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ON GLUING STABILITY CONDITIONS ON RULED SURFACES OF POSITIVE GENUS

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Abstract
We give a explicit description of gluing stability conditions on ruled surfaces by introducing gluing perversity. Moreover, we describe a destabilizing wall of skyscraper sheaves on ruled surfaces by deformation of stability conditions glued from $\widetilde{GL^+}(2,\mathbb{R})$-translates of the standard stability condition on the base curve.

1. Introduction

Bridgeland introduced the notion of a *stability condition* on a triangulated category in [4]. A *stability space* which is a set of stability conditions on a fixed triangulated category has a natural topology if one assumes *locally finiteness* for stability conditions. Especially, each connected component of the stability space is a complex manifold ([4] Theorem 1.2). In this paper we describe a *destabilizing wall* (Definition 2.8) of skyscraper sheaves on ruled surfaces in the stability space. A fundamental example of locally finite stability condition is *divisorial stability conditions* ([5]§6). However, the skyscraper sheaves are stable of the same phase with respect to divisorial stability conditions ([5]§6, [12] Proposition 3.6). Hence we need to ask if there is a stability condition with respect to which skyscraper sheaves are strictly semistable of the same phase.

Collins and Polishchuk [6] introduced *gluing stability conditions* on a triangulated category that has a semi-orthogonal decomposition. A derived category on a ruled surface has a semi-orthogonal decomposition that consists of its subcategories which are equivalent to the derived category on the base curve ([14]). Hence, one can hope to construct stability conditions glued from stability conditions on the base curve. In section 3, we introduce *gluing perversity* (Definition 3.6), which is the key notion to the following lemma:

Lemma 1.1 (Lemma 3.9)**.** *On ruled surfaces, a stability condition σ glued from $\widetilde{GL^+}(2,\mathbb{R})$-translates of the standard stability condition on the base curve is a locally finite stability condition if and only if the gluing perversity of σ is at least one.*

In this paper, we mean a stability condition glued from $\widetilde{GL^+}(2,\mathbb{R})$-translates of the *standard stability condition* on the base curve simply by a gluing stability condition. One can see from Theorem 1.1 that the existence of gluing stability conditions does not depend on genus of ruled surfaces. This means that the gluing stability conditions constitute a class of

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fundamental stability conditions on ruled surfaces. Furthermore, we describe the following lemma on the stability of skyscraper sheaves in the description of gluing perversity.

Lemma 1.2 (Lemma 3.10)**.** *Suppose that σ is a gluing stability condition on a ruled surface.*
(1) *If the gluing perversity of σ is equal to 1, the skyscraper sheaves are strictly semistable of the same phase for any point of the ruled surface in σ.*
(2) *If the gluing perversity is larger than 1, the skyscraper sheaves are not stable in for any point of the ruled surface in σ.*

In section 4, we describe a destabilizing wall of skyscraper sheaves on ruled surfaces. Lemma 1.2 already suggests that the set of gluing stability conditions with gluing perversity 1 is a destabilizing wall in the stability space. By deformation theory of stability conditions (see [4] §7.), we can prove the following lemma.

Lemma 1.3 (From Lemma 4.2)**.** *Let S be a ruled surface. Suppose that $\sigma_{gl} = (Z_{gl}, P_{gl})$ is a gluing stability condition with the gluing perversity 1 on S. Then there is an $\epsilon_0 > 0$ such that if $0 < \epsilon < \epsilon_0$ and $W : \mathcal{N}(S) \to \mathbb{C}$ is a group homomorphism satisfying*

- *the phase of $O_f(-C_0)$ is greater than the phase of O_f*
- *$|W(E) - Z(E)| < \sin(\pi\epsilon)|Z(F)|$ for any $E \in D^b(S)$ semistable in σ_{gl}*

then there is a unique locally finite Bridgeland stability condition $\tau = (W, \mathcal{Q})$ on S with $d(\mathcal{P}_{gl}, \mathcal{Q}) < \epsilon$ satisfying that O_x are stable of the same phase in τ for any $x \in S$.

From the above results, we can describe a certain destabilizing wall of skyscraper sheaves by simple calculation.

Theorem 1.4 (From Theorem 4.4)**.** *Let $p : S = \mathbb{P}_C(\mathcal{E}) \to C$ be a ruled surface, S_{geom} the set of divisorial stability conditions on S and $S_{div,p}$ the set of gluing stability conditions with gluing perversity p. Suppose that $A = \left(\begin{pmatrix} a & \frac{1}{2}a \deg \mathcal{E} \\ 0 & a \end{pmatrix}^{-1}, f\right) \in \widetilde{GL^+}(2, \mathbb{R})$ with $a < 0$. Then $\partial\overline{S_{div}} \cap S_{gl,1}$ is the set of $\widetilde{GL^+}(2, \mathbb{R})$-translates of a stability condition glued from $\sigma_{st}.A$ and σ_{st} where σ_{st} is a standard stability conition on C.*

2. Preliminaries : Geometric stability conditions

Bridgeland introduced the notion of a stability condition on a triangulated category in [4].

DEFINITION 2.1 ([4] DEFINITION 5.1). Let $\mathcal{D}$ be a trianguleted category and $K(\mathcal{D})$ Grothendieck group of $\mathcal{D}$. A *Bridgeland stability condition* on $\sigma = (Z, \mathcal{P})$ on $\mathcal{D}$ consists of a linear map $Z : K(\mathcal{D}) \to \mathbb{C}$ called the *central charge*, and full additive subcategories $\mathcal{P}(\phi) \subset \mathcal{D}$ for each $\phi \in \mathbb{R}$, satisfying the following axioms.
(1) for all $0 \neq E \in \mathcal{P}(\phi)$, if there exists some $m(E) > 0$ such that $Z(E) = m(E)\exp(i\pi\phi)$,
(2) $\mathcal{P}(\phi + 1) = \mathcal{P}(\phi)[1]$, for all $\phi \in \mathbb{R}$,
(3) if $\phi_1 > \phi_2$ and $A_j \in \mathcal{P}(\phi_j)$ $(j = 1, 2)$ then $\mathrm{Hom}(A_1, A_2) = 0$,
(4) for each nonzero object $E \in D$, there is a finite sequence of real number

$$\phi_1 > \phi_2 > \cdots > \phi_n$$

and a collection of triangles

$$E_j \to E_{j+1} \to A_{j+1} \to E_j[1]$$

with $E_0 = 0$, $E_n = E$, and $A_{j+1} \in \mathcal{P}(\phi_{j+1})$ for all $j = 0, \cdots, n-1$.

$\mathcal{P}$ is called the *slicing* of $\mathcal{D}$. An object E is defined to be *semistable* of phase ϕ in σ if $E \in \mathcal{P}(\phi)$. A semistable object $E \in \mathcal{P}(\phi)$ is *stable* if it has no nontrivial subobject in $\mathcal{P}(\phi)$.

Definition 2.2 ([4] Definition 5.7). A slicing $\mathcal{P}$ of a triangulated category $\mathcal{D}$ is *locally finite* if there exists a real number $\eta > 0$ such that the quasi abelian category $\mathcal{P}((t-\eta, t+\eta)) \subset \mathcal{D}$ is of finite length for all $t \in \mathbb{R}$. A Bridgeland stability condition $(Z, \mathcal{P})$ is *locally finite* if the corresponding slicing $\mathcal{P}$ is.

