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A POLYNOMIAL INVARIANT FOR KNOTOIDS

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Abstract

A polynomial invariant for multi-knotoids in S^2 is given by an elementary and combinatorial method. It is shown that the invariant is an extension of "a" HOMFLY polynomial for multi-knotoids and that there exist infinitely many non-trivial knotoids with trivial HOMFLY polynomial. Furthermore, formulas between the polynomials for a given multi-knotoid, its mirror and reverse images are given. By using the formulas, it is revealed that each knotoid with less than four crossings is invertible.

1. Introduction

The purpose of this paper is to provide a polynomial invariant for multi-knotoids as a tool for advancing the study of knotoids. The concept of a knotoid is introduced by Turaev [8] in 2012. Since the theory of knotoid diagrams suggests a new diagrammatic approach to knots, it can be expected that the study of knotoids is useful for that of knots. Hence, it is meaningful to create invariants for knotoids. Turaev gives several invariants for knotoids in his paper. One of them is a polynomial of "HOMFLY" type, which comes from a skein algebra of knotoids. In this paper, we define a polynomial which is an extension of a HOMFLY polynomial different from Turaev's one. It is given by an elementary and combinatorial method which is analogous to that by Lickorish-Millett [5].

First of all, we begin by introducing a multi-knotoid. Let I be the interval $[0, 1]$ and X a disjoint union of I and a finite number of circles. Let f be a generic immersion of X in S^2 whose singularities are transversal double points only. Let p be a double point on the image $f(X) \subset S^2$. Since two arcs form p, in other words, they share p, we can regard p as a point on each of them. We call the point corresponding to p on one arc the over crossing p_+ and the point corresponding to p on the other arc the under crossings p_-. A double point is said to have over/under crossing data if the over and the under crossings are assigned to the corresponding points on the arcs sharing the double point. A *multi-knotoid* diagram in S^2 is defined to be the image of X by f with over/under crossing data at each double point. If $X = I$, then such an image is said to be a *knotoid* diagram. A double point with over/under crossing data is called a crossing of the diagram. We depict a crossing on a diagram in the same way as in a link diagram. See Fig. 1. The right figure displays a crossing. The over and the under paths denote the arcs having the over and the under crossings, respectively.

Since the images of the points 0 and 1 in I under the immersion f are distinct from each other and from the crossings, the diagram is composed of a curve and knots, each of which is

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Fig. 1. A crossing on a diagram

called a component of the diagram. In particular, the curve is called the knotoid component and each knot is called a knot component.

A multi-knotoid diagram, to be precise, the knotoid component has just two endpoints. They are called the *leg* and the *head* of the component.

There exist specific local moves called Reidemeister moves on a multi-knotoid diagram. The moves are combinations of the three types of Reidemeister moves as shown in Fig. 2 [4, p. 4]. In particular, a move which does not increase the number of crossings is called a 1-*sided* Reidemeister move. A Reidemeister move is applied in a disk disjoint from the endpoints of the diagram. Such a disk is called a *stage* for the move.

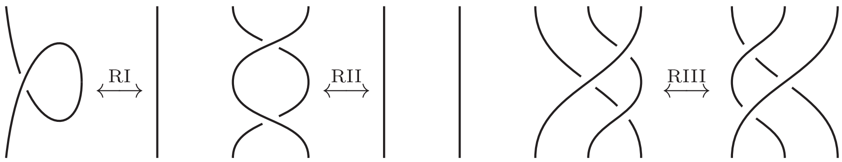

Fig. 2. Reidemeister moves

Two multi-knotoid diagrams are said to be equivalent if they are related by a finite sequence of Reidemeister moves or ambient isotopies of S^2. Note that an ambient isotopy of S^2 may displace the endpoints of the diagram. A multi-knotoid is defined to be an equivalence class of multi-knotoid diagrams.

A multi-knotoid diagram is said to be oriented if each component is oriented, provided that the knotoid component has the orientation from the leg to the head. Equivalence of two oriented multi-knotoid diagrams and an oriented multi-knotoid are similarly defined as above except that orientations are taken into consideration.

Let c be a crossing of a multi-knotoid diagram D. If c is composed of two arcs which belong to the same component, then it is said to be a *self* crossing. If different components form c, then c is called a *mixed* crossing. We denote the sets of the crossings, the self crossings and the mixed crossings of D by $\mathrm{C}(D), \mathrm{SC}(D)$ and $\mathrm{MC}(D)$, respectively. It is obvious that $\mathrm{C}(D) = \mathrm{SC}(D) \sqcup \mathrm{MC}(D)$.

Suppose that D is oriented. Then, we can define the signature of a crossing c denoted by $\mathrm{sign}(c)$ in a usual way; if c is positive (resp. negative), then $\mathrm{sign}(c)$ is $+1$ (resp. -1). Let S be a subset of $\mathrm{C}(D)$. We put $wr(\mathrm{S}) = \sum_{c \in \mathrm{S}} \mathrm{sign}(c)$. Then, $wr(\mathrm{S})$ is called the writhe of S. In particular, $wr(\mathrm{C}(D)), wr(\mathrm{SC}(D))$ and $wr(\mathrm{MC}(D))$ are said to be the writhe, the self writhe and the mixed writhe of D and are also denoted by $wr(D)$, $sw(D)$ and $mw(D)$, respectively.

If we smooth a crossing of an oriented multi-knotoid diagram according to its orientation, then we obtain an oriented multi-knotoid diagram again. The similar situation occurs even if we replace a null crossing with a crossing. These facts ensure existence of a triple (D_+, D_-, D_0) of oriented multi-knotoid diagrams D_+, D_- and D_0 which differ only in one place as shown in Fig. 3. Such a triple of diagrams is called a *skein triple*.

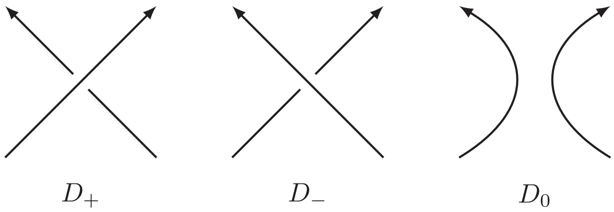

Fig. 3. A skein triple

Let D be a μ-component multi-knotoid diagram with $\mu \geq 1$. D is said to be *trivial* if D is equivalent to a disjoint union of a segment and $(\mu - 1)$ trivial circles. A trivial multi-knotoid diagram is called *canonical* if the diagram has no crossings, that is, the diagram itself is a disjoint union of a segment and trivial circles.

Theorem 1.1. *Let D be an oriented multi-knotoid diagram. Then there exists a polynomial $H(D; a, h, z) \in \mathbb{Z}[a^{\pm 1}, h^{\pm 1}, z^{\pm 1}]$ for D satisfying the following:*

(1) *$H(U_0; a, h, z) = 1$ for a canonical trivial knotoid diagram U_0.*
(2) *$H(D; a, h, z)$ is a regular isotopy invariant, that is, $H(D; a, h, z)$ is invariant under Reidemeister moves except the move of type I.*
(3) *If a multi-knotoid diagram D is obtained from a multi-knotoid diagram E by applying a 1-sided Reidemeister move of type I at a crossing c, then $H(D; a, h, z) = a^{-\mathrm{sign}(c)} H(E; a, h, z)$.*
(4) *For a skein triple (D_+, D_-, D_0),*

$$H(D_+; a, h, z) - H(D_-; a, h, z) = zH(D_0; a, h, z).$$

Remark 1.2. The polynomial $H(D; a, h, z)$ in Theorem 1.1 cannot be determined only by the above four properties.

By using the H-polynomial in Theorem 1.1, we define the R-polynomial for an oriented multi-knotoid diagram D by $R(D; a, h, z) = a^{-wr(D)} H(D; a, h, z)$. We also define the SR-polynomial for D by $SR(D; a, h, z) = a^{-sw(D)} H(D; a, h, z)$. Since $wr(D) = sw(D) + mw(D)$, there exists a relationship

$$R(D; a, h, z) = a^{-mw(D)} SR(D; a, h, z)$$

between the R- and the SR-polynomials. The two polynomials are identical for a knotoid diagram since such a diagram has no mixed crossings.

Remark 1.3. For a skein triple (D_+, D_-, D_0), it holds that

$$aR(D_+; a, h, z) - a^{-1} R(D_-, a, h, z) = zR(D_0; a, h, z).$$

Corollary 1.4. *Let L be an oriented multi-knotoid and D its diagram. Then, $R(L)$ and $SR(L)$, which are defined by $R(D)$ and $SR(D)$ respectively, are invariants for L.*

The paper is organized as follows. The following section is devoted to preliminaries for the definition of the H-polynomial. After that, we prove Theorem 1.1 in Section 3 and show that the R-polynomial is an extension of a HOMFLY polynomial for multi-knotoids in Section 4. In Section 5, we give formulas between the R-polynomials for a given multi-knotoid, its mirror and reverse images. In the final section, we classify knotoids with up to three crossings and show that each of them is invertible.

2. Preliminaries

Let D be a multi-knotoid diagram and $kt(D)$ the number of the knot components of D. Then, we denote D by $D_0 \cup D_1 \cup \cdots \cup D_{kt(D)}$, where D_0 is always assigned to the knotoid component.

A map $\pi : \{D_0, D_1, \ldots, D_{kt(D)}\} \to \{0, 1, \ldots, kt(D)\}$ is called an ordering of D if π is bijective and $\pi(D_0) = 0$. A multi-knotoid diagram D is said to be ordered if an ordering of D is given to D.

For an oriented multi-knotoid diagram $D = D_0 \cup D_1 \cup \cdots \cup D_{kt(D)}, kt(D) \geq 0$, let $P = (p_0, p_1, \ldots, p_{kt(D)}) \in \prod_{j=0}^{kt(D)} D_j$ be a $(kt(D)+1)$-tuple of points of D such that p_0 is the leg of the knotoid component D_0 and $p_j, 1 \leq j \leq kt(D)$, is distinct from the crossings of D. We call such a $(kt(D) + 1)$-tuple of points of D *standard*. D is said to have a basepoint if a standard $(kt(D) + 1)$-tuples of points is specified.

Suppose that an oriented multi-knotoid diagram D is ordered by an ordering π and has a basepoint $P = (p_0, p_1, \ldots, p_{kt(D)})$. We denote such a diagram by $D[\pi, P]$.

Since a crossing of the diagram $D[\pi, P]$ is a transversal intersection by two arcs, passing across the crossing may be supposed to go through on one of the arcs. Since there exist the over crossing on one arc and the under crossings on the other, when we travel the diagram $D[\pi, P]$, the order of passing the over and the under crossings at each crossing can be explicitly defined according to the oriention of D, the ordering π and the basepoint P. An oriented multi-knotoid diagram $D[\pi, P]$ is said to be *descending* if at each crossing the over crossing is first encountered when we travel the diagram $D[\pi, P]$ in the direction of the orientation according to the ordering π and the basepoint P. Note that an oriented multi-knotoid diagram with an ordering and a basepoint gives a unique descending one with respect to the ordering and the basepoint.

Let D be a multi-knotoid diagram and E a subdiagram of D with $0 < \mu(E) < \mu(D)$, where $\mu(F)$ denotes the number of the components of a diagram F. E is said to *overlie* $D - E$ if at each mixed crossing formed by a component of E and a component of $D - E$, its over and under crossings are on components of E and $D - E$ respectively. If E is disjoint from $D - E$, then E can be considered to overlie $D - E$.

Let k be a non-negative integer. An oriented multi-knotoid diagram $D[\pi, P]$ is said to be *descending of level* k, $k < kt(D)$, if the subdiagram $E_k = \pi^{-1}(0) \cup \pi^{-1}(1) \cup \cdots \cup \pi^{-1}(k)$ of D has the following two properties:

(1) E_k is descending with respect to the ordering and the basepoint induced from π and P.

(2) E_k overlies $D - E_k$.

Note that a descending diagram is also descending of level k for any $k, 0 \leq k < kt(D)$.

Let $\lambda_1, \ldots, \lambda_{m-1}$ and λ_m, $m \geq 1$, be different crossings of a multi-knotoid diagram D and

$\lambda = (\lambda_m, \lambda_{m-1}, \dots, \lambda_1)$ their ordered sequence. Then, a new diagram can be obtained from D by switching each crossing in λ. We denote it by $D(\lambda)$ and λ is called a *switching sequence* for D. We express the set of the crossings $\lambda_1, \dots, \lambda_{m-1}$ and λ_m by $\{\lambda\}$ and the diagram obtained from D by switching λ_1 by $S_{\lambda_1}D$. We denote by $Z_{\lambda_1}D$ the diagram obtained from D by smoothing λ_1. For example, $Z_{\lambda_2}S_{\lambda_1}D$ means the diagram obtained from D by smoothing λ_2 after changing λ_1. Thus, we see that $D(\lambda) = S_{\lambda_m}S_{\lambda_{m-1}}\cdots S_{\lambda_1}D$. We may write it as $S_mS_{m-1}\cdots S_1D$ for convenience of notation.

If an oriented multi-knotoid diagram is equipped with an ordering or a basepoint, then each equipment is called an *option* for the diagram.

Let D be an oriented multi-knotoid diagram with an ordering π and a basepoint P. Suppose that D is not descending with respect to the options (π, P). Then, there exists a switching sequence λ which changes D into the descending diagram $D(\lambda)$. Such a sequence is called a *descending sequence* for D. Note that λ is not unique, but $\{\lambda\}$ is unique and is determined by the options (π, P).

For an oriented multi-knotoid diagram D, we define the *weight* and the *determinant* of D as follows. The weight W_D of D is given by the monomial $a^{sw(D)}h^{mw(D)} \in \mathbb{Z}[a^{\pm1}, h^{\pm1}]$. The determinant of D denoted by d_D is defined to be the polynomial $\{(a - a^{-1})z^{-1}\}^{kt(D)} \in \mathbb{Z}[a^{\pm1}, z^{\pm1}]$.

Let D be an oriented multi-knotoid diagram with an ordering π and c a crossing of D composed of arcs of D_i and D_j, $0 \le i, j \le kt(D)$. The *classification* of c denoted by $cls(c)$ is defined to be $\delta(\pi(D_i), \pi(D_j)) \in \{0, 1\}$, where $\delta(m, n)$ denotes the Kronecker delta which takes the value 1 if $m = n$; otherwise, the value 0. In other words, if c is a self (resp. a mixed) crossing, then the classification $cls(c)$ of c is 1 (resp. 0). We define the *weight of c* denoted by $w(c)$ by the following monomial

$$w(c) = (a^{cls(c)}h^{1-cls(c)})^{\mathrm{sign}(c)} \in \mathbb{Z}[a^{\pm1}, h^{\pm1}].$$

Then, it is obvious that the weight of D is equal to the product of the weights of all crossings of D, that is, $W_D = \prod_{c\in \mathrm{C}(D)} w(c)$, where $\prod_{c\in \mathrm{C}(D)} w(c)$ is defined to be 1 if $\mathrm{C}(D) = \emptyset$.

At the end of this section, we introduce a local move on the diagram. Let D be a multi-knotoid diagram. Suppose that the connected component of D including the knotoid component D_0 has at least one crossing. Then, there exists the first crossing c of D encountered when traveling from the leg of D_0. We denote by T_cD the multi-knotoid diagram obtained from D by sliding the leg of D_0 just past the crossing c as in Fig. 4. We call the operation changing D into T_cD the *tug move* at c.

