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MONODROMIES OF SPLITTING FAMILIES FOR DEGENERATIONS OF RIEMANN SURFACES

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Abstract

When we study degenerations of Riemann surfaces from a topological viewpoint, the topological monodromies play a very important role. In this paper, as an analogy, we introduce the concept of "topological monodromies of splitting families" for degenerations of Riemann surfaces, and their "monodromy assortments". We show that the monodromy assortments of barking families associated with tame simple crusts act as a pseudo-periodic homeomorphism of negative twist on each irreducible component of the main fibers. As an application of our results, we show an interesting example of two splitting families for one degeneration that have different topological monodromies, although they give the same splitting.

1. Introduction

A *degeneration of Riemann surfaces* is a family of complex curves over an open disk in $\mathbb{C}$ such that the central fiber is singular and the other fibers are all smooth complex curves. When we classify degenerations of Riemann surfaces from a topological viewpoint, the topological monodromies play a very important role. It has been proved that the topological monodromy of a degeneration is always represented by a pseudo-periodic homeomorphism of negative twist. See [1], [4] and [10]. Matsumoto and Montesinos [7] showed the converse of this result. Namely, given a pseudo-periodic homeomorphism f of negative twist, they constructed a degeneration with singular fiber whose monodromy homeomorphism coincides with f up to conjugacy.

We are interested in "splittings of singular fibers of degenerations". For a degeneration of Riemann surfaces, we say that its singular fiber splits into several singular fibers if there exists a complex 1-parameter family of families of complex curves such that the family of complex curves over the origin coincides with the given degeneration and the other families have at least two singular fibers. Such a complex 1-parameter family of families of complex curves is called a *splitting family* for the degeneration of Riemann surfaces. In this paper, as an analogy of topological monodromies of degenerations of Riemann surfaces, we introduce the concept of "topological monodromies of splitting families" for degenerations of Riemann surfaces, and their "monodromy assortments" — the restrictions to the singular fibers. See Section 3 for the precise definitions.

In particular, this paper deals with the case of *barking families*. A barking family is a splitting family obtained by barking deformation method, which was introduced by Takamura in [11]. If the singular fiber of the given degeneration has a subdivisor satisfying certain condi-

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tions, then we have an associated barking family. Such subdivisors are called *simple crusts*. In a barking family, the original singular fiber X_0 is deformed to a simpler one in such a way that the simple crust looks "barked" off from X_0. We will review his theory in Sections 4 and 5. The main theorem of this paper is that the monodromy assortments of barking families associated with "tame" simple crusts act as a pseudo-periodic homeomorphism of negative twist on each irreducible component of the main singular fibers. See Theorem 6.2 for the more precise statement. Sections 7–9 are devoted to the proof of our results.

In Section 11, as an application of our results, we show an interesting example of two splitting families for one degeneration that have different topological monodromies, although they give the same splitting (that is, the types of the singular fibers appearing in respective splitting families coincide). This example indicates that the topological monodromy for splitting families plays a very important role when we classify the "topologically distinct" splitting families. Our theory on monodromies for more general splitting families will be developed in [9].

2. Preliminaries

Let $\pi : M \to \Delta$ be a family of complex curves of genus $g \geq 1$ over an open disk Δ in $\mathbb{C}$ centered at the origin, that is, a proper surjective holomorphic map from a smooth complex surface M to Δ such that all but finitely many fibers are connected smooth complex curves of genus g. We call such a π a *degeneration of Riemann surfaces* of genus g if the fiber $X_0 := \pi^{-1}(0)$ over the origin is singular and the other fibers $X_s := \pi^{-1}(s)$, $s \neq 0$, are all smooth.

Two degenerations $\pi_i : M_i \to \Delta$ $(i = 1, 2)$ are *topologically equivalent* if there exist two orientation preserving homeomorphisms $H : M_1 \to M_2$ and $h : \Delta \to \Delta$ such that $h(0) = 0$ and the following diagram commutes:

$$\begin{array}{ccc} M_1 & \xrightarrow{H} & M_2 \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \pi_2} \\ \Delta & \xrightarrow{h} & \Delta. \end{array}$$

On the topological classification of degenerations, the following is known.

Theorem 2.1 (Matsumoto-Montesinos [7])**.** *The topological equivalence classes of minimal degenerations of Riemann surfaces of genus $g \geq 2$ are in bijective correspondence with the conjugacy classes in MCG_g represented by pseudo-periodic homeomorphisms of negative type, via topological monodromy, where MCG_g denotes the mapping class group of an oriented closed real surface of genus g.*

For a given degeneration $\pi : M \to \Delta$, we define a splitting family as follows. Let $\mathcal{M}$ be a complex 3-dimensional manifold and set $\Delta^\dagger := \{t \in \mathbb{C} : |t| < \varepsilon\}$, an open disk with sufficiently small radius $\varepsilon > 0$. Consider a proper flat surjective holomorphic map $\Psi : \mathcal{M} \to \Delta \times \Delta^\dagger$ such that the composition $\mathrm{pr}_2 \circ \Psi : \mathcal{M} \to \Delta^\dagger$ with the second projection $\mathrm{pr}_2 : \Delta \times \Delta^\dagger \to \Delta^\dagger$ is a submersion. For each $t \in \Delta^\dagger$, set $\Delta_t := \Delta \times \{t\}$, $M_t := \Psi^{-1}(\Delta_t)$ and $\pi_t := \Psi\big|_{M_t} : M_t \to \Delta_t$. Note that M_t is a smooth complex surface, and $\pi_t : M_t \to \Delta_t$ is a family of complex curves over Δ_t. Suppose that $\pi_0 : M_0 \to \Delta_0$ coincides with $\pi : M \to \Delta$. Then we call $\Psi : \mathcal{M} \to \Delta \times \Delta^\dagger$ a *deformation family* for the degeneration $\pi : M \to \Delta$ and

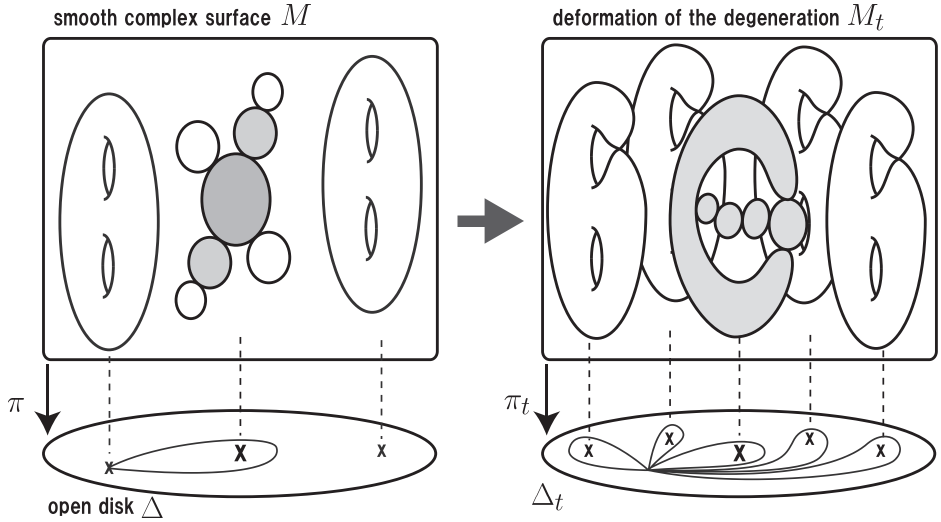

Fig. 1. The singular fiber of a degeneration of Riemann surfaces of genus two splits into some singular fibers.

each $\pi_t : M_t \to \Delta_t$, $t \in \Delta^\dagger \setminus \{0\}$, a *deformation* of $\pi : M \to \Delta$.

Let Sing Ψ be the set of singular points of Ψ, and set $\mathcal{D} := \Psi(\mathrm{Sing}\,\Psi)$, the singular value locus of Ψ, which is also called the *discriminant* of Ψ. From the assumption that $\Delta^\dagger$ is sufficiently small, it follows that $\mathcal{D}$ is a plane curve in $\Delta \times \Delta^\dagger$ with at most one singularity at $(0, 0)$.

In particular, $\Psi : \mathcal{M} \to \Delta \times \Delta^\dagger$ is called a *splitting family* if for some integer $N \geq 2$, every deformation $\pi_t : M_t \to \Delta_t$, $t \neq 0$, of the degeneration $\pi : M \to \Delta$ is a family of complex curves with N singular fibers. Set $X_{s,t} := \Psi^{-1}(s, t)\,(= \pi_t^{-1}(s))$ for each $(s, t) \in \Delta \times \Delta^\dagger$. For the deformation $\pi_t : M_t \to \Delta_t$ for a fixed $t \in \Delta^\dagger \setminus \{0\}$, denote the singular values of π_t by $s_1, s_2, \ldots, s_N$. Then we say that the singular fiber X_0 *splits* into the singular fibers $X_{s_1,t}, X_{s_2,t}, \ldots, X_{s_N,t}$, and we write

$$X_0 \longrightarrow X_{s_1,t} + X_{s_2,t} + \cdots + X_{s_N,t}.$$

See Figure 1. Here the singular values $s_1, s_2, \ldots, s_N$ themselves depend on t, while the topological types of the singular fibers $X_{s_1,t}, X_{s_2,t}, \ldots, X_{s_N,t}$ over them do not. We can see this from the following lemma.

Lemma 2.2. *Take any t, $t' \in \Delta^\dagger \setminus \{0\}$ and let $s_1, s_2, \ldots, s_N$ (resp. $s'_1, s'_2, \ldots, s'_N$) denote the singular values of the deformation $\pi_t : M_t \to \Delta_t$ (resp. $\pi_{t'} : M_{t'} \to \Delta_{t'}$). Then there exists a permutation $\sigma \in \mathfrak{S}_N$ such that for each $k = 1, 2, \ldots, N$, the singular fiber $X_{s_k,t}$ of π_t is topologically equivalent with the singular fiber $X_{s'_{\sigma(k)},t'}$ of $\pi_{t'}$.*

Proof. We will show that there exists a permutation $\sigma \in \mathfrak{S}_N$ such that for each $k = 1, 2, \ldots, N$, the topological monodromies around the singular fibers $X_{s_k,t}$ of π_t and $X_{s'_{\sigma(k)},t'}$ of $\pi_{t'}$ are conjugate. Take an oriented path $I \in \Delta^\dagger \setminus \{0\}$ from t to t'. Set $L := \mathcal{D} \cap (\Delta \times I)$. Then L forms a braid in the solid annulus $\Delta \times I$ of N strands joining $\{s_1, s_2, \ldots, s_N \in \Delta_t\}$ and $\{s'_1, s'_2, \ldots, s'_N \in \Delta_{t'}\}$. This naturally induces a permutation $\sigma \in \mathfrak{S}_N$ satisfying that s_k is joined with $s'_{\sigma(k)}$ by a strand.

Fix $k = 1, 2, \ldots, N$. For $s_k \in \Delta_t$, take a sufficiently small oriented loop γ in Δ_t going once around s_k in counterclockwise direction, and a base point b. For $s_{\sigma(k)} \in \Delta_{t'}$ take a sufficiently small oriented loop γ' in $\Delta_{t'}$ going around $s_{\sigma(k)}$ in the same way, and a base point b'. Moreover take an oriented path l in $(\Delta \times I) \setminus L$ in such a way that it starts at b, goes along the strand joining s_k and $s'_{\sigma(k)}$ and reaches $s'_{\sigma(k)}$. Then γ and $l \cdot \gamma' \cdot l^{-1}$ are loops in $(\Delta \times I) \setminus L$, so in $(\Delta \times \Delta^\dagger) \setminus \mathcal{D}$, and they have the same base point b.

Recall here that, for a smooth fibration whose general fibers are compact, the topological monodromy (that is, the isotopy class of monodromy diffeomorphisms) does not depend on the choice of a representative of a homotopy class of a loop in the base space.

Now the restriction $\Psi|_{(\Delta\times\Delta^\dagger)\setminus\mathcal{D}} : \Psi^{-1}((\Delta \times \Delta^\dagger) \setminus \mathcal{D}) \to (\Delta \times \Delta^\dagger) \setminus \mathcal{D}$ of the splitting family to the nonsingular part $(\Delta \times \Delta^\dagger) \setminus \mathcal{D}$ is a smooth fibration. Then we can define the topological monodromies $[F_\gamma]$ along γ and $[F_{l\cdot\gamma'\cdot l^{-1}}]$ along $l \cdot \gamma' \cdot l^{-1}$, which are isotopy classes of monodromy diffeomorphisms of a smooth fiber $X_{b,t}$. Since γ and $l \cdot \gamma' \cdot l^{-1}$ are homotopic in $(\Delta \times \Delta^\dagger) \setminus \mathcal{D}$, we have $[F_\gamma] = [F_{l\cdot\gamma'\cdot l^{-1}}]$. On the other hand, the topological monodromy $[F_{\gamma'}]$ along γ' is conjugate with $[F_{l\cdot\gamma'\cdot l^{-1}}]$ via parallel transformation along l. Thus the topological monodromies around $X_{s_k,t}$ and $X_{s'_{\sigma(k)},t'}$ are conjugate, and therefore $X_{s_k,t}$ and $X_{s'_{\sigma(k)},t'}$ are, by Theorem 2.1, topologically equivalent. □

3. Monodromies of splitting families

In this section, we introduce the new concept of "topological monodromies of splitting families."