Since the decomposition of a nonzero object $E \in \mathcal{D}$ given by Definition 2.1 (4) is unique up to isomorphisms, we can define $\phi^+_\sigma(E) = \phi_n$, $\phi^-_\sigma(E) = \phi_1$ and $m_\sigma(E) = \Sigma_j|Z(A_j)|$. There is a generalized metric on the space of locally finite stability conditions $\operatorname{Stab}\mathcal{D}$ on a triangulated category $\mathcal{D}$. The metric d is defined by

$$d(\sigma, \tau) = \sup_{0 \neq E \in D}\left\{|\phi^+_\sigma(E) - \phi^+_\tau(E)|, |\phi^-_\sigma(E) - \phi^-_\tau(E)|, \left|\log \frac{m_\tau(E)}{m_\sigma(E)}\right|\right\}.$$

Then $\phi^\pm$ and $m(E)$ are continuous functions on $\operatorname{Stab}\mathcal{D}$. It follows immediately from this that the subset of $\operatorname{Stab}\mathcal{D}$ consisting of those stability conditions in which a given object is semistable is a closed subset ([4] Proposition 8.1).

Let S be a smooth projective surface over $\mathbb{C}$. A Bridgeland stability condition $\sigma = (Z, \mathcal{P})$ is *numerical* if the central charge $Z : K(S) \to \mathbb{C}$ factors through the numerical Grothendieck group $\mathcal{N}(S)$. *Mukai pairing* is a symmetric bilinear form $(-,-)_S$ on $\mathcal{N}(S) \simeq \mathbb{Z} \oplus \mathrm{NS}(S) \oplus \frac{1}{2}\mathbb{Z}$ defined by the following formula

$$((r_1, D_1, s_1), (r_2, D_2, s_2))_S = D_1.D_2 - r_1 s_2 - r_2 s_1.$$

The set of numerical locally finite stability conditions $\operatorname{Stab}_{\mathcal{N}} S$ is called *stability space*. If $\sigma = (Z, \mathcal{P}) \in \operatorname{Stab}_{\mathcal{N}} S$, we can write $Z(E) = (\mathrm{pr}_1(\sigma), \mathrm{ch}(E))_S$. We remark that we always identify $\mathcal{N}(S) \simeq \mathrm{Hom}(\mathcal{N}(S), \mathbb{C})$ and abuse notation $\mathrm{pr}_1 : \mathrm{Stab}(S) \to \mathrm{Hom}(\mathcal{N}(S), \mathbb{C})$ via the non-degeneracy of the Mukai pairing in this paper.

Proposition 2.3 ([4] Corollary 1.3)**.** *For each connected component* $\operatorname{Stab}^\dagger S \subset \operatorname{Stab}_{\mathcal{N}} S$*, there are a subspace* $V(\operatorname{Stab}^\dagger S) \subset \mathrm{Hom}(\mathcal{N}(S), \mathbb{C})$ *and a local homeomorphism* $\mathrm{pr}_1 : \operatorname{Stab}^\dagger S \to V(\operatorname{Stab}^\dagger S)$ *which maps a stability condition to its central charge Z. In particular* $\operatorname{Stab}^\dagger S$ *is a finite dimensional complex manifold.*

A connected component $\operatorname{Stab}^\dagger S$ is *full* if the subspace $V(\operatorname{Stab}^\dagger S)$ is equal to $\mathrm{Hom}(\mathcal{N}(S), \mathbb{C})$. A stability condition $\sigma \in \operatorname{Stab}_{\mathcal{N}} S$ is *full* if it lies in a full component. On a derived category of coherent sheaves on a surface, one of fundamental examples of numerical locally finite stability conditions are *divisorial stability conditions* ([5] §6). We can construct a divisorial stability condition in the following way:

Definition 2.4 ([4] Definition 2.1). Let $\mathcal{A}$ be an abelian category and $K(\mathcal{A})$ Grothendieck group of $\mathcal{A}$. A *stability function* on $\mathcal{A}$ is a group homomorphism $Z : K(\mathcal{A}) \to \mathbb{C}$ such that for all $0 \neq E \in \mathcal{A}$ the complex number $Z(E)$ lies in the strict upper half plane $H = \{r\exp(i\pi\phi) \mid r > 0 \text{ and } 0 < \phi \leq 1\}$.

Let $\mathcal{A}$ be a heart of a bounded t-structure of a triangulated category $\mathcal{D}$. $\mathcal{A}$ is an abelian subcategory of $\mathcal{D}$ and one has an identification of Grothendieck group $K(\mathcal{D}) = K(\mathcal{A})$. To give a stability condition on $\mathcal{D}$ is equivalent to giving a bounded t-structure $\mathcal{D}$ and a stability function on its heart $\mathcal{A}$ with the Harder Narasimhan property ([4] Proposition 5.3). In this paper, stability function is also called *pre-stability condition*.

We denote $\mathrm{Amp}(S)$ ample cone of S and $\mathrm{NS}(S)$ Neron Severi group of S. Let $\omega \in \mathrm{Amp}(S)$. One defines the slope μ_ω of a torsion free sheaf $E \in \mathrm{Coh}\, S$ by

$$\mu_\omega(E) = \frac{c_1(E).\omega}{\mathrm{rank}(E)}.$$

For any $B, \omega \in \mathrm{NS}(S) \otimes \mathbb{R}$ with $\omega \in \mathrm{Amp}(S)$ there is a unique torsion pair $(\mathcal{T}_{B,\omega}, \mathcal{F}_{B,\omega})$ on the category $\mathrm{Coh}\, S$ such that $\mathcal{T}_{B,\omega}$ consists of sheaves whose torsion free parts have μ_ω-semistable Harder Narasimhan factors of slope $\mu_\omega > B.\omega$ and $\mathcal{F}_{B,\omega}$ consists of torsion free sheaves on S all of whose μ_ω-semistable Harder Narasimhan factors have slope $\mu_\omega \le B.\omega$ ([5] Lemma 6.1).

Definition 2.5 ([1] §2 Our Charges, [12] Definition 3.3). $\sigma_{B,\omega} = (Z_{B,\omega}, \mathcal{A}_{B,\omega})$ is defined by the stability function

$$Z_{B,\omega}(E) = (\exp(B + i\omega), \mathrm{ch}(E))_S$$

and the heart of the bounded t-structure $\mathcal{A}_{B,\omega}$ which is obtained from $\mathrm{Coh}\, S$ by tilting with respect to the torsion pair $(\mathcal{T}_{B,\omega}, \mathcal{F}_{B,\omega})$. If $\sigma_{B,\omega}$ is a stability condition, we call $\widetilde{GL^+}(2, \mathbb{R})$-translate of $\sigma_{B,\omega}$ a *divisorial stability condition*.

For each pair $B, \omega \in \mathrm{NS}(S) \otimes \mathbb{Q}$ with $\omega \in \mathrm{Amp}(S)$, $\sigma_{B,\omega}$ is a numerical locally finite stability condition ([12] Proposition 3.4). $\mathrm{Stab}\,\mathcal{D}$ carries a right action of the group $\widetilde{GL^+}(2, \mathbb{R})$, the universal covering space of $GL^+(2, \mathbb{R})$ ([5] Lemma 8.2).

Definition 2.6 ([9] Definition 1.7). Let σ be a stability condition. If skyscraper sheaves $\mathcal{O}_x$ are stable of the same phase in σ for all $x \in S$, then we call σ a *geometric stability condition*.

The following proposition is useful for later calculations. This claims that a divisorial stability condition is a geometric stability condition with a certain stability function.