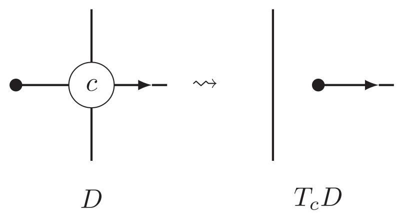

Fig. 4. The tug move

3. Proof of Theorem 1.1

In this section, we will provide the definition of the H-polynomial and the proof of its invariance. The H-polynomial will be defined under an inductive assumption. Its well-definedness, which is the independence of choices of the options, and invariance will be given by a combinatorial method based on an induction on the number of the crossings of a multi-knotoid diagram. It is similar to approach to the HOMFLY polynomial introduced by Lickorish and Millett [5]. After that, we will show that the R-polynomial is invariant for multi-knotoids.

Let D be an oriented multi-knotoid diagram. We denote the number of the crossings of D by $cr(D)$.

If $cr(D) = 0$, then D is a canonical trivial multi-knotoid diagram. The H-polynomial $H(D) = H(D; a, h, z)$ for D is defined to be $W_D d_D$ for any ordering and any basepoint.

If $cr(D) = n \geq 1$, then we assume the following.

Inductive hypothesis $(n - 1)$:

For all multi-knotoid diagrams with less than n crossings,

(a) The H-polynomial is well defined, that is independent of choices of the basepoint and the ordering.
(b) The H-polynomial has the properties (1) to (4) in Theorem 1.1.
(c) The H-polynomial is independent of the position of a disjoint component.
(d) If a multi-knotoid diagram has a descending function of level 0 and its connected component including the knotoid component has a crossing, then the H-polynomial for the diagram coincides with the H-polynomial for the multi-knotoid diagram applied the tug move at a crossing multiplied by the weight of the crossing.

We define the H-polynomial $H(D)$ for any multi-knotoid diagram D with n crossings, which has an ordering π and a basepoint P, as follows.

Recursive definition (n):

(a) If $D[\pi, P]$ is descending with respect to the options (π, P), then $H(D[\pi, P]) = W_D d_D$.
(b) If $D[\pi, P]$ is not descending, then by a descending sequence $\lambda = (\lambda_m, \ldots, \lambda_1)$ for $D[\pi, P]$, we define

$$H(D[\pi, P]) = H(D(\lambda)[\pi, P]) + z \sum_{k=1}^{m} \operatorname{sign}(\lambda_k) H(A_k^\lambda D),$$

where $A_k^\lambda D = Z_k S_{k-1} \ldots S_1 D$.

For a multi-knotoid diagram D and its switching sequence $\lambda = (\lambda_m, \ldots, \lambda_1)$, we denote the polynomial $\sum_{k=1}^{m} \operatorname{sign}(\lambda_k) H(A_k^\lambda D)$ by $\sum_D \lambda$. Thus, the H-polynomial of case (b) in Recursive definition (n) can be written as

$$H(D[\pi, P]) = H(D(\lambda)[\pi, P]) + z \sum_D \lambda.$$

Remark 3.1. According to the definition of $H(D)$ for a diagram D with $cr(D) = 0$, it is clear that Inductive hypothesis (0) is satisfied.

We realize the following on Inductive hypothesis (1) (d) . It will be referred to in the proof for independence of the choice of the ordering.

Lemma 3.2. *Let D be a multi-knotoid diagram with a unique crossing c. Suppose that D is a descending diagram of level 0 with respect to options (π, P). If the connected component of D including the knotoid component has the crossing c, then $D[\pi, P]$ is a descending diagram.*

Proof. We have two cases according to the classification of c. First, we assume that c is a self crossing. Then, c is on the knotoid component D_0. Since D is a descending diagram of level 0, D_0 is descending. It follows that D is a descending diagram. Next, we suppose that c is a mixed crossing. Then, c is composed of two arcs of D_0 and D_j, $1 \leq j \leq kt(D)$. Since D is a descending diagram of level 0, D_0 is above D_j. It implies that D is a descending diagram. Hence, we see that D is descending regardless of the classification of c. □

In the following six steps, we will show that the H-polynomials for multi-knotoid diagrams with at most n crossings satisfy Inductive hypothesis (n), on condition that Inductive hypothesis ($n-1$) is true for multi-knotoid diagrams with less than n crossings. It completes most of the proof of Theorem 1.1. Throughout the steps, we suppose that multi-knotoid diagrams are oriented.

Step 1: The H-polynomial is independent of the choice of the descending sequence.

Lemma 3.3. *Let D be a multi-knotoid diagram with n crossings, $n \geq 2$. Let $\lambda = (\lambda_2, \lambda_1)$ and $\varphi = (\lambda_1, \lambda_2)$ be two switching sequences for D. Then, $\sum_D \lambda = \sum_D \varphi$.*

Proof. Let $\varepsilon_i = \mathrm{sign}(\lambda_i)$. From the definition, we have

$$\begin{aligned}
-\sum_D \lambda + \sum_D \varphi &= -(\varepsilon_1 H(A_1^{\lambda} D) + \varepsilon_2 H(A_2^{\lambda} D)) + (\varepsilon_2 H(A_1^{\varphi} D) + \varepsilon_1 H(A_2^{\varphi} D)) \\
&= -(\varepsilon_1 H(Z_1 D) + \varepsilon_2 H(Z_2 S_1 D)) + (\varepsilon_2 H(Z_2 D) + \varepsilon_1 H(Z_1 S_2 D)) \\
&= -\varepsilon_1 (H(Z_1 D) - H(S_2 Z_1 D)) + \varepsilon_2 (H(Z_2 D) - H(S_1 Z_2 D)),
\end{aligned}$$

where we use $Z_2 S_1 D = S_1 Z_2 D$ and $Z_1 S_2 D = S_2 Z_1 D$. Since $cr(Z_1 D) = cr(Z_2 D) = n-1$, Inductive hypothesis ($n-1$) gives the two equalities $H(Z_1 D) - H(S_2 Z_1 D) = \varepsilon_2 z H(Z_2 Z_1 D)$ and $H(Z_2 D) - H(S_1 Z_2 D) = \varepsilon_1 z H(Z_1 Z_2 D)$. Since $Z_1 Z_2 D = Z_2 Z_1 D$, we have the result. □

Lemma 3.4. *Let $\lambda = (\lambda_m, \lambda_{m-1}, \ldots, \lambda_1)$ be a switching sequence for a multi-knotoid diagram D with n crossings, $n \geq 2$, and λ' the switching sequence for D obtained from λ by changing the order of two crossings λ_j and λ_{j+1}, $1 \leq j < m$, where $2 \leq m \leq n$. Then, $\sum_D \lambda = \sum_D \lambda'$.*

Proof. Let $\lambda' = (\lambda'_m, \lambda'_{m-1}, \ldots, \lambda'_1)$. We put $\varepsilon_k = \mathrm{sign}(\lambda_k)$ and $\varepsilon'_k = \mathrm{sign}(\lambda'_k)$, $1 \leq k \leq m$. Since $\lambda_k = \lambda'_k$ and $A_k^{\lambda} D = A_k^{\lambda'} D$, $1 \leq k < j$, $j+1 < k \leq m$, we have

$$\sum_D \lambda - \sum_D \lambda' = \varepsilon_j H(A_j^{\lambda} D) + \varepsilon_{j+1} H(A_{j+1}^{\lambda} D)) - \varepsilon'_j H(A_j^{\lambda'} D) - \varepsilon'_{j+1} H(A_{j+1}^{\lambda'} D)).$$

Put $E = S_{j-1} S_{j-2} \cdots S_1 D$ and let $\varphi_1 = (\lambda_{j+1}, \lambda_j)$ and $\varphi_2 = (\lambda_j, \lambda_{j+1})$ be two switching sequences for E. Since $A_j^{\lambda} D = A_1^{\varphi_1} E, A_{j+1}^{\lambda} D = A_2^{\varphi_1} E, A_j^{\lambda'} D = A_1^{\varphi_2} E$, and $A_{j+1}^{\lambda'} D = A_2^{\varphi_2} E$, by these equalities and Lemma 3.3, we have $\sum\limits_D \lambda - \sum\limits_D \lambda' = \sum\limits_E \varphi_1 - \sum\limits_E \varphi_2 = 0$. □

Lemma 3.5. *Let D be a multi-knotoid diagram which is not descending with respect to an ordering and a basepoint. Let λ and λ' be descending sequences for D with respect to the same options. Then, the H-polynomials of D obtained by using λ and λ' coincide.*

Proof. Let (π, P) be options for D. Let $H(D[\pi, P])[\lambda]$ and $H(D[\pi, P])[\lambda']$ be H-polynomials of $D[\pi, P]$ obtained by using λ and λ', respectively. Then, we have

$$H(D[\pi, P])[\lambda] = H(D(\lambda)[\pi, P]) + z \sum_D \lambda$$

and

$$H(D[\pi, P])[\lambda'] = H(D(\lambda')[\pi, P]) + z \sum_D \lambda'.$$

Since $\{\lambda\} = \{\lambda'\}$ and the two descending sequences λ and λ' are related by a finite sequence of changing order of adjacent crossings, Lemma 3.4 gives $\sum\limits_D \lambda = \sum\limits_D \lambda'$. Since $D(\lambda)$ is the same diagram as $D(\lambda')$, we obtain $H(D(\lambda)[\pi, P]) = H(D(\lambda')[\pi, P])$, which yields $H(D[\pi, P])[\lambda] = H(D[\pi, P])[\lambda']$. □

Step 2: The H-polynomial is independent of the choice of the position of the basepoint.

Lemma 3.6. *Let D be a multi-knotoid diagram with n crossings and $kt(D) > 0$ given an ordering π and a basepoint $P = (p_0, p_1, \ldots, p_{kt(D)})$. Let c denote the first crossing of D encountered when traveling from p_k on a knot component D_k, $1 \leq k \leq kt(D)$. Let q_k be a point on D_k obtained by sliding p_k just past the crossing c and Q the standard $(kt(D) + 1)$-tuple of points of D obtained from P by replacing p_k with q_k. Suppose that c is a self crossing of D_k and D is descending with respect to (π, P). Then,*

$$H(D[\pi, P]) - H(S_c D[\pi, Q]) = \mathrm{sign}(c) z H(Z_c D).$$

Proof. Since $D[\pi, P]$ is descending, we easily find that $S_c D[\pi, Q]$ is descending and $Z_c D$ is also descending with respect to suitable options. Note that $cr(Z_c D) = n - 1$. Recursive definition (n) and Inductive hypothesis $(n - 1)$ give $H(D[\pi, P]) = W_D d_D$, $H(S_c D[\pi, Q]) = W_{S_c D} d_{S_c D}$ and $H(Z_c D) = W_{Z_c D} d_{Z_c D}$.

Since c is a self crossing of D_k, the diagram $Z_c D_k$ has two knot components E_k and F_k, that is, $Z_c D_k = E_k \cup F_k$. Since D_k is descending with respect to p_k, we may assume that E_k is above F_k. Then, $Z_c D_k$ is splittable and the linking number $lk(E_k, F_k)$ of the components E_k and F_k is zero. Hence, $\sum\limits_{r \in \mathrm{MC}(Z_c D_k)} \mathrm{sign}(r) = 2lk(E_k, F_k) = 0$. Since the classification of each crossing $r \in \mathrm{MC}(Z_c D_k)$ is zero, we have $w(r) = h^{\mathrm{sign}(r)}$. These two equalities give $\prod\limits_{r \in \mathrm{MC}(Z_c D_k)} w(r) = h^{2lk(E_k, F_k)} = 1$.

It is easy to see that $\mathrm{SC}(D) = \mathrm{SC}(Z_cD) \sqcup \mathrm{MC}(Z_cD_k) \sqcup \{c\}$. We also find that

$$\begin{aligned}\mathrm{MC}(D) &= \mathrm{C}(D) - \mathrm{SC}(D)\\ &= (\mathrm{C}(Z_cD) \sqcup \{c\}) - (\mathrm{SC}(Z_cD) \sqcup \mathrm{MC}(Z_cD_k) \sqcup \{c\})\\ &= (\mathrm{C}(Z_cD) - \mathrm{SC}(Z_cD)) - \mathrm{MC}(Z_cD_k)\\ &= \mathrm{MC}(Z_cD) - \mathrm{MC}(Z_cD_k),\end{aligned}$$

and thus, $\mathrm{MC}(Z_cD) = \mathrm{MC}(D) \sqcup \mathrm{MC}(Z_cD_k)$. So, we obtain

$$\prod_{r\in\mathrm{MC}(Z_cD)} w(r) = \left(\prod_{r\in\mathrm{MC}(D)} w(r)\right)\left(\prod_{r\in\mathrm{MC}(Z_cD_k)} w(r)\right) = \prod_{r\in\mathrm{MC}(D)} w(r)$$

and

$$\begin{aligned}\prod_{r\in\mathrm{SC}(D)} w(r) &= \left(\prod_{r\in\mathrm{SC}(Z_cD)} w(r)\right)\left(\prod_{r\in\mathrm{MC}(Z_cD_k)} w(r)\right)w(c)\\ &= w(c)\left(\prod_{r\in\mathrm{SC}(Z_cD)} w(r)\right).\end{aligned}$$

It follows that

$$\begin{aligned}W_D = \prod_{r\in\mathrm{C}(D)} w(r) &= \left(\prod_{r\in\mathrm{SC}(D)} w(r)\right)\left(\prod_{r\in\mathrm{MC}(D)} w(r)\right)\\ &= w(c)\left(\prod_{r\in\mathrm{SC}(Z_cD)} w(r)\right)\left(\prod_{r\in\mathrm{MC}(Z_cD)} w(r)\right) = w(c)W_{Z_cD}.\end{aligned}$$

Similarly, we have $W_{S_cD} = w(c^*)W_{Z_cD}$, where c^* denotes the crossing obtained from c by switching. Since $kt(D) = kt(S_cD) = kt(Z_cD) - 1$, it is obvious that $d_{Z_cD} = \rho d_D = \rho d_{S_cD}$, where $\rho = (a - a^{-1})z^{-1}$. Hence, since $\mathrm{sign}(c^*) = -\mathrm{sign}(c)$, we obtain

$$\begin{aligned}&H(D[\pi, P]) - H(S_cD[\pi, Q]) - \mathrm{sign}(c)zH(Z_cD)\\ &\quad= W_D\, d_D - W_{S_cD}\, d_{S_cD} - \mathrm{sign}(c)zW_{Z_cD}\, d_{Z_cD}\\ &\quad= W_{Z_cD}\, d_D(w(c) - w(c^*) - \mathrm{sign}(c)z\rho)\\ &\quad= W_{Z_cD}\, d_D(a^{\mathrm{sign}(c)} - a^{-\mathrm{sign}(c)} - \mathrm{sign}(c)(a - a^{-1}))\\ &\quad= 0.\end{aligned}$$

□

Lemma 3.7. *Let D be a multi-knotoid diagram of n crossings with an ordering π. Then, the H-polynomial of D is independent of the position of the basepoint, that is, for any basepoints P and Q, $H(D[\pi, P]) = H(D[\pi, Q])$.*

Proof. If D is a knotoid diagram, then D has a unique basepoint. Suppose that $kt(D) > 0$. It is enough to show that the H-polynomial of D does not depend on the position of the basepoint on a knot component D_k, $1 \le k \le kt(D)$. Let q be a point on D_k and c the last crossing of D to the point q. We choose a new point p on D_k such that c is the first crossing of D from p. Using the points, we give the two basepoints $P = (p_0, p_1, \dots, p, \dots, p_{kt(D)})$ and $Q = (p_0, p_1, \dots, q, \dots, p_{kt(D)})$ to D. Note that P and Q differ only the points on D_k.