Let $\pi : M \to \Delta$ be a degeneration of Riemann surfaces of genus $g \geq 1$ and $\Psi : \mathcal{M} \to \Delta \times \Delta^\dagger$ be a splitting family for the degeneration $\pi : M \to \Delta$. Recall that the discriminant $\mathcal{D}$ of Ψ is a plane curve in $\Delta \times \Delta^\dagger$ with at most one singularity at $(0, 0)$. Suppose that each deformation $\pi_t : M_t \to \Delta_t$ of the degeneration $\pi : M \to \Delta$ has N singular fibers, then the natural projection $\mathrm{pr}_2 : \mathcal{D} \setminus \{(0, 0)\} \to \Delta^\dagger \setminus \{0\}$ is an N-fold covering map.

We first take a point $t_0 \in \Delta^\dagger \setminus \{0\}$, which will be fixed. Note that $\pi_{t_0} : M_{t_0} \to \Delta_{t_0}$ is a family of complex curves with at least two singular fibers. Let D_{t_0} denote its singular value locus, that is, $D_{t_0} = \mathcal{D} \cap \Delta_{t_0}$. Before proceeding, we define the mapping class group of $\pi_{t_0} : M_{t_0} \to \Delta_{t_0}$ as follows. Let $F : M_{t_0} \to M_{t_0}$ and $f : \Delta_{t_0} \to \Delta_{t_0}$ be orientation preserving homeomorphisms that make the diagram

$$\begin{array}{ccc} M_{t_0} & \xrightarrow{F} & M_{t_0} \\ {\scriptstyle \pi_{t_0}}\downarrow & & \downarrow{\scriptstyle \pi_{t_0}} \\ \Delta_{t_0} & \xrightarrow{f} & \Delta_{t_0} \end{array}$$

commutative. Clearly $f(D_{t_0}) = D_{t_0}$. Then we call the pair (F, f) a *topological automorphism* of $\pi_{t_0} : M_{t_0} \to \Delta_{t_0}$. Denote by $\mathcal{H}$ the group of topological automorphisms of π_{t_0}, that is,

$$\mathcal{H} := \{(F, f) \in \mathrm{Homeo}^+(M_{t_0}) \times \mathrm{Homeo}^+(\Delta_{t_0}) \ : \ f \circ \pi_{t_0} = \pi_{t_0} \circ F\},$$

where $\mathrm{Homeo}^+(M_{t_0})$ (resp. $\mathrm{Homeo}^+(\Delta_{t_0})$) is the group of orientation preserving homeomorphisms of M_{t_0} (resp. Δ_{t_0}). The group $\mathcal{H}$ naturally has the structure of a topological group with respect to the compact open topology. Now we define the (*fiber preserving*) *mapping* *class group* $\mathrm{MCG}(\pi_{t_0})$ of π_{t_0} as the group

$$\mathrm{MCG}(\pi_{t_0}) := \pi_0(\mathcal{H}).$$

In other words, $\mathrm{MCG}(\pi_{t_0})$ is the group of isotopy classes of topological automorphisms in $\mathcal{H}$.

Take a smooth simple closed curve γ in $\Delta^\dagger \setminus \{0\}$ with base point t_0 that goes once around the origin in the counterclockwise direction. Then $\Delta \times \gamma$ is an open solid torus. Setting $L := \mathcal{D} \cap (\Delta \times \gamma)$, we see that L is a closed braid in the open solid torus $\Delta \times \gamma$. In fact, the natural projection $\mathrm{pr}_2 : L \to \gamma$ is an unramified N-fold covering map.

Note that $\Psi^{-1}(\Delta \times \gamma)$ $(= (\mathrm{pr}_2 \circ \Psi)^{-1}(\gamma))$ is a smooth real 5-dimensional manifold. Now we consider the diagram

$$\Psi^{-1}(\Delta \times \gamma) \xrightarrow{\Psi} \Delta \times \gamma \xrightarrow{\mathrm{pr}_2} \gamma$$

of smooth real manifolds. For the closed braid $L \subset \Delta \times \gamma$, we set $W := \Psi^{-1}(L)$, which is nothing but the union of all singular fibers over $\Delta \times \gamma$. We see that there exists a stratification $(\mathcal{S}, \mathcal{S}')$ for the smooth map $\Psi : \Psi^{-1}(\Delta \times \gamma) \to \Delta \times \gamma$ such that (i) $\mathrm{pr}_2 : \Delta \times \gamma \to \gamma$ maps each stratum of $\mathcal{S}'$ into γ submersively, and that (ii) for any strata $V \in \mathcal{S}$ and $K \in \mathcal{S}'$, the restrictions $\Psi : \overline{V} \cap \Psi^{-1}(\Delta \times \gamma) \to \Delta \times \gamma$ and $\mathrm{pr}_2 : \overline{K} \cap (\Delta \times \gamma) \to \gamma$ are proper. Here we can take such a stratification as follows: $\mathcal{S}$ consists of

- the connected components of the locus of the singular points of (the reduced scheme of) the singular fibers,
- the connected components of the locus of the smooth part of the singular fibers, and
- the union W of all the smooth fibers,

while $\mathcal{S}'$ consists of the connected components of L and the complement $(\Delta \times \gamma) \setminus L$. Then the stratified map $\Psi : \Psi^{-1}(\Delta \times \gamma) \to \Delta \times \gamma$ is topologically locally trivial over γ. By pasting these trivializations along the simple closed curve γ, we obtain two orientation preserving homeomorphisms $F : M_{t_0} \to M_{t_0}$ and $f : \Delta_{t_0} \to \Delta_{t_0}$ such that F (resp. f) maps $M_{t_0} \cap W$ (resp. D_{t_0}) to itself homeomorphically and that the following diagram is commutative:

$$\begin{array}{ccc} M_{t_0} & \xrightarrow{F} & M_{t_0} \\ \pi_{t_0}\downarrow & & \downarrow\pi_{t_0} \\ \Delta_{t_0} & \xrightarrow{f} & \Delta_{t_0}. \end{array}$$

Thus the pair (F, f) is a topological automorphism of $\pi_{t_0} : M_{t_0} \to \Delta_{t_0}$ and it is uniquely determined up to isotopy. We call the isotopy class $[F, f]$ in $\mathrm{MCG}(\pi_{t_0})$ represented by (F, f) the *topological monodromy* of $\Psi : \mathcal{M} \to \Delta \times \Delta^\dagger$. See Figure 2.

Let us consider the restriction of F to the singular fibers. Since $f(D_{t_0}) = D_{t_0}$, a singular fiber is mapped to some singular fiber (possibly to itself). Recall that $L = \mathcal{D} \cap (\Delta \times \gamma)$ is a closed braid in the open solid torus $\Delta \times \gamma$. Let $K_1, K_2, \ldots, K_c$ be the connected components of L, where c is a positive integer. For each i $(i = 1, 2, \ldots, c)$, the intersection $\Delta_{t_0} \cap K_i$ is contained in D_{t_0}, and consequently it consists of singular values of π_{t_0}, say,

$$\Delta_{t_0} \cap K_i = \left\{s_1^{(i)}, s_2^{(i)}, \ldots, s_{l_i}^{(i)}\right\} \subset D_{t_0}.$$

Then f cyclically permutes $s_1^{(i)}, s_2^{(i)}, \ldots, s_{l_i}^{(i)}$, while F cyclically permutes the corresponding

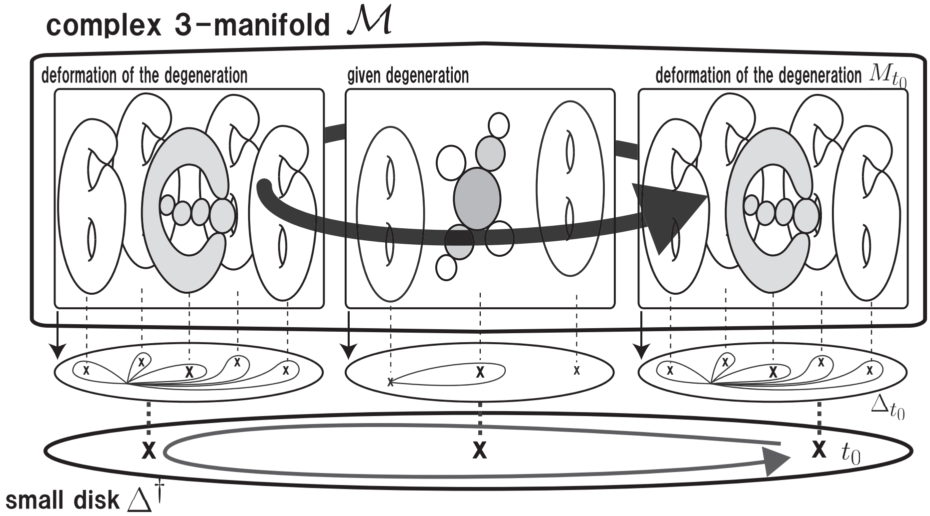

Fig. 2. The topological monodromy of a splitting family of a degeneration of Riemann surfaces.

singular fibers $X_{s_1^{(i)},t_0}, X_{s_2^{(i)},t_0}, \ldots, X_{s_{l_i}^{(i)},t_0}$. Let $\mathbb{X}_i$ denote the disjoint union of the singular fibers over $\Delta_{t_0} \cap K_i$:

$$\mathbb{X}_i := X_{s_1^{(i)},t_0} \sqcup X_{s_2^{(i)},t_0} \sqcup \cdots \sqcup X_{s_{l_i}^{(i)},t_0}.$$

We call the union $\mathbb{X}_i$ the *tassel*[1] over K_i. Consider the restriction

$$F_i := F\Big|_{\mathbb{X}_i} : \mathbb{X}_i \to \mathbb{X}_i$$

of F to the tassel $\mathbb{X}_i$. Clearly for each $j = 1, 2, \ldots, l_i$, we have $F_i^{l_i}(X_{s_j^{(i)},t_0}) = X_{s_j^{(i)},t_0}$. We call the $(c+1)$-tuple $(F_1, F_2, \ldots, F_c; f)$ a *monodromy assortment* of the splitting family $\Psi : \mathcal{M} \to \Delta \times \Delta^\dagger$. A monodromy assortment is uniquely determined up to isotopy.

4. Linear degenerations

This section reviews the concept of a linear degeneration, which is a representative of an equivalence class of degenerations. We mainly follow the terminology given in Sections 15, 18, 19 of [11].

Let $\pi : M \to \Delta$ be a degeneration of complex curves of genus $g \geq 1$ with singular fiber $X_0 = \sum_i m_i\Theta_i$, where each Θ_i is an irreducible component of X_0 with multiplicity m_i. Denote by X_0^{red} the *underlying reduced curve* of X_0, that is, $X_0^{\mathrm{red}} := \sum_i \Theta_i$. We say that the singular fiber X_0 (or more precisely, its underlying reduced curve X_0^{red}) has at most *simple normal crossings* if (i) every singularity of X_0^{red} is a node and (ii) none of the irreducible components Θ_i intersects itself (and therefore, each Θ_i is smooth). It is known that an arbitrary degeneration of Riemann surfaces, by successive blowing-ups, can be arranged so that its singular fiber has at most simple normal crossings.

In what follows, we assume that *the singular fibers of any given degenerations have at*

[1]This terminology is introduced only for barking families by Takamura [11].

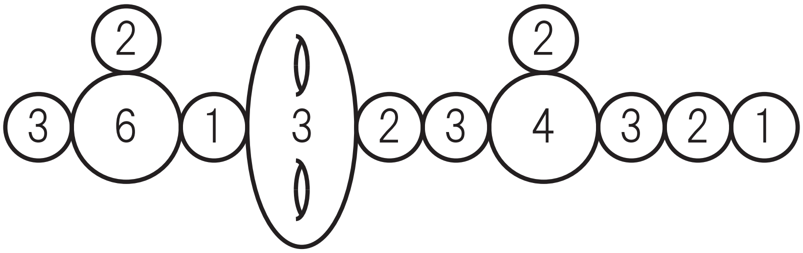

Fig. 3. This singular fiber has three cores, four branches and two trunks. The numbers stand for the multiplicity of each irreducible component and each intersection point is a node.

most simple normal crossings.

Let $\Theta_1, \Theta_2, \ldots, \Theta_\lambda$ ($\lambda \geq 1$) be Riemann spheres contained in X_0 as irreducible components that satisfy the following conditions.

- Θ_i intersects Θ_{i-1} and Θ_{i+1} at exactly one point, respectively, and does not intersect other irreducible components of X_0, $i = 2, 3, \ldots, \lambda - 1$.
- Θ_1 intersects at most one irreducible component other than Θ_2, and if it exists, say Θ_0, then Θ_1 intersects Θ_0 at exactly one point.
- Θ_λ intersects at most one irreducible component other than $\Theta_{\lambda-1}$, and if it exists, say $\Theta_{\lambda+1}$, then Θ_λ intersects $\Theta_{\lambda+1}$ at exactly one point.