Proposition 2.7 ([12] Proposition 3.6)**.** *$\sigma \in \mathrm{Stab}_{\mathcal{N}}\, S$ is divisorial if and only if*
(1) *for all $x \in S$, skyscraper sheaves $\mathcal{O}_x$ are stable of the same phase in σ,*
(2) *there exist $M \in GL^+(2, \mathbb{R})$ and $B, \omega \in \mathrm{NS}(S) \otimes \mathbb{R}$ such that $\omega^2 > 0$ and $M^{-1}\,\mathrm{pr}_1(\sigma) = \exp(B + i\omega)$.*

A *ruled surface* is a smooth projective surface S, together with a surjective morphism $p : S \to C$ to a smooth projective curve of genus g, such that the fibre S_x is isomorphic to $\mathbb{P}^1$ for any point $x \in C$, and such that p admits a section $s : C \to S$ ([7] §V.2). Furthermore, let C_0 be $s(C)$, $\mathcal{E}$ the direct image sheaf $p_*\mathcal{O}_S(C_0)$ and f a fibre of p. Then S is isomorphic to the projective bundle $\mathbb{P}_C(\mathcal{E})$ of $\mathcal{E}$, and we can calculate the intersection numbers as

$$C_0^2 = \deg \mathcal{E},\ C_0.f = 1,\ f^2 = 0,$$

and the canonical divisor $K_S = -2C_0 + (2g - 2 + \deg \mathcal{E})f$. $\mathrm{NS}(S)$ is generated by C_0 and

f, and hence $\dim_{\mathbb{R}} \mathrm{Hom}(\mathcal{N}(S), \mathbb{C}) = 8$. Our interesting is destabilizing wall of skyscraper sheaves on ruled surface S.

Definition 2.8. $W \subset \mathrm{Stab}_{\mathcal{N}}(S)$ is a *destabilizing wall of skyscraper sheaves* if W satisfies the following properties:
(1) W is a real codimension one connected submanifold of $\mathrm{Stab}_{\mathcal{N}}(S)$.
(2) For any $\sigma = (Z, \mathcal{P}) \in W$ and any point $x \in S$, there exists an exact sequence $0 \to E \to \mathcal{O}_x \to F \to 0$ of semistable objects in $\mathcal{P}(\phi)$ for some $\phi \in \mathbb{R}$.
(3) There is an $\epsilon_0 > 0$ and $\sigma = (Z, \mathcal{P}) \in W$ such that if $0 < \epsilon < \epsilon_0$ and $W : \mathcal{N}(S) \to \mathbb{C}$ satisfying

$$|W(E) - Z(E)| \quad < \quad \sin(\pi\epsilon)|Z(E)|$$

for all E semistable in σ, then there is a geometric stability condition $\tau = (W, \mathcal{Q})$.

Property (3) requests that a destabilizing wall intersects the boundary of the set of geometric stability condition. We will proof (1) in Lemma 4.3, (2) in Lemma 3.10, and (3) in Lemma 4.2.

Proposition 2.9. *For any f, $\mathcal{O}_f$ is stable of the same phase in a divisorial stability condition.*

Proof. (c.f. [5] Lemma 6.3) For any f, $Z(\mathcal{O}_f)$ always take in the same value in $\mathbb{C}$. It follows immediately that for any f the phase of $\mathcal{O}_f$ is the same if O_f is stable. First, we show that a subobject of torsion sheaf is also torsion sheaf in $\mathcal{A}_{B,\omega}$. Suppose T is torsion sheaf. Recall that T lies in the torsion subcategory $\mathcal{T}_{B,\omega}$ and hence in the abelian category $\mathcal{A}_{B,\omega}$. Suppose that

$$0 \to A \to T \to B \to 0$$

is a short exact sequence in $\mathcal{A}_{B,\omega}$ with $A \in \mathcal{T}_{B,\omega}$. Taking cohomology gives an exact sequence in $\mathrm{Coh}\, S$

$$0 \to \mathcal{H}^{-1}(B) \to \mathcal{H}^0(A) \to T \to \mathcal{H}^0(B) \to 0.$$

Since $\mathcal{H}^{-1}(B) \in \mathcal{T}_{B,\omega}$, $\mathcal{H}^{-1}(B)$ is torsion free sheaf. It follows that the μ_ω-semistable facotrs of $\mathcal{H}^{-1}(B)$ and $\mathcal{H}^0(A)$ have the same slope. The contradicts the definition of the category $\mathcal{A}_{B,\omega}$ unless $\mathcal{H}^{-1}(B) = 0$, in which case either A and B must be torsion sheaf.

Second, we show that subobjects of $\mathcal{O}_f$ are $\mathcal{O}_f(-p_1 - \cdots - p_n)$ with $p_1, \cdots, p_n \in f$. Let $i : f \hookrightarrow S$ and F a subobject of $\mathcal{O}_f$. Then F is a torsion sheaf and hence i^*F is a subsheaf of the structure sheaf of f, which is $\mathcal{O}_f(-p_1 - \cdots - p_n)$ with $p_1, \cdots, p_n \in f$. It follows that

$$F \simeq Ri_* i^* F = \mathcal{O}_f(-p_1 - \cdots - p_n)$$

with $p_1, \cdots, p_n \in f$. Hence, $\mathcal{O}_f$ is stable by comparison of these phases. □

3. Constructing gluing stability conditions on ruled surfaces

This section is concerned with the construction and the existence of the gluing stability conditions on ruled surfaces, and the stability of skyscraper sheaves in gluing stability conditions.

Since p is a flat morphism, p^* is an exact functor, and hence Lp^* can be simply denoted by p^*. Since $\mathcal{O}_S(-C_0)$ is locally free sheaf, $\otimes^L \mathcal{O}_S(-C_0)$ is ordinary tensor product $\otimes \mathcal{O}_S(-C_0)$. Orlov [14] showed that a derived category of a ruled surface has *Orlov's semi-orthogonal decomposition* $D^b(S) = \langle p^*D^b(C) \otimes \mathcal{O}_S(-C_0), p^*D^b(C)\rangle$. Recall that $p^*D^b(C) \otimes \mathcal{O}_S(-C_0)$ and $p^*D^b(C)$ are equivalent to the triangulated category $D^b(C)$. There exist the following canonical isomorphisms of Grothendieck groups (c.f. [11] section 2),

$$F_1 : K(C) \simeq K(p^*D^b(C) \otimes \mathcal{O}_S(-C_0)),$$
$$F_2 : K(C) \simeq K(p^*D^b(C)).$$

Furthermore, we can describe the space of stability conditions on the both categories,

$$\mathrm{Stab}(p^*D^b(C) \otimes \mathcal{O}_S(-C_0)) = \left\{ (Z_1, \mathcal{P}_1) \;\middle|\; \begin{array}{l} (Z, \mathcal{P}) \in \mathrm{Stab}\, C,\ Z_1 = Z \circ F_1^{-1} \\ \text{for all } \phi \in \mathbb{R},\ \mathcal{P}_1(\phi) = p^*\mathcal{P}(\phi) \otimes \mathcal{O}_S(-C_0) \end{array} \right\},$$
$$\mathrm{Stab}(p^*D^b(C)) = \left\{ (Z_2, \mathcal{P}_2) \;\middle|\; \begin{array}{l} (Z, \mathcal{P}) \in \mathrm{Stab}\, C,\ Z_2 = Z \circ F_2^{-1} \\ \text{for all } \phi \in \mathbb{R},\ \mathcal{P}_2(\phi) = p^*\mathcal{P}(\phi) \end{array} \right\}.$$

$\mathrm{Stab}\, C$ is completely determined in [4], [10] and [13]. $\sigma_{st} = (Z_{st}, \mathcal{P}_{st})$ with $Z_{st}(E) = -\deg E + i \operatorname{rank} E$ and $\mathcal{P}(0, 1] = \mathrm{Coh}\, C$ is a stability condition on $\mathrm{Stab}\, C$. It is called *standard stability condition*. Especially, the following result is remarkable.

