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MODELING OF MINIMAL SURFACE BASED ON AN ISOTROPIC BEZIER CURVE OF FIFTH ORDER

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Abstract. A method is proposed for constructing minimal surfaces based on fifth-order Bezier isotropic curves specified in a vector-parametric form, allowing control of the guide curve and the surface in user mode. The coefficients of the basic quadratic forms were calculated and it was shown that the surfaces would be minimal. An example of a surface constructed by the proposed method is given.

MSC: 49Q05, 65D17
Keywords: Bezier curve, isotropic curve, minimal surface, quadratic forms

1. Introduction

The problem of building a smooth surface containing specified points or curves is particularly relevant due to the intensive development of mechanical engineering, the construction industry and computer technologies [12, 19]. Previously used for such problems shells of zero Gaussian curvature [22,23], although simple in design and manufacture, do not always give an optimal result when covering complex structures, and the carrying capacity of such shells significantly depends on small deviations of their overall contour from ideal shape [23]. The elimination of these drawbacks in the most natural way is possible by using minimal surfaces [16], whose theory has been successfully developed for a long time.

The minimal surface is a surface whose average curvature H is zero at all its points. The minimal surface is thus a surface of negative Gaussian curvature. The first studies of minimal surfaces were performed by Lagrange, who investigated the variation problem of finding the surface of the smallest area stretched over a given contour [7, 29]. Later, Monge established that the minimalism of the area leads to the condition that the mean curvature is zero. The classical problem of Plato [10] is also known, which consists in finding the surface of the smallest area passing through a given curve. The physical implementation of the task is achieved by immersing a rigid wire frame of some given shape into soapy water and then removing it – the shape of the resulting soap film is a solution to Plato's problem.

Later on minimal surfaces were studied by many researchers [8, 9, 11, 12, 21, 23, 24]. For example, in [9], the authors introduced minimal surfaces in the form of isotropic curves in $\mathbb{C}^3$. By specifying such a curve, they define a connected surface and a family of minimal surfaces associated with a minimal surface, which is the real part of an isotropic curve. In [11] discusses the use of the theory of surfaces of constant mean curvature, including submanifolds of minimal and constant mean curvature, in geometric measure theory and hypothesis of a double bubble, Lagrange geometry, numerical modeling of geometric phenomena, in general theory of relativity, in Riemannian geometry, in isoperimetric problem, in the theory of completely nonlinear elliptic equations and in the topology of three-dimensional varieties. In [8], minimal surfaces in $\mathbb{R}^3$ and classes of embedded complete minimal surfaces of finite topological type are discussed. The study [24] covers parametric and non-parametric surfaces, isothermal parameters, Bernstein theorem, new results on the Plateau problem and isoperimetric inequalities. In [25], it was shown that among correctly constructed minimal surfaces in Euclidean three-dimensional space, those that have finite total curvature form a natural and important subclass.

Applications of minimal surfaces in a free architecture are considered in [12], where surfaces consisting of smoothly connected bilinear portions are considered. These surfaces turn out to be discrete versions of negatively curved affine minimal surfaces and have many common properties with their classical smooth analogs. The paper proposes design approaches to design and examines special cases that should be interesting for architectural applications.

In the theory of minimal surfaces is the key issue of finding ways of constructing them. The classical method of constructing a minimal surface using an isotropic curve constructed on the basis of a given analytic function was suggested by Weierstrass [5]. A complex variable was substituted into the parametric equation of the isotropic curve instead of the parameter. When the real part was selected in the resulting equation, the minimal surface equations were obtained. The Weierstrass algorithm is strictly dependent on the specified coefficients of the analytical function and does not allow for a dynamic change of the surface in space based on changes of the guide curve changes. Agaltsev [1] built minimal and close to minimal surfaces along predetermined contours by defining an ordered point frame. Kurek designed the construction of some minimal surfaces with a linear frame, consisting of special lines based on approximation of surfaces by polynomials [17, 18]. In the works of Kovalev [15], the main approaches to the formation of orthogonal grids and minimal surfaces were analyzed. In [6], it is proposed to model the minimal and attached minimal surfaces using an isotropic curve, the equation of which is determined on the basis of a flat parametric curve. The author of [14] considers the construction of screw minimal surfaces and suggests ways of finding isotropic spatial curves. A number of papers [27, 28] offer an analytical description of minimal surfaces constructed on the basis of isotropic curves, which lie on different surfaces and are related to isometric networks.

All the proposed construction methods do not provide the possibility of promptly changing the contour of the surface, which makes it impossible to use them effectively in modern computer graphics. In our works [2, 3], a new research direction was started, which allows to solve this problem, build minimal surfaces based on isotropic curves specified in a vector-parametric form, and control the guide curve and the surface in user mode. These studies were mainly carried out with curves of the third order, but in this case, surfaces of the third order were obtained, which reduces the possibilities of shaping and their application. To expand the scope of the proposed method, it is advisable to construct with its help minimal surfaces that are obtained on the basis of curves of higher orders.

