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THE HAUSDORFF DIMENSION OF THE REGION OF MULTIPLICITY ONE OF OVERLAPPING ITERATED FUNCTION SYSTEMS ON THE INTERVAL

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Abstract

We consider iterated function systems on the unit interval generated by two contractive similarity transformations with the same similarity ratio. When the ratio is greater than or equal to $1/2$, the limit set is the interval itself and the code map is not one-to-one. We study the set of points of the limit set having unique addresses. We obtain a formula for the Hausdorff dimension of the set when the similarity ratio belongs to certain intervals by applying the concept of graph directed Markov system.

1. Introduction

The Hausdorff dimension of the limit set of iterated function system is studied well when the iterated function system satisfies the open set condition. However when it does not satisfy the open set condition, it is difficult to evaluate the Hausdorff dimension of the limit set in general. To understand the structure of the limit set of overlapping iterated function system, we focus our attention on the region of multiplicity one of the limit set in this paper.

Let us consider iterated function systems on the unit interval $I = [0, 1]$ generated by two contractive similarity transformations

$$f_0(x) = ax, \qquad f_1(x) = ax + (1 - a) \tag{1}$$

with similarity ratio $0 < a < 1$. If a is grater than or equal to $1/2$, the limit set of the iterated function system $S(a) = \{f_0, f_1\}$ is the interval itself and we say that such an iterated function system is overlapping. We consider overlapping iterated function systems, and study the subset of points of the limit set having unique addresses which we denote by $J_1(S(a))$. Fig.1 shows $J_1(S(a))$ for values of a between $1/2$ and the golden ratio $g = (\sqrt{5} - 1)/2$. Note that $J_1(S(a)) = \{0, 1\}$ for $a \geq g$ (Proposition 2.2).

In this paper, we explicitly determine the Hausdorff dimension of $J_1(S(a))$ for values of a described below. For $k = 1, 2, \dots$, let b_k denote the unique value of $1/2 < a < 1$ satisfying

$$f_0 f_1^k f_0(1) = f_1(0). \tag{2}$$

Likewise, let c_k denote the unique value of $1/2 < a < 1$ satisfying

$$f_0 f_1^{k+1}(0) = f_1(0). \tag{3}$$

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Table 1. b_k,c_k,λ_k,$\log\lambda_k$ and $\log(2^{k+2}-6)/(k+3)$

k	b_k	c_k	λ_k	$\log\lambda_k$	$\log(2^{k+2}-6)/(k+3)$
1	0.5698402822	0.6180339754	1.0	0.0	
2	0.5356873572	0.5436890423	1.6180339887	0.4812118251	0.4605170186
3	0.5172810853	0.5187900364	1.8392867552	0.6093778634	0.5430160897
4	0.5083449185	0.5086604059	1.9275619755	0.6562559792	0.5800632872
5	0.5040674508	0.5041382611	1.9659482366	0.6759746921	0.6005026306
6	0.5020004213	0.5020170510	1.9835828434	0.6849047264	0.6134956575
7	0.5009901822	0.5009941757	1.9919641966	0.6891211854	0.6226536669
8	0.5004921257	0.5004931390	1.9960311797	0.6911607989	0.6295995634
9	0.5002452433	0.5002454817	1.9980294703	0.6921614300	0.6351404166
10	0.500122398	0.5001224577	1.9990186327	0.6926563765	0.6397154038

We will prove in Lemma 3.1 that

$$\frac{1}{2} < \cdots < b_k < c_k < \cdots < b_2 < c_2 < b_1 < c_1,$$

and that the sequences $\{b_k\}$ and $\{c_k\}$ converge to $1/2$ as k increases. The main theorem of this paper is the following.

Theorem 1.1. *For any a with $b_k \le a \le c_k$ $(k \ge 2)$, the Hausdorff dimension of $J_1(S(a))$ is given by*

$$\dim_H J_1(S(a)) = -\frac{\log\lambda_k}{\log a},$$

where λ_k is the largest eigenvalue of the matrix A_k given in Section 3.

We also have a simple formula which gives a lower bound of the Hausdorff dimension of $J_1(S(a))$.

Theorem 1.2. *For any a with $b_k \le a \le c_k$ $(k \ge 2)$, the Hausdorff dimension of $J_1(S(a))$ satisfies*

$$\dim_H J_1(S(a)) \ge -\frac{\log(2^{k+2}-6)}{(k+3)\log a}.$$

Table 1 shows the values of b_k, c_k, λ_k, $\log\lambda_k$ and $\log(2^{k+2}-6)/(k+3)$ for k up to 10. To prove the theorem, we define a graph directed Markov system. The matrix A_k is its incidence matrix.

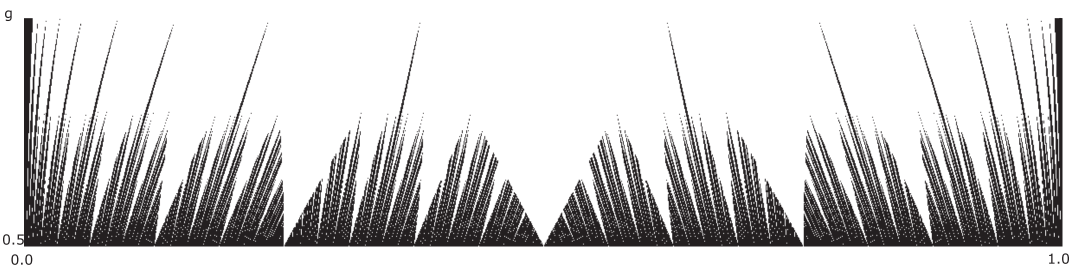

Fig. 1. $J_1(S(a))$ for a between $1/2$ and the golden ratio g

2. Preliminary

2.1. Multiplicity function. Let Σ be a finite set of symbols. We denote by Σ^n the set of codes of length n of symbols in Σ. The set of all finite codes is denoted by $\Sigma^* = \bigcup_{n=0}^{\infty} \Sigma^n$. The length of $\omega \in \Sigma^*$ is denoted by $|\omega|$. Given an infinite code

$$\omega = \omega_1\omega_2 \cdots \in \Sigma^{\infty},$$

we denote the finite code consisting of the first n symbols of ω by

$$\omega|_n = \omega_1\omega_2 \cdots \omega_n.$$

We deal with iterated function systems (IFS). Let X be a non-empty compact subset of the Euclidean space $\mathbf{R}^d$. A similarity iterated function system is a family of contracting similarity transformations

$$f_i : X \to X \quad (i \in \Sigma).$$

Let $S = \{f_i : I \to I \mid i \in \Sigma\}$ be a similarity iterated function system of the unit interval. Given a code $\omega = \omega_1\omega_2 \cdots \omega_n \in \Sigma^n$, we define $f_\omega : I \to I$ by

$$f_\omega = f_{\omega_1} \circ f_{\omega_2} \circ \cdots \circ f_{\omega_n}.$$

The code map

$$\pi : \Sigma^{\infty} \to I$$

is defined by

$$\pi(\omega) = \bigcap_{n=1}^{\infty} f_{\omega|_n}(I) \quad (\omega \in \Sigma^{\infty}).$$

Its image $\pi(\Sigma^{\infty})$ is called the limit set of the iterated function system, which we denote by $J(S)$.

