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SOME GEOMETRICAL ASPECTS OF EINSTEIN, RICCI AND YAMABE SOLITONS

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Abstract. Under certain assumptions, we characterize the almost η-Einstein, η-Ricci and η-Yamabe solitons on a pseudo-Riemannian manifold when the potential vector field of the soliton is infinitezimal harmonic or torse-forming. Moreover, in the second case, if the manifold is Ricci symmetric of constant scalar curvature, then the soliton is completely determined.

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1. Preliminaries

Einstein, Ricci and Yamabe solitons may be regarded as generalized fixed points of Einstein, Ricci and Yamabe flow respectively, being important tools in particular areas of theoretical physics and mathematical physics. Recently, some generalizations have been considered: almost η-Einstein, almost η-Ricci and almost η-Yamabe solitons, having the previous ones as particular cases. We shall briefly recall these notions.

Let g be a pseudo-Riemannian metric on the smooth manifold M, ξ a vector field, η is a one-form, and λ and μ two smooth functions on M. If we denote by $\pounds_\xi$ the Lie derivative operator along the vector field ξ, Ric the Ricci curvature tensor field, r the scalar curvature, then the data $(g, \xi, \eta, \lambda, \mu)$ define

i) *an almost η-Einstein soliton* on M if they satisfy the equation

$$\frac{1}{2}\pounds_\xi g + \mathrm{Ric} + \left(\lambda - \frac{r}{2}\right) g + \mu\eta \otimes \eta = 0 \tag{1}$$

ii) *an almost η-Ricci soliton* on M if they satisfy the equation

$$\frac{1}{2}\pounds_\xi g + \mathrm{Ric} + \lambda g + \mu\eta \otimes \eta = 0 \tag{2}$$

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iii) *an almost η-Yamabe soliton* on M if they satisfy the equation

$$\frac{1}{2}\pounds_{\xi}g + (\lambda - r)g + \mu\eta \otimes \eta = 0. \tag{3}$$

A soliton $(g, \xi, \eta, \lambda, \mu)$ is said to be *steady* if $\lambda = 0$, *shrinking* if $\lambda < 0$, *expanding* if $\lambda > 0$ or undefined, otherwise.

In particular: if $\mu = 0$, we call the solitons respectively, almost Einstein, almost Ricci and almost Yamabe solitons. If λ and μ are constant, we call the solitons respectively, η-Einstein, η-Ricci and η-Yamabe solitons. If $\mu = 0$ and λ is constant, we call the solitons respectively, Einstein, Ricci and Yamabe solitons.

In the present paper, we treat the above mentioned classes of solitons for two types of potential vector fields, namely, infinitezimal harmonic and torse-forming. If the manifold is Ricci symmetric, we provide necessary and sufficient conditions for the potential vector field of an almost Einstein and almost Ricci soliton to be infinitezimal harmonic. Also, we prove that in a Ricci symmetric manifold M of constant scalar curvature, if the potential vector field is torse-forming, then the manifold is of scalar curvature $r = -n(n-1)$, with $n = \dim(M)$, and the η-Einstein and η-Ricci solitons are completely determined. Infinitezimal harmonic vector fields (analytically characterized by $\mathrm{tr}_g(\pounds_{\xi}\nabla^g) = 0$, for ∇^g the Levi-Civita connection of the Riemannian metric g) generalize infinitezimal isometries (i.e Killing vector fields characterized by $\pounds_{\xi}g = 0$). The two notions coincide on compact manifolds satisfying $\mathrm{tr}_g(\pounds_{\xi}\mathrm{Ric}) = 0$, condition used in general relativity [1], weaker than the Ricci collineation (i.e., $\pounds_{\xi}\mathrm{Ric} = 0$).

2. Solitons with Infinitezimal Harmonic Potential Vector Field

Definition 1. *We say that ξ is an infinitezimal harmonic vector field on the Riemannian manifold (M, g) if $\mathrm{tr}_g(\pounds_{\xi}\nabla^g) = 0$, where ∇^g is the Levi-Civita connection of g.*

According to Yano [4], since $\mathrm{tr}_g(\pounds_{\xi}\nabla^g) = -\Delta\xi + 2Q\xi$, the vector field ξ is infinitezimal harmonic if and only if $\Delta\xi = 2Q\xi$, where $\Delta\xi := \mathrm{tr}_g(\nabla^2\xi)$, with $\nabla^2_{X,Y}\xi := \nabla^g_X\nabla^g_Y\xi - \nabla^g_{\nabla^g_X Y}\xi$, for $X, Y \in \chi(M)$.

