gamma\Big]_t^{t+T} + \int_t^{t+T}\int_\Omega b(x)^2|\partial_t v^i|^2\,dxds &= \int_t^{t+T}\int_\Omega \frac{b(x)^2}{\gamma^2}|\partial_t u^i|^2 - ab(x)\partial_t u^i\partial_t v^i \\ &+ (m_1 - \frac{m_2}{\gamma^2})b(x)v^i\partial_t u^i)\,dxds - \int_t^{t+T}\int_\Omega b(x)(2\nabla u\nabla\psi + \Delta\psi u)\partial_t v^i \\ &+ \frac{b(x)}{\gamma^2}(2\nabla v\nabla\psi + \Delta\psi v)\partial_t u^i\,dxds + \int_t^{t+T}\int_\Omega (\frac{1}{\gamma^2} - 1)b(x)\partial_t^2 u^i\partial_t v^i\,dxds \\ &- \int_t^{t+T}\int_\Omega \partial_t u^i(\Delta b(x)v^i + 2\nabla b(x)\nabla v^i)\,dxds.\end{aligned}$$

From Young's inequality and using hypothesis $(\mathcal{A}_1)$, we infer that

$$\begin{aligned}&\Big[\mathcal{K}_\gamma\Big]_t^{t+T} + \int_t^{t+T}\int_\Omega b(x)^2|\partial_t v^i|^2\,dxds \\ &\lesssim C_\varepsilon\int_t^{t+T}\int_\Omega a(x)(|\partial_t u|^2 + (1-\frac{1}{\gamma^2})^2|\partial_t^2 u|^2)\,dxds \\ &+ \varepsilon\int_t^{t+T}\int_\Omega (m_1 - \frac{m_2}{\gamma^2})^2|v|^2 + |\partial_t u|^2 + |\partial_t v|^2 + |\nabla v|^2\,dxds \\ &+ C_\varepsilon\int_t^{t+T}\int_{\Omega_{R_4}} |u|^2 + |v|^2\,dxds + C_\varepsilon\int_t^{t+T}\int_{C_{R_3,R_4}} |\nabla u|^2 + |\nabla v|^2\,dxds.\end{aligned} \tag{3.4}$$

This implies (3.3). □

Proof of Proposition 3.1. First, we recall the following observability estimate for the wave equation (see Proposition 3, [11]).

Lemma 3.2. *Let $\gamma, T > 0$ and $\mathcal{O}$ a bounded domain. Let ϕ be a nonnegative function on $\mathcal{O}$ and setting*

$$\mathcal{V} = \{\phi(x) > 0\}.$$

We assume that $(\mathcal{V}, T)$ satisfies the ***GCC***. *There exists $C_T > 0$, such that for all $(u_0, u_1) \in H_0^1(\mathcal{O})\times L^2(\mathcal{O})$, $f \in L^2_{loc}(\mathbb{R}_+, L^2(\mathcal{O}))$, and all $t > 0$ the solution of*

$$\begin{cases}\partial_t^2 u - \gamma^2\Delta u + mu = f & \text{in } \mathbb{R}_+\times\mathcal{O},\\ u = 0 & \text{on } \mathbb{R}_+\times\partial\mathcal{O},\\ (u(0,.), \partial_t u(0,.)) = (u_0, u_1) & \text{in } \mathcal{O},\end{cases}$$

where $m \geqslant 0$, satisfies with

$$E_u(t) = \frac{1}{2}\int_{\mathcal{O}} |\partial_t u(t,x)|^2 + m|u(t,x)|^2 + \gamma^2|\nabla u(t,x)|^2\,dx,$$

the inequality

$$\int_t^{t+T} E_u(s)\,ds \leqslant C_T\int_t^{t+T}\int_{\mathcal{O}} \phi(x)|\partial_t u|^2 + |f|^2\,dxds. \tag{3.5}$$