First, we suppose that c is a mixed crossing. Let λ and λ' be descending sequences with respect to (π, P) and (π, Q), respectively. Since $\{\lambda\} = \{\lambda'\}$, we have $D(\lambda) = D(\lambda')$ and, by Lemma 3.4, $\sum_D \lambda = \sum_D \lambda'$. Thus, the H-polynomial remains unchanged even if we make a change of basepoint form P to Q.

Next, we assume that c is a self crossing of D_k. Let $\lambda = (\lambda_{m-1}, \dots, \lambda_1)$ be a descending sequence for D with respect to (π, Q). We have two cases. If $c \notin \{\lambda\}$, then $\varphi = (\lambda_m, \lambda_{m-1}, \dots, \lambda_1)$ with $\lambda_m = c$ is a descending sequence for D with respect to (π, P). Then,

$$H(D[\pi, Q]) = H(D(\lambda)[\pi, Q]) + z \sum_D \lambda$$

and

$$\begin{aligned} H(D[\pi, P]) &= H(D(\varphi)[\pi, P]) + z \sum_D \varphi \\ &= H(D(\varphi)[\pi, P]) + z \left(\sum_D \lambda + \operatorname{sign}(\lambda_m) H(A_m^{\varphi} D) \right) \\ &= H(D(\varphi)[\pi, P]) + z \sum_D \lambda + \operatorname{sign}(c) z H(Z_c(D(\lambda))). \end{aligned}$$

We denote $D(\varphi)$ by E. Then, we see that $D(\lambda) = S_{c^*}E$ and $Z_c(D(\lambda)) = Z_{c^*}E$. Let $\varepsilon_c = \operatorname{sign}(c)$ and $\varepsilon_{c^*} = \operatorname{sign}(c^*)$. Since E is descending with respect to (π, P) and $\varepsilon_{c^*} = -\varepsilon_c$, by Lemma 3.6, we have

$$H(D[\pi, P]) - H(D[\pi, Q]) = H(E[\pi, P]) - H(S_{c^*}E[\pi, Q]) - \varepsilon_{c^*} z H(Z_{c^*}E) = 0.$$

Suppose that $c \in \{\lambda\}$. Let $\lambda_i = c$. The switching sequence λ' which is defined by $(\lambda_i, \dots, \lambda_1, \lambda_{m-1}, \dots, \lambda_{i+1})$ is a descending sequence for D with respect to (π, Q). We choose $\varphi = (\lambda_{i-1}, \dots, \lambda_1, \lambda_{m-1}, \dots, \lambda_{i+1})$ as a descending sequence for D with respect to (π, P). Then,

$$H(D[\pi, P]) = H(D(\varphi)[\pi, P]) + z \sum_D \varphi$$

and

$$\begin{aligned} H(D[\pi, Q]) &= H(D(\lambda')[\pi, Q]) + z \sum_D \lambda' \\ &= H(D(\lambda')[\pi, Q]) + z \left(\sum_D \varphi + \varepsilon_c H(A_{m-1}^{\lambda'} D) \right) \\ &= H(D(\lambda)[\pi, Q]) + z \sum_D \varphi + \varepsilon_c z H(Z_c(D(\varphi))). \end{aligned}$$

We denote $D(\varphi)$ by E. Then, we see that $D(\lambda) = S_c E$. Since E is descending with respect to (π, P), by Lemma 3.6, we have

$$H(D[\pi, P]) - H(D[\pi, Q]) = H(E[\pi, P]) - H(S_c E[\pi, Q]) - \varepsilon_c z H(Z_c E) = 0,$$

completing the proof. □

By Lemma 3.7, we can express the H-polynomial of a multi-knotoid diagram D with options (π, P) as $H(D[\pi])$ instead of $H(D[\pi, P])$ if it is not necessary to specify the basepoint.

Step 3: The H-polynomials for a skein triple satisfy a skein relation.

Lemma 3.8. *Let D be a multi-knotoid diagram with n crossings and c a crossing of D. Suppose that D and S_cD have the same ordering π. Then,*

$$H(D[\pi]) - H(S_cD[\pi]) = \operatorname{sign}(c)zH(Z_cD).$$

Proof. We choose the same basepoint P on D and S_cD. Let λ and λ' be descending sequences for D and S_cD with respect to (π, P), respectively. There are two cases. First, we consider the case $c \in \{\lambda\}$. If $\lambda = (c)$, then $D(\lambda) = S_cD$. Recursive definition (n) ensures the claim. We may assume that λ and λ' are $(\lambda_m, \lambda_{m-1}, \dots, \lambda_1)$ and $(\lambda_{m-1}, \dots, \lambda_1)$, respectively, with $\lambda_m = c$ and $m > 1$. Let $\varphi = (\lambda_{m-1}, \dots, \lambda_1, \lambda_m)$ and $\varepsilon_c = \operatorname{sign}(c)$. Then, by Recursive definition (n) and Lemma 3.5,

$$H(S_cD[\pi]) = H(S_cD(\lambda')[\pi]) + z\sum_{S_cD}\lambda'$$

and

$$\begin{aligned} H(D[\pi]) &= H(D(\varphi)[\pi]) + z\sum_{D}\varphi \\ &= H(D(\varphi)[\pi]) + z\left(\varepsilon_c H(Z_cD) + \sum_{j=1}^{m-1}\operatorname{sign}(\lambda_j)H(A_j^{\lambda'}(S_cD))\right) \\ &= H(D(\varphi)[\pi]) + \varepsilon_c zH(Z_cD) + z\sum_{S_cD}\lambda'. \end{aligned}$$

Since $D(\varphi) = D(\lambda) = S_cD(\lambda')$, we have $H(D[\pi]) - H(S_cD[\pi]) = \varepsilon_c zH(Z_cD)$.

Next, we suppose that $c \notin \{\lambda\}$. If $\lambda' = (c^*)$, then $S_cD(\lambda') = D$. The claim follows from Recursive definition (n). So, we may assume that λ and λ' are $(\lambda_{m-1}, \dots, \lambda_1)$ and $(\lambda_m, \lambda_{m-1}, \dots, \lambda_1)$, respectively, with $\lambda_m = c^*$ and $m > 1$. We put $\varphi = (\lambda_{m-1}, \dots, \lambda_1, \lambda_m)$ and $\varepsilon_{c^*} = \operatorname{sign}(c^*)$. Then, by Recursive definition (n) and Lemma 3.5,

$$H(D[\pi]) = H(D(\lambda)[\pi]) + z\sum_{D}\lambda$$

and

$$\begin{aligned} H(S_cD[\pi]) &= H(S_cD(\varphi)[\pi]) + z\sum_{S_cD}\varphi \\ &= H(S_cD(\varphi)[\pi]) + z\left(\varepsilon_{c^*} H(Z_{c^*}(S_cD)) + \sum_{j=1}^{m-1}\operatorname{sign}(\lambda_j)H(A_j^{\lambda}D)\right) \\ &= H(S_cD(\varphi)[\pi]) + \varepsilon_{c^*} zH(Z_cD) + z\sum_{D}\lambda. \end{aligned}$$

Since $D(\lambda) = S_cD(\varphi)$ and $\varepsilon_{c^*} + \varepsilon_c = 0$, we have the same formula as the previous case. This completes the proof. □

Step 4: The H-polynomial is independent of the position of the disjoint component.

Lemma 3.9. *Let D be a multi-knotoid diagram of n crossings with an ordering π. If D is a disjoint union of a multi-knotoid diagram D_1 and a link diagram D_2 and E is another disjoint union of D_1 and D_2 obtained from D by changing the position of D_2, then $H(D[\pi]) = H(E[\pi])$.*

Proof. Let P be a basepoint on D. By Step 2, we may choose P as a basepoint on E. If D is descending with respect to (π, P), then E is also descending with respect to (π, P). Since it is clear that $W_D = W_E$ and $d_D = d_E$, the claim is true. Suppose that $D[\pi, P]$ is not descending. Let $\lambda = (\lambda_m, \dots, \lambda_1)$ be a descending sequence for $D[\pi, P]$. Then, λ is also descending sequence for $E[\pi, P]$. By Recursive definition (n) and Step 2, we have

$$H(D[\pi]) = H(D(\lambda)[\pi]) + z\sum_D \lambda \quad \text{and} \quad H(E[\pi]) = H(E(\lambda)[\pi]) + z\sum_E \lambda.$$

Since $cr(A_k^\lambda D) = cr(A_k^\lambda E) = n-1, 1 \le k \le m$, Inductive hypothesis $(n-1)$ gives $H(A_k^\lambda D) = H(A_k^\lambda E), 1 \le k \le m$. It follows that $\sum_D \lambda = \sum_E \lambda$. Since $D(\lambda)[\pi, P]$ and $E(\lambda)[\pi, P]$ are descending diagrams, we have $H(D(\lambda)[\pi]) = H(E(\lambda)[\pi])$ according to the previous case. Thus, we obtain $H(D[\pi]) = H(E[\pi])$, completing the proof. □

Step 5: The H-polynomial has the properties (2) and (3) in Theorem 1.1.

We discuss invariance of the H-polynomial under Reidemeister moves. We begin with the Reidemeister move of type I.

Lemma 3.10. *Let D be a multi-knotoid diagram of n crossings with an ordering π and E a diagram obtained from D by applying a 1-sided Reidemeister move of type I at a crossing c of D. Then, $H(D[\pi]) = w(c)H(E[\pi])$.*

Proof. Let B be a stage for the local move. We choose a basepoint P on D outside B. We may regard P as a basepoint for E.

If $D[\pi, P]$ is descending, then $E[\pi, P]$ is also descending. By Recursive definition (n), we have $H(D[\pi]) = W_D d_D$ and $H(E[\pi]) = W_E d_E$. It is obvious that $d_D = d_E$ and $W_D = w(c)W_E$, which lead to $H(D[\pi]) = w(c)H(E[\pi])$.

We assume that $D[\pi, P]$ is not descending. Let $\lambda = (\lambda_m, \dots, \lambda_1)$ be a descending sequence for D with respect to (π, P). We have two cases.

First, we suppose that $c \notin \{\lambda\}$. Then, λ is also a descending sequence for E with respect to (π, P). Recursive definition (n) and Step 2 give

$$H(D[\pi]) = H(D(\lambda)[\pi]) + z\sum_D \lambda \quad \text{and} \quad H(E[\pi]) = H(E(\lambda)[\pi]) + z\sum_E \lambda.$$

Since $H(A_k^\lambda D) = w(c)H(A_k^\lambda E), 1 \le k \le m$, by Inductive hypothesis $(n-1)$, we have $\sum_D \lambda = w(c)\sum_E \lambda$. Since $D(\lambda)[\pi, P]$ and $E(\lambda)[\pi, P]$ are descending diagrams, by the result of the previous case, we have $H(D(\lambda)[\pi]) = w(c)H(E(\lambda)[\pi])$, and thus, $H(D[\pi]) = w(c)H(E[\pi])$.

Next, we suppose that $c \in \{\lambda\}$. By Step 1, we may assume that $\lambda_m = c$. Let $\varphi = (\lambda_{m-1}, \dots, \lambda_1)$. φ is a descending sequence for E with respect to (π, P). Then, we obtain

$$H(E[\pi]) = H(E(\varphi)[\pi]) + z\sum_{E}\varphi$$

by Recursive definition (n). Let $\varepsilon_c = \mathrm{sign}(c)$. We also have

$$\begin{aligned} H(D[\pi]) &= H(D(\lambda)[\pi]) + z\sum_{D}\lambda \\ &= H(D(\lambda)[\pi]) + z\left(\varepsilon_c H(A^{\lambda}_{m}D) + \sum_{j=1}^{m-1}\mathrm{sign}(\lambda_j)H(A^{\lambda}_{j}D)\right). \end{aligned}$$

Since $A^{\lambda}_{m}D = Z_c(D(\varphi))$ and

$$\sum_{D}\varphi = \sum_{j=1}^{m-1}\mathrm{sign}(\lambda_j)H(A^{\varphi}_{j}D) = \sum_{j=1}^{m-1}\mathrm{sign}(\lambda_j)H(A^{\lambda}_{j}D),$$

we obtain

$$H(D[\pi]) = H(D(\lambda)[\pi]) + \varepsilon_c zH(Z_c(D(\varphi)) + z\sum_{D}\varphi.$$

Since $E(\varphi)$ and $Z_c(D(\varphi)) = E(\varphi) \sqcup U_1$ are descending diagrams of $(n-1)$ crossings with respect to suitable options,

$$H(Z_c(D(\varphi))) = W_{E(\varphi)\sqcup U_1}d_{E(\varphi)\sqcup U_1} = W_{E(\varphi)}d_{E(\varphi)}\rho = \rho H(E(\varphi)),$$

where U_1 denotes a trivial circle and $\rho = (a - a^{-1})z^{-1}$. It is obvious that $W_{D(\lambda)} = w(c^*)W_{E(\varphi)}$ and $d_{D(\lambda)} = d_{E(\varphi)}$. It follows that $H(D(\lambda)[\pi]) = w(c^*)H(E(\varphi)[\pi])$, and then,

$$\begin{aligned} H(D(\lambda)[\pi]) + \varepsilon_c zH(Z_c(D(\varphi))) &= H(E(\varphi)[\pi])(w(c^*) + \varepsilon_c z\rho) \\ &= H(E(\varphi)[\pi])(a^{-\varepsilon_c} + \varepsilon_c(a - a^{-1})) \\ &= w(c)H(E(\varphi)[\pi]). \end{aligned}$$

We also find that $\sum_{D}\varphi = w(c)\sum_{E}\varphi$ because $H(A^{\varphi}_{j}D) = w(c)H(A^{\varphi}_{j}E), 1 \leq j \leq m-1$, by Inductive hypothesis $(n-1)$. Hence, we obtain $H(D[\pi]) = w(c)H(E[\pi])$. It completes the proof. □

Next, we deal with the Reidemeister move of type II.

Lemma 3.11. *Let D be a multi-knotoid diagram of n crossings with an ordering π and E a diagram obtained from D by applying a 1-sided Reidemeister move of type II. Then, $H(D[\pi]) = H(E[\pi])$.*

Proof. Let c_1 and c_2 be the crossings of D eliminated by the Reidemeister move of type II which changes D into E and B a stage for the local move. We choose a basepoint P on D outside B. We may regard P as a basepoint for E.

If $D[\pi, P]$ is descending, then $E[\pi, P]$ is also descending. By Recursive definition (n), we have $H(D[\pi]) = W_D d_D$ and $H(E[\pi]) = W_E d_E$. It is obvious that $d_D = d_E$. Since classification of c_1 is equal to that of c_2 and the signature of c_1 is different from that of c_2, we obtain $w(c_1)w(c_2) = 1$ and $W_D = W_E$. It follows that $H(D[\pi]) = H(E[\pi])$.

We assume that $D[\pi, P]$ is not descending. Let $\lambda = (\lambda_m, \ldots, \lambda_1)$ be a descending sequence for D with respect to (π, P).