Let m_i is the multiplicity of Θ_i in X_0 ($i = 1, 2, \ldots, \lambda$). Then the divisor

$$\mathbf{Ch} := m_1\Theta_1 + m_2\Theta_2 + \cdots + m_\lambda\Theta_\lambda$$

is called a *chain of Riemann spheres*. In what follows, we assume that, when we express a chain of Riemann spheres in this form, the Riemann spheres are arranged in this order. If Θ_1 (resp. Θ_λ) intersects an irreducible component Θ_0 (resp. $\Theta_{\lambda+1}$), then let m_0 (resp. $m_{\lambda+1}$) denote the multiplicity of Θ_0 (resp. $\Theta_{\lambda+1}$), and otherwise set $m_0 := 0$ (resp. $m_{\lambda+1} := 0$). Then we call the sequence of nonnegative integers $m_0, m_1, \ldots, m_{\lambda+1}$ the *multiplicity sequence* associated with the chain **Ch**.

An irreducible component of the singular fiber X_0 is called a *core* if it intersects the other irreducible components at at least three points or its genus is positive. A *branch* is a chain of Riemann spheres attached with a core on one end, while a *trunk* is a chain of Riemann spheres attached with cores on both ends. The singular fiber X_0 consists of cores, branches and trunks. See Figure 3. We say that X_0 is a *stellar* singular fiber if X_0 consists of exactly one core and some branches emanating from the core. Otherwise X_0 is said to be *constellar*. It is known that a constellar singular fiber is obtained from stellar singular fibers by “Matsumoto-Montesinos bonding” — Matsumoto-Montesinos bonding yields a trunk from two branches.

For an irreducible component Θ_i of X_0, we denote by N_i the normal bundle of Θ_i in M. Let $\{p_1, p_2, \ldots, p_h\}$ be the set of the intersection points on Θ_i with the other irreducible components of X_0 and m_j be the multiplicity of the irreducible component intersecting Θ_i at p_j ($j = 1, 2, \ldots, h$). Note that

$$r_i := \frac{\sum_{j=1}^{h} m_j}{m_i}$$

is a positive integer. In fact, the self-intersection number of Θ_i in M is equal to $-r_i$, which follows from the adjunction formula. Then there exists a holomorphic section σ_i of the line bundle $N_i^{\otimes(-m_i)}$ over Θ_i such that

$$\mathrm{div}(\sigma_i) = \sum_{j=1}^{h} m_j p_j,$$

where $\mathrm{div}(\sigma_i)$ denotes the divisor defined by σ_i. Here σ_i has a zero of order m_j at p_j. Note that σ_i is uniquely determined up to multiplication by a constant. We call σ_i the *standard section* of the line bundle $N_i^{\otimes(-m_i)}$ over Θ_i.

For each i, take an open covering $\Theta_i = \bigcup_\alpha U_\alpha$ such that $U_\alpha \times \mathbb{C}$ is a local trivialization of the normal bundle N_i. We denote by (z_α, ζ_α) coordinates of $U_\alpha \times \mathbb{C}$. Now define the holomorphic functions $\pi_{i,\alpha} : U_\alpha \times \mathbb{C} \to \mathbb{C}$ by

$$\pi_{i,\alpha}(z_\alpha, \zeta_\alpha) := \sigma_{i,\alpha}(z_\alpha)\zeta_\alpha^{m_i},$$

where $\sigma_{i,\alpha}$ is the local expression of σ_i on U_α. Then we see that the set $\{\pi_{i,\alpha}\}_\alpha$ of holomorphic functions defines a global holomorphic function $\pi_i : N_i \to \mathbb{C}$.

Definition 4.1. A degeneration $\pi : M \to \Delta$ is said to be *linear* if for every irreducible component Θ_i of its singular fiber X_0,

(i) a tubular neighborhood $N(\Theta_i)$ of Θ_i in M is biholomorphic to a tubular neighborhood of the zero section of the normal bundle N_i, and

(ii) under the identification by the biholomorphic map of (i), the following conditions are satisfied:

(a) the restriction $\pi\big|_{N(\Theta_i)}$ coincides with the holomorphic function π_i defined above, and

(b) if Θ_i intersects Θ_j at a point p, $j \neq i$, then there exist local trivializations $U_\alpha \times \mathbb{C}$ of N_i and $U_\beta \times \mathbb{C}$ of N_j around p such that neighborhoods of p in $N(\Theta_i)$ and in $N(\Theta_j)$ are identified by *plumbing*, $(z_\alpha, \zeta_\alpha) = (\zeta_\beta, z_\beta)$, and π is locally expressed as

$$\pi\big|_{N(\Theta_i)}(z_\alpha, \zeta_\alpha) = z_\alpha^{m_j}\zeta_\alpha^{m_i}, \quad \pi\big|_{N(\Theta_j)}(z_\beta, \zeta_\beta) = z_\beta^{m_i}\zeta_\beta^{m_j},$$

where $(z_\alpha, \zeta_\alpha) \in U_\alpha \times \mathbb{C}$ and $(z_\beta, \zeta_\beta) \in U_\beta \times \mathbb{C}$.

In a linear degeneration, tubular neighborhoods of the branches and the trunks can be constructed explicitly:

Lemma 4.2. *Let* $m_0, m_1, \ldots, m_{\lambda+1}$ $(\lambda \geq 1)$ *be nonnegative integers such that*

- $m_0, m_1, \ldots, m_\lambda$ *are positive integers, and*
- $r_i := \dfrac{m_{i-1} + m_{i+1}}{m_i}$ *is a positive integer* $(i = 1, 2, \ldots, \lambda)$.

Then there exist a smooth complex surface T *and a linear degeneration* $\pi : T \to \Delta$ *with the singular fiber*

$$X_0 = m_0V_0 + m_1\Theta_1 + m_2\Theta_2 + \cdots + m_\lambda\Theta_\lambda + m_{\lambda+1}U_{\lambda+1},$$

where V_0 and $U_{\lambda+1}$ are copies of $\mathbb{C}$, $m_1\Theta_1+m_2\Theta_2+\cdots+m_\lambda\Theta_\lambda$ is a chain of Riemann spheres, and V_0 (resp. $U_{\lambda+1}$) *intersects* Θ_1 (resp. Θ_λ) *at exactly one point.*

Proof. We take λ copies $\Theta_1, \Theta_2, \ldots, \Theta_\lambda$ of $\mathbb{C}P^1$. For each $i = 1, 2, \ldots, \lambda$, let $\Theta_i = U_i \cup V_i$ be an open covering by two copies U_i, V_i of $\mathbb{C}$ with coordinates $w_i \in U_i \setminus \{0\}$ and $z_i \in V_i \setminus \{0\}$ satisfying $z_i = 1/w_i$. Then we obtain a line bundle N_i over Θ_i of degree $-r_i$ from $U_i \times \mathbb{C}$ and $V_i \times \mathbb{C}$ by identifying $(z_i, \zeta_i) \in (V_i \setminus \{0\}) \times \mathbb{C}$ with $(w_i, \eta_i) \in (U_i \setminus \{0\}) \times \mathbb{C}$ via

$$g_i : z_i = \frac{1}{w_i}, \quad \zeta_i = w_i^{r_i}\eta_i.$$

Now patch N_i and N_{i+1} by plumbing, $(\zeta_i, z_i) = (w_{i+1}, \eta_{i+1})$, for each $i = 1, 2, \ldots, \lambda - 1$, then we obtain a smooth complex surface $\widehat{T}$.

Let us define the holomorphic functions $\pi_i : N_i \to \mathbb{C}$ by

$$\pi_i = \begin{cases} w_i^{m_{i-1}}\eta_i^{m_i}, & \text{on } U_i \times \mathbb{C}, \\ z_i^{m_{i+1}}\zeta_i^{m_i}, & \text{on } V_i \times \mathbb{C}. \end{cases}$$

The holomorphic functions $\{\pi_i\}$ together define a holomorphic function $\pi : \widehat{T} \to \mathbb{C}$ and the central fiber is

$$\pi^{-1}(0) = m_0V_0 + m_1\Theta_1 + m_2\Theta_2 + \cdots + m_\lambda\Theta_\lambda + m_{\lambda+1}U_{\lambda+1},$$

where $V_0 := \{0\} \times \mathbb{C} \subset U_1 \times \mathbb{C}$ and $U_{\lambda+1} := \{0\} \times \mathbb{C} \subset V_\lambda \times \mathbb{C}$. Thus, setting $T := \pi^{-1}(\Delta)$ for an open disk Δ in $\mathbb{C}$ centered at the origin, the restriction $\pi : T \to \Delta$ of the holomorphic function $\pi : \widehat{T} \to \mathbb{C}$ is the desired linear degeneration. □

REMARK 4.3. To be precise, since $\pi : T \to \Delta$ obtained in Lemma 4.2 is not proper, it is not a degeneration. However, it can be identified with the restriction of some degeneration to a tubular neighborhood T of a chain $m_1\Theta_1 + m_2\Theta_2 + \cdots + m_\lambda\Theta_\lambda$ contained in the singular fiber.

5. Tame simple crusts and barking families

Let us review Takamura's theory of barking families. For a degeneration, Takamura defined a simple crust as a subdivisor of its singular fiber that satisfies certain conditions, and constructed a splitting family associated to each such simple crust. A splitting family constructed by his method is called a barking family. For details see [11]. In this paper, we consider only simple crusts that satisfy some additional conditions and call them *tame* simple crusts. See Definition 5.4.

Let $\pi : M \to \Delta$ be a linear degeneration of Riemann surfaces with the singular fiber $X_0 = \sum_i m_i\Theta_i$. Let Y be an effective subdivisor of $X_0 = \sum_i m_i\Theta_i$. We express Y as

$$Y = \sum_i n_i\Theta_i,$$

where n_i is a nonnegative integer less than or equal to m_i. We define the *underlying reduced curve* of Y as $Y^{\text{red}} := \sum_i \Theta_i$, where the sum runs over all i with $n_i \geq 1$. Namely, an irreducible component Θ_i of X_0 is contained in Y^{red} if and only if $n_i \geq 1$. Let $\mathrm{Core}(X_0)$ denote the set of all cores of X_0 and $\mathrm{Core}(Y)$ denote the set of the cores of X_0 that are contained in Y^{red}. We first assume that

- Y (or, more precisely, Y^{red}) is connected, and
- at least one irreducible component of Y is a core of X (or equivalently, $\mathrm{Core}(Y) \neq \emptyset$).

Let **Br** be a branch of X_0 attached with a core Θ_0, and express it as

$$\mathbf{Br} = m_1\Theta_1 + m_2\Theta_2 + \cdots + m_\lambda\Theta_\lambda,$$

where Θ_1 is attached with the core Θ_0. Namely, denoting by m_0 the multiplicity of Θ_0, the branch **Br** is a chain associated with the multiplicity sequence $m_0, m_1, m_2, \ldots, m_\lambda, m_{\lambda+1} := 0$. For each $i = 1, 2, \ldots, \lambda$, we set

$$r_i := \frac{m_{i-1} + m_{i+1}}{m_i},$$

which is a positive integer. Recall that the self-intersection number of Θ_i in M is equal to $-r_i$. Let n_i be the multiplicity of Θ_i in Y, $i = 0, 1, \ldots, \lambda$. Now set

$$\mathbf{br} := n_1\Theta_1 + n_2\Theta_2 + \cdots + n_\lambda\Theta_\lambda.$$

From the above assumptions for Y, if $n_i = 0$ for some i, then $n_{i'} = 0$ for any $i' \geq i$. We thus may express as

$$\mathbf{br} = \emptyset, \quad \text{or} \quad \mathbf{br} = n_1\Theta_1 + n_2\Theta_2 + \cdots + n_\nu\Theta_\nu,$$

where ν is the least positive integer with $n_\nu \neq 0$ among $1, 2, \ldots, \lambda$. By convention, we set $\nu := 0$ if $\mathbf{br} = \emptyset$. We call **br** a *subbranch* of **Br** if one of the following conditions is satisfied.

- $\nu = 0$ or 1.
- $\nu \geq 2$, and $r_i = \dfrac{n_{i-1} + n_{i+1}}{n_i}$ for each $i = 1, 2, \ldots, \nu - 1$.

Set $\overline{n}_{\nu+1} := r_\nu n_\nu - n_{\nu-1}$. If $\nu = 0$ (that is, $\mathbf{br} = \emptyset$), then we set $\overline{n}_{\nu+1} := 0$.

Definition 5.1. Let l be a positive integer.

(A) A subbranch **br** of **Br** is of *type A_l* if $ln_i \leq m_i$ for each $i = 0, 1, \ldots, \nu$, and $\overline{n}_{\nu+1} \leq 0$.

(B) A subbranch **br** of **Br** is of *type B_l* if $ln_i \leq m_i$ for each $i = 0, 1, \ldots, \nu$, $n_\nu = 1$ and $m_\nu = l$.

(C) A subbranch **br** of **Br** is of *type C_l* if $ln_i \leq m_i$ for each $i = 0, 1, \ldots, \nu$, $n_\nu = \overline{n}_{\nu+1}$ and $m_\nu - m_{\nu+1}$ divides l.