Let $\omega_{b,1} = \omega_b \cap B_{R_4} = \{x \in \Omega \cap B_{R_4},\ b(x) > b^- > 0\}$. Since (ω_b, T) satisfies the **GCC**, $B_{R_1^c} \subset \omega_b$ and $R_4 > R_1$, we conclude that $(\omega_{b,1}, T)$ geometrically controls Ω_{R_4}.
So, according to Lemma 3.2 and using hypothesis $(\mathcal{A}_1)$, we have

$$\begin{aligned}
\int_t^{t+T} E_{v^i}(s)ds &\lesssim \int_t^{t+T}\int_{\omega_{b,1}} |\partial_t v^i|^2\, dxds + \int_t^{t+T}\int_\Omega b(x)|\partial_t u^i|^2 dxds \\
&\quad + \int_t^{t+T}\int_{C_{R_3,R_4}} |\nabla v|^2\, dxds + \int_t^{t+T}\int_{\Omega_{R_4}} |v|^2\, dxds \\
&\lesssim \int_t^{t+T}\int_\Omega b^2(x)|\partial_t v^i|^2\, dxds + \int_t^{t+T}\int_\Omega a(x)|\partial_t u|^2 dxds \\
&\quad + \int_t^{t+T}\int_{C_{R_3,R_4}} |\nabla v|^2\, dxds + \int_t^{t+T}\int_{\Omega_{R_4}} |v|^2\, dxds, \quad t > 0,
\end{aligned} \tag{3.6}$$

where

$$E_{v^i}(t) = \frac{1}{2}\int_\Omega \gamma^2|\nabla v^i(t,x)|^2 + |\partial_t v^i(t,x)|^2 + m_2|v^i(t,x)|^2\, dx.$$

We have also

$$\begin{aligned}
\int_t^{t+T} E_{u^i}(s)ds &\lesssim \int_t^{t+T}\int_\Omega a(x)|\partial_t u|^2 + b^2(x)|\partial_t v^i|^2 dxds \\
&+ \int_t^{t+T}\int_{C_{R_3,R_4}} |\nabla u|^2\, dxds + \int_t^{t+T}\int_{\Omega_{R_4}} |u|^2\, dxds, \quad t > 0,
\end{aligned} \tag{3.7}$$

where

$$E_{u^i}(t) = \frac{1}{2}\int_\Omega |\nabla u^i(t,x)|^2 + |\partial_t u^i(t,x)|^2 + m_1|u^i(t,x)|^2\, dx.$$

Adding the two estimates above and using (3.3), we deduce that

$$\begin{aligned}
\int_t^{t+T} E_{u^i,v^i}(s)ds &\lesssim C_\varepsilon \int_t^{t+T}\int_\Omega a(x)(|\partial_t u|^2 + (1-\frac{1}{\gamma^2})^2|\partial_t^2 u|^2)dxds \\
+\, \varepsilon\int_t^{t+T} E_{u,v}(s)\, ds &+ C_\varepsilon\int_t^{t+T} E^{R_3}(u,v,s)ds + C_\varepsilon\int_t^{t+T}\int_{\Omega_{R_4}} |u|^2 + |v|^2 dxds + \Big[\mathcal{K}_\gamma\Big]_t^{t+T}.
\end{aligned} \tag{3.8}$$

Since $\psi \equiv 1$ for $|x| \leqslant R_3$, we get

$$\int_t^{t+T} E_{R_3}(u,v,s)\, ds \leqslant \int_t^{t+T} E_{u^i,v^i}(s)ds.$$

Combining this estimate with (3.8), we conclude (3.2). □

4. Weak observability estimate

In this section, we prove the following Proposition.