First, we suppose that $c_1 \notin \{\lambda\}$. Then, $c_2 \notin \{\lambda\}$ and λ is also a descending sequence for E with respect to (π, P). Thus,

$$H(D[\pi]) = H(D(\lambda)[\pi]) + z\sum_D \lambda \quad \text{and} \quad H(E[\pi]) = H(E(\lambda)[\pi]) + z\sum_E \lambda.$$

Since $H(A_k^\lambda D) = H(A_k^\lambda E), 1 \leq k \leq m$, by Inductive hypothesis $(n-1)$, we have $\sum_D \lambda = \sum_E \lambda$. Since $D(\lambda)[\pi, P]$ and $E(\lambda)[\pi, P]$ are descending diagrams, by the above fact on the descending diagrams, we have $H(D(\lambda)[\pi]) = H(E(\lambda)[\pi])$ and thus, $H(D[\pi]) = H(E[\pi])$.

Next, we suppose that $c_1 \in \{\lambda\}$. Then, c_2 is also an element of $\{\lambda\}$. The signatures of c_1 and c_2 are different, and so we may assume that c_1 is positive. By Lemma 3.8, we have

$$H(S_{c_2}D[\pi]) - H(D[\pi]) = \mathrm{sign}(c_2^*)zH(Z_{c_2^*}D) = zH(Z_{c_2}D)$$

and

$$H(S_{c_2}D[\pi]) - H(S_{c_1}S_{c_2}D[\pi]) = \mathrm{sign}(c_1)zH(Z_{c_1}S_{c_2}D) = zH(Z_{c_1}S_{c_2}D).$$

Since $Z_{c_2}D$ and $Z_{c_1}S_{c_2}D$ are either identical or related by Reidemeister moves of type I by which the numbers of the crossings of diagrams do not exceed n, we find that $H(Z_{c_2}D)$ and $H(Z_{c_1}S_{c_2}D)$ coincide. The result of the previous case gives $H(S_{c_1}S_{c_2}D[\pi]) = H(E[\pi])$, which yields the claim. □

We focus on the Reidemeister move of type III. The Reidemeister move of type III as in Fig. 2 can be regarded as a passage of one arc over a crossing c between the others. The arc which pass over c is called the *top arc* and the remaining arcs are called the *middle* and the *bottom arcs*, where the middle arc and the bottom arc correspond to the overpath and the underpath at c, respectively.

Lemma 3.12. *Let D be a multi-knotoid diagram of n crossings with an ordering π and E a diagram obtained from D by applying a Reidemeister move of type III. Then, $H(D[\pi]) = H(E[\pi])$.*

Proof. Let B be a stage for the Reidemeister move of type III which changes D into E. Let c_1, c_2 and c_3 be the crossings of D in B which are composed of the top and the middle arcs, the top and the bottom arcs, and the middle and the bottom arcs, respectively. We denote the three crossings of E in B by r_1, r_2 and r_3, similarly. Let $\varepsilon_{c_i} = \mathrm{sign}(c_i)$ and $\varepsilon_{r_i} = \mathrm{sign}(r_i), 1 \leq i \leq 3$. Note that $\varepsilon_{c_i} = \varepsilon_{r_i}$. We choose a basepoint P on D outside B. We may regard P as a basepoint for E.

If $D[\pi, P]$ is descending, then $E[\pi, P]$ is also descending. Since $w(c_i) = w(r_i), 1 \leq i \leq 3$, we see that $W_D = W_E$ and $d_D = d_E$. By Recursive definition (n), we have $H(D[\pi]) = W_D d_D = W_E d_E = H(E[\pi])$.

We assume that $D[\pi, P]$ is not descending. Let $\lambda = (\lambda_m, \ldots, \lambda_1)$ be a descending sequence for D with respect to (π, P).

If $D[\pi, P]$ is descending in B, then λ is also a descending sequence for E with respect to (π, P). Then,

$$H(D[\pi]) = H(D(\lambda)[\pi]) + z\sum_D \lambda \quad \text{and} \quad H(E[\pi]) = H(E(\lambda)[\pi]) + z\sum_E \lambda.$$

Since $H(A_k^\lambda D) = H(A_k^\lambda E), 1 \le k \le m$, by Inductive hypothesis $(n-1)$, we have $\sum_D \lambda = \sum_E \lambda$. Since $D(\lambda)[\pi, P]$ and $E(\lambda)[\pi, P]$ are descending diagrams, we find that $H(D(\lambda)[\pi]) = H(E(\lambda)[\pi])$. Hence, we obtain $H(D[\pi]) = H(E[\pi])$.

Suppose that $D[\pi, P]$ is not descending in B. Let N be the set of the crossings in λ switched to change the diagram in B into descending one. Then, the five cases as in Table 1 can occur. The crossing change in Case 1 (resp. Case 2) can be realized by transposing the top and the middle (resp. the middle and the bottom) arcs in B. Then, we see that the crossing changes in any other case can be regarded as a combination of the above two transpositions. The claim is shown by induction on the number of the crossings in N. By the above description, it is enough to check Cases 1 and 2.

First, we consider Case 1, where c_1 and r_1 are changed. By Lemma 3.8, we have

$$H(D[\pi]) - H(S_{c_1}D[\pi]) = \varepsilon_{c_1} z H(Z_{c_1}D)$$

and

$$H(E[\pi]) - H(S_{r_1}E[\pi]) = \varepsilon_{r_1} z H(Z_{r_1}E).$$

Note that $cr(Z_{c_1}D) = cr(Z_{r_1}E) = n-1$. Since $Z_{c_1}D$ and $Z_{r_1}E$ are identical or they are related by two Reidemeister moves of type II by which the numbers of the crossings of diagrams do not exceed n, we obtain $H(Z_{c_1}D) = H(Z_{r_1}E)$. This result and the equality $H(S_{c_1}D[\pi]) = H(S_{r_1}E[\pi])$, which comes from the inductive hypothesis, provide $H(D[\pi]) = H(E[\pi])$.

Next, we consider Case 2, where c_3 and r_3 are changed. We may assume that $H(S_{c_3}D[\pi]) = H(S_{r_3}E[\pi])$ by the inductive hypothesis. The rest of the proof is similar to that of the previous case. □

Table 1. The set N in each case

Case	1	2	3	4	5
N	$\{c_1\}$	$\{c_3\}$	$\{c_1, c_2\}$	$\{c_2, c_3\}$	$\{c_1, c_2, c_3\}$

Step 6: The H-polynomial is independent of the choice of the ordering.

Lemma 3.13. *Let D be a multi-knotoid diagram of n crossings with options (π, P). Let c denote the first crossing of D encountered when traveling from the leg of the knotoid component. Let T_cD be the multi-knotoid diagram obtained from D by applying the tug move at c. If $D[\pi, P]$ is a descending diagram of level 0, then $H(D[\pi]) = w(c)H(T_cD)$.*

Proof. Note that by Lemma 3.2, $D[\pi, P]$ is descending if $n = 1$. If $D[\pi, P]$ is descending, then T_cD is also descending with respect to (π, P). Recursive definition (n) and Inductive hypothesis $(n-1)$ give $H(D[\pi]) = W_D d_D$ and $H(T_cD) = W_{T_cD} d_{T_cD}$, respectively. It is obvious that $d_D = d_{T_cD}$. Since $\mathrm{C}(D) = \mathrm{C}(T_cD) \cup \{c\}$, $W_D = w(c)W_{T_cD}$. Thus, the claim is true.

Suppose that $D[\pi, P]$ is not a descending diagram. Let $\lambda = (\lambda_m, \ldots, \lambda_1)$ be a descending sequence for D with respect to (π, P). Then, $c \notin \{\lambda\}$ and each $\lambda_j, 1 \leq j \leq m$, is a self crossing of a knot component or a mixed crossing composed of two different knot components because $D[\pi, P]$ is descending of level 0. Thus, $A^\lambda_j D, 1 \leq j \leq m$, is descending of level 0 with respect to suitable options. By Inductive hypothesis $(n-1)$, we have $H(A^\lambda_j D) = w(c)H(T_c(A^\lambda_j D))$. Since $D(\lambda)[\pi, P]$ is descending, the previous case gives $H(D(\lambda)[\pi]) = w(c)H(T_c(D(\lambda)))$, and thus,

$$\begin{aligned} H(D[\pi]) &= H(D(\lambda)[\pi]) + z\sum_{j=1}^{m} \operatorname{sign}(\lambda_j)H(A^\lambda_j D) \\ &= w(c)\left(H(T_c(D(\lambda))) + z\sum_{j=1}^{m} \operatorname{sign}(\lambda_j)H(T_c(A^\lambda_j D))\right). \end{aligned}$$

Since $c \notin \{\lambda\}$ and λ is a descending sequence for T_cD with respect to (π, P), we see that $T_c(D(\lambda)) = (T_cD)(\lambda)$, $T_c(A^\lambda_j D) = A^\lambda_j(T_cD), 1 \leq j \leq m$, and

$$\begin{aligned} H(T_cD) &= H((T_cD)(\lambda)) + z\sum_{T_cD} \lambda \\ &= H(T_c(D(\lambda))) + z\sum_{j=1}^{m} \operatorname{sign}(\lambda_j)H(A^\lambda_j(T_cD)) \\ &= H(T_c(D(\lambda))) + z\sum_{j=1}^{m} \operatorname{sign}(\lambda_j)H(T_c(A^\lambda_j D)). \end{aligned}$$

It follows that $H(D[\pi]) = w(c)H(T_cD)$. □

Lemma 3.14. *Let D be a multi-knotoid diagram with n crossings and π and ξ different orderings of D. If the connected component of D including the knotoid component has a crossing, then $H(D[\pi]) = H(D[\xi])$.*

Proof. Let c be the first crossing of D encountered when traveling from the leg of the knotoid component. We give a basepoint P on D.

If D is descending with respect to (π, P), then $D[\pi, P]$ and $D[\xi, P]$ are descending of level 0. By Lemma 3.13, we have $H(D[\pi]) = w(c)H(T_cD) = H(D[\xi])$.

Suppose that $D[\pi, P]$ is not descending. Let $\lambda = (\lambda_m, \ldots, \lambda_1)$ be a descending sequence for D with respect to (π, P). We have two cases.

First, we suppose that $c \notin \{\lambda\}$. Recursive definition (n) gives

$$H(D[\pi]) = H(D(\lambda)[\pi]) + z\sum_{j=1}^{m} \operatorname{sign}(\lambda_j)H(A^\lambda_j D).$$

On the other hand, the skein relation in Lemma 3.8 shows that

$$H(D[\xi]) = H(D(\lambda)[\xi]) + z\sum_{j=1}^{m} \operatorname{sign}(\lambda_j)H(A^\lambda_j D).$$

Note that c is a crossing of $D(\lambda)$ because of $c \notin \{\lambda\}$. Since $D(\lambda)[\pi, P]$ is descending, both $D(\lambda)[\pi, P]$ and $D(\lambda)[\xi, P]$) are descending of level 0. The result of the previous case implies $H(D(\lambda)[\pi]) = H(D(\lambda)[\xi])$, and thus, $H(D[\pi]) = H(D[\xi])$.

Next, we consider the case $c \in \{\lambda\}$. By using the recursive formula in Lemma 3.8, we obtain

$$H(D[\pi]) = H(S_c D[\pi]) + \mathrm{sign}(c) z H(Z_c D)$$

and

$$H(D[\xi]) = H(S_c D[\xi]) + \mathrm{sign}(c) z H(Z_c D).$$

So, we only have to show that $H(S_c D[\pi]) = H(S_c D[\xi])$. If $m = 1$, that is , $\lambda = (c)$, then $S_c D[\pi, P]$ is descending. It follows that $H(S_c D[\pi]) = H(S_c D[\xi])$. We assume that $m \geq 2$. By Step 1, we may assume that $\lambda_1 = c$. Let $\lambda' = (\lambda_m, \ldots, \lambda_2)$. Then, λ' is a descending sequence for $S_c D$ with respect to (π, P) and $c^* \notin \{\lambda'\}$. It comes down to the previous case. Hence, we have $H(S_c D[\pi]) = H(S_c D[\xi])$, completing the proof. □

Let D and E be link diagrams in a disk F. Then, E is said to be a *reduced* diagram of D if D can be changed into E by a finite sequence of 1-sided Reidemeister moves and the number of the crossings of D is strictly greater than that of the crossings of E.

The following lemma forms the core of the proof on independence of ordering in combinatorial composition of the HOMFLY polynomial for the link by Lickorish and Millett. Here, we adopt the claim by Rong; see Lemma 3 in [7].

Lemma 3.15. *Let D be a connected link diagram of n crossings, $n > 0$, with an ordering π in a disk F. Then, with some choice of the basepoint P on D, there exists a reduced diagram of $D(\lambda)$, where λ denotes a descending sequence for D with respect to (π, P).*

The reader will be familiar with the proof, so we omit it.

Lemma 3.16. *Let D be a multi-knotoid diagram of n crossings, $n > 0$, and π and ξ different orderings of D. If D is a disjoint union of a canonical trivial knotoid diagram and link diagrams, then $H(D[\pi]) = H(D[\xi])$.*

Proof. Let $D = D_0 \sqcup E_1 \sqcup \cdots \sqcup E_s$, where D_0 denotes the canonical trivial knotoid diagram and $E_k, 1 \leq k \leq s$, denotes a connected component of $D - D_0$. Then, by Step 4, we may change D into a disjoint union E of $D_0, E_1, \ldots, E_{s-1}$ and E_s such that there exist mutually disjoint disks $F_0, F_1, \ldots, F_{s-1}$ and F_s with $D_0 \subset F_0$ and $E_k \subset F_k, 1 \leq k \leq s$. We consider the polynomials of $E[\pi]$ and $E[\xi]$. Since D_0 has no crossings, there exists a connected component of $E - D_0$ whose crossing number is positive. We may assume that E_1 is such a component. Then, by Lemma 3.15, there exist a switching sequence $\lambda = (\lambda_m, \ldots, \lambda_1)$ for E such that each $\lambda_j, 1 \leq j \leq m$, is a crossing of E_1, $E_1(\lambda)$ is descending with respect to the ordering induced from π and some basepoint, and $E_1(\lambda)$ has a reduced diagram J_1. By the recursive formula in Step 3, we have

$$H(E[\pi]) = H(E(\lambda)[\pi]) + z \sum_E \lambda \quad \text{and} \quad H(E[\xi]) = H(E(\lambda)[\xi]) + z \sum_E \lambda.$$

Let J be the diagram obtained from $E(\lambda)$ by replacing $E_1(\lambda)$ with J_1. Then, it is obvious that J is a reduced diagram of $E(\lambda)$. And we also see that the 1-sided Reidemeister moves to change $E(\lambda)$ into J can be applied without regard to the ordering. Since $cr(J) < n$, Inductive hypothesis $(n-1)$ and Step 5 ensure that $H(E(\lambda)[\pi]) = H(E(\lambda)[\xi])$, which implies

$H(E[\pi]) = H(E[\xi])$. It follows that $H(D[\pi]) = H(D[\xi])$ from Step 4. □

Lemmas 3.14 and 3.16 give the following proposition, completing Step 6.

Proposition 3.17. *Let D be a multi-knotoid diagram with n crossings, $n > 0$, and π and ξ different orderings of D. Then, $H(D[\pi]) = H(D[\xi])$.*

Now, we are ready to prove the main theorem. Steps 1, 2 and 6 give condition (a) of Inductive Hypothesis (n). The definition of the H-polynomial for a canonical multi-knotoid diagram and Steps 3 and 5 provide condition (b). Conditions (c) and (d) are shown by Step 4 and Lemma 3.13, respectively. Hence, we find that Inductive hypothesis (n) is true for multi-knotoid diagrams with at most n crossings. We finish the proofs of existence and invariance of the H-polynomial.