Assume that $(g, \xi, \eta, \lambda, \mu)$ defines an almost η-Einstein soliton on M. Taking the covariant derivative of $\pounds_{\xi}g$ with respect to Z in equation (1), we get

$$\begin{aligned}\frac{1}{2}(\nabla^g_Z\pounds_{\xi}g)(X, Y) &= -(\nabla^g_Z\mathrm{Ric})(X, Y) - \left(Z(\lambda) - \frac{Z(r)}{2}\right)g(X, Y)\\ &\quad - Z(\mu)\eta(X)\eta(Y) - \mu\left(\eta(X)g(Y, \nabla^g_Z\xi) + \eta(Y)g(X, \nabla^g_Z\xi)\right).\end{aligned} \tag{4}$$

The following formula [3]

$$(\pounds_\xi \nabla^g_X g - \nabla^g_X \pounds_\xi g - \nabla^g_{[\xi,X]} g)(Y,Z) = -g((\pounds_\xi \nabla^g)(X,Y),Z) - g((\pounds_\xi \nabla^g)(X,Z),Y)$$

for any $X, Y, Z \in \chi(M)$, reduces to

$$(\nabla^g_X \pounds_\xi g)(Y,Z) = g((\pounds_\xi \nabla^g)(X,Y),Z) + g((\pounds_\xi \nabla^g)(X,Z),Y) \tag{5}$$

for any $X, Y, Z \in \chi(M)$. Since $\pounds_\xi \nabla^g$ is a symmetric $(1,2)$-tensor field, i.e., $(\pounds_\xi \nabla^g)(X,Y) = (\pounds_\xi \nabla^g)(Y,X)$, from (5) follows

$$g((\pounds_\xi \nabla^g)(X,Y),Z) = \frac{1}{2}(\nabla^g_X \pounds_\xi g)(Y,Z) + \frac{1}{2}(\nabla^g_Y \pounds_\xi g)(X,Z) - \frac{1}{2}(\nabla^g_Z \pounds_\xi g)(X,Y). \tag{6}$$

Using (4) in (6) yields

$$\begin{aligned} g((\pounds_\xi \nabla^g)(X,Y),Z) &= -\big((\nabla^g_X \mathrm{Ric})(Y,Z) + (\nabla^g_Y \mathrm{Ric})(X,Z) - (\nabla^g_Z \mathrm{Ric})(X,Y)\big) \\ &- \left(X(\lambda) - \frac{X(r)}{2}\right) g(Y,Z) - \left(Y(\lambda) - \frac{Y(r)}{2}\right) g(X,Z) + \left(Z(\lambda) - \frac{Z(r)}{2}\right) g(X,Y) \\ &-X(\mu)\eta(Y)\eta(Z) - Y(\mu)\eta(X)\eta(Z) + Z(\mu)\eta(X)\eta(Y) \\ &+\mu(\eta(X)\left(g(Y,\nabla^g_Z \xi) - g(Z,\nabla^g_Y \xi)\right) + \eta(Y)\left(g(X,\nabla^g_Z \xi) - g(Z,\nabla^g_X \xi)\right) \\ &-\eta(Z)\left(g(X,\nabla^g_Y \xi) + g(Y,\nabla^g_X \xi)\right)). \end{aligned} \tag{7}$$

If (g,ξ,λ) defines an almost Einstein soliton (i.e., $\mu = 0$) in the n-dimensional Ricci symmetric Riemannian manifold (M,g) (i.e., $\nabla^g \mathrm{Ric} = 0$), from (7) we get

$$\begin{aligned} (\pounds_\xi \nabla^g)(X,Y) &= -\left(X(\lambda) - \frac{X(r)}{2}\right) Y - \left(Y(\lambda) - \frac{Y(r)}{2}\right) X \\ &+g(X,Y)\left(\nabla\lambda - \frac{\nabla r}{2}\right) \end{aligned}$$

where ∇ denotes the gradient operator with respect to g, and

$$\mathrm{tr}_g(\pounds_\xi \nabla^g) = (n-2)\left(\nabla\lambda - \frac{\nabla r}{2}\right)$$

and we have

Theorem 2. *If (g,ξ,λ) defines an almost Einstein soliton in the n-dimensional Ricci symmetric Riemannian manifold (M,g) with $n \geq 3$, then the potential vector field ξ of the soliton is infinitezimal harmonic if and only if $\nabla\lambda = \dfrac{\nabla r}{2}$.*