Proposition 4.1. *Let $\gamma \in \mathbb{R}_+^*$ and $m_1, m_2 \in \mathbb{R}_+$. Let $R_1 > 0$ be such that $(\mathcal{A}_2)$ is satisfied and $R_5 > R_1$. We assume that the assumption $(\mathcal{A}_1)$ holds. Then for every $T > T_{\omega_b}$ and $\alpha > 0$, there exists $C_{T,\alpha} > 0$, such that for all $(u_0, u_1, v_0, v_1) \in (H_0^1(\Omega) \times L^2(\Omega))^2$, and all $t > 0$, the solution of the system* (1.7) *satisfies the following inequality*

$$\int_t^{t+T}\int_{\Omega_{R_5}}|v|^2+|u|^2\,dxds\leqslant C_{T,\alpha}\int_t^{t+T}\int_\Omega a(x)|\partial_t u|^2\,dxds+\alpha\int_t^{t+T}E_{u,v}(s)\,ds. \tag{4.1}$$

Proof of Proposition 4.1. We note that for each $(u_0,u_1,v_0,v_1)\in(H_1^0(\Omega)\times L^2(\Omega))^2$, the solution (u,v) are given as the limit of smooth solutions $(u_n,v_n)(t)$ with $(u_n,v_n)(0)=(u_{n,0},v_{n,0})\in(C_0^\infty(\Omega))^2$ and $(\partial_t u_n,\partial_t v_n)(0)=(u_{n,1},v_{n,t})\in(C_0^\infty(\Omega))^2$ such that $(u_{n,0},v_{n,0})\to(u_0,v_0)\in(H_0^1(\Omega))^2$ and $(u_{n,1},v_{n,1})\to(u_1,v_1)\in(L^2(\Omega))^2$. Note that

$$\|u_n(t,.)-u(t,.)\|_{H^1}+\|\partial_t u_n(t,.)-\partial_t u(t,.)\|_{L^2}\xrightarrow[n\to+\infty]{}0,$$
$$\|v_n(t,.)-v(t,.)\|_{H^1}+\|\partial_t v_n(t,.)-\partial_t v(t,.)\|_{L^2}\xrightarrow[n\to+\infty]{}0,$$

uniformly on the each closed interval $[0,T]$ for any $T>0$. Therefore we may assume that (u,v) is smooth.

To prove the estimate (4.1), we argue by contradiction. We assume that there exist a positive sequence (t_n) and a sequence

$$\mathcal{U}_n=(u_n,\partial_t u_n,v_n,\partial_t v_n)$$

of solution of the system (1.7) with initial data $(u_{n,0},u_{n,1},v_{n,0},v_{n,1})\in(H_0^1(\Omega)\times L^2(\Omega))^2$, such that

$$\int_{t_n}^{t_n+T}\int_{\Omega_{R_5}}|u_n|^2+|v_n|^2\,dxds\geqslant n\int_{t_n}^{t_n+T}\int_\Omega a(x)|\partial_t u_n|^2\,dxdt+\alpha\int_{t_n}^{t_n+T}E_{u_n,v_n}\,ds.$$

Set

$$\beta_n^2=\int_{t_n}^{t_n+T}\int_{\Omega_{R_5}}|u_n|^2+|v_n|^2\,dxds$$

and

$$(y_n,\partial_t y_n,z_n,\partial_t z_n)(t):=\frac{\mathcal{U}_n(t+t_n)}{\beta_n}.$$

We infer that

$$\int_0^T\int_{\Omega_{R_5}}|y_n|^2+|z_n|^2\,dxds=1, \tag{4.2}$$

$$\int_0^T\int_\Omega a(x)|\partial_t y_n|^2\,dxds\leqslant\frac{1}{n}, \tag{4.3}$$

$$\int_0^T E_{y_n,z_n}(s)\,ds\leqslant\frac{1}{\alpha}. \tag{4.4}$$

Therefore

$$(y_n,z_n)\rightharpoonup(y,z)\text{ in }\;L^2((0,T),H_0^1(\Omega))\cap W^{1,2}((0,T),L^2(\Omega)),$$