The goal of this paper is to construct a fifth-order isotropic Bezier curve in a vector-parametric form and to create on its basis a mathematical model of a minimal surface in a finite algebraic form, which allows changing the surface shape in real time while maintaining its zero mean curvature.

2. Modeling of Isotropic Fifth-Order Bezier Curve

Isotropic curves, or, in Lie's terminology [13, 20], "minimal curves" are curved lines whose arc differential in rectilinear coordinates is zero. Geometrically, this means that the tangent curves intersect the absolute conic section, that is, it lies on the surface adjoining the given curve. The length of the arc of such an isotropic curve, based on the definition, is zero. The contiguous plane of the curve contains two infinitely close tangent curves, that is, it will always be the tangent plane of absolute conic section at the same time. The simplest example of isotropic curves is the isotropic straight lines, for which the concept of a "touching plane" loses its meaning.

Isotropic curves can be represented using arbitrary analytic functions $f(t)$ and their derivatives [5] in the form

$$f(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + \ldots + a_n t^n \tag{1}$$

where $a_k, k = 0, \ldots, n$ – some complex values.

If the third derivative of such a function does not identically vanish

$$f^{'''}(t) \neq 0$$

then such a function corresponds to an isotropic line with a parametric equation of the form

$$\begin{aligned}
x_1 &= \mathrm{i}\cdot\left(f(t) - t\cdot f^{'}(t) - \frac{1-t^2}{2}f^{''}(t)\right)\\
x_2 &= \left(f(t) - t\cdot f^{'}(t) + \frac{1+t^2}{2}f^{''}(t)\right)\\
x_3 &= -\mathrm{i}\cdot\left(f^{'}(t) - t\cdot f^{''}(t)\right).
\end{aligned} \tag{2}$$

For any function $f(t)$, the curve defined by equation (2) will have zero length, that is, will be isotropic

$$\sum_{k=1}^{3}\left(x_k^{'}(t)\right)^2 = 0. \tag{3}$$

Let us apply the isotropy equation (3) to find the Bezier curves [2, 3, 16]. Let the nth order Bezier curve be given as

$$\mathbf{r}(t) = \sum_{j=0}^{n}\mathbf{r}_j J_{n,j}(t), \qquad J_{n,j}(t) = \frac{n!}{j!(n-j)!}t^j(1-t)^{(n-j)} \tag{4}$$

where $\mathbf{r}_j = [x_{1j}, x_{2j}, x_{3j}]$ are the reference point vectors.

Let us consider Bezier curves as complex analytic functions of a real argument for given isotropic characteristics. As isotropic characteristics, the sides of the characteristic polygon and the chord can be considered. Let us find the constraints that must be imposed on the reference points of the Bezier curve. To do this, take the derivative of expression (4) in the form

$$\begin{aligned}
\mathbf{r}^{'}(t) &= n\cdot\sum_{j=0}^{n-1}(\mathbf{r}_{j+1} - \mathbf{r}_j)J_{n-1,j}(t)\\
J_{n-1,j}(t) &= \frac{(n-1)!}{j!(n-1-j)!}t^j(1-t)^{(n-1-j)}.
\end{aligned} \tag{5}$$

We take the square of the expression (5) and substitute the isotropy condition of the curves. Let us denote

$$\begin{aligned}
S_{1k} &= \sum_{j=0}^{n-1}(x_{k(j+1)} - x_{kj})^2 J_{n-1,j}^2(t)\\
S_{2k} &= \sum_{j=0}^{n-1}\sum_{l=j+1}^{n-1}(x_{k(j+1)} - x_{kj})J_{n-1,j}(t)(x_{k(l+1)} - x_{kl})J_{n-1,l}(t).
\end{aligned}$$

In this way we get

$$\sum_{k=1}^{3} n^2(S_{1k} + 2S_{2k}) = 0. \tag{6}$$

Expression (6) determines the isotropy condition for the nth-order Bezier spatial curve. In order for this condition to be fulfilled and not dependent on the parameter values, it is necessary to equate the coefficients with t^j.

Thus, if in the isotropy condition for the n-th order Bezier curve we select the coefficients at t^j and equate them to zero, then we obtain equations for determining the coefficients of the functions $t^j(1-t)^{n-1-j}$ in the isotropy condition in expression (6). Let us turn to the fifth order curve

$$\begin{aligned}\mathbf{r}(t) =& \mathbf{r}_0(1-t)^5 + 5\mathbf{r}_1(1-t)^4 t + 10\mathbf{r}_2(1-t)^3t^2 \\ &+ 10\mathbf{r}_3(1-t)^2t^3 + 5\mathbf{r}_4(1-t)t^4 + \mathbf{r}_5 t^5.\end{aligned} \tag{7}$$