Corollary 3. *If (M, g) is an n-dimensional Ricci symmetric Riemannian manifold with $n \geq 3$ and (g, ξ, λ) defines an Einstein soliton, then ξ is infinitezimal harmonic if and only if $\nabla r = 0$ (which is equivalent to $\nabla(\mathrm{div}(\xi)) = 0$).*

Corollary 4. *Under the hypotheses of* Corollary 3*, if (g, ξ, λ) defines an Einstein soliton of magnetic type (i.e., $\mathrm{div}(\xi) = 0$), then ξ is infinitezimal harmonic and $\nabla r = 0$.*

Similarly, in the almost Ricci soliton case we get $\mathrm{tr}_g(\pounds_\xi \nabla^g) = (n-2)\nabla\lambda$ and in the almost Yamabe soliton case we get $\mathrm{tr}_g(\pounds_\xi \nabla^g) = (n-2)(\nabla\lambda - \nabla r)$, and we can state

Theorem 5. *Let (M, g) be an n-dimensional Riemannian manifold with $n \geq 3$.*

i) *If M is Ricci symmetric and (g, ξ, λ) defines an almost Ricci soliton, then ξ is infinitezimal harmonic if and only if $\nabla\lambda = 0$.*

ii) *If (g, ξ, λ) defines an almost Yamabe soliton, then ξ is infinitezimal harmonic if and only if $\nabla\lambda = \nabla r$.*

Corollary 6. *Let (M, g) be an n-dimensional Riemannian manifold with $n \geq 3$.*

i) *If M is Ricci symmetric and (g, ξ, λ) defines a Ricci soliton, then ξ is infinitezimal harmonic.*

ii) *If (g, ξ, λ) defines a Yamabe soliton, then ξ is infinitezimal harmonic if and only if $\nabla r = 0$.*

A weaker condition than $\nabla(\mathrm{div}(\xi)) = 0$ is $\xi(\mathrm{div}(\xi)) \geq 0$ which we will further impose.

Theorem 7. *Let (M, g) be a complete Riemannian manifold, ξ an infinitezimal harmonic vector field satisfying $\xi(\mathrm{div}(\xi)) \geq 0$ and $\mathrm{Ric}(\xi, \xi) = 0$. If $(g, \xi, \eta, \lambda, \mu)$ defines an almost η-Einstein soliton, then $(\lambda, \mu) = \left(\frac{r(n-3)}{2(n-1)}, \frac{r}{(n-1)|\xi|^2}\right)$. Moreover, if $\int_M |\xi|^p < \infty$, for any $p \neq 1$, then $|\xi|$ is constant.*

Proof: Taking the trace in the almost η-Einstein soliton equation (1) we get

$$\mathrm{div}(\xi) + r + n\left(\lambda - \frac{r}{2}\right) + \mu|\xi|^2 = 0.$$

Also, the condition $\mathrm{Ric}(\xi,\xi)=0$ implies

$$\lambda-\frac{r}{2}+\mu|\xi|^2=0.$$

Taking into account that on a complete pseudo-Riemannian manifold, an infinitezimal harmonic vector field ξ satisfying $\xi(\mathrm{div}(\xi))\geq 0$ is divergence-free and Killing vector field [2], we get $(n-3)r=2(n-1)\lambda$.

Moreover, replacing the value of the rough Laplacian $\bar{\Delta}$ on $\eta=i_\xi g$ from Weitzenböck formula into Kato's inequality, we have

$$|\xi|\Delta(|\xi|)\geq -g(\bar{\Delta}\eta,\eta)=-g(\tilde{\Delta}\eta,\eta)+g(i_{Q\xi}g,i_\xi g)$$

where $\tilde{\Delta}$ is Hodge-de Rham Laplacian, and taking into account that an infinitezimal harmonic vector field satisfies [2]

$$\tilde{\Delta}\eta=2i_{Q\xi}g$$

we obtain

$$|\xi|\Delta(|\xi|)\geq \mathrm{Ric}(\xi,\xi)=0$$

and therefore, under the hypotheses of completeness and $\int_M|\xi|^p<\infty$, for any $p\neq 1$, the non negative function $|\xi|$ must be constant according to Yau's theorem [5]. ■

Remark that if $n=3$, the soliton is steady. Also, if $|\xi|$ is constant, then λ and μ are constant if and only if M is of constant scalar curvature (in this case, the soliton reduces to an η-Einstein soliton).