with respect to the weak topology. By Rellich's lemma, we can assume that

$$(y_n,z_n)\to(y,z)\text{ in }\;(L^2((0,T)\times\Omega_{R_5}))^2.$$

It is easy to see that the limit (y,z) satisfies the system

$$
\begin{cases}
\partial_t^2 y - \Delta y + m_1 y + b(x)\partial_t z = 0 & \text{in } (0,T)\times\Omega,\\
\partial_t^2 z - \gamma^2\Delta z + m_2 z = 0 & \text{in } (0,T)\times\Omega,\\
y = z = 0 & \text{on } (0,T)\times\Gamma,\\
a(x)\partial_t y = 0 & \text{in } (0,T)\times\Omega
\end{cases}
\tag{4.5}
$$

and

$$
\int_0^T \int_{\Omega_{R_5}} |y|^2 + |z|^2 \, dxds = 1. \tag{4.6}
$$

It is clear that $(\partial_t y, \partial_t z)$ satisfies the following system

$$
\begin{cases}
\partial_t^2(\partial_t y) - \Delta(\partial_t y) + m_1\partial_t y + b(x)\partial_t(\partial_t z) = 0 & \text{in } (0,T)\times\Omega,\\
\partial_t^2(\partial_t z) - \gamma^2\Delta(\partial_t z) + m_2\partial_t z = 0 & \text{in } (0,T)\times\Omega,\\
\partial_t y = \partial_t z = 0 & \text{on } (0,T)\times\partial\Omega,\\
a(x)\partial_t y = 0 & \text{in } (0,T)\times\Omega.
\end{cases}
\tag{4.7}
$$

From the first and previous equations in (4.7), we deduce that $b(x)\partial_t^2 z = 0$ on $supp(a)$. But $supp(b) \subset supp(a)$, so $\partial_t^2 z = 0$ on $supp(b)$. Setting $w = \partial_t z$, we have

$$
\begin{cases}
\partial_t w = 0 & \text{in } (0,T)\times\omega_b,\\
\partial_t^2 w - \gamma^2\Delta w + m_2 w = 0 & \text{in } (0,T)\times\Omega,\\
w = 0 & \text{on } (0,T)\times\partial\Omega,\\
w \in L^2((0,T)\times\Omega).
\end{cases}
\tag{4.8}
$$

Using the first and second equations in (4.8), we can see that $WF^1(w)\cap(0,T)\times\omega_b\times\mathbb{R}\times R^d$ is a subset of

$$
\{(t,x,\tau,\xi)\in(0,T)\times\Omega\times\mathbb{R}\times\mathbb{R}^n;\tau^2-\gamma^2|\xi|^2=\tau=0\}=(0,T)\times\Omega\times\{0\}\times\{0\},
$$

where $WF^1(w)$ denotes the H^1-wavefront set of w. Since $B_{R_1}^c \subset \omega_b$, we deduce that $w \in H^1_{loc}((0,T)\times B_{R_1}^c)$. Next, we will show that $w \in H^1_{loc}([0,T]\times B_{R_1})$. Let $\rho_0 = (t_0, x_0, \tau_0, \xi_0) \in T^*([0,T]\times B_{R_1})$ and Γ_0 be the generalized bicharacteristic issued from ρ_0. Set $\{\rho_1 := (0, x_1, \tau_1, \xi_1)\} = \Gamma_0 \cap \{t = 0\}$ and $\{\rho_2 := (T, x_2, \tau_2, \rho_2)\} = \Gamma_0 \cap \{t = T\}$, so we distinguish two cases,

1st **case:** x_1 or $x_2 \notin B_{R_1}$. In this case ρ_1 or $\rho_2 \notin WF^1(w)$. Since $T > T_{\omega_b}$, then using the propagation of regularity along the bicharacteristic flow of the operator $\partial_t^2 - \gamma^2\Delta$ (see [17, 18]), we obtain $\rho_0 \notin WF^1(w)$.