Proof of Corollary 1.4. The writhe (resp. self writhe) of a multi-knotoid diagram is invariant under the Reidemeister moves of type II and of type III. A 1-sided Reidemeister move of type I at a crossing changes the writhe (resp. self writhe) by the signature of the crossing. These facts and the properties (2) and (3) on the H-polynomial in Theorem 1.1 show that the R-polynomial (resp. SR-polynomial) is invariant under Reidemeister moves. It completes the proof of invariance of the R-polynomial (resp. SR-polynomial). □

4. HOMFLY polynomial

A reduced polynomial derived from the R-polynomial is studied in this section.

Here, we review the HOMFLY polynomial for links [2, 5, 6]. The HOMFLY polynomial $P(L; v, z) \in \mathbb{Z}[v^{\pm 1}, z^{\pm 1}]$ of an oriented link L is an invariant of the isotopy type of L, which is defined by the following formulas:

(1) $P(U; v, z) = 1$ for the trivial knot U.

(2) For a skein triple (L_+, L_-, L_0) as in Fig. 3,

$$v^{-1}P(L_+; v, z) - vP(L_-; v, z) = zP(L_0, v, z).$$

An embedded arc in S^2 is called *simple* if it has no crossings. Let D and α be a multi-knotoid diagram and a simple arc in S^2, respectively. The arc α is called a *shortcut* for D if α connects the endpoints of D and meets D transversely at a finite set of points distinct from the crossings of D. In particular, a shortcut α is said to be *over* (resp. *under*) if α passes over (resp. under) D. We denote over and under shortcuts by α^+ and α^-, respectively. For a multi-knotoid diagram D and an over shortcut α^+ for D, we denote by $D(\alpha^+)$ the link diagram obtained from D by joining α^+ at the endpoints of D. Then, the link type of $D(\alpha^+)$ does not depend on the choice of the over shortcut α^+ because any two over shortcuts for D are isotopic in the class of embedded arcs in S^2 connecting the endpoints of D. The diagram $D(\alpha^-)$ is defined similarly. If D is oriented, we suppose that $D(\alpha)$ has the induced orientation from D, that is, α has the orientation from the head to the leg of D.

Proposition 4.1. *Let D and α^+ be an oriented multi-knotoid diagram and an over shortcut for D in S^2, respectively. Then, the R-polynomial provides the equation $R(D; a, a, z) = P(D(\alpha^+); a^{-1}, z)$.*

Proof. The proof is by induction on the number $cr(D)$ of the crossings of D.

If $cr(D) = 0$, then D is a canonical trivial multi-knotoid diagram. The definition of the H-polynomial gives $H(D; a, h, z) = W_D d_D = \rho^{kt(D)}$, where $\rho = (a - a^{-1})z^{-1}$. Since $wr(D) = 0$, we have $R(D; a, a, z) = \rho^{kt(D)}$. Since we see that $D(\alpha^+)$ is a trivial link diagram with $(kt(D) + 1)$ components, we obtain $P(D(\alpha^+); a^{-1}, z) = \rho^{kt(D)}$. Hence, the claim is true for this case.

Suppose that $cr(D) > 0$. We give D an ordering π and a basepoint P. If D is descending with respect to (π, P), then $H(D; a, h, z) = W_D d_D = a^{sw(D)} h^{mw(D)} \rho^{kt(D)}$, and thus, $R(D; a, a, z) = \rho^{kt(D)}$. Since D is descending, $D(\alpha^+)$ is a trivial link diagram with $(kt(D) + 1)$ components. Hence, $P(D(\alpha^+); a^{-1}, z) = \rho^{kt(D)}$, which is equal to $R(D; a, a, z)$. We assume that D is not descending with respect to (π, P). Let $\lambda = (\lambda_m, \dots, \lambda_1)$ be a descending sequence for D with respect to (π, P). Then, by using the recursive formula in Remark 1.3 repeatedly, we have

$$R(D; a, h, z) = a^{-2\tau_m} R(D(\lambda); a, h, z) + z \sum_{k=1}^{m} \operatorname{sign}(\lambda_k) a^{\operatorname{sign}(\lambda_k) - 2\tau_k} R(A_k^\lambda D; a, h, z),$$

where $\tau_k = \sum_{j=1}^{k} \operatorname{sign}(\lambda_j)$, $1 \leq k \leq m$. Since λ_j, $1 \leq j \leq m$, is also a crossing of $D(\alpha^+)$, by the recursive formula on the HOMFLY polynomial, we obtain

$$\begin{aligned} P(D(\alpha^+); a^{-1}, z) = a^{-2\tau_m} & P((D(\alpha^+))(\lambda); a^{-1}, z) \\ & + z \sum_{k=1}^{m} \operatorname{sign}(\lambda_k) a^{\operatorname{sign}(\lambda_k) - 2\tau_k} P(A_k^\lambda (D(\alpha^+)); a^{-1}, z). \end{aligned}$$

Since $A_k^\lambda(D(\alpha^+)) = (A_k^\lambda D)(\alpha^+)$, $1 \leq k \leq m$, the inductive hypothesis shows that $R(A_k^\lambda D; a, a, z) = P(A_k^\lambda(D(\alpha^+)); a^{-1}, z)$. Since $(D(\alpha^+))(\lambda) = (D(\lambda))(\alpha^+)$, we see that $R(D(\lambda); a, a, z) = P((D(\alpha^+))(\lambda); a^{-1}, z)$ from the result of the previous case. It follows that $R(D; a, a, z) = P(D(\alpha^+); a^{-1}, z)$. This completes the proof. □

A 2-variable polynomial invariant $P(L; v, z) \in \mathbb{Z}[v^{\pm 1}, z^{\pm 1}]$ of an oriented multi-knotoid L is defined to be of HOMFLY type if it satisfies the following two identities:

(1) $P(U; v, z) = 1$ for the trivial knotoid U.

(2) For a skein triple (L_+, L_-, L_0) as in Fig. 3,

$$v^{-1} P(L_+; v, z) - v P(L_-; v, z) = z P(L_0, v, z).$$

By Remark 1.3, the reduced polynomial $R(D; a, a, z)$ satisfies the following skein relation:

$$a R(D_+; a, a, z) - a^{-1} R(D_-; a, a, z) = z R(D_0; a, a, z).$$

We put $Y(D; a, z) = R(D; a^{-1}, a^{-1}, z)$. Then, we have

$$a^{-1} Y(D_+; a, z) - a Y(D_-; a, z) = z Y(D_0; a, z).$$

This implies that the polynomial $Y(D; a, z)$ is of HOMFLY type. We also see that $Y(D; a, z) = P(D(\alpha^+); a, z)$. The "HOMFLY" polynomial for a multi-knotoid proposed by Turaev [8] is also of HOMFLY type from its definition. Since Turaev's HOMFLY polynomial corresponds to the HOMFLY polynomial for the link obtained from the knotoid and an under short cut, the two HOMFLY polynomials are distinct. It indicates that there exist not a unique but at least two polynomials of HOMFLY type.

Corollary 4.2. *There exist infinitely many non-trivial oriented knotoids with trivial polynomial of HOMFLY type.*

Proof. Let $D_n, n > 0$, be the oriented knotoid diagram as in Fig. 5, where T_n denotes the 2-string tangle which has horizontal n positive crossings. Then, we have

$$R(D_n; a, h, z) = \begin{cases} a^{-n-1} + (a^{-1} - a^{-n-2})h & \text{if } n \text{ is odd,} \\ a^{-n} + (a - a^{-n+1})h^{-1} & \text{if } n \text{ is even.} \end{cases}$$

Since $R(D_n; a, h, z) \neq 1$, D_n is not trivial. Since $R(D_n; a, a, z) = 1$, and thus $Y(D; a, z) = 1$, D_n has trivial HOMFLY polynomial. For any distinct positive integers m and l, we have $R(D_m; a, h, z) \neq R(D_l; a, h, z)$. It follows that D_m and D_l are distinct. This completes the proof. □

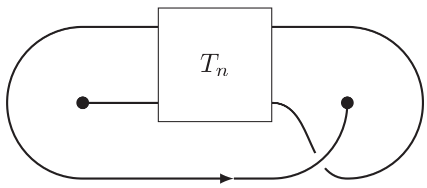

Fig. 5. The diagram D_n

5. Mirror and reverse images

Let D be a multi-knotoid diagram in $S^2 = \mathbb{R}^2 \cup \{\infty\}$. Suppose that $D \subset \mathbb{R}^2$. Let $\overline{D}$ be the diagram obtained from D by the reflection of $\mathbb{R}^2$ with respect to a line on $\mathbb{R}^2$ and $D!$ the diagram obtained from D by switching all crossings of D. $\overline{D}$ and $D!$ are usually said to be mirror images of D. Here, we distinguish them. We call $\overline{D}$ the horizontal mirror image of D. $D!$ is called the vertical mirror image of D. For example, in Fig. 9 shown in the following section, the middle two diagrams are the horizontal and the vertical mirror images of the left diagram.

REMARK 5.1. In [8], Turaev calls $\overline{D}$ and $D!$ the *symmetry* and the *mirror* images of D, respectively. He uses the notations sym(D) and mir(D) for them.

REMARK 5.2. For a multi-knotoid diagram D, $\overline{D}$ is uniquely determined without regard to choosing an axis of reflection.

The following proposition specifies a relationship between the polynomials for a multi-knotoid diagram and its horizontal mirror image.

Proposition 5.3. *Let D be an oriented multi-knotoid diagram and $\overline{D}$ the horizontal mirror image of D. Then, $H(\overline{D}; a, h, z) = H(D; a^{-1}, h^{-1}, -z)$, and thus, $R(\overline{D}; a, h, z) = R(D; a^{-1}, h^{-1}, -z)$ and $SR(\overline{D}; a, h, z) = SR(D; a^{-1}, h^{-1}, -z)$.*

Proof. Let c be a crossing of D and $\overline{c}$ the crossing of $\overline{D}$ corresponding to c. Note that the signatures of c and $\overline{c}$ are opposite and over/under informations at c and $\overline{c}$ coincide.

The proof is by induction on the number $cr(D)$ of the crossings of D. If $cr(D) = 0$, then $\overline{D} = D$, which implies $H(\overline{D}) = H(D) = \rho^{kt(D)}$, where $\rho = \rho(a, z) = (a - a^{-1})z^{-1}$. Since $\rho(a^{-1}, -z) = \rho(a, z)$, the claim is true.

Suppose that $cr(D) > 0$. Let π and P be an ordering and a basepoint for D, respectively. Let $\overline{P}$ be the basepoint on $\overline{D}$ corresponding to P. We give $\overline{D}$ the same ordering π as D.

First, we suppose that D is descending with respect to (π, P). Then, $\overline{D}$ is also descending with respect to $(\pi, \overline{P})$. By the definition of the H-polynomial, we see that $H(D) = W_D d_D$ and $H(\overline{D}) = W_{\overline{D}} d_{\overline{D}}$. Since $W_{\overline{D}} = W_{\overline{D}}(a, h) = a^{sw(\overline{D})} h^{mw(\overline{D})} = a^{-sw(D)} h^{-mw(D)} = W_D(a^{-1}, h^{-1})$ and $d_{\overline{D}} = d_{\overline{D}}(a, z) = \{(a - a^{-1})z^{-1}\}^{kt(\overline{D})} = \{(a^{-1} - a)(-z)^{-1}\}^{kt(D)} = d_D(a^{-1}, -z)$ because of $kt(\overline{D}) = kt(D)$, we obtain

$$H(\overline{D}; a, h, z) = W_{\overline{D}}(a, h)\, d_{\overline{D}}(a, z) = W_D(a^{-1}, h^{-1})\, d_D(a^{-1}, -z) = H(D; a^{-1}, h^{-1}, -z).$$

Next, we suppose that D is not descending with respect to (π, P). Let $\lambda = (\lambda_m, \dots, \lambda_1)$ be a descending sequence for D with respect to (π, P). Then, $\overline{\lambda} = (\overline{\lambda_m}, \dots, \overline{\lambda_1})$ is a descending sequence for $\overline{D}$ with respect to $(\pi, \overline{P})$, where $\overline{\lambda_j}$, $1 \le j \le m$, denotes the crossing of $\overline{D}$ corresponding to λ_j. Hence,

$$H(D; a, h, z) = H(D(\lambda); a, h, z) + z \sum_{j=1}^{m} \mathrm{sign}(\lambda_j) H(A_j^{\lambda} D; a, h, z)$$

and

$$H(\overline{D}; a, h, z) = H(\overline{D}(\overline{\lambda}); a, h, z) + z \sum_{j=1}^{m} \mathrm{sign}(\overline{\lambda_j}) H(A_j^{\overline{\lambda}} \overline{D}; a, h, z).$$

Since $A_j^{\overline{\lambda}} \overline{D} = \overline{A_j^{\lambda} D}$, $1 \le j \le m$, the inductive hypothesis gives

$$H(A_j^{\overline{\lambda}} \overline{D}; a, h, z) = H(A_j^{\lambda}; a^{-1}, h^{-1}, -z).$$

Since $\overline{D}(\overline{\lambda}) = \overline{D(\lambda)}$, it follows from the previous case that $H(\overline{D}(\overline{\lambda}); a, h, z) = H(D(\lambda); a^{-1}, h^{-1}, -z)$. Since $\mathrm{sign}(\overline{\lambda_j}) = -\mathrm{sign}(\lambda_j)$, we have

$$\begin{aligned} H(\overline{D}; a, h, z) &= H(D(\lambda); a^{-1}, h^{-1}, -z) + (-z) \sum_{j=1}^{m} \mathrm{sign}(\lambda_j) H(A_j^{\lambda} D; a^{-1}, h^{-1}, -z) \\ &= H(D; a^{-1}, h^{-1}, -z). \end{aligned}$$

This completes the proof. □

Remark 5.4. For an oriented multi-knotoid diagram D and its vertical mirror image $D!$, we see that $(D!)(\alpha^+) = D(\alpha^-)!$. Then, by Proposition 4.1 and [3, Theorem 8.4.1], we have $R(D; a, a, z) = P(D(\alpha^+); a^{-1}, z)$ and $R(D!; a, a, z) = P((D!)(\alpha^+); a^{-1}, z) = P(D(\alpha^-)!; a^{-1}, z) = (-1)^{kt(D)} P(D(\alpha^-); -a, z)$. Since the two links $D(\alpha+)$ and $D(\alpha^-)$ in general are quite different, it looks like there is no relationship between the R-polynomials of D and $D!$.

Next, we show a relationship between the polynomials for a multi-knotoid diagram and its reverse image defined later. To do that, we explore multi-knotoid diagrams in a disk or an annulus.

Let F be a disk or an annulus in S^2 or $\mathbb{R}^2$ and t a generic immersed arc in F. We call t an arc shortly and call each double point of t a crossing of t. An arc t is called *proper* if $t \cap \partial F = \partial t$. In particular, t in an annulus A is called 2-*sided* proper if t connects the two boundaries of A. If both endpoints of t are on one of the two boundaries of A, then t is said to be 1-*sided* proper.