Similarly, we obtain in the almost η-Ricci and almost η-Yamabe soliton cases, the followings

Theorem 8. *Let (M,g) be a complete Riemannian manifold, ξ an infinitezimal harmonic vector field satisfying $\xi(\mathrm{div}(\xi))\geq 0$.*

i) *If (g,ξ,η,λ,μ) defines an almost η-Ricci soliton and $\mathrm{Ric}(\xi,\xi)=0$, then $(\lambda,\mu)=\left(-\frac{r}{n-1},\frac{r}{(n-1)|\xi|^2}\right)$.*

ii) *If (g,ξ,η,λ,μ) defines an almost η-Yamabe soliton, then $(\lambda,\mu)=(r,0)$.*

3. Solitons with Torse-Forming Potential Vector Field

We shall further assume that the potential vector field of the soliton is torse-forming. It is known that torse-forming vector fields naturally arise in different geometries, for instance, the characteristic vector field of an almost paracontact metric structure is a torse-forming vector field.

Let ξ be a *torse-forming vector field* on the n-dimensional pseudo-Riemannian manifold (M, g), i.e., $\nabla^g\xi = \mathrm{Id} - \eta \otimes \xi$, with $\eta := i_\xi g$ the g-dual one-form of ξ. Then for any $X, Y \in \chi(M)$

$$\begin{aligned} \pounds_\xi g &= 2(g - \eta \otimes \eta) \\ R^{\nabla^g}(X, Y)\xi &= \eta(X)Y - \eta(Y)X \\ \mathrm{Ric}(X, \xi) &= -(n-1)\eta(X) \\ Q\xi &= -(n-1)\xi. \end{aligned}$$

Remark that

$$\Delta\xi = (2|\xi|^2 - n - 1)\xi$$

for $n = \dim(M)$. Also, $Q\xi = -(n-1)\xi$ and therefore

$$\mathrm{tr}_g(\pounds_\xi \nabla^g) = -(2|\xi|^2 + n - 3)\xi$$

so that one can deduce that in a Riemannian manifold there exists no infinitezimal harmonic torse-forming vector field.

Consider now the case of three-dimensional manifolds. It is known that in a three-dimensional pseudo-Riemannian manifold (M, g), the curvature tensor field is given by

$$\begin{aligned} R^{\nabla^g}(X, Y)Z = g(Y, Z)\left(QX - \frac{r}{2}X\right) - g(X, Z)\left(QY - \frac{r}{2}Y\right) \\ + \mathrm{Ric}(Y, Z)X - \mathrm{Ric}(X, Z)Y \end{aligned} \tag{8}$$

for any $X, Y, Z \in \chi(M)$.

If we assume that ξ is a torse-forming vector field with $\nabla^g\xi = \mathrm{Id} - \eta \otimes \xi$ for $\eta := i_\xi g$, from (8) we obtain

$$Q = \frac{r+2}{2}\mathrm{Id} - \frac{r+6}{2|\xi|^2}\eta \otimes \xi \tag{9}$$

and

$$\mathrm{Ric} = \frac{r+2}{2}g - \frac{r+6}{2|\xi|^2}\eta \otimes \eta.$$

It follows that $\mathrm{Ric}(\xi, \xi) = -2|\xi|^2$ and also

$$|\mathrm{Ric}|^2 = \frac{(r+2)^2}{2} + 4. \tag{10}$$

In this case

i) the almost η-Einstein soliton equation (1) reduces to

$$(\lambda+2)g+\left(\mu-1-\frac{r+6}{2|\xi|^2}\right)\eta\otimes\eta=0$$

ii) the almost η-Ricci soliton equation (2) reduces to

$$\left(\lambda+2+\frac{r}{2}\right)g+\left(\mu-1-\frac{r+6}{2|\xi|^2}\right)\eta\otimes\eta=0$$