2nd **case:** $x_1, x_2 \in B_{R_1}$. Since $\rho_1, \rho_2 \in T^*([0,T]\times B_{R_1})$ and ω_b controls geometrically $[0,T]\times\Omega$, then Γ_0 intersects the region $[0,T]\times(\omega_b\cap\Omega_{R_1})$. But $w \in H^1_{loc}([0,T]\times(\omega_b\cap\Omega_{R_1}))$, then applying again the regularity propagation theorem, we deduce that $\rho_0 \notin WF^1(w)$. Therefore, we conclude that $w \in H^1_{loc}((0,T)\times\Omega)$. Now, set $\tilde{w} = \partial_t w$. Since $R^d \setminus \Omega_{R_5} \subset \omega_b$, so $\tilde{w} = 0$ on $R^d \setminus \Omega_{R_5}$ and satisfies

$$(4.9)\qquad \begin{cases} \partial_t^2 \tilde{w} - \gamma^2 \Delta \tilde{w} + m_2 \tilde{w} = 0 & \text{in } (0,T)\times\Omega_{R_5}, \\ \tilde{w} = 0 & \text{on } (0,T)\times\partial\Omega_{R_5}, \\ \tilde{w} = 0 & \text{in } (0,T)\times(\omega_b \cap \Omega_{R_5}), \\ \tilde{w} \in L^2((0,T)\times\Omega_{R_5}). \end{cases}$$

Since $\omega_b \cap \Omega_{R_5}$ controls geometrically Ω_{R_5}, then using the classical unique continuation result (see [7, 8]), we infer that $\tilde{w} \equiv 0$ on $(0,T)\times\Omega_{R_5}$. Therefore, the function z satisfies

$$(4.10)\qquad \begin{cases} -\gamma^2 \Delta z + m_2 z = 0 & \text{in } (0,T)\times\Omega, \\ z = 0 & \text{in } (0,T)\times\partial\Omega. \end{cases}$$

This implies that $z = 0$ on $(0,T)\times\Omega$. Now, from (4.5) we obtain

$$(4.11)\qquad \begin{cases} \partial_t^2 y - \Delta y + m_1 y = 0 & \text{in } (0,T)\times\Omega, \\ a(x)\partial_t y = 0 & \text{on } (0,T)\times\Omega, \\ y = 0 & \text{on } (0,T)\times\partial\Omega, \\ y \in H^1((0,T)\times\Omega). \end{cases}$$

Arguing as for z, we can prove that $y = 0$. This is in contradiction with (4.6). □

5. Proof of Theorem 1.1

Let $R_2 > R_1$. According to (2.1) for $t = nT$, $n \in \mathbb{N}^*$, we have

$$(5.1)\qquad \int_0^{nT} E^{R_2}(u,v,s)ds \lesssim C_\varepsilon\Big(E_{u,v}(0) + (1-\frac{1}{\gamma^2})^2 E_{\partial_t u,\partial_t v}(0) + \int_0^{nT}\int_{\Omega_{R_2}} |u|^2 + |v|^2 dxds\Big) + \varepsilon\int_0^{nT} E_{u,v}(s)\, ds + \|(u,v)(0)\|^2_{L^2 * L^2}.$$

Next, using (3.2) with $R_3 = 2R_2$ and $R_4 = 3R_2$, we get

$$(5.2)\qquad \int_{kT}^{(k+1)T} E_{2R_2}(u,v,s)ds \lesssim C_\varepsilon \int_{kT}^{(k+1)T}\int_\Omega a(x)(|\partial_t u|^2 + (1-\frac{1}{\gamma^2})^2|\partial_t^2 u|^2)dxds + \varepsilon\int_{kT}^{(k+1)T} E_{u,v}(s)\, ds + C_\varepsilon\int_{kT}^{(k+1)T} E^{2R_2}(u,v,s)ds + C_\varepsilon\int_{kT}^{(k+1)T}\int_{\Omega_{3R_2}} |u|^2 + |v|^2 dxds - \Big[\mathcal{K}_\gamma\Big]_{kT}^{(k+1)T}, \forall\, k \in \mathbb{N}.$$