Let t be a proper arc in a disk or an annulus F and $\partial t = \{t_0, t_1\}$. Let $\mathrm{C}(t)$ be the set of the crossings of t and $cr(t)$ the number of the crossings of t. Suppose that $cr(t) > 0$. We define a surjection $f_t\colon \{1, 2, \ldots, 2cr(t)\} \to \mathrm{C}(t)$ by sending each $n \in \{1, 2, \ldots, 2cr(t)\}$ to the n-th crossing encountered when traveling t from t_0. f_t is said to be the *presentation map* of t. Then, we call the sequence $f_t(1)f_t(2)\cdots f_t(2cr(t))$ of the $2cr(t)$ crossings the *presentation word* of t and denote it by PW_t. Since the same crossing appears twice in PW_t, for any crossing $c \in \mathrm{C}(t)$, the preimage $f_t^{-1}(c) \subset \{1, 2, \ldots, 2cr(t)\}$ consists of just two positive integers which are denoted by $n_1(c)$ and $n_2(c)$, where $n_1(c) < n_2(c)$.

For a crossing $c \in \mathrm{C}(t)$, the subword $f_t(n_1(c))\cdots f_t(n_2(c))$ of PW_t is called the *cutoff* word by c and is denoted by $PW_t(c)$. The cutoff word $PW_t(c)$ determines a unique closed subarc of t which starts from and arrives at c. Then, the closed subarc is said to be *associated with* $PW_t(c)$. The cutoff word $PW_t(c)$ is said to be *simple* if $PW_t(c)$ includes no cutoff word by any distinct crossing $p \in \mathrm{C}(t)$ from c. If $PW_t(c)$ is simple, then the closed subarc of t associated with $PW_t(c)$ forms a loop without self crossings. It is called the 1-gon with the vertex c.

Lemma 5.5. *Let t be a proper arc in a disk or an annulus and PW_t the presentation word of t. If $cr(t) > 0$, then there exists a crossing c of t such that $PW_t(c)$ is simple.*

Proof. We suppose that for any crossing c of t, $PW_t(c)$ is not simple. We put $|PW_t(c)| = n_2(c) - n_1(c) + 1$. For any $c \in \mathrm{C}(t)$, $2 \leq |PW_t(c)| \leq 2cr(t)$. Then, there exists a crossing p of t with $|PW_t(p)| = \min\{\, |PW_t(c)|\, ; c \in \mathrm{C}(t)\}$. Since $PW_t(p)$ is not simple, there exists a crossing q of t such that $PW_t(q)$ is a subword of $PW_t(p)$. Then, $|PW_t(q)| \leq |PW_t(p)| - 2 < |PW_t(p)|$, which is a contradiction. $\square$

Let D be a disk and B a disk in D with $B \cap \partial D = \emptyset$. We denote the annulus $D - intB$ in D by $A(D, B)$, where $intB$ denotes the interior of B. Let t be an arc in D or $A(D, B)$ and $\Gamma \subset t - \partial t$ a 1-gon. The 1-gon Γ bounds a disk in D. We denote it by D_Γ. Let p be the vertex of Γ. Then, the 1-gon Γ is called *standard* if there exists a neighborhood U_p of p such that $(t - \Gamma) \cap D_\Gamma = \emptyset$ in U_p. If Γ is not standard, then it is said to be *non-standard*. The left and the right drawings in Fig. 6 show standard and non-standard 1-gons, respectively.

Fig. 6. Standard and non-standard 1-gons

Lemma 5.6. *Let D be a disk and t a proper arc in D. If t has a non-standard 1-gon, then $cr(t) \geq 3$.*

Proof. We choose a non-standard 1-gon Γ. Since Γ is non-standard, $(D_\Gamma - \Gamma) \cap t \neq \emptyset$. Since $\partial t \subset \partial D$, both endpoints of t are outside D_Γ. Hence, there should exist at least two crossings on Γ except the vertex of Γ. It completes the proof. □

Let t be an arc in S^2 and p and q points on t. We denote by $a_t[p,q]$ a subarc of t whose endpoints are p and q. We also denote by $a_t(p,q)$ the open arc obtained from the arc $a_t[p,q]$ by removing the endpoints p and q.

Lemma 5.7. *Let D be a disk and t a proper arc in D. If $cr(t) > 0$, then t has a standard 1-gon in D.*

Proof. The proof is by induction on the number $cr(t)$ of the crossings of t. By Lemma 5.5, there exists a crossing p of t such that $PW_t(p)$ is simple. Hence, there exists the 1-gon Γ with the vertex p associated with $PW_t(p)$. If Γ is non-standard, then $cr(t) \geq 3$ by Lemma 5.6. This means that the claim is true for $cr(t) \leq 2$. Suppose that $cr(t) > 2$. We assume that t has no standard 1-gons. Then, Γ is non-standard. Let $\partial t = \{t_0, t_1\}$. We divide t into the three subarcs $a_t[t_0, p], \Gamma$ and $a_t[p, t_1]$ associated with the subword ahead of $PW_t(p)$, $PW_t(p)$ and the subword behind $PW_t(p)$ in PW_t, respectively. Note that $t - (\Gamma \cup \partial t) = a_t(t_0, p) \cup a_t(p, t_1)$. Let q be the first crossing where $a_t[p, t_1]$ meets Γ when traveling $a_t[p, t_1]$ from p to t_1. Note that there exists such a crossing since Γ is non-standard. Then, the subarc $a_t[p,q]$ of the arc $a_t[p, t_1]$ is proper in D_Γ. If $cr(a_t[p,q]) = 0$, then t has the standard 1-gon whose vertex is q. It is a contradiction. We suppose that $cr(a_t[p,q]) > 0$. Since $cr(a_t[p,q]) \leq cr(t) - 2 < cr(t)$, the inductive hypothesis ensures that the arc $a_t[p,q]$ has a standard 1-gon in D_Γ. It follows that t has a standard 1-gon in D, which is a contradiction. Hence, the claim is true for $cr(t) > 0$. □

An arc in a disk D or an annulus $A \subset D$ is said to be *simple* if it has no self crossings. Let t be a non-simple arc and $\mathrm{V}_1(t)$ the set of the vertices of 1-gons on the arc t. For a crossing c of t, it is clear that $PW_t(c)$ is simple if and only if $c \in \mathrm{V}_1(t)$. By Lemma 5.5, we see that $\mathrm{V}_1(t) \neq \emptyset$.

Lemma 5.8. *Let t be a non-simple 2-sided proper arc in an annulus $A(D,B)$ with $t_0 = \partial t \cap \partial B$ and f_t the presentation map of t. Let $m_t = \min\{n_1(c); c \in \mathrm{V}_1(t)\}$ and Γ the 1-gon in $A(D,B)$ associated with $PW_t(f_t(m_t))$. Then,*

(i) *If Γ is standard, then $D_\Gamma \subset A(D,B)$, otherwise $D_\Gamma \supset B$,*

(ii) *$a_t[t_0, f_t(m_t)] \cap \Gamma = \{f_t(m_t)\}$ and $cr(a_t[t_0, f_t(m_t)]) = 0$,*

where $a_t(t_0, f_t(m_t)) \subset t - \Gamma$.

Proof. Let $p = f_t(m_t)$. First, we assume that Γ is standard. Suppose that $B \subset D_\Gamma$. Since $t_0 \in B$, $a_t[t_0, p] \cap \Gamma \neq \{p\}$. Let q be a crossing in $a_t[t_0, p] \cap \Gamma - \{p\}$. Then, we see that $n_1(q) < n_1(p) = m_t < n_2(q) < n_2(p)$. Since $n_1(q) < m_t$, $PW_t(q)$ is not simple, and thus, there exists a crossing $r \in \mathrm{C}(t)$ such that $PW_t(r)$ is simple and a subword of $PW_t(q)$. Since $n_2(q) < n_2(p)$, we have $n_2(r) < n_2(p)$. Then, the following three cases can occur among the four integers $n_1(r), n_2(r), n_1(p)$ and $n_2(p)$.

Case 1. $n_1(r) < n_2(r) < n_1(p) = m_t < n_2(p)$.
Case 2. $n_1(r) < n_1(p) = m_t < n_2(r) < n_2(p)$.
Case 3. $n_1(p) = m_t < n_1(r) < n_2(r) < n_2(p)$.

The first two cases show $n_1(r) < m_t$, which is a contradiction. The last one implies that $PW_t(p)$ is not simple, which is a contradiction. Hence, $B \not\subset D_\Gamma$, that is, $B \cap D_\Gamma = \emptyset$. It follows that $D_\Gamma \subset A(D, B)$.

Next, we assume that Γ is non-standard. Suppose that $D_\Gamma \cap B = \emptyset$. Since $D_\Gamma \cap (t - \Gamma) \neq \emptyset$, $a_t[t_0, p] \cap \Gamma \neq \{p\}$. Let q be a crossing in $a_t[t_0, p] \cap \Gamma - \{p\}$. Then, we find that $n_1(q) < n_1(p) = m_t < n_2(q) < n_2(p)$. The rest of the proof is similar to that of the previous case. We conclude that $B \subset D_\Gamma$ and complete the proof of the first claim.

We proceed to the proof of the second claim. Suppose that $a_t[t_0, p] \cap \Gamma \neq \{p\}$. Then, there exists a crossing $q \in a_t[t_0, p] \cap \Gamma$ with $n_1(q) < n_1(p) = m_t < n_2(q) < n_2(p)$. It leads to a contradiction. To show the second claim of (ii), we suppose that $cr(a_t[t_0, p]) > 0$. Then, there exists a self crossing c of the arc $a_t[t_0, p]$ with $n_1(c) < n_2(c) < n_1(p) = m_t$. Hence, $PW_t(c)$ is not simple, and thus, there exists a crossing $r \in V(t)$ such that $PW_t(r)$ is a simple subword of $PW_t(c)$. It follows that $n_1(r) < n_2(r) < n_2(c) < n_1(p) = m_t$, a contradiction. □

A figure with two sides formed by two simple arcs meeting only at their endpoints in a disk D or an annulus $A \subset D$ as in Fig. 7 is called a 2-gon. A 2-gon Δ bounds a disk in D. We denote it by D_Δ. Similarly, a 3-gon denoted by Ω and a disk D_Ω bounded by Ω in D are defined.

Lemma 5.9. *Let $A(D, B)$ be an annulus. Let p and q be different points on ∂D and b a point on ∂B. Let s be a simple 2-sided proper arc in $A(D, B)$ with the endpoints b and p and t a 1-sided proper arc in $A(D, B)$ with the endpoints p and q. Suppose that $s \cap t - \{p\}$ consists of a finite number of crossings. Then, at least one of the following four claims holds:*

(1) *The arc t has a standard 1-gon Γ with $D_\Gamma \subset A(D, B)$.*
(2) *The arcs s and t form a 2-gon Δ with $D_\Delta \subset A(D, B)$.*
(3) *The arc t and ∂D form a 2-gon Δ with $D_\Delta \subset A(D, B)$.*
(4) *The arcs s and t and ∂D form a 3-gon Ω with $D_\Omega \subset A(D, B)$.*

Proof. We orient the arcs s and t from b and p to p and q, respectively. We assume that there are n crossings composed of s and t except p. We denote the crossings by $p_1, p_2, \dots, p_n$ in order of passage when traveling t from p according to its orientation.

If $n = 0$, then t is a proper arc in the disk E obtained from $A(D, B)$ by cutting $A(D, B)$ along s. If $cr(t) > 0$, then by Lemma 5.7, t has a standard 1-gon Γ in E. It follows that there exists a standard 1-gon Γ in $A(D, B)$ with $D_\Gamma \subset A(D, B)$. If $cr(t) = 0$, then there exists the 2-gon formed by t and an arc on ∂D whose endpoints are p and q.

Suppose that $n > 0$. We divide the arc $t = t[p, q]$ into the $(n + 1)$ subarcs which are denoted by $a_t[p, p_1], a_t[p_1, p_2], \dots, a_t[p_{n-1}, p_n]$ and $a_t[p_n, q]$. Let E be the disk obtained from $A(D, B)$ by cutting $A(D, B)$ along s. Then, $\partial E = \partial B \cup s^+ \cup \partial D \cup s^-$, where s^+ and s^- denote copies of s. Since each of the $(n+1)$ subarcs of t is proper in E, by Lemma 5.7 it has a standard 1-gon in E if it has a self crossing. It follows that there exists a standard 1-gon Γ in $A(D, B)$ with $D_\Gamma \subset A(D, B)$. We suppose that each of the $(n + 1)$ subarcs of t is simple. Each arc except $a_t[p_n, q]$ has the endpoints on $s^+ \cup s^-$. If there exists an arc $a_t[p_k, p_{k+1}], 0 \leq k < n$, where $p_0 = p$, whose endpoints are on either s^+ or s^-, then a 2-gon with vertices p_k and p_{k+1}

is formed in E by the arc $a_t[p_k, p_{k+1}]$ and the copy of s. It ensures that there exists a 2-gon Δ in $A(D, B)$ with $D_\Delta \subset A(D, B)$. If each arc except $a_t[p_n, q]$ has the endpoints on both s^+ and s^-, then we find a 3-gon in E whose sides are the arc $a_t[p_n, q]$, the subarc $a_s[p_n, p]$ of s and a subarc of ∂D whose endpoints are p and q. Hence, there exists a 3-gon Ω whose vertices are p, q and p_n with $D_\Omega \subset A(D, B)$. This completes the proof. □

Lemma 5.10. *Let t be a 2-sided proper arc in an annulus $A(D, B)$. If t has a non-standard 1-gon, then $cr(t) > 1$.*

Proof. Let Γ be a non-standard 1-gon and p the vertex of Γ. We divide t into the three arcs $a_t[t_0, p], \Gamma$ and $a_t[p, t_1]$, where $t_0 = t \cap \partial B, t_1 = t \cap \partial D$. Since Γ is non-standard and $D_\Gamma \subset D - \partial D$, the open arc $a_t(p, t_1)$ and Γ cross each other. It implies that $cr(t) \geq 2$. □

A 2-gon Δ formed by two subarcs of an arc t in a disk D or an annulus $A \subset D$ is *standard* if for each vertex p of Δ, there exists a neighborhood U_p of p such that $(t - \Delta) \cap D_\Delta = \emptyset$ in U_p. If Δ is not standard, then it is said to be *non-standard*. Fig. 7 illustrates 2-gons. The left drawing displays a standard 2-gon and the middle and the right ones exhibit non-standard 2-gons.

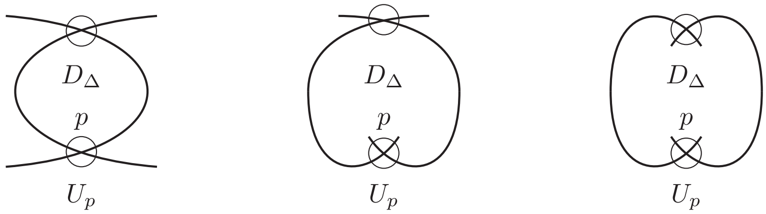

Fig. 7. Standard and non-standard 2-gons

Proposition 5.11. *Let t be a 2-sided proper arc in annulus $A(D, B)$. If t is not simple, then t has a standard 1- or 2-gon Γ with $D_\Gamma \subset A(D, B)$.*

Proof. The proof is by induction on the number $cr(t)$ of the crossings of t. If $cr(t) = 1$, then it is clear that t has a unique 1-gon Γ in $A(D, B)$. By Lemmas 5.8 and 5.10, Γ is standard and $D_\Gamma \subset A(D, B)$.

Suppose that $cr(t) > 1$. Let $t_0 = t \cap \partial B$ and $t_1 = t \cap \partial D$. Lemma 5.5 implies that $\mathrm{V}_1(t) \neq \emptyset$. We put $m_t = \min\{n_1(c); c \in \mathrm{V}_1(t)\}$ and $p = f_t(m_t) \in \mathrm{V}_1(t)$. Let Γ be the 1-gon in $A(D, B)$ associated with $PW_t(p)$.