iii) the almost η-Yamabe soliton equation (3) reduces to

$$(\lambda+2-r)g+(\mu-1)\eta\otimes\eta=0.$$

From the above relations, we obtain

Lemma 9. *Let (M,g) be a three-dimensional pseudo-Riemannian manifold and let ξ be a torse-forming vector field on M ($\nabla^g\xi=\mathrm{Id}-\eta\otimes\xi$ with $\eta:=i_\xi g$).*

i) *If* (1) *defines an almost η-Einstein soliton on M, then the scalar curvature of M is*
$$r=2\left(\lambda-1+(\mu-1)|\xi|^2\right). \tag{11}$$

ii) *If* (2) *defines an almost η-Ricci soliton on M, then $\lambda=1-(\mu-1)|\xi|^2$.*

iii) *If* (3) *defines an almost η-Yamabe soliton on M, then the scalar curvature of M is $r=\lambda+1+(\mu-1)|\xi|^2$.*

On the other hand, using (9) and the almost η-Einstein soliton equation (1), we compute
$$|\mathrm{Ric}|^2=\frac{1}{2}\left(r^2-2(\lambda+1)r+4(\mu-1)|\xi|^2\right) \tag{12}$$
and together with (10), it implies
$$r(\lambda+3)=2\left((\mu-1)|\xi|^2-3\right).$$
Using now (11), we get
$$(r+2)(\lambda+2)=0$$
from where we get either $r=-2$ (and $\lambda=(1-\mu)|\xi|^2$) or $\lambda=-2$ (and $r=2((\mu-1)|\xi|^2-3)$) we can state

Theorem 10. *Let (M, g) be a three-dimensional pseudo-Riemannian manifold and let ξ be a torse-forming vector field on M ($\nabla^g \xi = \mathrm{Id} - \eta \otimes \xi$ with $\eta := i_\xi g$). If* (1) *defines an almost η-Einstein soliton on M, then the manifold is of constant scalar curvature $r = -2$ or the soliton is shrinking.*

Remark also that (12) implies $\mu > 1 + \dfrac{(\lambda+1)^2}{4|\xi|^2}$.

Following the same steps, we obtain from the almost η-Ricci soliton equation

$$|\mathrm{Ric}|^2 = -(\lambda+1)r + 2(\mu-1)|\xi|^2$$

and we can state

Proposition 11. *Let (M, g) be a three-dimensional pseudo-Riemannian manifold and let ξ be a torse-forming vector field on M ($\nabla^g \xi = \mathrm{Id} - \eta \otimes \xi$ with $\eta := i_\xi g$). If* (2) *defines an almost η-Ricci soliton on M, then the scalar curvature of M is a solution of the following equation*

$$r^2 + 2(\lambda+3)r + 4(\lambda-1)|\xi|^2 + 12 = 0. \tag{13}$$

Remark 12. *From* (13) *we get $2\lambda(r + 2|\xi|^2) = -(r+3)^2 + 4|\xi|^2 - 3$. Remark the following facts*

i) *if the soliton is steady, then the scalar curvature is $r = -3 \pm \sqrt{4|\xi|^2 - 3}$ provided by $|\xi|^2 \geq \frac{3}{4}$. In particular, if ξ is a unitary vector field, then $r \in \{-4, -2\}$*

ii) *if $|\xi|^2 = -\frac{r}{2}$, then $|\xi|^2 \in \{1, 3\}$ and $r \in \{-2, -6\}$.*

Consider now the case of Ricci symmetric manifolds. If the potential vector field of the almost η-Einstein soliton (1) is torse-forming with $\nabla^g \xi = \mathrm{Id} - \eta \otimes \xi$ for $\eta := i_\xi g$ and M is Ricci symmetric (i.e., $\nabla^g \mathrm{Ric} = 0$), then removing Z from (7), it follows that

$$(\pounds_\xi \nabla^g)(X, Y) = g(X, Y)\left(\nabla\lambda - \frac{\nabla r}{2}\right) + \eta(X)\eta(Y)\nabla\mu - \left(X(\lambda) - \frac{X(r)}{2}\right)Y$$
$$- \left(Y(\lambda) - \frac{Y(r)}{2}\right)X - (X(\mu)\eta(Y) + Y(\mu)\eta(X) + 2\mu(g(X, Y) - \eta(X)\eta(Y)))\,\xi$$