Now let **Tk** be a trunk of X_0 and express it as

$$\mathbf{Tk} = m_1\Theta_1 + m_2\Theta_2 + \cdots + m_\lambda\Theta_\lambda.$$

Let Θ_0 (resp. $\Theta_{\lambda+1}$) be the core intersecting Θ_1 (resp. Θ_λ) and let m_0 (resp. $m_{\lambda+1}$) denote its multiplicity. Then the trunk **Tk** is a chain of Riemann sphere associated with the multiplicity sequence $m_0, m_1, m_2, \ldots, m_\lambda, m_{\lambda+1}$. Recall that, for each $i = 1, 2, \ldots, \lambda$, the self-intersection number of Θ_i in M is equal to $-r_i$, where

$$r_i := \frac{m_{i-1} + m_{i+1}}{m_i}.$$

Let n_i be the multiplicity of Θ_i in Y, $i = 0, 1, \ldots, \lambda + 1$. Now set

$$\mathbf{tk} := n_1\Theta_1 + n_2\Theta_2 + \cdots + n_\lambda\Theta_\lambda.$$

Since Y is connected and $\mathrm{Core}(Y) \neq \emptyset$, either n_0 or $n_{\lambda+1}$ or both must be positive.

Definition 5.2. Let l be a positive integer. We call $\mathbf{tk}$ a *tame subtrunk* of $\mathbf{Tk}$ with *barking multiplicity* l if the following condition is satisfied.

- $0 < ln_i \leq m_i$ and $r_i = \dfrac{n_{i-1} + n_{i+1}}{n_i}$ for each $i = 1, 2, \ldots, \lambda$.

We next consider the cores of X_0. Let Θ_0 be a core of X_0 and let N_0 denote the normal bundle of Θ_0 in M. Recall that there exists a holomorphic section σ_0 of the line bundle $N_0^{\otimes(-m_0)}$ over Θ_0 such that

$$\mathrm{div}(\sigma_0) = \sum_{j=1}^{h} m_j p_j,$$

where p_j are the points at which Θ_0 intersects the other irreducible components of X_0 and m_j are the corresponding multiplicities. Let n_0 denote the multiplicity of Θ_0 in Y. Now suppose that there exists a meromorphic section τ of the line bundle $N_0^{\otimes n_0}$ over Θ_0 such that

$$\mathrm{div}(\tau) = -\sum_{j=1}^{h} n_j p_j + D$$

for some nonnegative divisor $D = \sum_{j=h+1}^{h'} a_j p_j$ on Θ_0, where $p_1, p_2, \ldots, p_{h'}$ are all distinct points of Θ_0. Then we call the meromorphic section τ a *core section* over Θ_0 for Y. Note that τ is not uniquely determined by Y. It follows that $r_0 := (\sum_{j=1}^{h} m_j)/m_0$ is a positive integer, while $r'_0 := (\sum_{j=1}^{h} n_j)/n_0$ is not necessarily an integer. Furthermore, we have the following (see [11] Section 3.4).

Lemma 5.3. *Suppose that the core Θ_0 is a Riemann sphere. Then Y has a core section τ over the core Θ_0 if and only if $r_0 \leq r'_0$. Moreover, τ has no zero, that is, $D = 0$, exactly when $r_0 = r'_0$.*

Now we define tame simple crusts. Recall that $\mathrm{Core}(X_0)$ denotes the set of all cores of X_0 and $\mathrm{Core}(Y)$ denotes the set of the cores of X_0 that are contained in Y^{red}. Set

$$\mathrm{Adja}(Y) := \left\{\Theta \in \mathrm{Core}(X_0) \setminus \mathrm{Core}(Y) \ : \ \Theta \cap Y^{\mathrm{red}} \neq \emptyset\right\},$$

whose elements are said to be *adjacent* to Y.

Definition 5.4. Let Y be a connected subdivisor of X_0 such that $\mathrm{Core}(Y) \neq \emptyset$, and let l be a positive integer. We call Y a *tame simple crust* of X_0 with *barking multiplicity* l if the following conditions are satisfied.

- $ln_0 \leq m_0$ for each core $\Theta_0 \in \mathrm{Core}(Y)$, where m_0 and n_0 are the multiplicities of Θ_0 in X_0 and Y, respectively.
- The subdivisor $\mathbf{br}$ of each branch $\mathbf{Br}$ of X_0 for Y is a subbranch of type A_l, B_l or C_l.
- The subdivisor $\mathbf{tk}$ of each trunk $\mathbf{Tk}$ of X_0 for Y is a tame subtrunk with barking multiplicity l.

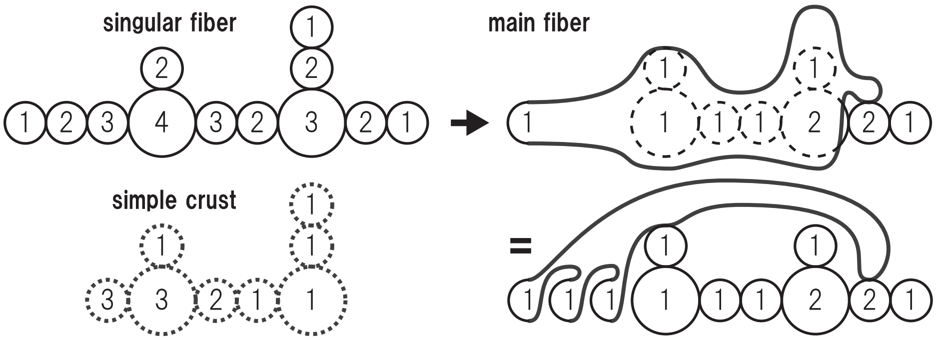

Fig. 4. In a barking family, the singular fiber is deformed to the main fiber in such a way that the simple crust looks "barked" off from the original singular fiber.

- For each core $\Theta_0 \in \mathrm{Core}(Y) \cup \mathrm{Adja}(Y)$, there exists a core section over Θ_0 for Y which has no zeros.

In fact, Takamura [11] defined *simple crusts* — subdivisors of X_0 satisfying more general conditions. (Note that tame simple crusts which we defined above are simple crusts in the sense of [11].) He constructed a deformation family of the given degeneration $\pi : M \to \Delta$ associated with the simple crust Y and its barking multiplicity l. We call a deformation family obtained by his method a *barking family*. In a barking family $\Psi\colon \mathcal{M} \to \Delta \times \Delta^\dagger$, the original singular fiber X_0 is deformed to a simpler singular fiber in such a way that the subdivisor Y looks "barked" off from X_0 as depicted in Figure 4. The resulting singular fiber appears over the origin of Δ_t, so we denote it by $X_{0,t}$ and call it the *main fiber*. For each irreducible component Θ of the main fiber $X_{0,t}$, exactly one of the following phenomena occurs while $X_{0,t}$ deforms to X_0 as $t \to 0$.

(1) The irreducible component Θ degenerates to a union of irreducible components of X_0 which contains Y^{red}. (More precisely, there exists a degenerating subfamily with fiber Θ that is naturally contained in the family $\mathcal{M} \to \Delta^\dagger$, as seen in Section 9.)

(2) The irreducible component Θ does not degenerate but trivially deforms to one irreducible component of X_0. (More precisely, there exists a trivial subfamily with fiber Θ that is naturally contained in the family $\mathcal{M} \to \Delta^\dagger$, as seen in Section 7.)

The irreducible component Θ of $X_{0,t}$ is called a *barked component* if (1) occurs, and a *stable component* if (2) occurs.

In a barking family, there necessarily appear not only the main fiber but also other singular fibers over some points away from the origin of Δ_t, which are called *subordinate fibers*. Under the deformation, the topological type of the singular fiber over the origin changes, so the local monodromy around it also changes. On the other hand, the global monodromies before and after the deformation — that is, the two monodromies each of which is induced by a loop in Δ (resp. Δ_t) parallel and closed to its boundary $\partial\Delta$ (resp. $\partial\Delta_t$) — coincide. We then deduce that there should necessarily appear other singular fibers with nontrivial monodromies, since we see that the monodromies before and after the deformation are distinct. Thus every barking family turns out to be a splitting family. Therefore, we have the following.

Theorem 5.5 (Takamura [11]). *Let $\pi : M \to \Delta$ be a linear degeneration with the singular fiber X_0. If X_0 has a simple crust Y, then $\pi : M \to \Delta$ admits a splitting family $\Psi : \mathcal{M} \to \Delta \times \Delta^\dagger$.*

REMARK 5.6. In this paper, for a degeneration which is *not* necessarily relatively minimal, a splitting family is defined in such a way that each deformation has at least two singular fibers. Thus some singular fibers of a deformation in a splitting family may possibly become smooth fibers by blowing-downs. Such singular fibers are said to be *fake*.

6. Monodromy assortments of barking families

Let $\pi : M \to \Delta$ be a linear degeneration of Riemann surfaces of genus $g \geq 1$ and $\Psi : \mathcal{M} \to \Delta \times \Delta^\dagger$ be a barking family for $\pi : M \to \Delta$ associated with a tame simple crust Y. Denote by $\mathcal{D}$ the discriminant of Ψ, that is, $\mathcal{D} := \Psi(\operatorname{Sing}\Psi)$.

Let us consider the deformation $\pi_{t_0} : M_{t_0} \to \Delta_{t_0}$ of $\pi : M \to \Delta$ for a fixed point $t_0 \in \Delta^\dagger \setminus \{0\}$. Take a smooth simple closed curve γ in $\Delta^\dagger \setminus \{0\}$ with base point t_0 that goes once around the origin in the counterclockwise direction: then $L := \mathcal{D} \cap (\Delta \times \gamma)$ is a closed braid in the open solid torus $\Delta \times \gamma$, and we obtain the topological monodromy $[F, f] \in \mathrm{MCG}(\pi_{t_0})$ of the barking family $\Psi : \mathcal{M} \to \Delta \times \Delta^\dagger$. Let $K_1, K_2, \ldots, K_c$ be the knot components of L, and $\mathbb{X}_j$ be the tassel over K_j, $j = 1, 2, \ldots, c$. Then we obtain the monodromy assortment $(F_1, F_2, \ldots, F_c; f)$ of Ψ. See Section 3 for details.

Since the main fiber of the deformation $\pi_t : M_t \to \Delta_t$ for any $t \in \gamma$ lies over the origin of Δ_t, the core curve $\{0\} \times \gamma$ in the open solid torus $\Delta \times \gamma$ is nothing but the knot component over which the main fibers lie. In what follows, we denote the knot component $\{0\} \times \gamma$ by K_1, and we call the tassel $\mathbb{X}_1$ over K_1 the *main tassel*. Clearly $\mathbb{X}_1 = X_{0,t_0}$. On the other hand, we call the other tassels $\mathbb{X}_2, \mathbb{X}_3, \ldots, \mathbb{X}_c$ *subordinate tassels*. Each subordinate fiber of the deformation $\pi_{t_0} : M_{t_0} \to \Delta_{t_0}$ is contained in one of the subordinate tassels.

By using results on determination of subordinate fibers, we see that each irreducible component of $\mathcal{D}$ is the hypersurface

$$\left\{(s, t) \in \Delta \times \Delta^\dagger \ : \ s^n = \kappa t^m\right\},$$

for some relatively prime positive integers m, n (arising from the multiplicities of X_0 and Y) and some complex number κ. See [8] for instance. Then the intersection of the irreducible component and $\Delta \times \gamma$ is one of the knot components of L, and moreover we see that it is an (m, n)-torus knot in the open solid torus $\Delta \times \gamma$ (more precisely, when the solid torus is embedded in the standard way in a 3-sphere). Hence we have the following.

Proposition 6.1. *Let K_1 be the knot component of L over which the main tassel lies, and let $K_2, K_3, \ldots, K_c$ be the knot components of L over which the subordinate tassels lie. Then, we have the following.*

- *The knot component K_1 is a trivial closed braid in $\Delta \times \gamma$.*
- *The knot component K_j ($j = 2, 3, \ldots, c$) is a torus knot in $\Delta \times \gamma$.*

We consider the monodromy homeomorphism $F_1 : X_{0,t_0} \to X_{0,t_0}$ on the main fiber, where $\mathbb{X}_1 = X_{0,t_0}$. We will show that F_1 acts as a pseudo-periodic homeomorphism of negative twist on each irreducible component of the main fiber. To be more precise, we have the following, which is a summary of Propositions 7.2, 9.2 and 9.3.