If an iterated function system S satisfies

$$f_i(J(S)) \cap f_j(J(S)) = \emptyset$$

for any i, j with $i \neq j$, we say that S is totally disconnected. If not, we say that S is overlapping. If S is totally disconnected, the code map π is one-to-one and every point $x \in J(S)$ has a unique address $\pi^{-1}(x)$. But in case of overlapping iterated function system, π is not one-to-one and some limit points $x \in J(S)$ have more than one address. The multiplicity function

$$m : I \to \mathbf{N} \cup \{\infty\}$$

is given by

$$m(x) = \sharp\{\omega \in \Sigma^{\infty} \mid \pi(\omega) = x\} \quad (x \in I).$$

For $k = 0, 1, \ldots,$ we define $J_k(S)$ by

$$J_k(S) = \{x \in I \mid m(x) = k\}.$$

Then the limit set decomposes into a disjoint union as

$$J(S) = J_1(S) \cup J_2(S) \cup \cdots \cup J_\infty(S).$$

For totally disconnected iterated function systems, we have $J_1(S) = J(S)$. Here we are interested in $J_1(S)$ for overlapping iterated function systems.

Now let us consider the iterated function system given by (1). If $a < 1/2$, the system is totally disconnected. The limit set $J(S(a)) = J_1(S(a))$ is the Cantor set, and its Hausdorff dimension is given by the Hutchinson's theorem ([3]). For $a = 1/2$, $J(S) = I = J_1(S(1/2)) \cup J_2(S(1/2))$ where $J_2(S(1/2))$ is countable. The Hausdorff dimension of $J_1(S(1/2))$ is therefore 1. When $a > 1/2$, the Hausdorff dimension of $J_1(S(a))$ is generally difficult to determine. But in the cases described in Theorem 1.1 we can determine the Hausdorff dimension.

Assume that $a > 1/2$. We define

$$F = f_0(I) \cap f_1(I) = [1 - a, a],$$

and

$$F^* = \bigcup_{\mu \in \{0,1\}^*} f_\mu(F).$$

Proposition 2.1. *Consider an iterated function system $S(a) = \{f_0, f_1\}$ given by* (1). *If $a > 1/2$, then we have*

$$\bigcup_{m \geq 2} J_m(S(a)) = F^*.$$

Proof. For any $x \in \bigcup_{m \geq 2} J_m(S(a))$, since $m(x) \geq 2$, there exist distinct codes $\omega, \omega' \in \{0,1\}^\infty$ such that $\pi(\omega) = \pi(\omega') = x$. Denote the maximal leading subcode common to ω and ω' by μ. Then we may assume that

$$\omega = \mu 0 \tilde{\omega} \qquad (\tilde{\omega} \in \{0,1\}^\infty),$$
$$\omega' = \mu 1 \tilde{\omega}' \qquad (\tilde{\omega}' \in \{0,1\}^\infty).$$

It follows that

$$x = f_\mu(\pi(0\tilde{\omega})) = f_\mu(\pi(1\tilde{\omega}')).$$

Therefore,

$$x \in f_\mu(f_0(I)) \cap f_\mu(f_1(I)) = f_\mu(F).$$

Conversely, if $x \in F^*$, there exists $\mu \in \{0,1\}^*$ such that

$$x \in f_\mu(F) = f_{\mu 0}(I) \cap f_{\mu 1}(I).$$

Therefore, there exist infinite codes $\omega = \mu 0 \cdots$ and $\omega' = \mu 1 \cdots$ such that $\pi(\omega) = \pi(\omega') = x$. □

Proposition 2.2. *If a is greater than or equal to the golden ratio g, then $J_1(S(a)) = \{0, 1\}$.*

Proof. Since the left end point of F is $1 - a$ and the right end point of $f_0(F)$ is a^2, $g < a$ implies

$$f_0(F) \cap F \neq \emptyset,$$

and hence

$$f_0^{i+1}(F) \cap f_0^i(F) \neq \emptyset$$

for all $i = 0, 1, \ldots$. Therefore we have

$$\bigcup_{i=0}^{\infty} f_0^i(F) = (0, a],$$

for the left end point $f_0^i(1-a) = a^i(1-a)$ of $f_0^i(F)$ converges to 0 as i increases. Similarly, we have

$$\bigcup_{i=0}^{\infty} f_1^i(F) = [1-a, 1).$$

Thus, the interval $(0, 1)$ contains only points of multiplicity 2 or more. On the other hand, 0 has a unique address, for if $\omega = 0^n 1\tilde{\omega}$, $\pi(\omega) \in f_0^n f_1(I)$ which does not contain 0. Likewise 1 also has a unique address. □

2.2. Graph directed Markov systems. In the proof of the main theorem we use the concept of graph directed Markov system. A graph directed Markov system is based on a directed multigraph and an associated incidence matrix, (V, E, A, i, t). The set of vertices V and the set of directed edges E are assumed to be finite. The function $A : E \times E \to \{0, 1\}$ is called an incidence matrix. It determines which edges may follow a given edge. For each edge e, $i(e)$ is the initial vertex of e and $t(e)$ is the terminal vertex of e. So, it holds that $A_{uv} = 1$ if and only if $t(u) = i(v)$. We will consider finite and infinite code spaces of edges consistent with the incidence matrix. We define the infinite code space by

$$E_A^{\infty} = \{\eta \in E^{\infty} \mid A_{\eta_i \eta_{i+1}} = 1 \text{ for all } i \geq 1\}.$$

We also define

$$E_A^n = \{\eta \in E^n \mid A_{\eta_i \eta_{i+1}} = 1 \text{ for all } i \geq 1\}.$$

The space of codes of finite length is denoted by $E_A^* = \bigcup_{n=1}^{\infty} E_A^n$.

We say that A is irreducible if for all $a, b \in E$, there exists $\eta \in E_A^*$ such that $a\eta b \in E_A^*$.

A graph directed Markov system (GDMS) consists of the following

- a directed multigraph, (V, E, i, t),
- an incidence matrix A,
- a set of nonempty compact spaces $\{X_v \subset \mathbf{R}^d \mid v \in V\}$,
- for every $e \in E$, a similarity transformation $f_e : X_{i(e)} \to X_{t(e)}$ with a Lipschitz constant K $(0 < K < 1)$.

Briefly the set

$$S = \{f_e : X_{t(e)} \to X_{i(e)} \mid e \in E\}$$

is called a GDMS. When the vertex set V is a singleton, S is an iterated function system. We can generalize the code map to GDMS.

DEFINITION 2.3. The code map $\pi : E_A^\infty \to \bigcup_{v\in V} X_v$ is defined by

$$\pi(\eta) = \bigcap_{l=1}^{\infty} f_{\eta|_l}(X_{t(\eta_l)}) \quad (\eta \in E_A^\infty).$$

The limit set of the graph directed Markov system S is defined to be the image of the code map,

$$J(S) = \bigcup_{\eta\in E_A^\infty} \pi(\eta).$$

With respect to the product topology, the code space E_A^∞ is compact and the code map π is continuous. Hence, the limit set $J(S)$ is compact. Since $f_{\eta|_l}(X_{t(\eta_l)})$ shrinks to a point uniformly as $l \to \infty$, we can also express the limit set as

$$J(S) = \bigcap_{l=1}^{\infty} \bigcup_{\eta\in E_A^l} f_\eta(X_{t(\eta_l)}).$$

2.3. The Hausdorff dimension. We need a couple of conditions to evaluate the Hausdorff dimension of the limit set. The first one is the open set condition.