Proposition 3.1 ([4] Theorem 9.1, [10] Theorem 2.7)**.** *If a smooth projective curve C has positive genus, then the action of $\widetilde{GL^+}(2, \mathbb{R})$ on* $\mathrm{Stab}\, C$ *is free and transitive, so that*

$$\mathrm{Stab}\, C \simeq \widetilde{GL^+}(2, \mathbb{R}).$$

Collins and Polishchuk [6] gave the definition of gluing stability conditions.

DEFINITION 3.2 ([6] §2. DEFINITION). Suppose $\mathcal{D}$ is a triangulated category that have a semi-orthogonal decomposition $\langle \mathcal{D}_1, \mathcal{D}_2\rangle$, λ_1 is the left adjoint functor of $\mathcal{D}_1 \to \mathcal{D}$ and ρ_2 is the right adjoint functor of $\mathcal{D}_2 \to \mathcal{D}$. $\sigma = (Z, \mathcal{A})$ is called *gluing pre-stability condition* of σ_1 and σ_2 if $\sigma_j = (Z_j, \mathcal{A}_j) \in \mathrm{Stab}\, \mathcal{D}_j$ $(j = 1, 2)$ satisfy the following conditions,
(1) $Z = Z_1 \circ \lambda_1 + Z_2 \circ \rho_2$,
(2) $\mathcal{A} = \{F \in \mathcal{D} \mid \lambda_1(\mathcal{F}) \in \mathcal{A}_1 \text{ and } \rho_2(\mathcal{F}) \in \mathcal{A}_2\}$,
(3) $\mathrm{Hom}(\mathcal{A}_1, \mathcal{A}_2[i]) = 0$ for any $i \leq 0$ (We call this *gluing property*.)

It is called *gluing stability condition* if it satisfies Harder-Narasimhan property. In the above definition, we set

$$\mathcal{D} = D^b(S),\ \mathcal{D}_1 = p^*D^b(C) \otimes \mathcal{O}_S(-C_0) \text{ and } \mathcal{D}_2 = p^*D^b(C).$$

Then we get explicit formulas of λ_1 and ρ_2.

Proposition 3.3. *Let F be an object of $D^b(S)$. We get*
(1) $\lambda_1(F) = p^*(Rp_*(F(-C_0 + (2g - 2 + \deg \mathcal{E})f)) \otimes \omega_C^*[1]) \otimes \mathcal{O}_S(-C_0)$,
(2) $\rho_2(F) = p^*Rp_*F$.

Proof. Recall that p^* and $\otimes \mathcal{O}_S(-C_0)$ are fully faithful. λ_1 can be calculated by the following calculation.

$$\begin{aligned} & \mathrm{Hom}(F, p^*G \otimes \mathcal{O}_S(-C_0)) \\ = \;& \mathrm{Hom}(F(C_0), p^*G) \\ = \;& \mathrm{Hom}(F(C_0), p^!G \otimes \omega_p^*[-1]) \\ = \;& \mathrm{Hom}(F(C_0) \otimes \omega_p[1], p^!G) \\ = \;& \mathrm{Hom}(Rp_*(F(C_0) \otimes \omega_p[1]), G) \end{aligned}$$

$$
\begin{aligned}
&= \mathrm{Hom}(Rp_*(F(C_0)\otimes \omega_S \otimes p^*\omega_C^*[1]), G)\\
&= \mathrm{Hom}(p^*(Rp_*(F(-C_0+(2g-2+\deg\mathcal{E})f)\otimes p^*\omega_C^*[1]))\otimes \mathcal{O}_S(-C_0), p^*G\otimes \mathcal{O}_S(-C_0))\\
&= \mathrm{Hom}(p^*(Rp_*(F(-C_0+(2g-2+\deg\mathcal{E})f))\otimes \omega_C^*[1])\otimes \mathcal{O}_S(-C_0), p^*G\otimes \mathcal{O}_S(-C_0))
\end{aligned}
$$

We can get ρ_2 by similar calculation. □

If one takes stability conditions on $\mathcal{D}_1$ and $\mathcal{D}_2$, the gluing of the stability conditions under the above definition is not a stability condition. Gluing procedure is compatible with the action of $\widetilde{GL^+}(2,\mathbb{R})$.

Proposition 3.4. *Suppose $A \in \widetilde{GL^+}(2,\mathbb{R})$ and σ_{gl} is a gluing pre-stability condition of σ_1 and σ_2. Then $\sigma_{gl}.A$ is equal to the gluing of $\sigma_1.A$ and $\sigma_2.A$.*

Proof. By Definition 3.2 (2), both gluing stability conditions have the central charge. We show that both have the same heart of the bounded t-structure. Let $A = (M, f) \in \widetilde{GL^+}(2,\mathbb{R})$. Suppose that $\sigma_{gl} = (Z_{gl}, \mathcal{P}_{gl})$ is a stability condition glued from $\sigma_1 = (Z_1, \mathcal{P}_1)$ and $\sigma_2 = (Z_2, \mathcal{P}_2)$. For any ϕ,

$$\mathcal{P}_1(f^{-1}(\phi)) \subset \mathcal{P}_{gl}(f^{-1}(\phi)) \text{ and } \mathcal{P}_2(f^{-1}(\phi)) \subset \mathcal{P}_{gl}(f^{-1}(\phi))$$

by [6] Proposition 2.2 (3). Then

$$
\begin{aligned}
\mathcal{P}_1(f^{-1}(0), f^{-1}(1)] &\subset \mathcal{P}_{gl}(f^{-1}(0), f^{-1}(1)],\\
\mathcal{P}_2(f^{-1}(0), f^{-1}(1)] &\subset \mathcal{P}_{gl}(f^{-1}(0), f^{-1}(1)].
\end{aligned}
$$

Furthermore, we get the inclusion

$$\langle \mathcal{P}_1(f^{-1}(0), f^{-1}(1)], \mathcal{P}_2(f^{-1}(0), f^{-1}(1)]\rangle \subset \mathcal{P}_{gl}(f^{-1}(0), f^{-1}(1)]$$

by extension closedness. Hence, both have the same heart of a bounded t-structure. □

From now on, let σ_1 and σ_2 $\widetilde{GL^+}(2,\mathbb{R})$-translates of a stability condition on $p^*D^b(C)\otimes \mathcal{O}(-C_0)$ and $p^*D^b(C)$ induced from the standard stability condition $D^b(C)$ respectively. We can calculate a central charge of such a gluing pre-stability condition.