Theorem 6.2. *Let $\Psi : \mathcal{M} \to \Delta \times \Delta^\dagger$ be a barking family for a linear degeneration $\pi : M \to \Delta$ associated with a tame simple crust Y. Let us consider the deformation $\pi_{t_0} : M_{t_0} \to \Delta_{t_0}$ of $\pi : M \to \Delta$ for a fixed $t_0 \in \Delta^\dagger$. We denote by $(F_1, F_2, \ldots, F_c; f)$ the monodromy assortment of Ψ, and let F_1 be the monodromy homeomorphism on the main fiber X_{0,t_0}. Let $\Theta_1, \Theta_2, \ldots, \Theta_a$ be the stable components of X_{0,t_0} and let $\Xi_1, \Xi_2, \ldots, \Xi_b$ be the barked components of X_{0,t_0}. Then we have the following.*

(1) *For each $i = 1, 2, \ldots, a$, we have $F_1(\Theta_i) = \Theta_i$, and the restriction $F_1\big|_{\Theta_i} : \Theta_i \to \Theta_i$ is isotopic to the identity map of Θ_i.*

(2) *If $b = 1$, then we have $F_1(\Xi_1) = \Xi_1$, and the isotopy class of the restriction $F_1\big|_{\Xi_1} : \Xi_1 \to \Xi_1$ is conjugate to the topological monodromy of a degeneration of Riemann surfaces whose singular fiber is the enlargement[2] of the tame simple crust Y. In particular, $F_1\big|_{\Xi_1} : \Xi_1 \to \Xi_1$ is isotopic to a pseudo-periodic homeomorphism of negative twist.*

(3) *If $b \geq 2$, then F_1 permutes $\Xi_1, \Xi_2, \ldots, \Xi_b$, and the restriction*

$$F_1\big|_{\Xi_1 \cup \Xi_2 \cup \cdots \cup \Xi_b} : \Xi_1 \cup \Xi_2 \cup \cdots \cup \Xi_b \to \Xi_1 \cup \Xi_2 \cup \cdots \cup \Xi_b$$

coincides with the monodromy homeomorphism of a degeneration of disjoint unions of b Riemann surfaces whose singular fiber is the enlargement of the tame simple crust Y, up to isotopy and conjugacy.

7. Degenerations of stable components

Let $\pi : M \to \Delta$ be a linear degeneration of Riemann surfaces of genus $g \geq 1$ and $\Psi : \mathcal{M} \to \Delta \times \Delta^\dagger$ be a barking family for $\pi : M \to \Delta$ associated with a tame simple crust Y with barking multiplicity l. For a base point $t_0 \in \Delta^\dagger \setminus \{0\}$, we denote the monodromy assortment of $\Psi : \mathcal{M} \to \Delta \times \Delta^\dagger$ by $(F_1, F_2, \ldots, F_c; f)$, where $F_j : \mathbb{X}_j \to \mathbb{X}_j$ is the monodromy homeomorphism on the j-th tassel $\mathbb{X}_j$, $j = 1, 2, \ldots, c$, and $f : \Delta_{t_0} \to \Delta_{t_0}$ is the associated homeomorphism on the open disk Δ_{t_0} over the base point t_0. We assume that $\mathbb{X}_1$ is the main tassel (so $\mathbb{X}_1 = X_{0,t_0}$).

In this section, we investigate the restriction of $F_1 : X_{0,t_0} \to X_{0,t_0}$ to a stable component Θ of X_{0,t_0}. For this purpose, we construct a degeneration such that the stable component Θ coincides with a smooth fiber of it and that its monodromy homeomorphism is isotopic to $F_1\big|_\Theta$.

For each $s \in \Delta$, we set $\Delta_s^\dagger := \{s\} \times \Delta^\dagger$ and $M_s^\dagger := \Psi^{-1}(\Delta_s^\dagger)$. Then we obtain the map

$$\pi_s^\dagger := \Psi\big|_{M_s^\dagger} : M_s^\dagger \to \Delta_s^\dagger.$$

It might be plausible that $\pi_s^\dagger$ is a family of complex curves if we regard the deformation parameter t of Ψ as a degeneration parameter. However, it is not the case unless $M_s^\dagger$ is a smooth complex surface. Note that, for the case $s = 0$, the central fiber $(\pi_0^\dagger)^{-1}(0)$ of $\pi_0^\dagger : M_0^\dagger \to \Delta_0^\dagger$, coincides with the singular fiber X_0 of the original degeneration $\pi : M \to \Delta$, while a general fiber $(\pi_0^\dagger)^{-1}(t)$, $t \neq 0$, coincides with the main fiber $X_{0,t}$ of the deformation $\pi_t : M_t \to \Delta_t$.

[2] See Section 8. Note that a subdivisor Y itself is not always realized as a singular fiber of a degeneration.

Now we consider the restriction of $\pi_0^\dagger : M_0^\dagger \to \Delta_0^\dagger$ to a certain smooth complex surface, which is a degeneration of Riemann surfaces. Let Θ be a stable component of the main fiber X_{0,t_0} of the deformation $\pi_{t_0} : M_{t_0} \to \Delta_{t_0}$. Recall that, as $t_0 \to 0$, the component Θ approaches to some irreducible component of the singular fiber X_0 of $\pi : M \to \Delta$, say Θ_0. Let N_0 be the normal bundle of Θ_0 in M, with coordinates (z, ζ), where z is the base coordinate and ζ is the fiber coordinate. We have the following (see [11] Section 16.2).

Lemma 7.1. *The complex* 3*-dimensional manifold* $\mathcal{M}$ *is locally expressed near the core* Θ_0 *as the hypersurface*

$$\left\{(z, \zeta, s, t) \in N_0 \times \Delta \times \Delta^\dagger \ : \ \sigma(z)\zeta^{m_0 - l n_0} \left(\zeta^{n_0} + t\tau(z)\right)^l - s = 0\right\},$$

where m_0 *and* n_0 *are the multiplicities of* Θ_0 *in* X_0 *and* Y*, respectively,* σ *is the standard section of* $N_0^{\otimes(-m_0)}$ *and* τ *is a core section of* $N_0^{\otimes n_0}$ *for* Y*. Furthermore,* $\Psi : \mathcal{M} \to \Delta \times \Delta^\dagger$ *locally coincides with the restriction of the projection* $N_0 \times \Delta \times \Delta^\dagger \to \Delta \times \Delta^\dagger$ *to the hypersurface* $\mathcal{M}$*.*

Note that the degeneration $\pi : M \to \Delta$ (that is, $\pi_0 : M_0 \to \Delta_0$) corresponds to the restriction of the projection $N_0 \times \Delta \times \{0\} \to \Delta \times \{0\}$ to the hypersurface given by

$$\sigma(z)\zeta^{m_0} - s = 0, \quad \text{in } N_0 \times \Delta \times \{0\},$$

while $\pi_0^\dagger : M_0^\dagger \to \Delta_0^\dagger$ corresponds to the restriction of the projection $N_0 \times \{0\} \times \Delta^\dagger \to \{0\} \times \Delta^\dagger$ to the hypersurface given by

$$\sigma(z)\zeta^{m_0 - l n_0} \left(\zeta^{n_0} + t\tau(z)\right)^l = 0, \quad \text{in } N_0 \times \{0\} \times \Delta^\dagger.$$

Now consider the hypersurface given by

$$\zeta = 0, \quad \text{in } N_0 \times \{0\} \times \Delta^\dagger,$$

which is contained in $M_0^\dagger$. This hypersurface is nothing but $\Theta_0 \times \Delta_0^\dagger$, and the restriction of $\pi_0^\dagger$: $M_0^\dagger \to \Delta_0^\dagger$ to $\Theta_0 \times \Delta_0^\dagger$ coincides with the trivial degeneration $\pi_0^\dagger : \Theta_0 \times \Delta_0^\dagger \to \Delta_0^\dagger$ of Riemann surfaces. Note that the fiber $(\pi_0^\dagger)^{-1}(t_0)$ over $t_0 \in \Delta_0^\dagger$ coincides with the stable component Θ. Since the restriction of F_1 to Θ coincides with the monodromy homeomorphism of this trivial degeneration, $F_1\big|_\Theta$ is isotopic to the identity map of Θ. Thus we have the following.

Proposition 7.2. *Let* $\Theta_1, \Theta_2, \ldots, \Theta_a$ *be the stable components of the main fiber* X_{0,t_0} *of the deformation* $\pi_{t_0} : M_{t_0} \to \Delta_{t_0}$*. Then, for each* $i = 1, 2, \ldots, a$*, we have*

$$F_1(\Theta_i) = \Theta_i,$$

and the restriction $F_1\big|_{\Theta_i} : \Theta_i \to \Theta_i$ *is isotopic to the identity map of* Θ_i*.*

REMARK 7.3. In fact, Proposition 7.2 holds for barking families associated with simple crusts (not necessarily tame simple crusts).

8. Enlargements of tame simple crusts

Let $\pi : M \to \Delta$ be a linear degeneration of Riemann surfaces of genus $g \geq 1$ and $\Psi : \mathcal{M} \to \Delta \times \Delta^\dagger$ be a barking family for $\pi : M \to \Delta$ associated with a tame simple crust Y with barking multiplicity l.

In this section, we study the restriction of the monodromy homeomorphism $F_1 : X_{0,t_0} \to X_{0,t_0}$ to a barked component Ξ of the main fiber X_{0,t_0}. Recall that, as $t_0 \to 0$, the component Ξ approaches to a certain union of irreducible components of X_0 which contains Y^{red}. Unfortunately, unlike the case of stable components in Section 7, we cannot always construct a degeneration of Riemann surfaces such that Ξ coincides with a smooth fiber of it and that the degeneration itself can be identified with a certain restriction of $\pi_0^\dagger : M_0^\dagger \to \Delta_0^\dagger$. However, we can construct a degeneration of Riemann surfaces such that its restriction to the complement of a thin subset coincides with a certain restriction of $\pi_0^\dagger$, and that its singular fiber is identified with an "enlargement" of Y.

We introduce the concept of "enlargements" of tame simple crusts as follows. First let us define the enlargements of subbranches. Let **Br** be a branch of the singular fiber X_0 of the degeneration $\pi : M \to \Delta$ and **br** be a subbranch of **Br** for the tame simple crust Y. Express them as

$$\begin{cases} \mathbf{Br} = m_1\Theta_1 + m_2\Theta_2 + \cdots + m_\lambda\Theta_\lambda, \text{ and} \\ \mathbf{br} = n_1\Theta_1 + n_2\Theta_2 + \cdots + n_\nu\Theta_\nu, \text{ or } \emptyset, \end{cases}$$

where $0 \le \nu \le \lambda$, and $0 \le n_i \le m_i$, $i = 1, 2, \ldots, \nu$.

First suppose that **br** is a subbranch of type A_l. Then, from the definition, it follows that $\overline{n}_{\nu+1}$ $(= r_\nu n_\nu - n_{\nu-1})$ may possibly be a negative integer. We define the decreasing sequence of nonnegative integers

$$n_\nu > \widetilde{n}_{\nu+1} > \widetilde{n}_{\nu+2} > \cdots > \widetilde{n}_\mu > \widetilde{n}_{\mu+1} = 0$$

by the Euclidean division algorithm of negative type: namely, we choose the integers so that

$$\begin{cases} \widetilde{r}_\nu := \dfrac{n_{\nu-1} + \widetilde{n}_{\nu+1}}{n_\nu}, \ \widetilde{r}_{\nu+1} := \dfrac{n_\nu + \widetilde{n}_{\nu+2}}{\widetilde{n}_{\nu+1}}, \text{ and} \\ \widetilde{r}_i := \dfrac{\widetilde{n}_{i-1} + \widetilde{n}_{i+1}}{\widetilde{n}_i} \ (i = \nu+2, \nu+3, \ldots, \mu) \end{cases}$$

are integers greater than or equal to 2. If $\nu = 0$ (that is, $\mathbf{br} = \emptyset$), then we set $\widetilde{n}_{\nu+1} := 0$. We now consider the sequence

$$n_0, n_1, n_2, \ldots, n_\nu, \widetilde{n}_{\nu+1}, \widetilde{n}_{\nu+2}, \ldots, \widetilde{n}_\mu, \widetilde{n}_{\mu+1} = 0.$$

By Lemma 4.2, there exists a linear degeneration $\pi : \widetilde{T} \to \Delta$ with the singular fiber

$$n_0V_0 + n_1\Theta_1 + n_2\Theta_2 + \cdots + n_{\nu-1}\Theta_{\nu-1} + n_\nu\widetilde{\Theta}_\nu + \widetilde{n}_{\nu+1}\widetilde{\Theta}_{\nu+1} + \cdots + \widetilde{n}_\mu\widetilde{\Theta}_\mu,$$

where V_0 is a copy of $\mathbb{C}$. We call the chain $\widetilde{\mathbf{br}} := n_1\Theta_1 + n_2\Theta_2 + \cdots + \widetilde{n}_\mu\widetilde{\Theta}_\mu$ the *enlargement* of the subbranch **br**.

Suppose that **br** is of type B_l or C_l. we then consider the sequence

$$n_0, n_1, n_2, \ldots, n_\nu, \widetilde{n}_{\nu+1} := 0.$$

Note that $\widetilde{r}_\nu := (n_{\nu-1} + \widetilde{n}_{\nu+1})/n_\nu = n_{\nu-1}/n_\nu$ is a positive integer. In fact, if **br** is of type B_l, then $\widetilde{r}_\nu = n_{\nu-1}$. On the other hand, if **br** is of type C_l, then $\widetilde{r}_\nu = r_\nu - 1$. By Lemma 4.2, there exists a linear degeneration $\pi : \widetilde{T} \to \Delta$ with the singular fiber

$$n_0V_0 + n_1\Theta_1 + n_2\Theta_2 + \cdots + n_{\nu-1}\Theta_{\nu-1} + n_\nu\widetilde{\Theta}_\nu,$$

where V_0 is a copy of $\mathbb{C}$. We call the chain $\widetilde{\mathbf{br}} := n_1\Theta_1 + n_2\Theta_2 + \cdots + n_\nu\widetilde{\Theta}_\nu$ the *enlargement* of the subbranch $\mathbf{br}$. By convention, we set $\mu := \nu$.