DEFINITION 2.4. We say that a GDMS $S = \{f_e : X_{t(e)} \to X_{i(e)} \mid e \in E\}$ satisfies the open set condition if there exists a nonempty open set $U \subset \bigcup_{v\in V} X_v$ such that for all $e, e' \in E$ $(e \neq e')$,

$$f_e(U \cap X_{t(e)}) \cap f_{e'}(U \cap X_{t(e')}) = \emptyset \qquad \text{and} \qquad \bigcup_{e\in E} f_e(U \cap X_{t(e)}) \subset U.$$

The second condition we need is the bounded distortion property. We denote the derivative of f at x by f'_x, and define $|f'(x)| = \max\{|f'_x(y)|\}$, where the maximum is taken over all unit vectors y in the tangent space.

DEFINITION 2.5. A GDMS $S = \{f_e : X_{t(e)} \to X_{i(e)} \mid e \in E\}$ satisfies the bounded distortion property if there exists $K \geq 1$ such that $\frac{|f'_\eta(x)|}{|f'_\eta(y)|} \leq K$ for all $\eta \in E_A^n$ $(n = 1, 2, \dots)$ and $x, y \in X_{t(\eta_n)}$.

If a GDMS satisfies these conditions and the incidence matrix is irreducible, we can evaluate the Hausdorff dimension of the limit set as follows ([2]).

Theorem 2.6 (Mauldin and Urbański)**.** *Suppose that a GDMS $S = \{f_e : X_{t(e)} \to X_{i(e)} \mid e \in E\}$ satisfies the open set condition and the bounded distortion property, and that the incidence matrix A is irreducible. Define $P(t)$ by*

$$P(t) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{\eta\in E_A^n} \|f'_\eta\|^t.$$

Then the Hausdorff dimension of the limit set is given by

$$\dim_H J(S) = \sup\{t > 0 \mid P(t) > 0\} = \inf\{t > 0 \mid P(t) < 0\}.$$

When all the transformations are similarity transformations, we note that it is also possible to evaluate the Hausdorff dimension by a method similar to the proof of Hutchinson's theorem ([3]).

3. The structure of the GDMS

We first prove the following.

Lemma 3.1. *Let b_k denote the unique value of $1/2 < a < 1$ satisfying* (2). *Likewise, let c_k denote the unique value of $1/2 < a < 1$ satisfying* (3). *Then,*

$$\frac{1}{2} < \cdots < b_k < c_k < \cdots < b_2 < c_2 < b_1 < c_1.$$

Furthermore, the sequences b_k and c_k converge to $1/2$ as k increases.

Proof. Define $P_k(a)$ and $Q_k(a)$ by

$$\begin{aligned} P_k(a) &= a^{k+2} - a^{k+1} + 2a - 1, \\ Q_k(a) &= -a^{k+2} + 2a - 1, \end{aligned}$$

and (2) is equivalent to $P_k(a) = 0$ and (3) is equivalent to $Q_k(a) = 0$.

First, we show that $b_k < c_k$. For all $k \geq 2$, we have

$$\begin{aligned} P_k'(a) &= (k+2)a^{k+1} - (k+1)a^k + 2, \\ P_k''(a) &= (k+2)(k+1)a^k - k(k+1)a^{k-1}. \end{aligned}$$

It follows that

$$P_k''(a) \begin{cases} \leq 0 & \text{if } 1/2 < a \leq k/(k+2) \\ > 0 & \text{if } k/(k+2) < a < 1. \end{cases}$$

Hence, for all a with $1/2 < a < 1$, we have

$$\begin{aligned} P_k'(a) &\geq P_k'\left(\frac{k}{k+2}\right) \\ &= (k+2)\left(\frac{k}{k+2}\right)^{k+1} - (k+1)\left(\frac{k}{k+2}\right)^k + 2 \\ &= 2 - \left(\frac{k}{k+2}\right)^k \\ &> 0. \end{aligned}$$

Therefore $P_k(a)$ is monotonically increasing for $1/2 < a < 1$. On the other hand, since $-c_k^{k+2} + 2c_k - 1 = 0$, we have

$$\begin{aligned} P_k(c_k) &= c_k^{k+2} - c_k^{k+1} + 2c_k - 1 \\ &= 2c_k^{k+2} - c_k^{k+1} \\ &= c_k^{k+1}(2c_k - 1) \\ &> 0, \end{aligned}$$

for $\frac{1}{2} < c_k < 1$. Since $P_k(b_k) = 0$, we have $b_k < c_k$.

Next, we show that $c_{k+1} < b_k$. Let us first see that $b_k < 0.536$. For all $0 < k < l$, since $b_l^{l+2} - b_l^{l+1} + 2b_l - 1 = 0$, we have

$$P_k(b_l) = b_l^{k+2} - b_l^{k+1} + 2b_l - 1$$

$$= (b_l^{k+2} - b_l^{k+1})(1 - b_l^{l-k}).$$

Thus if $0 < k < l$, we have $P_k(b_l) < 0$. Then we have $b_l < b_k \leq b_2 < 0.536$.(See Table 1.) Now we have

$$Q_k'(a) = -(k+2)a^{k+1} + 2,$$

and

$$Q_k''(a) = -(k+2)(k+1)a^k < 0,$$

for $\frac{1}{2} < a < 1$. Since $Q_k'(\frac{1}{2}) = -(k+2)(\frac{1}{2})^{k+1} + 2 > 0$ and $Q_k'(1) = -k < 0$, and the function $Q_k'(a)$ is continuous and monotonically decreasing, there exists a unique value a satisfying $Q_k'(a) = 0$, which we denote by $C(k)$. Note that $C(k)$ is greater than $2/3$ since $Q_k'(\frac{2}{3}) > 0$ for all $k \geq 2$. When $a < C(k)$, the function $Q_k(a)$ is increasing and $Q_k(C(k)) = \frac{2C(k)(k+1)}{k+2} - 1 > 0$. Thus, we see $\frac{1}{2} < c_k < C(k)$. Since $b_k^{k+2} - b_k^{k+1} + 2b_k - 1 = 0$, we have

$$\begin{aligned}
Q_{k+1}(b_k) &= -b_k^{k+3} + 2b_k - 1 \\
&= -b_k^{k+3} - b_k^{k+2} + b_k^{k+1} \\
&= -b_k^{k+1}\left\{\left(b_k + \frac{1}{2}\right)^2 - \frac{5}{4}\right\} \\
&> 0
\end{aligned}$$

for $b_k < 0.536$. Since b_k is smaller than $C(k+1)$, we have $c_{k+1} < b_k$.

Finally, to see that b_k and c_k converge to $1/2$, let us assume that c_k converged to $1/2 + \epsilon$ $(\epsilon > 0)$. We have

$$\frac{1}{2} + \epsilon < c_k < C(k)$$

for all k. Since $Q_k(a)$ is increasing for $a < C(k)$, and since $Q_k(c_k) = 0$, we have

$$\begin{aligned}
Q_k(1/2 + \epsilon) &= -(1/2 + \epsilon)^{k+2} + 2\epsilon \\
&< 0
\end{aligned}$$

for all k. This is a contradiction. This completes the proof. □

In the above proof, we have also shown:

REMARK 3.2. If $l < k$, then $b_k < c_k < b_l < c_l$ and $Q_l(a) < 0$ for all a with $b_k \leq a \leq c_k$.