Proposition 3.5. *Let $M^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL^+(2,\mathbb{R})$. Suppose that σ_1 is a stability condition on $p^*D^b(C)\otimes \mathcal{O}_S(-C_0)$ and σ_2 is a standard stability condition on $p^*D^b(C)$. Then a gluing stability conditions $\sigma_{gl} = (Z_{gl}, \mathcal{P}_{gl})$ glued from $\sigma_1.M$ and σ_2 satisfies*

$$\mathrm{pr}_1(\sigma_{gl}) = \Big((1-a) - ic, -C_0 + \Big[\Big\{\tfrac{1}{2}\deg\mathcal{E}(a+1) - b\Big\} + i\Big\{\tfrac{1}{2}c\deg\mathcal{E} + (1-d)\Big\}\Big] f, -i\Big).$$

Proof. By Definition 3.2 (2) and Proposition 3.3, all we need to calculate is $\mathrm{ch}\, Rp_*(F(-C_0 + (2g-2+\deg\mathcal{E})f))\otimes\omega_C^*[1]$ and $\mathrm{ch}\, Rp_*(F)$. Now, we calculate $\mathrm{ch}\, Rp_*(F(-C_0+(2g-2+\deg\mathcal{E})f))\otimes\omega_C^*[1]$. By Grothendieck-Riemann-Roch formula,

$$
\begin{aligned}
&\mathrm{ch}\, Rp_*(F(-C_0+(2g-2+\deg\mathcal{E})f))\otimes\omega_C^*[1]\\
=\ &-p_*(\mathrm{ch}\, F(-C_0+(2g-2+\deg\mathcal{E})f).\,\mathrm{td}\, S).\,\mathrm{td}\, C^{-1}.\,\mathrm{ch}\,\omega_C^{-1}.
\end{aligned}
$$

Suppose that $\mathrm{ch}\, F = (r, c_1, \mathrm{ch}_2)$, then we can get the following by simple calculation of Chern character:

$$\operatorname{ch} Rp_* F(-C_0 + (2g-2+\deg\mathcal{E})f) \otimes \omega_C^*[1] = \left(-c_1.f, -\operatorname{ch}_2 - \left(\tfrac{1}{2}\deg\mathcal{E}\right)c_1.f\right).$$

We can calculate $\operatorname{ch} Rp_*(F) = \left(c_1.f + r, \operatorname{ch}_2 + c_1.C_0 - \left(\tfrac{1}{2}\deg\mathcal{E}\right)c_1.f\right)$ similarly. Then we get

$$\begin{aligned}
\operatorname{Re} Z_{gl}(F) &= \left[a\left\{\operatorname{ch}_2 + \left(\frac{1}{2}\deg\mathcal{E}\right)c_1.f\right\} + b(-c_1.f)\right] - \left\{\operatorname{ch}_2 + c_1.C_0 - \left(\frac{1}{2}\deg\mathcal{E}\right)c_1.f\right\} \\
&= -c_1.C_0 + \left\{\frac{1}{2}\deg\mathcal{E}(a+1) - b\right\}c_1.f + (a-1)\operatorname{ch}_2 \\
\operatorname{Im} Z_{gl}(F) &= \left[c\left\{\operatorname{ch}_2 + \left(\frac{1}{2}\deg\mathcal{E}\right)c_1.f\right\} + d(-c_1.f)\right] + (c_1.f + r) \\
&= r + \left\{\left(\frac{1}{2}c\deg\mathcal{E}\right)c_1.f + (1-d)c_1.f\right\} + c\operatorname{ch}_2 .
\end{aligned}$$

□

Now, one cannot usually glue σ_1 and σ_2. For describing a necessary and sufficient condition of the existence of the gluing stability condition, we introduce *gluing perversity*.

Definition 3.6. Let $\sigma_{st} = (Z_{st}, \mathcal{P}_{st})$ be the standard stability condition on the base curve. Suppose that $\sigma_1 = (Z_1, \mathcal{P}_1) \in \operatorname{Stab}(p^*D^b(C) \otimes \mathcal{O}_S(-C_0))$ with $\mathcal{P}_1(0) = p^*\mathcal{P}_{st}(\phi_1) \otimes \mathcal{O}_S(-C_0)$ and $\sigma_2 = (Z_2, \mathcal{P}_2) \in \operatorname{Stab}(p^*D^b(C))$ with $\mathcal{P}_2(0) = p^*\mathcal{P}_{st}(\phi_2)$. Assume that σ is a gluing pre-stability condition of σ_1 and σ_2, then *gluing perversity of* σ is defined to be $\operatorname{per}(\sigma) = \phi_1 - \phi_2$.

Proposition 3.7. *Suppose σ_{gl} is a gluing pre-stability condition. A $\widetilde{GL^+}(2,\mathbb{R})$-translate of σ_{gl} has gluing perversity 1 if and only if* $\operatorname{per}(\sigma_{gl}) = 1$

Proof. Suppose $\sigma_{gl} = (Z_{gl}, \mathcal{P}_{gl})$ is a gluing pre-stability condition of σ_1 and σ_2, and $A = (M, f) \in \widetilde{GL^+}(2,\mathbb{R})$. If the heart of the bounded t-structure of σ_1 satisfies $\mathcal{P}_1(0) = p^*\mathcal{P}_{st}(\phi) \otimes \mathcal{O}(-C_0)$ and the heart of the bounded t-structure of σ_2 satisfies $\mathcal{P}_2(0) = p^*\mathcal{P}_{st}(\psi)$, then $\operatorname{per}(\sigma_{gl}.A) = f^{-1}(\phi) - f^{-1}(\psi)$. $\operatorname{per}(\sigma_{gl}) = \phi - \psi = 1$ if and only if $\operatorname{per}(\sigma_{gl}.A) = f^{-1}(\phi) - f^{-1}(\psi) = 1$ since f is bijective and $f(\phi+1) = f(\phi) + 1$. □

Proposition 3.8. *σ_1 and σ_2 satisfy the gluing property. Then* $\operatorname{per}(\sigma)$ *is not less than 1.*

Proof. By Proposition 3.7, we can assume that σ_2 is the standard stability condition on $p^*D^b(S)$. Suppose that $\phi < 1$ and $A_1 = p^*\mathcal{P}_{st}(\phi, \phi+1] \otimes \mathcal{O}_S(-C_0)$. It is enough to show that

$$\operatorname{Hom}(p^*\mathcal{P}_{st}(\phi, \phi+1]) \otimes \mathcal{O}_S(-C_0), p^*\operatorname{Coh} C[i]) \neq 0$$

for some $i \leq 0$. Recall that for all $q \in \frac{1}{\pi}\arctan\frac{1}{\mathbb{Z}}$ there is a line bundle L such that $L \in \mathcal{P}_{st}(q)$. (For example, $L = \mathcal{O}_C(-n)$ with $q = \frac{1}{\pi}\arctan\frac{1}{n}$.) If we take $q \in (\phi - \lfloor\phi\rfloor, 1)$, there is a line bundle $L \in \mathcal{P}_{st}(q)$ and we get

$$p^*L \otimes \mathcal{O}(-C_0)[\lfloor\phi\rfloor] \in p^*\mathcal{P}_{st}(\phi, \phi+1].$$

Hence, $\operatorname{Hom}(p^*L \otimes \mathcal{O}_S(-C_0)[\lfloor\phi\rfloor], p^*L[\lfloor\phi\rfloor]) \neq 0$. □

Lemma 3.9. *On ruled surfaces, a gluing pre-stability condition σ of $\widetilde{GL^+}(2,\mathbb{R})$-actions of the standard stability condition is a locally finite stability condition if and only if the gluing perversity of σ is at least 1.*

Proof. By Proposition 3.8, it would be sufficient to prove $\mathrm{Hom}(\mathcal{A}_1, \mathcal{A}_2[i])$ for $i \leq 0$ if $\phi = \mathrm{per}(\sigma) \geq 1$. By Proposition 3.7, we can assume $\mathcal{A}_1 = p^*\mathcal{P}_{st}(\phi, \phi+1] \otimes \mathcal{O}_S(-C_0)$ and $\mathcal{A}_2 = p^*\mathcal{P}_{st}(0,1]$. Suppose that $F \in \mathcal{P}_{st}(\phi, \phi+1]$, $G \in \mathcal{P}_{st}(0,1] = \mathrm{Coh}\, C$ and $1 \leq \phi$.