Remark 8.1. If a subbranch $\mathbf{br}$ is of both type A_l and B_l, then the enlargements of $\mathbf{br}$ defined above by the two methods coincide. In this case, the length ν of $\mathbf{br}$ is equal to λ. Note that subbranches of type A_l are not of type C_l.

Let $\mathbf{Tk}$ be a trunk of the singular fiber X_0 of the degeneration $\pi : M \to \Delta$ and $\mathbf{tk}$ be a subtrunk of $\mathbf{Tk}$ for the tame simple crust Y. Express them as

$$\begin{cases} \mathbf{Tk} = m_1\Theta_1 + m_2\Theta_2 + \cdots + m_\lambda\Theta_\lambda \text{ and} \\ \mathbf{tk} = n_1\Theta_1 + n_2\Theta_2 + \cdots + n_\lambda\Theta_\lambda. \end{cases}$$

We define the *enlargement* $\widetilde{\mathbf{tk}}$ of $\mathbf{tk}$ as $\mathbf{tk}$ itself, that is, we set $\widetilde{\mathbf{tk}} := \mathbf{tk}$. In fact, by Lemma 4.2, there exists a linear degeneration $\pi : \widetilde{T} \to \Delta$ with the singular fiber

$$n_0V_0 + n_1\Theta_1 + n_2\Theta_2 + \cdots + n_\lambda\Theta_\lambda + n_{\lambda+1}U_{\lambda+1},$$

where V_0 and $U_{\lambda+1}$ are copies of $\mathbb{C}$.

Recall that $\mathcal{M}$ is expressed near the core Θ_0 as the hypersurface given by

$$\sigma(z)\zeta^{m_0-ln_0}\,(\zeta^{n_0} + t\tau(z))^l - s = 0, \qquad \text{in } N_0 \times \Delta \times \Delta^\dagger,$$

and that Ψ coincides with the restriction of the projection map $N_0 \times \Delta \times \Delta^\dagger \to \Delta \times \Delta^\dagger$ to the hypersurface $\mathcal{M}$. Here σ is the standard section of the line bundle $N_0^{\otimes(-m_0)}$ satisfying

$$\operatorname{div}(\sigma) = \sum_{j=1}^{h} m_j p_j,$$

where $p_1, p_2, \ldots, p_h$ are the intersection points on Θ_0 with the other irreducible components of X_0. On the other hand, τ is a core section of the line bundle $N_0^{\otimes n_0}$ for Y satisfying

$$\operatorname{div}(\tau) = -\sum_{j=1}^{h} n_j p_j,$$

Note that, from the definition of tame simple crusts, the core section τ has no zeros. Then $\tau^{-1}\,(= 1/\tau)$ is a holomorphic section of $N_0^{\otimes(-n_0)}$ which has a zero of order n_j at p_j, $j = 1, 2, \ldots, h$, and $\tau^{-1}(z)\zeta^{n_0}$ defines a holomorphic function on N_0. Now consider the hypersurface W_0 in $N_0 \times \Delta_0^\dagger$ defined by

$$\tau^{-1}(z)\zeta^{n_0} + t = 0,$$

where $\Delta_0^\dagger := \{0\} \times \Delta^\dagger$. Then the restriction of the projection map $N_0 \times \Delta_0^\dagger \to \Delta_0^\dagger$ to the hypersurface W_0 is a degeneration of punctured Riemann surfaces whose singular fiber is

$$n_0\Theta_0 + \sum_{j=1}^{h} n_j U_j,$$

where U_j is the fiber of N_0 over the point p_j $(j = 1, 2, \ldots, h)$, that is, $U_j = \{(p_j, \zeta) \in N_0\}$.

Now we define the enlargement of the tame simple crust Y. Express Y as

$$Y := \sum_i n_i \Theta_i + \sum_j \mathbf{br}^{(j)} + \sum_k \mathbf{tk}^{(k)},$$

where Θ_i, $\mathbf{br}^{(j)}$ and $\mathbf{tk}^{(k)}$ are the cores, the subbranches and the subtrunks of Y, respectively. For each $\mathbf{br}^{(j)}$ (resp. $\mathbf{tk}^{(k)}$), let $\widetilde{\mathbf{br}}^{(j)}$ (resp. $\widetilde{\mathbf{tk}}^{(k)}$) be its enlargement defined as above. By the same argument as that for linear degenerations, we patch the tubular neighborhoods of the cores Θ_i and the enlargements $\widetilde{\mathbf{br}}^{(j)}$ and $\widetilde{\mathbf{tk}}^{(k)}$ to obtain a smooth complex surface $\widetilde{M}$. From the construction, it is easy to see that the holomorphic map $\widetilde{\pi} : \widetilde{M} \to \Delta^\dagger$ defined by the degeneration maps of these neighborhoods is a degeneration with the singular fiber

$$\widetilde{Y} := \sum_i n_i \Theta_i + \sum_j \widetilde{\mathbf{br}}^{(j)} + \sum_k \widetilde{\mathbf{tk}}^{(k)}.$$

We call $\widetilde{Y}$ the *enlargement* of Y.

9. Degenerations of barked components

Recall that the restriction $\pi_0^\dagger : M_0^\dagger \to \Delta_0^\dagger$ of $\Psi : \mathcal{M} \to \Delta \times \Delta^\dagger$ to the preimage $M_0^\dagger := \Psi^{-1}(\Delta_0^\dagger)$ of $\Delta_0^\dagger := \{0\}\times\Delta^\dagger$ is not a family of complex curves, but that the central fiber $(\pi_0^\dagger)^{-1}(0)$ coincides with the singular fiber X_0 of the original degeneration $\pi : M \to \Delta$ and the general fiber $(\pi_0^\dagger)^{-1}(t)$, $t \neq 0$, coincides with the main fiber $X_{0,t}$ of the deformation $\pi_t : M_t \to \Delta_t$.

We will show that the restriction of $\widetilde{\pi} : \widetilde{M} \to \Delta^\dagger$ to the complement of a thin subset of $\widetilde{M}$ coincides with a certain restriction of $\pi_0^\dagger : M_0^\dagger \to \Delta_0^\dagger$.

Lemma 9.1. *For the enlargement*

$$\widetilde{\mathbf{br}}^{(j)} = n_1\Theta_1 + n_2\Theta_2 + \cdots + n_{\nu-1}\Theta_{\nu-1} + n_\nu\widetilde{\Theta}_\nu + \widetilde{n}_{\nu+1}\widetilde{\Theta}_{\nu+1} + \cdots + \widetilde{n}_\mu\widetilde{\Theta}_\mu$$

of each subbranch $\mathbf{br}^{(j)}$ *of* Y*, set* $E^{(j)} := \widetilde{\Theta}_{\nu+1} \cup \widetilde{\Theta}_{\nu+2} \cup \cdots \cup \widetilde{\Theta}_\mu \cup \widetilde{U}_\mu$*, where* $\widetilde{U}_\mu$ *is the fiber over the end*[3] *of* $\widetilde{\mathbf{br}}^{(j)}$ *of the normal bundle* $\widetilde{N}_\mu$ *of* $\widetilde{\Theta}_\mu$*. Then*

$$\widetilde{M}^\times := \widetilde{M} \setminus \bigcup_j E^{(j)}$$

is naturally contained in $M_0^\dagger$*. Moreover, the restriction of* $\widetilde{\pi} : \widetilde{M} \to \Delta^\dagger$ *to* $\widetilde{M}^\times$ *can be identified with a certain restriction of* $\pi_0^\dagger : M_0^\dagger \to \Delta_0^\dagger$*.*

Proof. The degeneration $\widetilde{\pi} : \widetilde{M} \to \Delta^\dagger$ is locally expressed near a core Θ_i as the restriction of the projection map $N_i \times \Delta_0^\dagger \to \Delta_0^\dagger$ to the hypersurface W_i in $N_i \times \Delta_0^\dagger$ defined by

$$\tau^{-1}(z)\zeta^{n_i} + t = 0,$$

while $\pi_0^\dagger : M_0^\dagger \to \Delta_0^\dagger$ is locally expressed as the restriction of the projection map of $N_i \times \Delta_0^\dagger$ to the hypersurface given by

$$\sigma(z)\tau^l(z)\zeta^{m_i - l n_i}\left(\tau^{-1}(z)\zeta^{n_i} + t\right)^l = 0.$$

Thus $\widetilde{M}$ is contained in $M_0^\dagger$ near the core Θ_i, and the restrictions of $\widetilde{\pi} : \widetilde{M} \to \Delta^\dagger$ and $\pi_0^\dagger : M_0^\dagger \to \Delta_0^\dagger$ coincide.

[3]A point on $\widetilde{\Theta}_\mu$ away form the attachment point with $\widetilde{\Theta}_{\mu-1}$.

We next consider the enlargement $\widetilde{\mathbf{br}}^{(j)}$ of $\mathbf{br}^{(j)}$. Let Θ_i ($i = 1, 2, \ldots, \nu-1$) be the Riemann sphere contained in $\widetilde{\mathbf{br}}^{(j)}$ (or $\mathbf{br}^{(j)}$) as the irreducible component. Let $\Theta_i = U_i \cup V_i$ be an open covering by two copies U_i, V_i of $\mathbb{C}$ with coordinates $w_i \in U_i \setminus \{0\}$ and $z_i \in V_i \setminus \{0\}$ satisfying $z_i = 1/w_i$. Then we obtain a line bundle N_i over Θ_i of degree $-r_i$ from $U_i \times \mathbb{C}$ and $V_i \times \mathbb{C}$ by identifying $(z_i, \zeta_i) \in (V_i \setminus \{0\}) \times \mathbb{C}$ with $(w_i, \eta_i) \in (U_i \setminus \{0\}) \times \mathbb{C}$ via

$$g_i : z_i = \frac{1}{w_i}, \quad \zeta_i = w_i^{r_i} \eta_i,$$

where $r_i = (n_{i-1} + n_{i+1})/n_i = (m_{i-1} + m_{i+1})/m_i$.

The degeneration $\widetilde{\pi} : \widetilde{M} \to \Delta^\dagger$ is locally expressed near Θ_i as the restriction of the projection map $N_i \times \Delta_0^\dagger \to \Delta_0^\dagger$ to the hypersurface H_i in $N_i \times \Delta_0^\dagger$ defined by

$$\begin{cases} w_i^{n_{i-1}} \eta_i^{n_i} + t = 0, & (w_i, \eta_i, t) \in U_i \times \mathbb{C} \times \Delta_0^\dagger, \\ z_i^{n_{i+1}} \zeta_i^{n_i} + t = 0, & (z_i, \zeta_i, t) \in V_i \times \mathbb{C} \times \Delta_0^\dagger. \end{cases}$$

On the other hand, $\pi_0^\dagger : M_0^\dagger \to \Delta_0^\dagger$ is locally expressed as the restriction of the projection map of $N_i \times \Delta_0^\dagger$ to the hypersurface given by

$$\begin{cases} w_i^{m_{i-1} - l n_{i-1}} \eta_i^{m_i - l n_i} (w_i^{n_{i-1}} \eta_i^{n_i} + t)^l = 0, & (w_i, \eta_i, t) \in U_i \times \mathbb{C} \times \Delta_0^\dagger, \\ z_i^{m_{i+1} - l n_{i+1}} \zeta_i^{m_i - l n_i} (z_i^{n_{i+1}} \zeta_i^{n_i} + t)^l = 0, & (z_i, \zeta_i, t) \in V_i \times \mathbb{C} \times \Delta_0^\dagger. \end{cases}$$

Thus $\widetilde{M}$ is contained in $M_0^\dagger$ near $\widetilde{\mathbf{br}}^{(j)} \setminus \bigcup_j E^{(j)}$, and the restrictions of $\widetilde{\pi} : \widetilde{M} \to \Delta^\dagger$ and $\pi_0^\dagger : M_0^\dagger \to \Delta_0^\dagger$ coincide.

By the same argument as that for $\widetilde{\mathbf{br}}^{(j)}$, we see that $\widetilde{M}$ is contained in $M_0^\dagger$ near the subtrunk $\mathbf{tk}^{(k)}$ ($= \widetilde{\mathbf{tk}}^{(k)}$) and the restrictions of $\widetilde{\pi} : \widetilde{M} \to \Delta^\dagger$ and $\pi_0^\dagger : M_0^\dagger \to \Delta_0^\dagger$ coincide. This completes the proof of the assertion. □

Under the identification of $\widetilde{\pi} : \widetilde{M}^\times \to \Delta^\dagger$ with a certain restriction of $\pi_0^\dagger : M_0^\dagger \to \Delta_0^\dagger$ in Lemma 9.1, the central fiber of $\widetilde{\pi} : \widetilde{M}^\times \to \Delta^\dagger$ corresponds to the singular curve obtained by puncturing the simple crust Y at the end of each subbranch, while a general fiber over $t_0 \in \Delta^\dagger \setminus \{0\}$ is the disjoint union of punctured Riemann surfaces obtained from the barked components of the main fiber X_{0,t_0} of the deformation $\pi_{t_0} : M_{t_0} \to \Delta_{t_0}$.