REMARK 3.3. If $b_k \leq a$, we have $P_k(a) \geq 0$, which is equivalent to $f_0 f_1^k f_0(1) \geq f_1(0)$. Also if $a < c_k$, we have $Q_k(a) < 0$, which is equivalent to $f_0 f_1^{k+1}(0) < f_1(0)$. Hence if $b_k \leq a < c_k$, we have the right end point of $f_0 f_1^k(F)$ contained in F, but not the whole interval. We also have the left end point of $f_1 f_0^k(F)$ contained in F, but not the whole interval.

Now we consider the iterated function system $S(a) = \{f_0, f_1\}$ defined by (1). We denote its limit set by $J(S(a))$.

If $a = 1/2$, for any k, we have

$$J(S(a)) = I = \bigcup_{\omega \in \{0,1\}^k} f_\omega(I), \tag{4}$$

where the intervals $f_\omega(I)$ line up from 0 to 1 according to the order of binary integers $\omega_1\omega_2\cdots\omega_k$ (2). (See Fig. 2(top).)

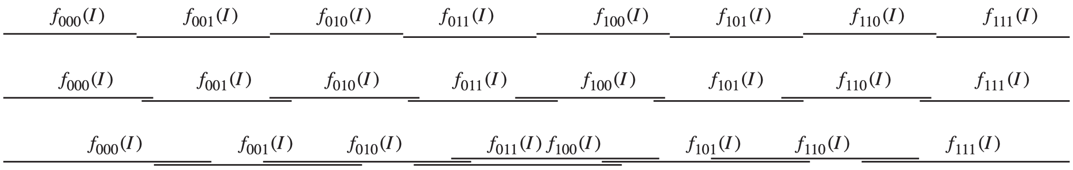

Fig. 2. The intervals $f_\omega(I)$ for $\omega \in \{0,1\}^3$ when $a = 0.5$ (top), $a = 0.52$ (middle), and $a = 0.58$ (bottom).

If a is slightly larger than $1/2$, (4) still holds with slightly overlapping intervals as shown in Fig. 2 (middle).

In the following, for any intervals I and I', we define $I < I'$ if and only if $x < y$ for any $x \in I$ and $y \in I'$.

Lemma 3.4. *Suppose that* $1/2 < a < b_{k-1}$. *Then, we have* $f_{\omega-1}(I) < f_{\omega+1}(I)$ *for all* $\omega \in \{0,1\}^{k+1} \setminus \{0\cdots0, 1\cdots1\}$.

Proof. It suffices to show that

$$f_{\omega-1}(1) < f_{\omega+1}(0) \tag{5}$$

for all $\omega \in \{0,1\}^{k+1} \setminus \{0\cdots0, 1\cdots1\}$. Denote by μ the maximal leading subcode common to $\omega - 1$ and $\omega + 1$. Since $(\omega+1) - (\omega-1) = 2$, we have either

$$\omega - 1 = \mu 01\cdots11, \qquad \omega + 1 = \mu 10\cdots01,$$

or

$$\omega - 1 = \mu 01\cdots10, \qquad \omega + 1 = \mu 10\cdots00.$$

The function f_μ is monotonically increasing. Hence the condition (5) is equivalent to either

$$f_0(1) = f_0 f_1^{n-1}(1) < f_1 f_0^{n-2} f_1(0)$$

or

$$f_0 f_1^{n-2} f_0(1) < f_1 f_0^{n-1}(0) = f_1(0)$$

for all $1 \le n \le k+1$ accordingly. Both inequalities hold if $a < b_{n-2}$ for all $1 \le n \le k$ by Lemma 3.1. □

We define X_ω to be the interval between $f_{\omega-1}(I)$ and $f_{\omega+1}(I)$. (See Fig. 3.)

Definition 3.5. Suppose that $1/2 < a < b_{k-1}$. For any $\omega \in \{0,1\}^{k+1}$, we define the interval X_ω by

$$\begin{aligned} X_\omega &= f_\omega(I) - \mathrm{int}(f_{\omega-1}(I)) - \mathrm{int}(f_{\omega+1}(I)) \quad (\omega \ne 0\cdots0, 1\cdots1), \\ X_{0\cdots0} &= f_{0\cdots0}(I) - \mathrm{int}(f_{0\cdots01}(I)), \end{aligned}$$

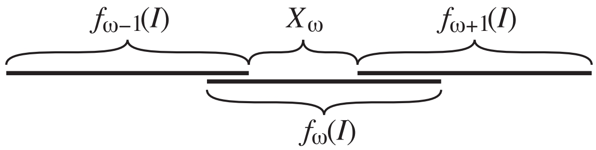

Fig. 3. X_ω

$$X_{1\cdots1} = f_{1\cdots1}(I) - \mathrm{int}(f_{1\cdots10}(I)).$$

In terms of end points, for $\omega \neq 0\cdots0, 1\cdots1$ we have

$$\begin{aligned} X_\omega &= [f_{\omega-1}(1), f_{\omega+1}(0)] \\ &= [f_{\alpha_1}\ldots f_{\alpha_{k+1}}(1),\ f_{\beta_1}\ldots f_{\beta_{k+1}}(0)] \end{aligned}$$

where $\omega - 1 = \alpha_1 \ldots \alpha_{k+1}$ and $\omega + 1 = \beta_1 \ldots \beta_{k+1}$, and

$$\begin{aligned} X_{0\cdots0} &= [0, f_0 \ldots f_0 f_1(0)], \\ X_{1\cdots1} &= [f_1 \ldots f_1 f_0(1), 1]. \end{aligned}$$

We note that X_ω is non-empty if $b_k \leq a \leq c_k$ by Lemma 3.4. We consider the lexicographical order of the code space $\{0,1\}^n$. For any $\omega, \tau \in \{0,1\}^n$ with $\omega \neq \tau$, take an integer i such that

$$i = \min\{1 \leq j \leq n \mid \omega_j \neq \tau_j\}.$$

Then we denote $\omega < \tau$ if $\omega_i < \tau_i$.

Lemma 3.6. *Suppose that $b_k \leq a \leq c_k$, and $l \leq k+2$. For any $\omega, \tau \in \{0,1\}^l$, if $\omega < \tau$ we have $f_\omega(x) < f_\tau(x)$ for all $x \in I$.*

Proof. We prove the lemma by induction. It is clear that $f_0(x) < f_1(x)$ for all $x \in I$. Assume that the statement holds for l and hence

$$f_{\omega_1} \cdots f_{\omega_l}(x) < f_{\tau_1} \cdots f_{\tau_l}(x)$$

for all $x \in I$ when $\omega < \tau$. Then we have

$$\begin{aligned} f_0 f_{\omega_1} \cdots f_{\omega_l}(x) < f_0 f_{\tau_1} \cdots f_{\tau_l}(x) < f_0 f_1 \cdots f_1(x), \\ f_1 f_0 \cdots f_0(x) < f_1 f_{\omega_1} \cdots f_{\omega_l}(x) < f_1 f_{\tau_1} \cdots f_{\tau_l}(x). \end{aligned}$$

Since $b_k \leq a \leq c_k$, we have

$$f_{10\ldots0}(x) - f_{01\ldots1}(x) = a^{l+1} - 2a + 1 = -Q_{l-1}(a) > 0$$

by Remark 3.2. Therefore the statement holds for $l+1$. This completes the proof. □

Lemma 3.7. *Suppose that $1/2 < a < b_{k-1}$. Then for all $\omega \in \{0,1\}^k$, we have*

$$f_\omega(I) \cap f_{\omega+1}(I) = f_\mu(F),$$

where μ is the maximal leading subcode common to ω and $\omega + 1$.