If Γ is standard, then Lemma 5.8 implies $D_\Gamma \subset A(D, B)$.

Suppose that Γ is non-standard. We divide t into the three subarcs $a_t[t_0, p], \Gamma$ and $a_t[p, t_1]$. Then, by Lemma 5.8, we have $B \subset D_\Gamma$, $a_t[t_0, p] \cap \Gamma = \{p\}$ and $cr(a_t[t_0, p]) = 0$. Since Γ is non-standard, the open arc $a_t(p, t_1)$ and Γ cross each other. Let q be the first crossing where we meet Γ when traveling $a_t[p, t_1]$ from p to t_1. We divide $s = a_t[p, t_1]$ into the two subarcs $a_s[p, q]$ and $a_s[q, t_1]$. Let u be the subarc of t obtained from the three arcs $a_t[t_0, p]$, Γ and $a_s[p, q]$ by connecting them at p. First, we assume that $cr(u) < cr(t)$. If we choose an appropriate disk E which is a small neighborhood of D_Γ, then by the inductive hypothesis, we find that u has a standard 1- or 2-gon Δ in $A(E, B)$ with $D_\Delta \subset A(E, B)$. Hence, the claim is true since $E \subset D$. Next, we suppose that $cr(u) = cr(t)$. Then, we have the desired result by Lemma 5.9 with regarding D_Γ as the disk D in Lemma 5.9. Note that each 2-gon in cases (2) and (3) in Lemma 5.9 is standard and the 3-gon Ω appearing in case (4) can be regarded as a standard 2-gon in this case because appropriate two sides of Ω can be replaced with one side. The proof is complete. □

A multi-knotoid diagram D in S^2 is called *normal* if D is in $\mathbb{R}^2 = S^2 - \{\infty\}$ and the head of D can be connected to the point ∞ by a path avoiding the rest of D.

Remark 5.12. Any multi-knotoid diagram D can be changed into a normal diagram with the same number of the crossings as D by using a finite sequence of isotopies of S^2. Hence, any multi-knotoid can be represented by a normal diagram.

A normal diagram of a multi-knotoid can be regarded as a 2-sided proper arc with over/under information at each crossing and a link diagram in an annulus.

The following proposition is an immediate consequence of Proposition 5.11.

Proposition 5.13. *Let D be a normal diagram of a multi-knotoid in S^2. If the knotoid component D_0 of D has a self crossing, then there exists a standard 1- or 2-gon on D_0 which bounds a disk in $\mathbb{R}^2$ away from the endpoints of D.*

For a multi-knotoid diagram D in S^2, we denote by $r(D)$ the diagram obtained from D by reversing the orientations of all components of D and call it the reverse image of D. For a crossing c of D, we denote by $r(c)$ the crossing of $r(D)$ corresponding to c. We will give a relationship between the H-polynomials for a multi-knotoid diagram and its reverse image.

Lemma 5.14. *Suppose that a multi-knotoid diagram with less than n crossings, $n > 0$, and its reverse image have the same H-polynomial. If a multi-knotoid diagram D of n crossings is a disjoint union of its knotoid component D_0 with $cr(D_0) = 0$ and the link diagram $D - D_0$, then $H(D) = H(r(D))$.*

Proof. We give D an ordering π. By Lemmas 3.9 and 3.15, with some choice of the basepoint P and the descending sequence $\lambda = (\lambda_m, \dots . \lambda_1)$ for D, we have a reduced diagram J of $D(\lambda)$. Let $r(\lambda)$ be the switching sequence $(r(\lambda_m), \dots, r(\lambda_1))$ for $r(D)$. Note that $\mathrm{sign}(\lambda_j) = \mathrm{sign}(r(\lambda_j))$, $1 \leq j \leq m$. By the recursive formula in Lemma 3.8, we obtain

$$\begin{aligned} H(D) &= H(D(\lambda)) + z \sum_{D} \lambda \\ &= H(D(\lambda)) + z \sum_{j=1}^{m} \mathrm{sign}(\lambda_j) H(A_j^\lambda D) \end{aligned}$$

and

$$\begin{aligned} H(r(D)) &= H(r(D)(r(\lambda))) + z \sum_{r(D)} r(\lambda) \\ &= H(r(D)(r(\lambda))) + z \sum_{j=1}^{m} \mathrm{sign}(r(\lambda_j)) H(A_j^{r(\lambda)} r(D)) \end{aligned}$$

$$= H(r(D)(r(\lambda))) + z\sum_{j=1}^{m} \mathrm{sign}(\lambda_j)H(A_j^{r(\lambda)} r(D))$$

Since $A_j^{r(\lambda)} r(D) = r(A_j^{\lambda} D)$, $1 \leq j \leq m$, the hypothesis of the lemma gives

$$H(A_j^{r(\lambda)} r(D)) = H(r(A_j^{\lambda} D)) = H(A_j^{\lambda} D),\ 1 \leq j \leq m,$$

and thus, $\sum_{D} \lambda = \sum_{r(D)} r(\lambda)$.

Let χ be a finite sequence of 1-sided Reidemeister moves to change $D(\lambda)$ into J. Then, there exists the corresponding sequence $r(\chi)$ of 1-sided Reidemeister moves which yields $r(J)$ from $r(D)(r(\lambda))$ because the two diagrams $D(\lambda)$ and $r(D)(r(\lambda))$ differ only their orientations. The sequence χ has at most one Reidemeiseter move of type I, which comes from the proof of Lemma 3.15. Suppose that such a move does not appear in the sequence χ. Then, the sequence $r(\chi)$ also has no Reidemeister move of type I. Hence, by the hypothesis of the lemma, we obtain $H(D(\lambda)) = H(J) = H(r(J)) = H(r(D)(r(\lambda)))$. We assume that there exists a unique 1-sided Reidemeister move of type I, which eliminates a crossing c of $D(\lambda)$, in the sequence χ. Then, the sequence $r(\chi)$ has the corresponding local move which eliminates the crossing $r(c)$. Since the contribution of a 1-sided Reidemeister move of type I to the H-polynomial depends only on the signature of the crossing eliminated by the local move, we see that $H(D(\lambda)) = w(c)H(J) = w(r(c))H(r(J)) = H(r(D)(r(\lambda)))$, completing the proof. □

It is easy to see the following.

Lemma 5.15. *Let D be a multi-knotoid diagram with options (π, P). Let c denote the first crossing where is encountered when traveling the knotoid component from its leg. If $D[\pi, P]$ is a descending diagram of level 0, then $(T_c D)[\pi, P]$ is also a descending diagram of level 0.*

Lemma 5.16. *Suppose that a multi-knotoid diagram with less than n crossings, $n > 0$, and its reverse image have the same H-polynomial. If a multi-knotoid diagram D of n crossings with options (π, P) is descending of level 0 and its knotoid component has no self crossings, then $H(D) = H(r(D))$.*

Proof. Let $P = (p_0, p_1, \ldots, p_{kt(D)})$ and $Q = (q, p_1, \ldots, p_{kt(D)})$, where q denotes the head of the knotoid component D_0. Note that p_0 denotes the leg of the knotoid component D_0. We choose Q as a basepoint for $r(D)$. If D is a disjoint union of D_0 and the link diagram $D - D_0$, then Lemma 5.14 ensures the claim.

Suppose that $D_0 \cap (D - D_0) \neq \emptyset$. Let $c_1, \ldots, c_{m-1}$ and c_m, $m \geq 1$, be all mixed crossings of D which belong to $D_0 \cap (D - D_0)$. By repeating the tug move at each crossing c_j, $1 \leq j \leq m$, in appropriate order, D can be changed into a disjoint union of D_0 and $D - D_0$. By Lemmas 3.13 and 5.15, we see that

$$H(D) = \left(\prod_{j=1}^{m} w(c_j)\right) H(D_0 \sqcup (D - D_0)).$$

Since the knotoid component D_0 has no self crossings, $r(D)[\pi, Q]$ is descending of level 0. Thus, by Lemmas 3.9, 3.13 and 5.15 and the equalities $w(c_j) = w(r(c_j))$, $1 \leq j \leq m$, we also obtain the following similar result for $r(D)$:

$$H(r(D)) = \left(\prod_{j=1}^{m} w(r(c_j))\right) H(r(D_0) \sqcup (r(D) - r(D_0)))$$
$$= \left(\prod_{j=1}^{m} w(c_j)\right) H(r(D_0 \sqcup (D - D_0))).$$

Since $cr(D_0 \sqcup (D - D_0)) = cr(D) - m < cr(D)$, the hypothesis of the lemma shows $H(r(D_0 \sqcup (D - D_0))) = H(D_0 \sqcup (D - D_0))$. It follows that $H(D) = H(r(D))$. □

Proposition 5.17. *For a multi-knotoid diagram D, $H(D) = H(r(D))$ and thus, $R(D) = R(r(D))$ and $SR(D) = SR(r(D))$.*

Proof. The proof is by induction on the number $cr(D)$ of the crossings of D. We may assume that D is normal by Remark 5.12.

If $cr(D) = 0$, then by the definition of the H-polynomial, $H(D)$ depends only on the number of knot components. It implies that the claim is true.

Suppose that $cr(D) > 0$. First, we consider the case $cr(D_0) = 0$. We choose an ordering π and a basepoint P for D. Let $\lambda = (\lambda_m, \ldots, \lambda_1)$ be the descending sequence for D with respect to (π, P). Then, we have $H(D) = H(D(\lambda)) + z \sum_D \lambda$. For a switching sequence $r(\lambda) = (r(\lambda_m), \ldots, r(\lambda_1))$ for $r(D)$, by the recursive formula in Lemma 3.8, we obtain $H(r(D)) = H(r(D)(r(\lambda))) + z \sum_{r(D)} r(\lambda)$. Since $A_j^{r(\lambda)} r(D) = r(A_j^{\lambda} D)$ and $\mathrm{sign}(r(\lambda_j)) = \mathrm{sign}(\lambda_j)$, $1 \leq j \leq m$, the inductive hypothesis ensures $\sum_D \lambda = \sum_{r(D)} r(\lambda)$. Since $r(D)(r(\lambda)) = r(D(\lambda))$ and $D(\lambda)$ is descending of level 0 with respect to (π, P), by Lemma 5.16, we see that $H(D(\lambda)) = H(r(D)(r(\lambda)))$. It follows that $H(D) = H(r(D))$.

Next, we suppose that $cr(D_0) > 0$. Then, by Proposition 5.13, D_0 has a standard 1- or 2-gon Γ in a disk F away from the endpoints of D. If necessary, we may move disjoint components from D_Γ outside F.

Suppose that Γ is a local curl. Since D can be changed into the diagram E with $cr(E) = cr(D) - 1$ by the 1-sided Reidemeister move of type I which eliminates the vertex c of Γ, we have $H(D) = w(c)H(E)$. Since the Reidemeister move can be applied to $r(D)$, we also have $H(r(D)) = w(r(c))H(r(E))$. The inductive hypothesis shows $H(E) = H(r(E))$. Since $w(c) = w(r(c))$, we obtain $H(D) = H(r(D))$.

We assume that Γ is not a local curl. We give D options (π, P).

First, we suppose that $D[\pi, P]$ is descending of level 0. If Γ is a standard 1-gon, then Γ can be reduced to a small one near the vertex of Γ by Reidemeister moves except the move of type I. Note that each of these Reidemeister moves is not necessarily 1-sided. Then, the resultant diagram E has smaller crossings than D and $H(E) = H(D)$. Since $r(D)$ is the same diagram as D if their orientations are ignored, the same deformation as for D can be applied to $r(D)$. Hence, we have $H(r(D)) = H(r(E))$. Since the inductive hypothesis gives $H(E) = H(r(E))$, we see that $H(D) = H(r(D))$. If Γ is a standard 2-gon, then we apply a 1-sided Reidemeister move of type II, which can be realized by a finite sequence of Reidemeister moves except the move of type I, to Γ. Hence, the H-polynomial for the resultant diagram E corresponds to that of D. Since the same deformation as for D can be applied to $r(D)$, we obtain $H(r(D)) = H(r(E))$. Since the number of the crossings of E is less than that of the crossings of D by two, the inductive hypothesis shows $H(E) = H(r(E))$. It follows that $H(D) = H(r(D))$.

Next, we suppose that $D[\pi, P]$ is not descending of level 0. Let $\lambda = (\lambda_m, \ldots, \lambda_1)$ be a descending sequence of level 0 for D with respect to (π, P), where a descending sequence of level 0 means a switching sequence for D changing D into a descending diagram of level 0. We also denote the switching sequence $(r(\lambda_m), \ldots, r(\lambda_1))$ for $r(D)$ by $r(\lambda)$. Then, by the recursive formula in Lemma 3.8, we have

$$H(D) = H(D(\lambda)) + z \sum_{D} \lambda$$

and

$$H(r(D)) = H(r(D)(r(\lambda))) + z \sum_{r(D)} r(\lambda) = H(r(D(\lambda)) + z \sum_{r(D)} r(\lambda).$$

By the same reason as the case $cr(D_0) = 0$, the equality $\sum_{D} \lambda = \sum_{r(D)} r(\lambda)$ holds. Since $D(\lambda)[\pi, P]$ is descending of level 0, it follows from the previous case that $H(D(\lambda)) = H(r(D(\lambda)))$. Hence, we have $H(D) = H(r(D))$. The proof is completed by the fact that the signature of a crossing c of a diagram D is equal to that of the corresponding crossing $r(c)$ of $r(D)$. □

6. Knotoids with up to 3 crossings

The *crossing number* of a knotoid K denoted by $cr(K)$ is defined to be the minimum number of the crossings of all diagrams of K.

Bartholomew gives a list of distinct knotoids with up to 5 crossings in [1]. The list is produced by a computer search. In this section, a theoretical approach is provided to determine and completely classify knotoids with up to 3 crossings. It is elementary, but explicit.

A knotoid K is called *trivial* if K is the equivalence class of a trivial knotoid diagram.

Let K be the trivial knotoid. Since a canonical trivial knotoid diagram has no crossings, we have $cr(K) \leq 0$. Since $cr(K) \geq 0$ by the definition of the crossing number of a knotoid, we obtain $cr(K) = 0$.

Let K be a knotoid with $cr(K) = 0$. Then, there exists a knotoid diagram D of K with $cr(D) = 0$. Since a knotoid diagram without crossings is equivalent to a segment which is a canonical trivial knotoid diagram, the diagram D is trivial, and thus, K is trivial. Hence, we have the following.

Lemma 6.1. *Let K be a knotoid. Then, K is trivial if and only if $cr(K) = 0$.*

Lemma 6.2. *Let D be a knotoid diagram. If $cr(D) = 1$, then D is trivial.*

Proof. By Remark 5.12, we may assume that D is normal. Since $1 \leq cr(D) < 2$, Proposition 5.13 shows that D has a standard 1-gon. Since $cr(D) = 1$, the 1-gon is a local curl. Applying a 1-sided Reidemeister move of type I to the 1-gon, we have a diagram without crossings. It follows that D is trivial. □

Corollary 6.3. *There is no knotoid with $cr(K) = 1$.*

For an arc t in an annulus, we denote by $\tilde{t}$ the arc t with over/under information at each crossing. Hence, $\tilde{t}$ represents a knotoid diagram.