Thus

$$(5.3)\qquad \sum_{k=0}^{n-1}\int_{kT}^{(k+1)T} E_{2R_2}(u,v,s)ds \lesssim \sum_{k=0}^{n-1}\Big(C_\varepsilon\int_{kT}^{(k+1)T}\int_\Omega a(x)(|\partial_t u|^2 + (1-\frac{1}{\gamma^2})^2|\partial_t^2 u|^2)dxds + \varepsilon\int_{kT}^{(k+1)T} E_{u,v}(s)\, ds - \Big[\mathcal{K}_\gamma\Big]_{kT}^{(k+1)T} + C_\varepsilon\Big(\int_{kT}^{(k+1)T} E^{2R_2}(u,v,s)ds + \int_{kT}^{(k+1)T}\int_{\Omega_{3R_2}} |u|^2 + |v|^2 dxds\Big)\Big), \forall\, n \in \mathbb{N}^*.$$

This gives

$$\int_0^{nT} E_{2R_2}(u,v,s)\,ds \lesssim C_\varepsilon \int_0^{nT}\int_\Omega a(x)(|\partial_t u|^2 + (1-\frac{1}{\gamma^2})^2|\partial_t^2 u|^2)dxds \tag{5.4}$$
$$+\,\varepsilon\int_0^{nT} E_{u,v}(s)\,ds + C_\varepsilon\int_0^{nT} E^{2R_2}(u,v,s)\,ds$$
$$+\,C_\varepsilon\int_0^{nT}\int_{\Omega_{3R_2}} |u|^2+|v|^2 dxds - \left[\mathcal{K}_\gamma\right]_0^{nT}, \forall\, n\in\mathbb{N}^*.$$

From the following estimate

$$\left|\mathcal{K}_\gamma(s)\right| \lesssim E_{u,v}(0), \forall s\geqslant 0,$$

and using (1.9) and (5.1), we deduce that

$$\int_0^{nT} E_{2R_2}(u,v,s)\,ds \lesssim C_\varepsilon(E_{u,v}(0) + (1-\frac{1}{\gamma^2})^2 E_{\partial_t u,\partial_t v}(0)) \tag{5.5}$$
$$+\,\varepsilon\int_0^{nT} E_{u,v}(s)\,ds + C_\varepsilon\int_0^{nT}\int_{\Omega_{3R_2}} |u|^2+|v|^2 dxds, \forall\, n\in\mathbb{N}^*.$$

So, combining (5.5) and (5.1), we conclude for small enough ε the following estimate

$$\int_0^{nT} E_{u,v}(s)ds \lesssim C_\varepsilon(E_{u,v}(0) + (1-\frac{1}{\gamma^2})^2 E_{\partial_t u,\partial_t v}(0)) \tag{5.6}$$
$$+\,\|(u,v)(0)\|_{L^2}^2 + C_\varepsilon\int_0^{nT}\int_{\Omega_{3R_2}} (|v|^2+|u|^2)\,dxds.$$

Next, from (4.1) with $R_5 = 3R_2$ we have

$$\sum_{k=0}^{n-1}\int_{kT}^{(k+1)T}\int_{\Omega_{3R_2}} |v|^2+|u|^2\,dxds \lesssim \sum_{k=0}^{n-1}\Big(\int_{kT}^{(k+1)T}\int_\Omega a(x)|\partial_t u|^2\,dxds$$
$$+\,\alpha\int_{kT}^{(k+1)T} E_{u,v}(s)\,ds\Big).$$

Thus

$$\int_0^{nT}\int_{\Omega_{3R_2}} |v|^2+|u|^2\,dxds \lesssim E_{u,v}(0) + \alpha\int_0^{nT} E_{u,v}(s)\,ds. \tag{5.7}$$