Proof. We have

$$\omega = \mu 01 \cdots 1, \qquad \omega + 1 = \mu 10 \cdots 0.$$

It follows that for $0 \le n \le k-1$,

$$f_\omega(I) \cap f_{\omega+1}(I) = f_\mu(f_0 f_1^n(I) \cap f_1 f_0^n(I)),$$

where

$$f_0 f_1^n(I) = [a - a^{n+1}, a],$$
$$f_1 f_0^n(I) = [1 - a, a^{n+1} + (1-a)].$$

Since $a^{n+1} + (1-a) - a = -Q_{n-1}(a) > 0$ by Remark 3.2, we conclude that

$$f_0 f_1^n(I) \cap f_1 f_0^n(I) = F = [1-a, a].$$

This completes the proof. □

Now we define the GDMS $S_k(a)$ for $b_k \le a \le c_k$ $(k = 1, 2, \dots)$. The multigraph with associated incidence matrix (V_k, E_k, A_k, i, t) is defined as follows. The vertex set is

$$V_k = \{0, 1\}^{k+1}.$$

Elements of V_k are codes of length $k+1$; we also regard them as $(k+1)$-digit binary numbers. The edge set E_k is then defined by

$$E_k = \{(\omega, \phi_0(\omega)) \in V_k \times V_k \mid \omega \ne 1 \cdots 1\} \cup \{(\omega, \phi_1(\omega)) \in V_k \times V_k \mid \omega \ne 0 \cdots 0\},$$

where the maps $\phi_0, \phi_1 : V_k \to V_k$ are defined by

$$\phi_0(\omega) = \left\lfloor \frac{\omega}{2} \right\rfloor, \qquad \phi_1(\omega) = \left\lfloor \frac{\omega}{2} \right\rfloor + 2^k.$$

Here, $\left\lfloor \frac{\omega}{2} \right\rfloor$ is the maximum integer not greater than $\omega/2$. The incidence matrix

$$A_k : E_k \times E_k \to \{0, 1\}$$

is given by

$$A_k((\omega, \phi_l(\omega)), (\omega', \phi_m(\omega'))) = \begin{cases} 1 & (\phi_l(\omega) = \omega') \\ 0 & (\text{otherwise}). \end{cases}$$

We define $S_k(a)$ by

$$S_k(a) = \{f_e : X_{t(e)} \to X_{i(e)} \mid e \in E_k\},$$

where

$$f_e = \begin{cases} f_0|_{X_{t(e)}} \text{ if } e = (\omega, \phi_0(\omega)) \\ f_1|_{X_{t(e)}} \text{ if } e = (\omega, \phi_1(\omega)). \end{cases}$$

That the image of f_e is contained in $X_{i(e)}$ is seen from the following lemma.

Lemma 3.8. *The GDMS $S_k(a)$ satisfies the open set condition. In fact, the open set*

$$U = \bigcup_{\omega \in V_k} \operatorname{int}(X_\omega),$$

satisfies

$$f_e(U \cap X_{t(e)}) \cap f_{e'}(U \cap X_{t(e')}) = \emptyset \tag{6}$$

for $e, e' \in E$ $(e \neq e')$, *and*

$$\bigcup_{e \in E_k} f_e(U \cap X_{t(e)}) \subset U. \tag{7}$$

Proof. We first show that (7) holds. From

$$X_{0\cdots 0} = [0, f_0 \ldots f_0 f_1(0)] = [0, a^k(1-a)],$$
$$X_{1\cdots 1} = [f_1 \ldots f_1 f_0(1), 1] = [1 - a^k(1-a), 1].$$

we have

$$f_0(\mathrm{int}(X_{0\cdots 0})) = [0, a^{k+1}(1-a)) \subset [0, a^k(1-a)) = \mathrm{int}(X_{0\cdots 0}),$$
$$f_1(\mathrm{int}(X_{1\cdots 1})) = (1 - a^{k+1}(1-a), 1] \subset (1 - a^k(1-a), 1] = \mathrm{int}(X_{1\cdots 1}).$$

Assume that $\omega \neq 0 \cdots 0, 1 \cdots 1$. For $j = 0, 1$, let us show that

$$f_j(\mathrm{int}(X_\omega)) \subset \mathrm{int}(X_p) \tag{8}$$

holds for some $p \in V_k$. Write $\omega - 1 = \alpha_1 \ldots \alpha_{k+1}$ and $\omega + 1 = \beta_1 \ldots \beta_{k+1}$. Since their difference is 2, there are two cases. First assume that $\alpha_{k+1} = \beta_{k+1} = 0$. We have

$$\begin{aligned} f_j(\mathrm{int}(X_\omega)) &= (f_j f_{\alpha_1} \ldots f_{\alpha_{k+1}}(1),\ f_j f_{\beta_1} \ldots f_{\beta_{k+1}}(0)) \\ &= (f_j f_{\alpha_1} \ldots f_{\alpha_k}(a),\ f_j f_{\beta_1} \ldots f_{\beta_k}(0)). \end{aligned}$$

Then we have

$$\begin{aligned} \beta_1 \ldots \beta_k - \alpha_1 \ldots \alpha_k &= \left\lfloor \frac{\beta_1 \ldots \beta_{k+1}}{2} \right\rfloor - \left\lfloor \frac{\alpha_1 \ldots \alpha_{k+1}}{2} \right\rfloor \\ &= \frac{\beta_1 \ldots \beta_{k+1} - \alpha_1 \ldots \alpha_{k+1}}{2} \\ &= 1. \end{aligned}$$

We put

$$\gamma_1 \ldots \gamma_k = \alpha_1 \ldots \alpha_k - 1.$$

Since $\gamma_1 \ldots \gamma_k 1 < \alpha_1 \ldots \alpha_k 0$, we see from Lemma 3.6 that,

$$f_{\gamma_1} \ldots f_{\gamma_k} f_1(1) < f_{\alpha_1} \ldots f_{\alpha_k} f_0(1),$$

$$f_{\gamma_1} \ldots f_{\gamma_k}(1) < f_{\alpha_1} \ldots f_{\alpha_k}(a).$$

Thus it follows that

$$f_j f_{\gamma_1} \ldots f_{\gamma_k}(1) < f_j f_{\alpha_1} \ldots f_{\alpha_k}(a),$$

and

$$f_j(\mathrm{int}(X_\omega)) \subset (f_j f_{\gamma_1} \ldots f_{\gamma_k}(1),\ f_j f_{\beta_1} \ldots f_{\beta_k}(0)).$$

The difference between $j\gamma_1 \ldots \gamma_k$ and $j\beta_1 \ldots \beta_k$ is 2. Take the number p with $j\gamma_1 \ldots \gamma_k < p < j\beta_1 \ldots \beta_k$, and it follows that $f_j(\mathrm{int}(X_\omega)) \subset \mathrm{int}(X_p)$. The existence of p satisfying (8) can be shown similarly.