$$\begin{aligned} & \mathrm{Hom}(p^*F \otimes \mathcal{O}_S(-C_0), p^*G[i]) \\ = \; & \mathrm{Hom}(p^*F, p^*G \otimes \mathcal{O}_S(C_0)[i]) \\ = \; & \mathrm{Hom}(F, Rp_*(p^*G \otimes \mathcal{O}_S(C_0)[i])) \\ = \; & \mathrm{Hom}(F, G \otimes Rp_*\mathcal{O}_S(C_0)[i]). \end{aligned}$$

Since $Rp_*\mathcal{O}_S(C_0)$ is a locally free sheaf, $G \otimes Rp_*\mathcal{O}_S(C_0)[i] \in \mathcal{P}(i, i+1]$. Therefore,

$$\mathrm{Hom}(F, G \otimes Rp_*\mathcal{O}_S(C_0)[i]) = 0$$

by the phase of F and $G \otimes Rp_*\mathcal{O}_S(C_0)$. Then by Definition 3.2 (2), the image of σ is discrete subgroup of $\mathbb{C}$. By [6] Proposition 3.5 (a), σ is a Bridgeland stability condition. Moreover, σ is locally finite by [5] Lemma 4.4. □

In the above theorem, we declare all gluing stability conditions on ruled surfaces with base curve of positive genus. From now on, we mean a Bridgeland stability condition glued from $\widetilde{GL}^+(2,\mathbb{R})$-translates of stanard stability conditions on the base curve simply by a gluing stability conditions.

Lemma 3.10. *Suppose that $\sigma = (Z, \mathcal{A})$ is a gluing stability condition. Then*
(1) *for any f, $\mathcal{O}_f$ and $\mathcal{O}_f(-C_0)[1]$ are stable of the same phase in σ respectively,*
(2) *the phase of $\mathcal{O}_f$ is larger than the phase of $\mathcal{O}_f(-C_0)[1]$,*
(3) *if $\mathrm{per}(\sigma) = 1$ skyscraper sheaves are strictly semistable of the same phase in σ, and also if $1 < \mathrm{per}(\sigma)$ skyscraper sheaves are destabilised by $\mathcal{O}_f$ with $x \in f$.*

Proof. By Proposition 3.7, we can assume that σ_2 is the standard stability condition on $p^*D^b(S)$.
(1) Since $\mathcal{O}_f = p^*\mathcal{O}_y$ with $y = p(f)$, $\mathcal{O}_f$ is semistable of the same phase 1 for any f by [6] Proposition 2.2 (3). Suppose that $\mathcal{F}$ is a subobject of $\mathcal{O}_f$ on $\mathcal{P}(1)$. $\mathcal{F}$ is also in $\mathcal{A}$. Hence, we have the following diagram in $\mathcal{P}(1)$.

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \rho_2(\mathcal{F}) & \longrightarrow & \mathcal{F} & \longrightarrow & \lambda_1(\mathcal{F}) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \rho_2(\mathcal{O}_f) & \longrightarrow & \mathcal{O}_f & \longrightarrow & \lambda_1(\mathcal{O}_f) & \longrightarrow & 0 \end{array}$$

Then $\mathcal{F} \simeq \rho_2(\mathcal{F}) \subset \rho_2(\mathcal{O}_f) = \mathcal{O}_f$ in $p^*D^b(C)$ by $\lambda_1(\mathcal{O}_f) = 0$. $\mathcal{O}_f$ is a minimal object in $p^*D^b(C)$. Hence, $\mathcal{F}$ is isomorphic to 0 or $\mathcal{O}_f$. $\mathcal{O}_f(-C_0)[1]$ can be proved similarly.
(2) $\mathcal{O}_f = p^*\mathcal{O}_y$ with $y = p(f)$, $\mathcal{O}_f(-C_0)[1] = p^*\mathcal{O}_y[1] \otimes \mathcal{O}_S(-C_0)$ with $y = p(f)$. Since $\mathrm{per}(\sigma) \geq 1$, the phase of $\mathcal{O}_f$ is larger than the phase of $\mathcal{O}_f(-C_0)[1]$ by [6] Proposition 2.2 (3).
(3) If $\mathcal{O}_x$ is semistable of the phase ϕ we have the following in $\mathcal{A}[\lceil\phi\rceil - 1]$. (c.f. [6] Lemma 2.1)

$$0 \to \rho_2(\mathcal{O}_x) \to \mathcal{O}_x \to \lambda_1(\mathcal{O}_x) \to 0 \text{ exact.}$$

Since $\rho_2(\mathcal{O}_x) = \mathcal{O}_f$ and $\lambda_1(\mathcal{O}_x) = \mathcal{O}_f(-C_0)[1]$, ϕ must be 1 by the phases, and hence if $1 < \mathrm{per}(\sigma)$ $\mathcal{O}_x$ is destabilized by $\mathcal{O}_f$ with $x \in f$. Now we assume that $\mathrm{per}(\sigma) = 1$. Since $\mathcal{O}_f \in \mathcal{P}(1)$ and $\mathcal{O}_f(-C_0) \in \mathcal{P}(1)$, $\mathcal{O}_x$ is strictly semistable in σ by extension closedness of $\mathcal{P}(1)$. □

4. A destabilizing wall of skyscraper sheaves on ruled surfaces

In this section, we describe a destabilizing wall of skyscraper sheaves on ruled surfaces. We start by the deformation theory of Bridgeland stability conditions.

For each $\sigma = (Z, \mathcal{P}) \in \mathrm{Stab}_{\mathcal{N}} S$, define a function

$$\|\cdot\|_\sigma : \mathrm{Hom}(\mathcal{N}(S), \mathbb{C}) \to [0, \infty)$$

by sending a group homomorphism $U : \mathcal{N}(S) \to \mathbb{C}$ to

$$\|U\|_\sigma = \sup \left\{ \frac{|U(E)|}{|Z(E)|} \mid E \text{ semistable in } \sigma \right\}.$$

Note that $\|\cdot\|_\sigma$ has all the properties of a norm on the complex vector space $\mathrm{Hom}(\mathcal{N}(S), \mathbb{C})$. A norm of a finite dimensional vector space is unique up to equivalence. Hence, this norm is equivalent to the standard norm of the finite dimensional vector space $\mathrm{Hom}(\mathcal{N}(S), \mathbb{C})$. If $\sigma = (Z, \mathcal{P})$ and $\tau = (W, \mathcal{Q})$ are stability conditions on a derived category $D^b(S)$ define

$$d(\mathcal{P}, \mathcal{Q}) = \sup \left\{ |\phi^+_\sigma(E) - \phi^+_\tau(E)|, |\phi^-_\sigma(E) - \phi^-_\tau(E)| \mid 0 \neq E \in D^b(S) \right\}.$$

It is a generalized metric on the space of slicings. Then an open basis of $\mathrm{Stab}_{\mathcal{N}} S$ consists of the following

$$B_\epsilon(\sigma) = \{ \tau = (W, \mathcal{Q}) \in \mathrm{Stab}_{\mathcal{N}} S \mid \|W - Z\|_\sigma < \sin(\pi\epsilon) , d(\mathcal{P}, \mathcal{Q}) < \epsilon \}.$$

Proposition 4.1 ([4] Theorem 7.1)**.** *Let $\sigma = (Z, \mathcal{P})$ be a numerical locally finite stability condition on a derived category $\mathcal{D}^b(S)$. Then there is an ϵ_0 such that if $0 < \epsilon < \epsilon_0$ and $W : \mathcal{N}(S) \to \mathbb{C}$ is a group homomorphism satisfying*

$$|W(E) - Z(E)| < \sin(\pi\epsilon)|Z(E)|$$

for all $E \in \mathcal{D}^b(S)$ semistable in σ, then there is a locally finite stability condition $\tau = (W, \mathcal{Q})$ on $\mathcal{D}^b(S)$ with $d(\mathcal{P}, \mathcal{Q}) < \epsilon$.