where ∇ stands for the gradient operator with respect to g. Taking $Y = \xi$ we get

$$(\pounds_\xi \nabla^g)(X, \xi) = \eta(X)\left(\nabla\lambda - \frac{\nabla r}{2} + |\xi|^2\nabla\mu\right) - \left(\xi(\lambda) - \frac{\xi(r)}{2}\right)X$$
$$- \left(X(\mu)|\xi|^2 + \xi(\mu)\eta(X) + 2\mu\eta(X) - 2\mu|\xi|^2\eta(X) + X(\lambda) - \frac{X(r)}{2}\right)\xi.$$

Assume now that λ, μ and r are constant and ξ is a unitary vector field. Then the previous relation gives

$$(\pounds_\xi \nabla^g)(X,\xi) = 0. \tag{14}$$

Taking the covariant derivative of (14) with respect to Y, we obtain

$$(\nabla^g_Y \pounds_\xi \nabla^g)(X,\xi) = -(\pounds_\xi \nabla^g)(X,Y). \tag{15}$$

Again from [3]

$$(\pounds_\xi R)(X,Y,Z) = (\nabla^g_X \pounds_\xi \nabla^g)(Y,Z) - (\nabla^g_Y \pounds_\xi \nabla^g)(X,Z). \tag{16}$$

Therefore (15) and (16) yields

$$(\pounds_\xi R)(X,Y,\xi) = 0$$

and hence

$$(\pounds_\xi \mathrm{Ric})(X,\xi) = 0$$

which together with $\nabla^g \mathrm{Ric} = 0$ implies

$$\mathrm{Ric}(X,\xi) = \eta(X)\mathrm{Ric}(\xi,\xi)$$

and

$$\mathrm{Ric} = \mathrm{Ric}(\xi,\xi)g + \mathrm{d}(\mathrm{Ric}(\xi,\xi)) \otimes \eta.$$

Setting $X = Y = \xi$ in (1) it follows that

$$\mathrm{Ric}(\xi,\xi) = \frac{r}{2} - (\lambda + \mu)$$

and therefore

$$\mathrm{Ric} = \mathrm{Ric}(\xi,\xi)g$$

produces

$$\mathrm{Ric}(\xi,\xi) = \frac{r}{n}\cdot$$

Replacing Ric in the soliton equation we get

$$(1-\mu)(g - \eta \otimes \eta) = 0. \tag{17}$$

Computing the soliton equation (1) in (ξ,ξ) we get

$$\lambda + \mu = n - 1 + \frac{r}{2} \tag{18}$$

and taking the trace in (1) we obtain

$$n\lambda + \mu = -n + 1 + \frac{r(n-2)}{2}\cdot \tag{19}$$

From (17), (18) and (19) we can state

Proposition 13. *Let $(g, \xi, \eta, \lambda, \mu)$ define an η-Einstein soliton on the n-dimensional pseudo-Riemannian manifold (M, g) with unitary torse-forming potential vector field ($\nabla^g \xi = \mathrm{Id} - \eta \otimes \xi$ with $\eta := i_\xi g$). If M is Ricci symmetric with constant scalar curvature, then the manifold is Einstein of scalar curvature $r = -n(n-1)$ and $(\lambda, \mu) = \Big(-\dfrac{n(n-3)}{2} - 2, 1\Big)$.*

Following the same steps one can prove also

Proposition 14. *If $(g, \xi, \eta, \lambda, \mu)$ defines an η-Ricci soliton on the n-dimensional pseudo-Riemannian manifold (M, g) with unitary torse-forming potential vector field ($\nabla^g \xi = \mathrm{Id} - \eta \otimes \xi$ with $\eta := i_\xi g$) and M is Ricci symmetric with constant scalar curvature, then the manifold is Einstein of scalar curvature $r = -n(n-1)$ and $(\lambda, \mu) = (n-2, 1)$.*

Conclusions

Geometric flows, with their corresponding solitons, still constitutes a very actual subject to study. The paper puts into light the connection between the properties of the potential vector field of a soliton (of being infinitezimal harmonic or torse-forming) and the scalar curvature of the manifold, any contribution in this direction bringing new points of view on the geometry of the manifold.

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