Finally, we show (6). For $j = 0, 1$, since f_j is injective, it is clear that

$$f_j(\mathrm{int}(X_\omega)) \cap f_j(\mathrm{int}(X_{\omega'})) = \emptyset$$

for any $\omega \neq \omega'$. We can also see that

$$f_0(\mathrm{int}(X_\omega)) \cap f_1(\mathrm{int}(X_{\omega'})) = \emptyset$$

for any $\omega \neq 1 \cdots 1$ and $\omega' \neq 0 \cdots 0$, because

$$f_0(\mathrm{int}(X_\omega)) \subset f_0(I) \setminus F,$$

and

$$f_1(\mathrm{int}(X_{\omega'})) \subset f_1(I) \setminus F.$$

This completes the proof. □

Lemma 3.9. *Suppose that $b_k \leq a < c_k$. Then, for $l = 0, 1, \ldots,$ we have*

$$\bigcup_{\eta \in E^l_{A_k}} f_\eta(X_{t(\eta_l)}) \cup \bigcup_{|\mu| \leq l+k} f_\mu(F) = I. \tag{9}$$

Moreover, $f_\eta(X_{t(\eta_l)})$ for $\eta \in E^l_{A_k}$ and $f_\mu(F)$ for $|\mu| \leq l + k$ can meet only at a boundary point.

Proof. The proof is by induction on l. By Lemma 3.7, the interval between X_ω and $X_{\omega+1}$ is written as

$$f_\omega(I) \cap f_{\omega+1}(I) = f_\mu(F)$$

for some code μ of length at most k. Therefore we have

$$\bigcup_{\omega \in \{0,1\}^{k+1}} X_\omega \cup \bigcup_{|\mu| \leq k} f_\mu(F) = I.$$

Here, X_ω and $f_\mu(F)$ can meet only at a boundary point, which proves the statement for $l = 0$.

Now let us assume that the statement is true for l so that (9) holds. Note that we have

$$\bigcup_{\tilde{\eta} \in E^{l+1}_{A_k}} f_{\tilde{\eta}}(X_{t(\tilde{\eta}_{l+1})}) = \bigcup_{\substack{\eta \in E^l_{A_k} \\ i(\eta_1) \neq 1 \cdots 1}} f_0 f_\eta(X_{t(\eta_l)}) \cup \bigcup_{\substack{\eta \in E^l_{A_k} \\ i(\eta_1) \neq 0 \cdots 0}} f_1 f_\eta(X_{t(\eta_l)}). \tag{10}$$

Applying f_0 to (9), we obtain

$$\bigcup_{\substack{\eta \in E^l_{A_k} \\ i(\eta_1) \neq 1 \cdots 1}} f_0 f_\eta(X_{t(\eta_l)}) \cup \bigcup_{\substack{\eta \in E^l_{A_k} \\ i(\eta_1) = 1 \cdots 1}} f_0 f_\eta(X_{t(\eta_l)}) \cup \bigcup_{|\mu| \leq l+k} f_0 f_\mu(F) = f_0(I).$$

For $\eta \in E^l_{A_k}$ satisfying $i(\eta_1) = 1 \cdots 1$, we have

$$f_0 f_\eta(X_{t(\eta_n)}) \subset f_0(X_{1 \cdots 1}) \subset F,$$

and therefore

$$\bigcup_{\substack{\eta\in E^l_{A_k}\\ i(\eta_1)\neq 1\cdots 1}} f_0 f_\eta(X_{t(\eta_l)}) \cup \bigcup_{|\mu|\leq l+k} f_0 f_\mu(F) = f_0(I). \tag{11}$$

By the assumption of induction hypothesis, $f_0 f_\eta(X_{t(\eta_l)})$ and $f_0 f_\mu(F)$ in (11) can meet only at a boundary point. Similarly we obtain

$$\bigcup_{\substack{\eta\in E^l_{A_k}\\ i(\eta_1)\neq 0\cdots 0}} f_1 f_\eta(X_{t(\eta_l)}) \cup \bigcup_{|\mu|\leq l+k} f_1 f_\mu(F) = f_1(I), \tag{12}$$

where $f_1 f_\eta(X_{t(\eta_l)})$ and $f_1 f_\mu(F)$ can meet only at a boundary point. An interval of the form $f_0 f_\eta(X_{t(\eta_l)})$ ($i(\eta_1) \neq 1\cdots 1$) and $f_1 f_\mu(F)$ do not meet, since the former does not intersect with $f_1(I)$ by Remark 3.3. Likewise, an interval of the form $f_1 f_\eta(X_{t(\eta_l)})$ ($i(\eta_1) \neq 0\cdots 0$) and $f_0 f_\mu(F)$ do not meet.

From (10), (11), (12), we have

$$\bigcup_{\tilde\eta\in E^{l+1}_{A_k}} f_{\tilde\eta}(X_{t(\tilde\eta_{l+1})}) \cup \bigcup_{|\mu|\leq l+1+k} f_\mu(F) = f_0(I)\cup f_1(I) = I,$$

which shows that the statement is true for $l+1$. This completes the proof. □

Lemma 3.10. *Suppose that $b_k \leq a < c_k$. Then we have $J(S_k(a)) = J_1(S(a))$.*

Proof. Since $J_1(S(a)) = (F^*)^c$ by Proposition 2.1, it suffices to show that

$$(J(S_k(a)))^c = F^*.$$

First assume that $x \notin J(S_k(a))$. Then, there exists l such that

$$x \notin \bigcup_{\eta\in E^l_{A_k}} f_\eta(X_{t(\eta_l)}).$$

Then, by Lemma 3.9, we have

$$x \in \bigcup_{|\mu|\leq l+k} f_\mu(F) \subset F^*.$$

Conversely, assume that $x \in F^*$. There is a code $\nu \in \{0,1\}^*$ such that $x \in f_\nu(F)$. Applying Lemma 3.9 to $l = |\nu|+1$, we see that $f_\eta(X_{t(\eta_l)})$ for $\eta \in E^l_{A_k}$ can meet $f_\nu(F)$, $f_\nu f_0 f_1^k(F)$, or $f_\nu f_1 f_0^k(F)$ only at a boundary point. Since

$$f_\nu(F) \subset \mathrm{int}(f_\nu(F) \cup f_\nu f_0 f_1^k(F) \cup f_\nu f_1 f_0^k(F))$$

for $b_k \leq a < c_k$ by Remark 3.3, $f_\eta(X_{t(\eta_l)})$ does not intersect with $f_\nu(F)$. Therefore, we have

$$x \notin \bigcup_{\eta\in E^l_{A_k}} f_\eta(X_{t(\eta_l)}),$$

and we see that $x \notin J(S_k(a))$. □

4. The region of multiplicity one

We start the evaluation of the Hausdorff dimension of $J_1(S(a))$ in case of $k = 1$.

Theorem 4.1. *For any $a \geq b_1$, we have* $\dim_H J_1(S(a)) = 0$.

Proof. If $a \geq c_1 = g$, $J_1(S(a)) = \{0, 1\}$. So we have $\dim_H J_1(S(a)) = 0$.