Suppose that X_{0,t_0} has exactly one barked component, and denote it by Ξ. Then the restriction of the monodromy homeomorphism F_1 of the barking family to the punctured barked component (that is, the Riemann surface obtained by puncturing the barked component Ξ at the attachment points with other irreducible components) forms a self-homeomorphism, and it coincides with the restriction of the monodromy homeomorphism of the degeneration of Riemann surfaces $\widetilde{\pi} : \widetilde{M} \to \Delta^\dagger$ to the punctured general fiber of $\widetilde{\pi} : \widetilde{M}^\times \to \Delta^\dagger$ up to isotopy. Since a self-homeomorphism of a punctured real surface uniquely induces a self-homeomorphism of its compactification, the monodromy homeomorphism of the degeneration of Riemann surfaces $\widetilde{\pi} : \widetilde{M} \to \Delta^\dagger$ is isotopic to $F_1\big|_\Xi : \Xi \to \Xi$. Hence, we have the following.

Proposition 9.2. *We denote by* $(F_1, F_2, \ldots, F_c; f)$ *the monodromy assortment of* Ψ. *Suppose that the main fiber* X_{0,t_0} *of the deformation* $\pi_{t_0} : M_{t_0} \to \Delta_{t_0}$ *has exactly one barked* *component, say* Ξ*. Then we have* $F_1(\Xi) = \Xi$*, and the isotopy class of* $F_1\big|_{\Xi} : \Xi \to \Xi$ *is conjugate to the topological monodromy of a degeneration of Riemann surfaces whose singular fiber is the enlargement of the tame simple crust* Y*. In particular,* $F_1\big|_{\Xi} : \Xi \to \Xi$ *is isotopic to a pseudo-periodic homeomorphism of negative twist.*

Now, let us consider the general case: let $\Xi_1, \Xi_2, \ldots, \Xi_b$ be the barked components of X_{0,t_0}. Note that $\widetilde{\pi} : \widetilde{M} \to \Delta^\dagger$ is not necessarily a degeneration of *Riemann surfaces* but is a degeneration of *disjoint unions of Riemann surfaces* (see Remark 9.4 for example). In other words, each general fiber is a disjoint union of Riemann surfaces. In this case, the fiber $\widetilde{\pi}^{-1}(t_0)$ over $t_0 \in \Delta^\dagger$ consists of $\Xi_1, \Xi_2, \ldots, \Xi_b$. By an argument similar to the above, the monodromy homeomorphism of $\widetilde{\pi} : \widetilde{M} \to \Delta^\dagger$ is isotopic to

$$F_1\big|_{\Xi_1\cup\Xi_2\cup\cdots\cup\Xi_b} : \Xi_1 \cup \Xi_2 \cup \cdots \cup \Xi_b \to \Xi_1 \cup \Xi_2 \cup \cdots \cup \Xi_b.$$

Hence, we have the following.

Proposition 9.3. *We denote by* $(F_1, F_2, \ldots, F_c; f)$ *the monodromy assortment of* Ψ*. Let* $\Xi_1, \Xi_2, \ldots, \Xi_b$ *be the barked components of the main fiber* X_{0,t_0} *of the deformation* $\pi_{t_0} : M_{t_0} \to \Delta_{t_0}$*. Then* F_1 *permutes* $\Xi_1, \Xi_2, \ldots, \Xi_b$*, and*

$$F_1\big|_{\Xi_1\cup\Xi_2\cup\cdots\cup\Xi_b} : \Xi_1 \cup \Xi_2 \cup \cdots \cup \Xi_b \to \Xi_1 \cup \Xi_2 \cup \cdots \cup \Xi_b$$

coincides with the monodromy homeomorphism of a degeneration of disjoint unions of b Riemann surfaces whose singular fiber is the enlargement of the tame simple crust Y, up to isotopy and conjugacy.

Remark 9.4. Given a degeneration $\pi : M \to \Delta$ of Riemann surfaces of genus g with singular fiber $X_0 = \sum_i m_i\Theta_i$, for a positive integer $b \geq 2$, the composition $\rho\circ\pi : M \to \Delta$ with the holomorphic map $\rho : \Delta \ni s \mapsto s^b \in \Delta$ is a "degeneration" of disjoint unions of b Riemann surfaces of genus g with singular fiber $bX_0 = \sum_i(m_i b)\Theta_i$. Furthermore, its monodromy homeomorphism F cyclically permutes the b Riemann surfaces, and the restriction of F^b to one of the Riemann surfaces coincides with the monodromy homeomorphism of $\pi : M \to \Delta$ up to isotopy and conjugacy.

Genera of barked components. Let us determine the genera of barked components. In the above argument, we showed that the restriction $\widetilde{\pi}|_{\widetilde{M}^\times} : \widetilde{M}^\times \to \Delta^\dagger$ of $\widetilde{\pi}$ to $\widetilde{M}^\times$ can be identified with a certain restriction of $\pi_0^\dagger : M_0^\dagger \to \Delta_0^\dagger$. In particular, the compactification of a general fiber of the latter coincides with that of a general fiber of the former. Thus the union of barked components coincides with a general fiber of $\widetilde{\pi}$: $\widetilde{M} \to \Delta^\dagger$.

Suppose that the tame simple crust Y is not multiple, that is, the greatest common divisor of the multiplicities of Y is one. Then the enlargement $\widetilde{Y}$ of Y is not multiple either. Thus a general fiber of $\widetilde{\pi}$: $\widetilde{M} \to \Delta^\dagger$ is connected (otherwise, the singular fiber $\widetilde{Y}$ of $\widetilde{\pi}$ would be multiple), which implies that the main fiber X_{0,t_0} of a deformation $\pi_{t_0} : M_{t_0} \to \Delta_{t_0}$ has exactly one barked component Ξ. Since both $\widetilde{Y}$ and the barked component Ξ are regarded as fibers of the same family $\widetilde{\pi}$, their arithmetic genera coincide.

Proposition 9.5. *Let* $\pi : M \to \Delta$ *be a degeneration and* $\Psi\colon \mathcal{M} \to \Delta \times \Delta^\dagger$ *be its barking family associated with a tame simple crust* Y*. If* Y *is not multiple, then the main fiber* X_{0,t_0} *of a deformation* $\pi_{t_0} : M_{t_0} \to \Delta_{t_0}$ *in* Ψ *has exactly one barked component, and its genus*

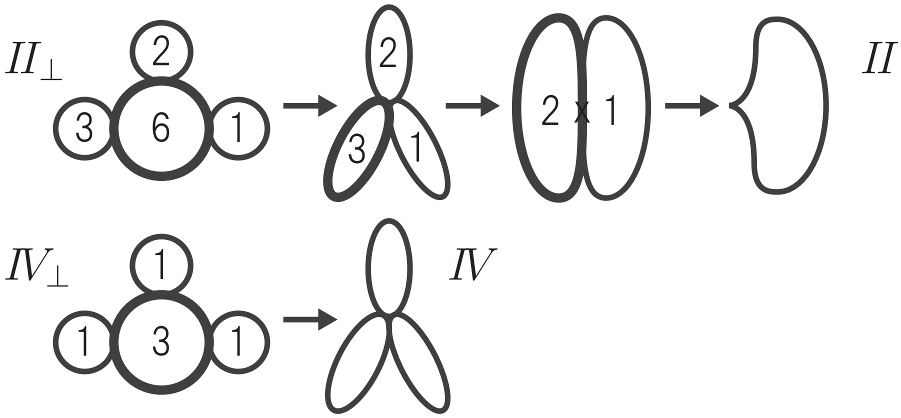

Fig. 5. Blowing-ups that make the singular fibers II and IV have at most normal crossings.

coincides with the arithmetic genus of the enlargement $\widetilde{Y}$ of Y.

Remark 9.6. In the case that Y is multiple, it cannot be determined whether the main fiber has one barked component or more.

Example 9.7. Recall that Kodaira [5] completely classified the singular fibers appearing in degenerations of elliptic curves — to be precise, relatively minimal degenerations of elliptic curves. For instance, the singular fiber II (Kodaira's well-known notation) is a rational curve with one cusp, and the singular fiber IV is a union of three nonsingular rational curves intersecting at one point. Note that they have worse singular points than nodes. For the singular fiber II (resp. IV), the three successive blowing-ups at the singular points (resp. the one blowing-up at the singular point) gives us a degeneration of elliptic curves with a singular fiber that has at most simple normal crossings, which we denote by $II_\perp$ (resp. $IV_\perp$), as seen in Figure 5. The arithmetic genera of $II_\perp$ and $IV_\perp$ are one.

Now, let Y be a tame simple crust for a degeneration $\pi\colon M \to \Delta$ and $\Psi\colon \mathcal{M} \to \Delta \times \Delta^\dagger$ be its barking family associated with Y. If Y is not multiple and if the enlargement of Y coincides with either $II_\perp$ or $IV_\perp$, then the barked component of the main fiber is unique and its genus is one by Proposition 9.5.

10. Useful lemmas for determination of singular fibers

In this section, we state some lemmas which help us to determinate the topological types of singular fibers appearing in splitting families. The proofs of the lemmas can be found in [8].

The first lemma is for general splitting families. For a singular fiber X, we denote by $\mathcal{E}(X)$ the *Euler contribution* of X, that is, we set $\mathcal{E}(X) := e(X) - 2(1 - g)$, where $e(X)$ is the topological Euler characteristic of the underlying reduced curve of X.

Lemma 10.1. *Let $\pi : M \to \Delta$ be a degeneration of Riemann surfaces of genus $g \geq 1$ with the singular fiber X_0 and let $\Psi : \mathcal{M} \to \Delta \times \Delta^\dagger$ be a splitting family of $\pi : M \to \Delta$ such that X_0 splits into singular fibers $X_1, X_2, \ldots, X_N$ $(N \geq 2)$. Then, we have the following.*

(1) $\mathcal{E}(X_i) \geq 0$ *for each* $i = 0, 1, \dots, N$.

(2) $\mathcal{E}(X_0) = \sum_{i=1}^{N} \mathcal{E}(X_i)$.

Now let $\pi : M \to \Delta$ be a linear degeneration of Riemann surfaces with singular fiber X_0. Suppose that X_0 has a simple crust Y with barking multiplicity l. Then, we have a barking family $\Psi : \mathcal{M} \to \Delta \times \Delta^\dagger$ of $\pi : M \to \Delta$ associated with Y.

Lemma 10.2. *Every subordinate fiber X appearing in $\Psi : \mathcal{M} \to \Delta \times \Delta^\dagger$ is a reduced curve at most with A-singularities*[4]. *In particular,* $\mathcal{E}(X) \geq 1$, *where the equality holds exactly when X is a Lefschetz fiber.*

Let **br** be a subbranch of a branch **Br** of X_0 for Y. Express them as

$$\begin{cases} \mathbf{Br} = m_1\Theta_1 + m_2\Theta_2 + \cdots + m_\lambda\Theta_\lambda, \text{ and} \\ \mathbf{br} = n_1\Theta_1 + n_2\Theta_2 + \cdots + n_\nu\Theta_\nu, \text{ or } \emptyset, \end{cases}$$

where[5] $0 \leq \nu \leq \lambda$, and $0 \leq ln_i \leq m_i$, $i = 1, 2, \dots, \nu$. Let Θ_0 be the core intersecting the component Θ_1 and let m_0 (resp. n_0) be the multiplicity of Θ_0 in X_0 (resp. Y). We say that **br** is *proportional* if $\nu = \lambda$ and

$$\frac{n_0}{m_0} = \frac{n_1}{m_1} = \cdots = \frac{n_\lambda}{m_\lambda}.$$

Now we assume that the singular fiber X_0 is stellar, that is, X_0 consists of exactly one core Θ_0 and some branches emanating from Θ_0. The following two lemmas give us the number of the subordinate fibers and that of their singularities.

Lemma 10.3. *Suppose that* (i) *the core* Θ_0 *is a Riemann sphere,* (ii) X_0 *has three branches,* (iii) *the core section* τ *for Y has no zero, and* (iv) *Y has no proportional subbranches. Let* m_0 (*resp.* n_0) *denote the multiplicity of the core* Θ_0 *in* X_0 (*resp.* Y)*. Then we have the following.*

(a) *Each deformation of the degeneration has exactly* $\bar{n}_0$ *subordinate fibers.*

(b) *Each subordinate fiber in* (a) *has c singularities.*

Here $c := \gcd(m_0, n_0)$*, the greatest common divisor of* m_0 *and* n_0*, and* $\bar{n}_0 := n_0/c$.

Lemma 10.4. *Suppose that* (i) *the core* Θ_0 *is a Riemann sphere,* (ii) X_0 *has three branches,* (iii) *the core section* τ *for Y has no zero, and* (iv) *Y has a proportional subbranch* $\mathbf{br} = n_1\Theta_1 + n_2\Theta_2 + \cdots + n_\lambda\Theta_\lambda$ *of a branch* $\mathbf{Br} = m_1\Theta_1 + m_2\Theta_2 + \cdots + m_\lambda\Theta_\lambda$ *of* X_0*. Then no other subbranches are proportional, and moreover we have the following.*

(a) *Each deformation of the degeneration has exactly* $\bar{n}_\lambda$ *subordinate fibers.*

(b) *Each subordinate fiber in* (a) *has c singularities.*

Here $c := \gcd(m_\lambda, n_\lambda)$*, the greatest common divisor of* m_λ *and* n_λ*, and* $\bar{n}_\lambda := n_\lambda/c$.