The above $\mathcal{Q}$ is constructed as follows. A *thin subcategory* of $\mathcal{D}^b(S)$ is a full subcategory of the form $\mathcal{P}((a, b)) \subset \mathcal{D}^b(S)$ where a and b are real numbers with $0 < b - a < 1 - 2\epsilon$. Suppose $\psi(E)$ is the phase of E on W. A nonzero object $E \in \mathcal{P}((a, b))$ is defined to be *enveloped* by $\mathcal{P}((a, b))$ if $\mathcal{P}((a, b))$ is a thin subcategory satisfying $a + \epsilon \leq \psi(E) \leq b - \epsilon$. Then for each $\psi \in \mathbb{R}$ define $\mathcal{Q}(\psi)$ to be the full additive subcategory $\mathcal{D}^b(S)$ consisting of the zero objects of $\mathcal{D}^b(S)$ together with those object $E \in \mathcal{D}^b(S)$ which are W-semistable of phase ψ in some thin enveloping subcategory $\mathcal{P}((a, b))$.

First, the following lemma plays an important role of the proof that gluing stability conditions with the gluing perversity 1 are a destabilizing wall of skyscraper sheaves.

Lemma 4.2. *Let S be a ruled surface. Suppose that $\sigma_{gl} = (Z_{gl}, P_{gl})$ is a gluing stability condition with the gluing perversity 1 on S. Then there is an $\epsilon_0 > 0$ such that if $0 < \epsilon < \epsilon_0$ and $W : \mathcal{N}(S) \to \mathbb{C}$ is a group homomorphism satisfying*

- *the phase of* $O_f(-C_0)$ *is greater than the phase of* O_f
- $|W(E) - Z(E)| < \sin(\pi\epsilon)|Z(F)|$ *for any* $E \in D^b(S)$ *semistable in* σ_{gl}

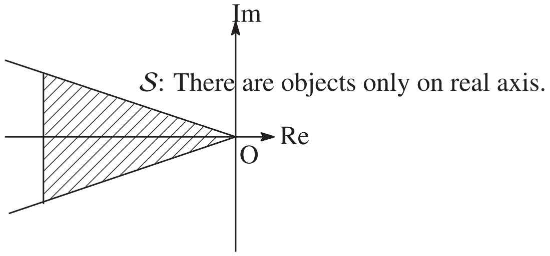
then there is a unique locally finite Bridgeland stability condition $\tau = (W, Q)$ *on* S *with* $d(\mathcal{P}_{gl}, Q) < \epsilon$ *satisfying that* O_x *are stable of the same phase in* τ *for any* $x \in S$.

Proof. By Proposition 3.7, we can assume that σ_2 is the standard stability condition on $p^* D^b(C)$. Then the phase of $\mathcal{O}_x$ is equal to 1. By the construction of Q,we can construct the following slicing Q of τ

$$Q(\psi) = \left\{ F \;\middle|\; \begin{array}{l} F \text{ is enveloped by } \mathcal{P}_{gl}(a,b), \\ \text{and semistable of phase } \psi \text{ in some } (W, \mathcal{P}_{gl}(a,b)) \end{array} \right\}.$$

We show that $\mathcal{O}_x$ is a minimal object in $Q(\psi)$. Since σ_{gl} is discrete, we can take such an $\epsilon_0 < \frac{1}{6}$ that

$$\mathcal{S} := \{F \mid \operatorname{Re} Z_{gl}(\mathcal{O}_x) < \operatorname{Re} Z_{gl}(F) < 0,\ F \in \mathcal{P}_{gl}(1-2\epsilon, 1+2\epsilon)\} \subset \mathcal{P}_{gl}(1).$$



It is sufficient to show that $\mathcal{O}_x$ is stable in $(W, \mathcal{P}_{gl}(1-2\epsilon, 1+2\epsilon))$. Suppose $\mathcal{O}_x$ is not stable in $\mathcal{P}_{gl}(1-2\epsilon, 1+2\epsilon)$. Then we can take F a proper stable subobject of $\mathcal{O}_x$ in $\mathcal{P}_{gl}(1-2\epsilon, 1+2\epsilon)$. We take an exact sequence in $\mathcal{P}_{gl}(1-2\epsilon, 1+2\epsilon)$:

$$0 \to F \to \mathcal{O}_x \to \mathcal{O}_x/F \to 0.$$

We assume $F \notin \mathcal{P}_{gl}(1)$. Since $Z(\mathcal{O}_x) = Z(F) + Z(\mathcal{O}_x/F)$, $\operatorname{Re} Z(\mathcal{O}_x) = \operatorname{Re} Z(F) + \operatorname{Re} Z(\mathcal{O}_x/F)$. Then we get $\operatorname{Re} Z(\mathcal{O}_x/F) > 0$ since $\operatorname{Re} Z(F) \le \operatorname{Re} Z(\mathcal{O}_x) \le 0$. This is contradictory to $\mathcal{O}_x/F \in \mathcal{P}_{gl}(1-2\epsilon, 1+2\epsilon)$. Hence, we get $F \in \mathcal{P}_{gl}(1)$. We take $\alpha : F \hookrightarrow \mathcal{O}_x \to \mathcal{O}_f(-C_0)[1]$.

- If $\alpha = 0$, there exists a morphism $F \to \mathcal{O}_f$.

$$\begin{array}{ccccccc} F & \longrightarrow & \mathcal{O}_x & \longrightarrow & \mathcal{O}_x/F & \longrightarrow & F[1] \\ \downarrow & & \downarrow & & & & \\ \mathcal{O}_f & \longrightarrow & \mathcal{O}_x & \longrightarrow & \mathcal{O}_f(-C_0)[1] & \longrightarrow & \mathcal{O}_f[1] \end{array}$$

  Since $\mathcal{O}_f$ is a minimal object in $\mathcal{P}_{gl}(1)$, we get $F \simeq \mathcal{O}_f$.
- If $\alpha \neq 0$, α is surjective. Moreover, we get $\ker \alpha \simeq 0$ since F is stable in $\mathcal{P}_{gl}(1)$. Hence α is ismorphism. So $F \simeq \mathcal{O}_f(-C_0)[1]$. Since $\operatorname{Hom}(F, \mathcal{O}_x) = \operatorname{Hom}(\mathcal{O}_f(-C_0)[1], \mathcal{O}_x) = 0$, then this is contradictory to $F \subset \mathcal{O}_x$.

Hence, we get $F \simeq \mathcal{O}_f$. Since $W(\mathcal{O}_x) = W(\mathcal{O}_f) + W(\mathcal{O}_f(-C_0)[1])$ and $\psi(\mathcal{O}_f) < \psi(\mathcal{O}_f(-C_0)[1])$, $\psi(\mathcal{O}_f) < \psi(\mathcal{O}_x) = \psi$. Namely, $\mathcal{O}_x$ is stable in $(W, \mathcal{P}_{gl}(1-2\epsilon, 1+2\epsilon))$. □

Second, the set of gluing stability conditions are connected submanifold of $\operatorname{Stab}_{\mathcal{N}} S$. We prove the following lemma.