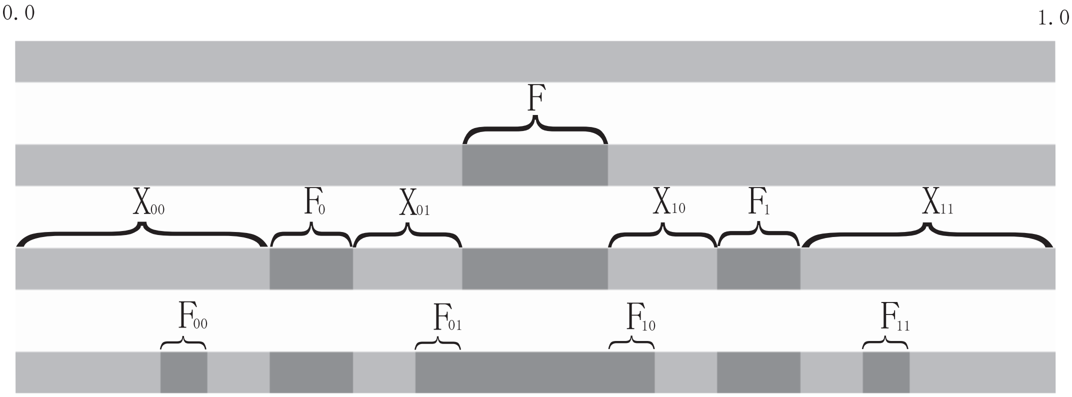

Fig.4. The structure of $S_1(a)$.

Suppose that $b_1 \leq a < c_1$ (See Fig. 4). By Lemma 3.10, we have $J_1(S(a)) = J(S_1(a))$. The GDMS $S_1(a)$ is given as follows. The multigraph and the associated incidence matrix are given by

$$V_1 = \{00, 01, 10, 11\},$$

$$E_1 = \{(00, 00), (00, 01), (01, 10), (10, 01), (11, 10), (11, 11)\},$$

$$A_1 = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}.$$

Elements of $E_{A_1}^{\infty}$ correspond to the codes

$$(00, 00)^{\infty}, (11, 11)^{\infty}, \{(01, 10)(10, 01)\}^{\infty}, \{(10, 01)(01, 10)\}^{\infty},$$

$$(00, 00)^n(00, 01)\{(01, 10)(10, 01)\}^{\infty} \qquad (n = 0, 1, \ldots),$$

$$(11, 11)^n(11, 10)\{(10, 01)(01, 10)\}^{\infty} \qquad (n = 0, 1, \ldots).$$

In particular, $J(S_1(a))$ is countable. So $\dim_H J_1(S(a)) = 0$. □

Corollary 4.2. *For any $a \geq b_1$, we have* $\dim_H \bigcup_{i=2}^{\infty} J_i(S(a)) = 1$.

Theorem 4.3. *Suppose that $b_2 \leq a \leq c_2$. Then, we have*

$$\dim_H J_1(S(a)) = -\frac{\log \frac{1+\sqrt{5}}{2}}{\log a}.$$

Proof. Suppose that $b_2 \le a \le c_2$ (See Fig. 5). By Lemma 3.10, if $b_2 \le a < c_2$, then we have $J_1(S(a)) = J(S_2(a))$. If $a = c_k$, $J_1(S(a)) \subset J(S_2(a))$ and their difference $J(S_2(a)) \setminus J_1(S(a))$ consists of boundary points of $f_\nu(F)$ ($\nu \in \{0,1\}^*$), and is countable. In either case, we have

$$\dim_H J_1(S(a)) = \dim_H J(S_2(a)).$$

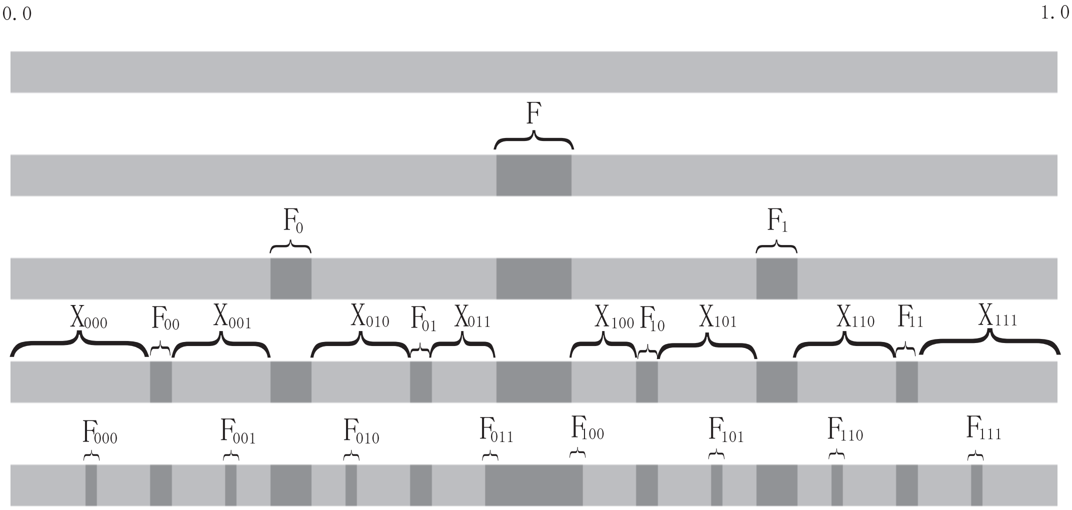

Fig. 5. The structure of $S_2(a)$.

The GDMS $J(S_2(a))$ is given as follows. The multigraph with the associated incidence matrix, (V_2, E_2, A_2, i, t), is given by

$$V_2 = \{000, 001, 010, 011, 100, 101, 110, 111\},$$

$$E_2 = \{(000,000), (000,001), (001,010), (001,011), (010,100), (010,101), (011,110), (100,001), (101,010), (101,011), (110,100), (110,101), (111,110), (111,111)\},$$

$$A_2 = \begin{pmatrix} 1&1&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&1&1&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&1&1&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&1&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&1&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&1&1&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&1&1&0&0\\ 0&0&1&1&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&1&1&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&1&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&1&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&1&1&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&1&1&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&1&1 \end{pmatrix}.$$

We have

$$\dim_H J(S_2(a)) \le -\frac{\log \lambda_2}{\log a}, \tag{13}$$

where $\lambda_2 = (1+\sqrt{5})/2$ is the largest eigenvalue of A_2. This can be shown as follows. Since $\{f_\eta(X_{t(\eta_l)}) \mid \eta \in E^l_{A_k}\}$ is a a^l-cover of $J(S_2(a))$, for $s > 0$, we have

$$\begin{aligned}\mathcal{H}^s_{a^l}(J(S_2(a))) &\le \sum_{\eta \in E^l_{A_k}} |f_\eta(X_{t(\eta_l)})|^s \\ &= a^{ls} \sharp E^l_{A_k} \\ &\le a^{ls} c \lambda_2^l \quad (c \text{ is a const.}).\end{aligned}$$

Thus, if $a^s \lambda_2 < 1$, we have

$$\mathcal{H}^s_{a^l}(J(S_2(a))) \to 0 \quad \text{as} \quad l \to \infty,$$

and

$$\mathcal{H}^s(J(S_2(a))) = 0.$$

Since $\dim_H J(S_2(a)) \le s$ for any s satisfying $a^s \lambda_2 < 1$, we have (13).

The GDMS satisfies the open set condition by Lemma 3.8. It also satisfies the bounded distortion property since all of the transformations are similarity transformations. Theorem 2.6 asserts that the equality would hold in (13) if A_2 were irreducible.