[4] An A-singularity is a singularity analytically equivalent to $y^2 = x^{e+1}$ for some positive integer e.

[5] We set $\nu := 0$ if $\mathbf{br} = \emptyset$. See the paragraph above Definition 5.1.

11. Barking families giving the same splitting

As an application of our results, we show an interesting example of two splitting families for one degeneration which give the same splitting (that is, the topological types of the singular fibers appearing in the two splitting families coincide) and which have, nevertheless, the different topological monodromies. This example indicates that the topological monodromies of splitting families play a very important role when we classify "topologically distinct" splitting families.

Let us consider the linear degeneration $\pi : M \to \Delta$ of Riemann surfaces of genus two whose singular fiber is stellar and is of the form

$$X_0 = 10\Theta_0 + \sum_{j=1}^{3} \mathbf{Br}^{(j)},$$

where the core Θ_0 is a Riemann sphere and the three branches, attached to Θ_0, are as follows:

$$\begin{cases} \mathbf{Br}^{(1)} = 5\Theta_1^{(1)}, \\ \mathbf{Br}^{(2)} = 4\Theta_1^{(2)} + 2\Theta_2^{(2)}, \\ \mathbf{Br}^{(3)} = 1\Theta_1^{(3)}. \end{cases}$$

Here the core Θ_0 intersects $\Theta_1^{(1)}$, $\Theta_1^{(2)}$, and $\Theta_1^{(3)}$.

Barking 1. We first define the connected subdivisor Y_1 as

$$Y_1 = 6\Theta_0 + \sum_{j=1}^{3} \mathbf{br}_1^{(j)}, \quad \text{where} \quad \begin{cases} \mathbf{br}_1^{(1)} = 3\Theta_1^{(1)}, \\ \mathbf{br}_1^{(2)} = 2\Theta_1^{(2)}, \\ \mathbf{br}_1^{(3)} = 1\Theta_1^{(3)}. \end{cases}$$

See Figure 6. Lemma 5.3 ensures that Y_1 has a core section over Θ_0 which has no zeros, and

- $\mathbf{br}_1^{(1)}$ is a subbranch of $\mathbf{Br}_1^{(1)}$ of type A_1,
- $\mathbf{br}_1^{(2)}$ is a subbranch of $\mathbf{Br}_1^{(2)}$ of type A_1,
- $\mathbf{br}_1^{(3)}$ is a subbranch of $\mathbf{Br}_1^{(3)}$ of type B_1.

Therefore Y_1 is a tame simple crust of X_0 and we can see that the enlargement $\widetilde{Y}_1$ of Y_1 is Y_1 itself. Now Y_1 and induces a barking family $\Psi_1 : \mathcal{M}_1 \to \Delta \times \Delta^\dagger$ with barking multiplicity 1, in which the singular fiber X_0 is deformed to the main fiber X_0' as depicted in Figure 6. Here X_0' has exactly one barked component and its genus is one, which follows from Proposition 9.5 together with the fact that $\widetilde{Y}_1$ $(= Y_1)$ is not multiple and its arithmetic genus is one (or Example 9.6).

The set of the subordinate fibers in each deformation of the degeneration $\pi : M \to \Delta$ for the barking family $\Psi_1 : \mathcal{M}_1 \to \Delta \times \Delta^\dagger$ consists of three Lefschetz fibers. In fact, since $\mathbf{br}^{(1)}$ is proportional and Y_1 satisfies the assumptions of Lemma 10.4, we see that there are exactly three subordinate fibers and each of them has exactly one singularity. On the other hand, we have

$$\mathcal{E}(X_0) = 8 \quad \text{and} \quad \mathcal{E}(X_0') = 5,$$

where $\mathcal{E}(X)$ denotes the Euler contribution[6] of a singular fiber X. Thus, by Lemma 10.1, the

[6]That is, $\mathcal{E}(X) := e(X) - 2(1 - g)$, where $e(X)$ is the topological Euler characteristic of the underlying reduced

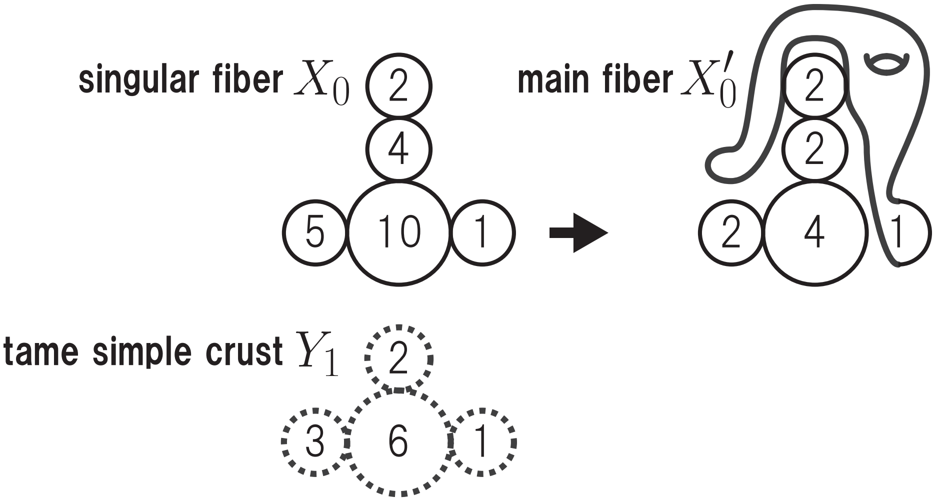

Fig. 6. The barking family associated with the tame simple crust Y_1 with barking multiplicity 1.

sum of the Euler contributions of the subordinate fibers is equal to three. Furthermore, from Lemma 10.2, we see that the Euler contribution of each of the three subordinate fibers is equal to one, and that the three subordinate fibers are all Lefschetz fibers.

Barking 2. We next define the connected subdivisor Y_2 as

$$Y_2 = 3\Theta_0 + \sum_{j=1}^{3} \mathbf{br}_2^{(j)}, \quad \text{where} \quad \begin{cases} \mathbf{br}_2^{(1)} = 1\Theta_1^{(1)}, \\ \mathbf{br}_2^{(2)} = 1\Theta_1^{(2)}, \\ \mathbf{br}_2^{(3)} = 1\Theta_1^{(3)}. \end{cases}$$

See Figure 7. Lemma 5.3 ensures that Y_2 has a core section over Θ_0 which has no zeros, and

- $\mathbf{br}_2^{(1)}$ is a subbranch of $\mathbf{Br}_2^{(1)}$ of type A_1,
- $\mathbf{br}_2^{(2)}$ is a subbranch of $\mathbf{Br}_2^{(2)}$ of type A_1,
- $\mathbf{br}_2^{(3)}$ is a subbranch of $\mathbf{Br}_2^{(3)}$ of type B_1.

Therefore Y_2 is a tame simple crust of X_0 and we can see that the enlargement $\widetilde{Y}_2$ of Y_2 is Y_2 itself. Now Y_2 induces a barking family $\Psi_1 : \mathcal{M}_1 \to \Delta \times \Delta^\dagger$ with barking multiplicity 1, in which the singular fiber X_0 is deformed to the main fiber X''_0 as depicted in Figure 7. Here X''_0 has exactly one barked component and its genus is one, which follows from Proposition 9.5 together with the fact that $\widetilde{Y}_2$ $(= Y_2)$ is not multiple and its arithmetic genus is one (or Example 9.6).

The set of the subordinate fibers in each deformation of the degeneration $\pi : M \to \Delta$ for the barking family $\Psi_2 : \mathcal{M}_2 \to \Delta \times \Delta^\dagger$ consists of three Lefschetz fibers. In fact, since Y_2 satisfies the assumptions of Lemma 10.3, we see that there are exactly three subordinate fibers and each of them has exactly one singularity. On the other hand, we have

$$\mathcal{E}(X_0) = 8 \quad \text{and} \quad \mathcal{E}(X''_0) = 5.$$

Thus by the same argument as that for Y_1, we see that the three subordinate fibers are all

curve of X.

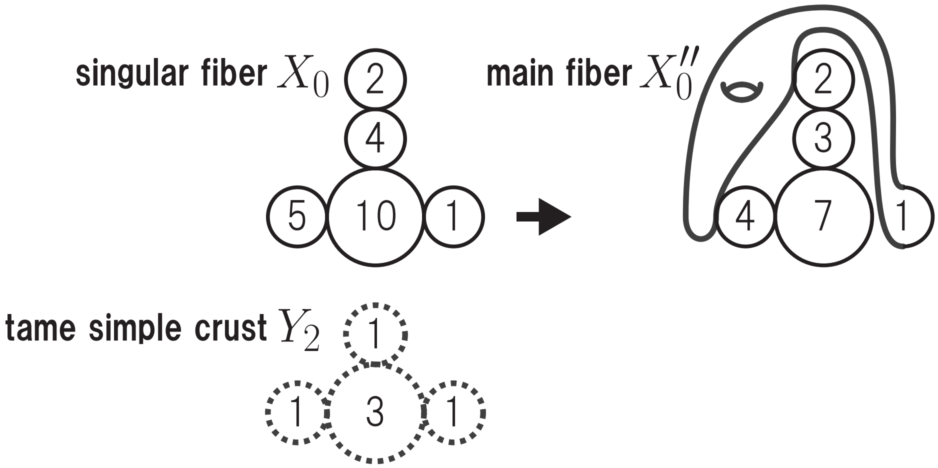

Fig. 7. The barking family associated with the tame simple crust Y_2 with barking multiplicity 1.

Lefschetz fibers.

Comparison. The main fibers X_0' and X_0'' appearing in the deformations of the barking families $\Psi_1 : \mathcal{M}_1 \to \Delta \times \Delta^\dagger$ and $\Psi_2 : \mathcal{M}_2 \to \Delta \times \Delta^\dagger$, respectively, apparently have different topological types. However, both of them turn into Lefschetz fibers, by successive blowing-downs. To be more precise, recall that every (-1)-curve in a complex surface M is preserved under an arbitrary deformation of M by Kodaira's stability theorem [6]. Namely, there exists an analytic family of (-1)-curves in $\mathcal{M}_i$ for each $i = 1, 2$. Furthermore, by [2], we can blow down them simultaneously and then the resulting family is a splitting family of the degeneration obtained from $\pi : M \to \Delta$ by a blowing-down. Repeating this process four times, we obtain a splitting family $\overline{\Psi}_i : \overline{\mathcal{M}}_i \to \Delta \times \Delta^\dagger$ of the relatively minimal degeneration $\overline{\pi} : \overline{M} \to \Delta$ $(i = 1, 2)$. In the splitting family $\overline{\Psi}_1$ (resp. $\overline{\Psi}_2$), the singular fiber $\overline{X_0'}$ (resp. $\overline{X_0''}$), which is obtained by blowing down the main fiber X_0' (resp. X_0''), is a Lefschetz fiber. Hence, both $\overline{\Psi}_1$ and $\overline{\Psi}_2$ split the singular fiber of the minimal degeneration $\overline{\pi} : \overline{M} \to \Delta$ into four Lefschetz fibers. In particular, they give the same splitting.

Now we investigate the monodromy assortment of these splitting families. Note that the Lefschetz fiber $\overline{X_0'}$ is obtained from the unique barked component Ξ of the main fiber X_0' by identifying the two attachment points on it in the above blowing-down process. Since the complement of the family of (-1)-curves is preserved under the simultaneous blowing-downs, the monodromy homeomorphism F' on the singular fiber $\overline{X_0'}$ of the splitting family $\overline{\Psi}_1 : \overline{\mathcal{M}}_1 \to \Delta \times \Delta^\dagger$ is induced from that on the the barked component Ξ of the main fiber X_0'. From Theorem 6.2, we see that the monodromy homeomorphism on the the barked component Ξ corresponds to the topological monodromy of the degeneration whose singular fiber is the enlargement $\widetilde{Y}_1$ $(= Y_1)$. Here, the multiplicity of the core Θ_0 in Y_1 is six. Since the topological monodromy of a degeneration with stellar singular fiber whose core has multiplicity m is periodic of order m, the monodromy homeomorphism F' corresponds to a periodic mapping classes of order six. Similarly, the monodromy homeomorphism F'' of the singular fiber $\overline{X_0''}$ corresponds to the topological monodromy of the degeneration whose singular fiber is the enlargement $\widetilde{Y}_2$ $(= Y_2)$, that is, a periodic mapping class of order three.

Thus, the monodromy homeomorphisms F' and F'' are distinct up to isotopy and conjugacy. On the other hand, since the three subordinate fibers in the respective barking family form one subordinate tassel, the monodromy homeomorphism F' does not correspond to the monodromy homeomorphism of the subordinate tassel of $\overline{\Psi}_2$. Thus the monodromy assortment of the splitting families $\overline{\Psi}_1$ and $\overline{\Psi}_2$ are distinct up to isotopy and conjugacy. Hence, we have the following.

Proposition 11.1. *There exist two splitting families for one degeneration that have different topological monodromies, although they give the same splitting.*

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