Finally, using (5.7) for α small enough in (5.6), we find

$$\int_0^{nT} E_{u,v}(s)ds \lesssim C_\varepsilon(E_{u,v}(0) + (1-\frac{1}{\gamma^2})^2 E_{\partial_t u,\partial_t v}(0)) + \|(u,v)(0)\|_{L^2*L^2}^2. \tag{5.8}$$

Therefore

$$\int_0^{+\infty} E_{u,v}(s)ds \lesssim E_{u,v}(0) + (1-\frac{1}{\gamma^2})^2 E_{\partial_t u,\partial_t v}(0) + \|(u,v)(0)\|_{L^2*L^2}^2.$$

As the energy is decreasing then

$$(1+t)E_{u,v}(t) \leqslant \int_0^{+\infty} E_{u,v}(s)ds + E_{u,v}(0) \tag{5.9}$$
$$\lesssim E_{u,v}(0) + (1-\frac{1}{\gamma^2})^2 E_{\partial_t u,\partial_t v}(0) + \|(u,v)(0)\|^2_{L^2*L^2}, \text{ for all } t>0.$$

On the other hand, using (2.1), (5.7) and (5.8), we deduce that

$$\int_\Omega \varphi(|u(t)|^2 + |v(t)|^2)\, dx \lesssim E_{u,v}(0) + (1-\frac{1}{\gamma^2})^2 E_{\partial_t u,\partial_t v}(0) + \|(u,v)(0)\|^2_{L^2*L^2}. \tag{5.10}$$

Since $\varphi \equiv 1$ for $|x| \geqslant R_2$,

$$\int_\Omega \varphi(|u(t)|^2 + |v(t)|^2)\, dx \geqslant \int_{\Omega^c_{R_2}} |u(t)|^2 + |v(t)|^2\, dx, \tag{5.11}$$

therefore

$$\int_{\Omega^c_{R_2}} |u(t)|^2 + |v(t)|^2\, dx \lesssim E_{u,v}(0) + (1-\frac{1}{\gamma^2})^2 E_{\partial_t u,\partial_t v}(0) + \|(u,v)(0)\|^2_{L^2*L^2}. \tag{5.12}$$

Poincare's inequality and the fact that the energy of (u, v) is decreasing give

$$\int_{\Omega_{3R_2}} |u(t)|^2 + |v(t)|^2\, dx \leqslant C_\Omega \int_{\Omega_{3R_2}} |\nabla u(t)|^2 + |\nabla v(t)|^2\, dx \lesssim E_{u,v}(0) \tag{5.13}$$

for all $t > 0$.
Adding (5.13) and (5.12), we infer that

$$\int_\Omega |u(t)|^2 + |v(t)|^2\, dx \lesssim E_{u,v}(0) + (1-\frac{1}{\gamma^2})^2 E_{\partial_t u,\partial_t v}(0) + \|(u,v)(0)\|^2_{L^2*L^2}, \tag{5.14}$$

for all $t > 0$. This finishes the proof of (1.12).
Now, using the density of $\mathcal{H}_\gamma$ in $\mathcal{H}$, we deduce from the first estimate in (1.12) that the energy $E_{u,v}(t)$ converges to zero as t goes to infinity, for every $(u_0, u_1, v_0, v_1) \in \mathcal{H}$. This achieves the proof of Theorem 1.1.

Proof of Corollary 1. Let $\gamma = 1$. From (5.9), we deduce that

$$E_{u,v}(t) \leqslant \frac{C}{t} E_{u,v}(0), \qquad \text{for all } t > 0.$$

Now, using the semi-group property, we conclude the estimate (1.13).
In the case $\gamma \neq 1$, we have

$$E_{u,v}(t) \leqslant \frac{C}{t}(E_{u,v}(0) + (1-\frac{1}{\gamma^2})^2 E_{\partial_t u,\partial_t v}(0)), \qquad \text{for all } t > 0.$$

So, according to [1, Theorem 2.1], we infer the estimate (1.14). □

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