Lemma 4.3. *Let $S_{gl,p}$ be the set of gluing stability conditions with gluing perversity p. $S_{gl,1}$ is connected submanifold of* $\mathrm{Stab}_{\mathcal{N}} S$ *with real dimension 7. Moreover, $S_{gl} := \bigcup_p S_{gl,p}$ is also a submanifold with real dimension 8, especially the subset of full components.*

Proof. We show that the action of $\widetilde{GL^+}(2,\mathbb{R})$ on $S_{gl,1}$ is free. Suppose $\sigma_{gl} \in S_{gl,1,st}$ and $A = (M, f) \in \widetilde{GL^+}(2,\mathbb{R})$. If $\sigma_{gl}.A = \sigma_{gl}$, then we get

$$M^{-1}(Z_{gl}(\mathcal{O}_S)) = Z_{gl}(\mathcal{O}_S)$$

and

$$M^{-1}(Z_{gl}(\mathcal{O}_f)) = Z_{gl}(\mathcal{O}_f).$$

By Proposition 3.5, $Z_{gl}(\mathcal{O}_S) = i$ and $Z_{gl}(\mathcal{O}_f) = -1$. Hence, M is the identity matrix by comparison of both values of central charges. $f = \mathrm{id}$ can be get by the comparison of both hearts of the bounded t-structures. Suppose that $S_{gl,1,st}$ consists of the element of $S_{gl,1}$ that σ_2 is the standard stability condition on $p^*D^b(C)$. Then by [4] Theorem 9.1,

$$S_{gl,1,st} \simeq \left\{ \left(\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}, f \right) \middle| \ a > 0, b \in \mathbb{R}, d > 0 \text{ and } f(0) = 0 \right\}.$$

Especially, $S_{gl,1,st}$ is a connected submanifold with real dimension 3 since pr_1 is a local homeomorphism. Hence, $S_{gl,p}$ is connected submanifold of $\mathrm{Stab}_{\mathcal{N}} S$ with real dimension 7. We can prove in the case of S_{gl} similarly. □

Finally, we describe a concrete description between divisorial stability conditions and gluing stability conditions on the stability space. This is the end of the proof of Theorem 1.4.

Theorem 4.4. *Let S_{div} be the set of divisorial stability conditions on S. Suppose that $A = \left(\begin{pmatrix} a & \frac{1}{2} a \deg \mathcal{E} \\ 0 & a \end{pmatrix}^{-1}, f \right) \in \widetilde{GL^+}(2,\mathbb{R})$ with $a < 0$. Then $\partial \overline{S_{div}} \cap S_{gl,1}$ is the set of $\widetilde{GL^+}(2,\mathbb{R})$-translates of a stability condition glued from $\sigma_{st}.A$ and σ_{st}.*

Proof. We can assume that $\sigma_{gl} = (Z_{gl}, P_{gl})$ is a gluing stability condition that σ_2 is a standard stabilty condition. It is sufficient to show that $Z_{gl} = M^{-1} \exp(B + i\omega)$ if and only if $Z_{gl} = \begin{pmatrix} a & \frac{1}{2} a \deg \mathcal{E} \\ 0 & a \end{pmatrix} Z_{st} \circ \lambda_1 + Z_{st} \circ \rho_2$ with $a < 0$. Let

$$M^{-1} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, B = xC_0 + yf \text{ and } \omega = zC_0 + wf.$$

We denote

$$\begin{aligned} I &= \frac{1}{2}\alpha\{(x^2 - z^2)\deg \mathcal{E} + 2(xy - zw)\} + \beta\{xz \deg \mathcal{E} + (yz + xw)\}, \\ J &= \frac{1}{2}\gamma\{(x^2 - z^2)\deg \mathcal{E} + 2(xy - zw)\} + \delta\{xz \deg \mathcal{E} + (yz + xw)\}. \end{aligned}$$

Then

$$\begin{aligned} & \exp(B + i\omega) \\ = \ & (1, x + iz, y + iw, \frac{1}{2}\{(x^2 - z^2)\deg \mathcal{E} + 2(xy - zw)\} + i\{xz \deg \mathcal{E} + (yz + xw)\}) \end{aligned}$$

and

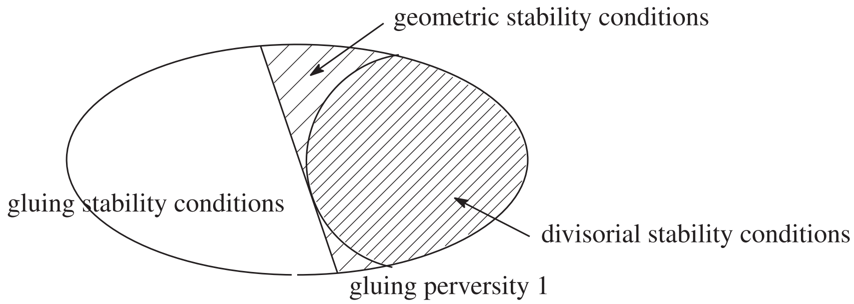
$$\begin{aligned} & M^{-1}\exp(B+i\omega) \\ = \ & (\alpha+i\gamma, \{(\alpha x+\beta z)+i(\gamma x+\delta z)\}C_0+\{(\alpha y+\beta w)+i(\gamma y+\delta w)\}f, I+iJ). \end{aligned}$$

We compare it to Proposition 3.5. Recall that σ_{gl} has gluing perversity 1. So $a < 0$ and $c = 0$. Then

$$\mathrm{pr}_1(\sigma_{gl}) = (1-a, -C_0+[\{\tfrac{1}{2}\deg\mathcal{E}(a+1)-b\}+i(1-d)]f, -i).$$

From $\alpha+i\gamma = 1-a$, we get $\alpha = 1-a$ and $\gamma = 0$. Then we get $z = 0$ from $\gamma x+\delta z = 0$ since $\det M = \alpha\delta \neq 0$. And then we get $x = \frac{1}{a-1}$ from $\alpha x+\beta z = -1$. And then we get $a = d$ from $J = \delta x w = -1$ and $\gamma y+\delta w = 1-d$. From $I = -\frac{1}{2}(\frac{1}{a-1}\deg\mathcal{E}+2y)+\beta\frac{1}{a-1}w = 0$ and $\alpha y+\beta w = (1-a)y+\beta w = \frac{1}{2}\deg\mathcal{E}(a+1)-b$, we get $b = \frac{1}{2}a\deg\mathcal{E}$. □

The set of gluing stability conditions is a codimension one submanifold of the full stability space (Lemma 4.3). Lemma 3.10 (3) and Lemma 4.2 suggest that the set of gluing stability conditions neighbors on the set of geometric stability conditions on the stability space. Especially, the set of gluing stability conditions with the gluing perversity 1 is a destabilizing wall of skyscraper sheaves. In addition, the boundary of the set of divisorial stability conditions only contacts the destabilizing wall (Theorem 4.4). The following picture of $\mathrm{Stab}_{\mathcal{N}}\, S$ is convenient for understanding.



Remark 4.5. *Let $\overline{\mathcal{M}^\sigma}([\mathcal{O}_x])$ be the variety of S-equivalent classes of objects $E \in \mathcal{P}(\phi(\mathcal{O}_x))$.*

- *If σ is a divisorial stability condition, then $\overline{\mathcal{M}^\sigma}([\mathcal{O}_x]) \simeq S$.*
- *If σ is a gluing stability condition with gluing perversity 1, then $\overline{\mathcal{M}^\sigma}([\mathcal{O}_x]) \simeq C$.*
- *If σ is a gluing stability condition with gluing perversity > 1, then $\overline{\mathcal{M}^\sigma}([\mathcal{O}_x])$ is empty.*

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