Lemma 6.4. *Let t be a 2-sided proper arc in an annulus $A(D, B)$. If $cr(t) = 2$ and t has no local curls which bound disks in $A(D, B)$, then $\tilde{t}$ forms either of the two diagrams as in Fig. 8, where p and q denote crossings.*

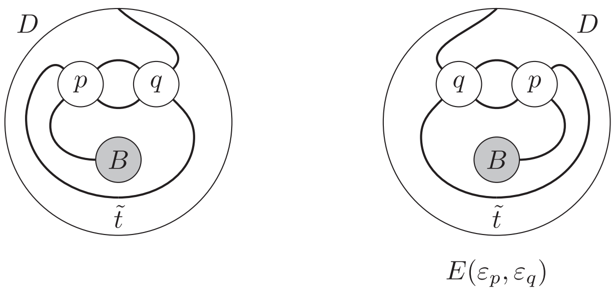

Fig. 8. Knotoid diagrams with 2 crossings

Proof. Let $t_0 = t \cap \partial B$ and $t_1 = t \cap \partial D$. Let $m_t = \min\{n_1(c); c \in \mathrm{V}_1(t)\}$ and Γ the 1-gon in $A(D, B)$ associated with $PW_t(p)$, where $p = f_t(m_t)$. Suppose that Γ is standard. By Lemma 5.8, $D_\Gamma \subset A(D, B)$. Then, by the assumption of the lemma, $\Gamma \cap (t - \Gamma)$ consists of just one point. It follows that $\partial t \cap D_\Gamma \neq \emptyset$, a contradiction. Hence, Γ should be non-standard. We divide t into the three subarcs $a_t[t_0, p]$, Γ and $a_t[p, t_1]$. Since Γ is non-standard and $cr(t) = 2$, $a_t[p, t_1] \cap \Gamma$ consists of just one point, which is denoted by q. Hence, we have the diagrams as in Fig. 8. □

We can regard each $\tilde{t}$ in Fig. 8 as an oriented normal knotoid diagram with the leg on ∂B and the head on ∂D.

Let $E(\varepsilon_p, \varepsilon_q)$ be the right diagram of Fig. 8, where ε_c denotes the signature sign(c) of a crossing c. Then, $E(-\varepsilon_p, -\varepsilon_q) = E(\varepsilon_p, \varepsilon_q)!$ and $\overline{(E(\varepsilon_p, \varepsilon_q)!)} = \overline{E(\varepsilon_p, \varepsilon_q)}!$. We also find that each diagram on the left of Fig. 8 may be considered as the horizontal mirror image of one of the right diagrams.

It is easy to see the following.

Lemma 6.5. *If $E(\varepsilon_p, \varepsilon_q)$ is a diagram of a knotoid K with $cr(K) = 2$, then $\varepsilon_p = \varepsilon_q$.*

Computing the R-polynomials for the four knotoid diagrams $E(1, 1)$, $\overline{E(1, 1)}$, $E(1, 1)!$ and $\overline{E(1, 1)}!$, we obtain the following.

Lemma 6.6.

$$R(E(1, 1); a, h, z) = a^{-2} + (a^{-1} - a^{-3})h,$$
$$R(\overline{E(1, 1)}; a, h, z) = a^2 - (a^3 - a)h^{-1},$$
$$R(E(1, 1)!; a, h, z) = a^2 - (a^3 - a)h + a^2 z^2,$$

$$R(\overline{E(1,1)}!; a, h, z) = a^{-2} + (a^{-1} - a^{-3})h^{-1} + a^{-2}z^2.$$

Proposition 6.7. *There exist just four knotoids with two crossings.*

Proof. Let K be a knotoid with $cr(K) = 2$. Then, K has a normal diagram D with $cr(D) = 2$. By Lemmas 6.4 and 6.5 and the above observation on diagrams in Fig. 8, D is equivalent to one of the four diagrams $E(1,1)$, $\overline{E(1,1)}$, $E(1,1)!$ and $\overline{E(1,1)}!$. Lemma 6.6 implies that the equivalence classes whose representatives are the four diagrams are mutually distinct. This completes the proof. □

We put the four knotoids in Proposition 6.7 on display in Fig. 9. The drawn diagrams, which are not normal, are called *bifoils* [8].

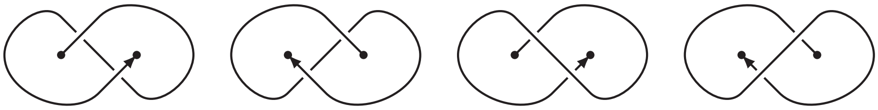

Fig. 9. Knotoids with 2 crossings

A 2-string tangle diagram $T = (B, t)$ is a pair of a disk B and two proper arcs t. Let $T_1 = (B, t_1)$ and $T_2 = (B, t_2)$ be the left and the right tangles in Fig. 10, respectively, where p, q and r denote crossings. Then, T_1 can be changed into T_2 by a finite sequence of Reidemeister moves in B, provided that the boundaries of t_1 are fixed. It implies the following.

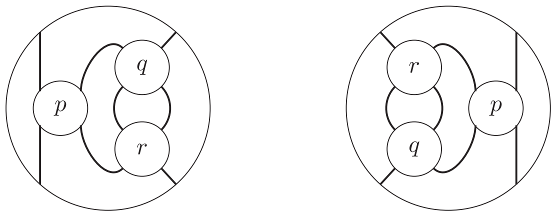

Fig. 10. Equivalent tangles

Lemma 6.8. *Two knotoid diagrams which differ only in one* 2*-string tangle as in* Fig. 10 *are equivalent.*

Lemma 6.9. *Let t be a* 2*-sided proper arc in an annulus $A(D, B)$. If $cr(t) = 3$ and t has no local curls which bound disks in $A(D, B)$, then $\tilde{t}$ forms one of the four diagrams except the right column as in* Fig. 11.

Proof. Let $t_0 = t \cap \partial B$ and $t_1 = t \cap \partial D$. Let $m_t = \min\{n_1(c); c \in \mathrm{V}_1(t)\}$ and Γ the 1-gon in $A(D, B)$ associated with $PW_t(p)$, where $p = f_t(m_t)$. We divide t into the three subarcs $a_t[t_0, p], \Gamma$ and $a_t[p, t_1]$. Then, By Lemma 5.8, we have $a_t[t_0, p] \cap \Gamma = \{p\}$ and $cr(a_t[t_0, p]) = 0$.

First, we suppose that Γ is standard. The assumption of the lemma and the fact $D_\Gamma \subset A(D, B)$ coming from Lemma 5.8 give $\Gamma \cap a_t(p, t_1) \neq \emptyset$. Since Γ is standard, the arc $a_t(p, t_1)$ meets Γ at even points. Since $cr(t) = 3$, $a_t(p, t_1) \cap \Gamma$ consists of two points, which are denoted by q and r, and $a_t[t_0, p] \cap a_t[p, t_1] = \{p\}$. Realizing such conditions, we can obtain the four

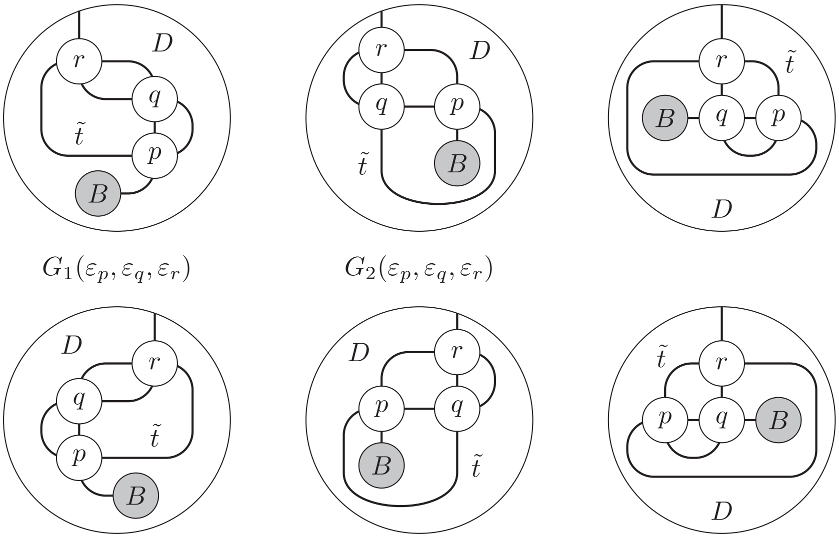

Fig. 11. Knotoid diagrams with 3 crossings

types of knotoid diagrams depicted in two columns from the left of Fig. 11.

Next, we suppose that Γ is non-standard. Since $a_t(p, t_1) \cap D_\Gamma \neq \emptyset$ and $t_1 \notin D_\Gamma$, $a_t(p, t_1)$ meets Γ at odd points. Since $cr(t) = 3$, $a_t(p, t_1) \cap \Gamma$ consists of only one point, which is denoted by q. Assume that $a_t[t_0, p] \cap a_t[p, t_1] = \{p\}$. Then, $cr(a_t[p, t_1]) = 1$ since $cr(a_t[t_0, p]) = cr(\Gamma) = 0$. We divide $s = a_t[p, t_1]$ into the two subarcs $a_s[p, q]$ and $a_s[q, t_1]$. Since $a_s[p, q] \subset D_\Gamma$ and $a_s[q, t_1] \cap D_\Gamma = \{q\}$, we see that $a_s[p, q] \cap a_s[q, t_1] = \{q\}$. It follows that $cr(a_s[p, q]) = 1$ or $cr(a_s[q, t_1]) = 1$. By Lemma 5.7 and Proposition 5.11, we have a local curl which bounds a disk in $A(D, B)$. It is a contradiction. Hence, $a_t(t_0, p)$ and $a_t(p, t_1)$ cross only once. Drawing a diagram with these conditions, we have the two types of knotoid diagrams described in the right column of Fig. 11. Since these diagrams are equivalent to the two types of diagrams in the middle column of Fig. 11 by Lemma 6.8, we have the result. □

We can regard each $\tilde{t}$ in Fig. 11 as an oriented normal knotoid diagram with the leg on ∂B and the head on ∂D.

Let $G_1(\varepsilon_p, \varepsilon_q, \varepsilon_r)$ and $G_2(\varepsilon_p, \varepsilon_q, \varepsilon_r)$ be the two diagrams from the left in the top row of Fig. 11, respectively. Then, the two diagrams from the left in the bottom row of Fig. 11 may be considered as the horizontal mirror images of the two diagrams from the left in the top row, respectively.

It is easy to see the following lemmas.

Lemma 6.10. *If $G_1(\varepsilon_p, \varepsilon_q, \varepsilon_r)$ is a diagram of a knotoid K with $cr(K) = 3$, then $\varepsilon_p, \varepsilon_q$ and ε_r have the same value.*

Lemma 6.11. *$\overline{G_1(1, 1, 1)}$ and $\overline{G_1(-1, -1, -1)}$ are equivalent to $G_1(-1, -1, -1)$ and $G_1(1, 1, 1)$, respectively.*

Lemma 6.12. *If $G_2(\varepsilon_p, \varepsilon_q, \varepsilon_r)$ is a diagram of a knotoid K with $cr(K) = 3$, then $(\varepsilon_p, \varepsilon_q, \varepsilon_r) = (1, -1, -1)$ or $(-1, 1, 1)$.*

Proof. If $\varepsilon_q \neq \varepsilon_r$, then we have a reduced diagram. It follows that $\varepsilon_q = \varepsilon_r$. Suppose that $\varepsilon_p = \varepsilon_q$. Then, we can obtain a reduced diagram by applying Reidemeister moves of type I and of type III. Hence, $\varepsilon_p \neq \varepsilon_q$. □

REMARK 6.13. $G_1(-1,-1,-1) = G_1(1,1,1)!$ and $G_2(-1,1,1) = G_2(1,-1,-1)!$.

Computing the R-polynomials for the six knotoid diagrams $G_1(1,1,1)$, $G_1(1,1,1)!$, $G_2(1,-1,-1)$, $\overline{G_2(1,-1,-1)}$, $G_2(1,-1,-1)!$ and $\overline{G_2(1,-1,-1)}!$, we obtain the following.

Lemma 6.14.

$$\begin{aligned}
R(G_1(1,1,1);a,h,z) &= (2a^{-2} - a^{-4}) + a^{-2}z^2,\\
R(G_1(1,1,1)!;a,h,z) &= (2a^2 - a^4) + a^2z^2,\\
R(G_2(1,-1,-1);a,h,z) &= a^2 - (a - a^{-1})h^{-1} - z^2,\\
R(\overline{G_2(1,-1,-1)};a,h,z) &= a^{-2} + (a - a^{-1})h - z^2,\\
R(G_2(1,-1,-1)!;a,h,z) &= a^{-2} + (a - a^{-1})h^{-1},\\
R(\overline{G_2(1,-1,-1)}!;a,h,z) &= a^2 - (a - a^{-1})h.
\end{aligned}$$

Proposition 6.15. *There exist just six knotoids with three crossings.*

Proof. Let K be a knotoid with $cr(K) = 3$. Then, K has a normal diagram D with $cr(D) = 3$. By Lemmas 6.9 – 6.12 and Remark 6.13, D is equivalent to one of the six diagrams which are $G_1(1,1,1)$, $G_1(1,1,1)!$, $G_2(1,-1,-1)$, $\overline{G_2(1,-1,-1)}$, $G_2(1,-1,-1)!$ and $\overline{G_2(1,-1,-1)}!$. Lemma 6.14 reveals that the equivalence classes whose representatives are the six diagrams are mutually distinct. This completes the proof. □

At the end of the paper, we deal with inverse of a knotoid in brief. A knotoid K is said to be *invertible* if K is equivalent to $r(K)$, that is, for a digram D of K, D is equivalent to $r(D)$. It is clear that the trivial knotoid is invertible.

Proposition 6.16. *A knotoid K with $cr(K) = 2$ is invertible.*

Proof. For each diagram D as in Fig. 9, its reverse image $r(D)$ can be obtained by rotating D through angle π around an axis perpendicular to the projection plane. It shows that D is equivalent to $r(D)$, completing the proof. □

Proposition 6.17. *A knotoid K with $cr(K) = 3$ is invertible.*

Proof. Since $cr(K) = 3$, there exists a diagram D of K with $cr(D) = 3$. Then, $cr(r(D)) = 3$ and thus, $cr(r(K)) \leq 3$. If $cr(r(K)) < 3$, then there exists a diagram E of $r(K)$ with $cr(E) < 3$. Since E is equivalent to $r(D)$, $r(E)$ is equivalent to $r(r(D)) = D$. It follows that $r(E)$ is a diagram of K with less than three crossings. This contradicts the assumption $cr(K) = 3$. Hence, $cr(r(K)) = 3$. Since $R(D) = R(r(D))$ by Proposition 5.17, Lemma 6.14 gives $K = [D] = [r(D)] = r(K)$ because the six polynomials given by the lemma are different one another, where $[D]$ denotes the equivalence class of D. Hence, K is invertible. □

In fact, $G_2(1,-1,-1)$ can be changed into $r(G_2(1,-1,-1))$ as in Fig. 12.

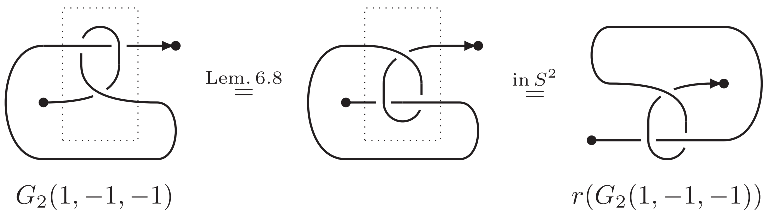

Fig. 12. $G_2(1,-1,-1)$ to $r(G_2(1,-1,-1))$

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