Since A_2 is not irreducible, we modify our GDMS slightly. We define the modified multigraph with the associated incidence matrix by

$$V_2' = \{001, 010, 011, 100, 101, 110\},$$

$$\begin{aligned}E_2' = \{&(001,010),(001,011),(010,100),(010,101),(011,110),\\ &(100,001),(101,010),(101,011),(110,100),(110,101)\},\end{aligned}$$

$$A_2' = \begin{pmatrix} 0&0&1&1&0&0&0&0&0&0\\ 0&0&0&0&1&0&0&0&0&0\\ 0&0&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&1&1&0&0\\ 0&0&0&0&0&0&0&0&1&1\\ 1&1&0&0&0&0&0&0&0&0\\ 0&0&1&1&0&0&0&0&0&0\\ 0&0&0&0&1&0&0&0&0&0\\ 0&0&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&1&1&0&0 \end{pmatrix}.$$

Let us denote this modified GDMS by $S_2'(a)$. We have

$$\dim_H J(S_2'(a)) \le \dim_H J(S_2(a)). \tag{14}$$

It is easy to check that A_2' is irreducible, and we can now apply Theorem 2.6. The Hausdorff dimension of the limit set $J(S_2'(a))$ is the zero point of the topological pressure function

$$
\begin{aligned}
P(t) &= \lim_{n\to\infty} \frac{1}{n} \log \sum_{\omega\in E^n_{A'_2}} \|f'_\omega\|^t \\
&= \lim_{n\to\infty} \frac{1}{n} \log \sharp E^n_{A'_2} + t\log a \\
&= \log \lambda'_2 + t\log a,
\end{aligned}
$$

where λ'_2 is the largest eigenvalue of A'_2. Thus, we have

$$
\dim_H J(S'_2(a)) = -\frac{\log \lambda'_2}{\log a}. \tag{15}
$$

We now show that $\lambda'_2 = \lambda_2$. Computing the characteristic polynomial of A_2, we obtain

$$
\begin{aligned}
&\det(A_2 - sE) \\
&= \det \left(\begin{array}{cc|cccccccccc|cc}
1-s & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -s & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & -s & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -s & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -s & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -s & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -s & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & -s & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & -s & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -s & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -s & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & -s & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & -s & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1-s
\end{array} \right) \\
&= s^2(1-s)^2 \det(A'_2 - sE').
\end{aligned}
$$

Thus the eigenvalues of A_2 are $0, 1$ and the eigenvalues of A'_2. Combining (13), (14), and (15), we obtain

$$
-\frac{\log \lambda_2}{\log a} = -\frac{\log \lambda'_2}{\log a} = \dim_H J(S'_2(a)) \le \dim_H J(S_2(a)) \le -\frac{\log \lambda_2}{\log a}.
$$

This completes the proof of Theorem 4.3. □

5. Proof of the theorems

We generalize Theorem 4.3 and prove Theorem 1.1. Suppose that $b_k \le a \le c_k$. By Lemma 3.10, if $b_k \le a < c_k$, then we have $J_1(S(a)) = J(S_k(a))$. If $a = c_k$, $J_1(S(a)) \subset J(S_k(a))$ and their difference $J(S_k(a)) \setminus J_1(S(a))$ consists of boundary points of $f_\nu(F)$ ($\nu \in \{0,1\}^*$), and is countable. In either case, we have

$$
\dim_H J_1(S(a)) = \dim_H J(S_k(a)).
$$

Recall that we define the multigraph with the associated incidence matrix (V_k, E_k, A_k, i, t)

in Section 3. First, we can show that

$$\dim_H J(S_k(a)) \leq -\frac{\log \lambda_k}{\log a} \tag{16}$$

in the same way as we showed (13).

The GDMS satisfies the open set condition by Lemma 3.8. It also satisfies the bounded distortion property since all of the transformations are similarity transformations. Theorem 2.6 asserts that the equality would hold in (16) if A_k were irreducible.

Since the incidence matrix A_k is not irreducible, we modify the GDMS. The multigraph with the associated incidence matrix, (V'_k, E'_k, A'_k, i, t), is defined as follows. The vertex set is given by $V'_k = V_k \setminus \{0 \cdots 0, 1 \cdots 1\}$. The edges of E'_k are those of E_k not involving the vertices $0 \cdots 0, 1 \cdots 1$. The incidence matrix A'_k is the restriction of A_k to $E'_k \times E'_k$.

Given $\omega \in \{0,1\}^{k+1}$, the map ϕ_0 (resp. ϕ_1) shifts the digits of ω to the right and append 0 (resp. 1) to the left:

$$\phi_0(\omega_1 \ldots \omega_k \omega_{k+1}) = 0\omega_1 \ldots \omega_k,$$
$$\phi_1(\omega_1 \ldots \omega_k \omega_{k+1}) = 1\omega_1 \ldots \omega_k.$$

To see that the modified incidence matrix A'_k is irreducible, we show for any $p, q \in V'_k$, there exists a path from p to q within $E^*_{A'_k}$. Define $r_0, r_1 \in \{0,1\}$ by

$$r_0 \neq p_1, \qquad r_1 \neq q_{k+1}.$$

Then we have

$$\phi_{q_1} \cdots \phi_{q_{k+1}} \phi_{r_1} \phi_{r_0}(p) = q,$$

and for all $i = 1, \ldots, k$, we have

$$\phi_{q_i} \cdots \phi_{q_{k+1}} \phi_{r_1} \phi_{r_0}(p) \in V'_k.$$

This shows that A'_k is irreducible.

Similaly to (14) and (15), we have

$$\dim_H J(S'_k(a)) \leq \dim_H J(S_k(a)), \tag{17}$$

and

$$\dim_H J(S'_k(a)) = -\frac{\log \lambda'_k}{\log a}, \tag{18}$$

where λ'_k is the largest eigenvalue of A'_k.

The eigenvalues of A_k are 0, 1, and the eigenvalues of A'_k. This can be seen by

$$\det(A_k - sE) = \det\left(\begin{array}{cc|cccccc|cc} 1-s & 1 & 0 & \cdots & & 0 & 0 & & 0 & 0 \\ 0 & -s & 1 & & 1 & 0 & \cdots & & 0 & 0 \\ \hline 0 & 0 & & & & & & & 0 & 0 \\ \vdots & \vdots & & & A'_k - sE' & & & & 0 & 0 \\ \vdots & \vdots & & & & & & & \vdots & \vdots \\ 0 & 0 & & & & & & & 0 & 0 \\ \hline 0 & 0 & \cdots & & 0 & 1 & 1 & & -s & 0 \\ 0 & 0 & 0 & & \cdots & 0 & 0 & & 1 & 1-s \end{array}\right)$$

$$= s^2(1-s)^2 \det(A'_k - sE').$$

Combining (16), (17), and (18), we obtain

$$-\frac{\log \lambda_k}{\log a} = -\frac{\log \lambda'_k}{\log a} = \dim_H J(S'_k(a)) \leq \dim_H J(S_k(a)) \leq -\frac{\log \lambda_k}{\log a}.$$

This completes the proof of Theorem 1.1.

The proof of Theorem 1.2 is as follows. Since all elements of $(A'_k)^{(k+3)}$ are greater than or equal to 1, for every integer m, we have

$$\mathrm{tr}(A'_k)^{(k+3)m} \geq (2^{k+2} - 6)^m.$$

Therefore, by the Perron-Frobenius theorem, we obtain

$$\begin{aligned}(k+3)\log \lambda'_k &= \lim_{m\to\infty} \frac{1}{m} \log \mathrm{tr}(A'_k)^{(k+3)m} \\ &\geq \log(2^{k+2} - 6).\end{aligned}$$

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