Condition (6) with $n = 5$ for each coordinate will be in form

$$\begin{aligned}
&25\sum_{k=1}^{3}[(x_{k1}-x_{k0})^2(1-t)^8 + (x_{k5}-x_{k4})^2t^8 \\
&+16(x_{k2}-x_{k1})^2(1-t)^6t^2 + 36(x_{k3}-x_{k2})^2(1-t)^4t^4 \\
&+16(x_{k4}-x_{k3})^2(1-t)^2t^6 + 48(x_{k2}-x_{k1})(x_{k3}-x_{k2})(1-t)^5t^3 \\
&+32(x_{k2}-x_{k1})(x_{k4}-x_{k3})(1-t)^4t^4 + 48(x_{k3}-x_{k2})(x_{k4}-x_{k3})(1-t)^3t^5 \\
&+8(x_{k1}-x_{k0})(x_{k2}-x_{k1})(1-t)^7t + 12(x_{k1}-x_{k0})(x_{k3}-x_{k2})(1-t)^6t^2 \\
&+8(x_{k1}-x_{k0})(x_{k4}-x_{k3})(1-t)^5t^3 + 2(x_{k1}-x_{k0})(x_{k5}-x_{k4})(1-t)^4t^4 \\
&+8(x_{k2}-x_{k1})(x_{k5}-x_{k4})(1-t)^3t^5 + 12(x_{k3}-x_{k2})(x_{k5}-x_{k4})(1-t)^2t^6 \\
&+8(x_{k4}-x_{k3})(x_{k5}-x_{k4})(1-t)t^7] = 0
\end{aligned}$$

where x_{kj} are the corresponding coordinates of the jth reference point. Let us represent this equation in the form

$$\sum_{j=0}^{n} B_j t^j(1-t)^{n-1-j} = 0. \tag{8}$$

Equating the coefficients B_j to zero, we get

$$\begin{aligned}
\sum_{k=1}^{3}(x_{k1}-x_{k0})^2=0\\
\sum_{k=1}^{3}8(x_{k1}-x_{k0})(x_{k2}-x_{k1})=0\\
\sum_{k=1}^{3}[16(x_{k2}-x_{k1})^2+12(x_{k1}-x_{k0})(x_{k3}-x_{k2})]=0\\
\sum_{k=1}^{3}[48(x_{k2}-x_{k1})(x_{k3}-x_{k2})+8(x_{k1}-x_{k0})(x_{k41}-x_{k3})]=0\\
\sum_{k=1}^{3}[36(x_{k3}-x_{k2})^2+32(x_{k2}-x_{k1})(x_{k4}-x_{k3})\\
+2(x_{k1}-x_{k0})(x_{k5}-x_{k4})]=0\\
\sum_{k=1}^{3}[48(x_{k3}-x_{k2})(x_{k4}-x_{k3})+8(x_{k2}-x_{k1})(x_{k5}-x_{k4})]=0\\
\sum_{k=1}^{3}[16(x_{k4}-x_{k3})^2+12(x_{k3}-x_{k2})(x_{k5}-x_{k4})]=0\\
\sum_{k=1}^{3}8(x_{k4}-x_{k3})(x_{k5}-x_{k4})=0\\
\sum_{k=1}^{3}(x_{k5}-x_{k4})^2=0.
\end{aligned} \tag{9}$$

The system of equations (9) determines the isotropic conditions of the fifth-order spatial Bezier curve.

Analysis of expression (9) shows that the lengths of the first and the last links of the characteristic polygon should be zero, that is, be isotropic.

If

$$\sum_{k=1}^{3}(x_{k(j+1)}-x_{kj})^2=0$$

where $j=0\dots4$, then it will be isotropic and the chord that pulls together the curve segment, that is

$$\sum_{k=1}^{3}(x_{k5}-x_{k0})^2=0.$$

In this case, we will have an isotropic Bezier curve, which lies in a three-dimensional plane in three-dimensional complex space.

The system (9) is a system of nonlinear (quadratic) equations. To solve this system, one can use one of the nonlinear methods of conjugate gradients, for example, the Polak and Ribiera methods [26]. This is an improved method of Fletcher and Reeves, which is based on a more general, compared to a quadratic, assumption about the approximation of the objective function and is less sensitive to round-off errors. An example of the implementation of the Polak and Ribiera methods can be found in [30].

In order to get rid of nonlinearity to find an isotropic curve, we use the method of deformation of a flat curve in the complex space [4]. Let the fifth-order flat isotropic Bezier curve be constructed using expressions (7) for the plane and the isotropy condition of the links of the characteristic polygon from the system

$$(x_{1(j+1)} - x_{1j}) = \pm\mathrm{i}(x_{2(j+1)} - x_{2j}), \qquad (x_{10} - x_{1n}) = \pm\mathrm{i}(x_{20} - x_{2n}) \quad (10)$$

where $j = 0 \ldots 4$ and the signs of both equations are identical.

We will deform the flat isotropic curve so that the length of the curve in the complex space remains unchanged, that is, isotropic. For a flat Bezier curve, all sides of the characteristic polygon and the chord are equal to zero, and for a spatial curve this condition should not be preserved. Therefore, we will change the points of the characteristic polygon with preservation of the condition of isotropy of the length of the curve.

For a fifth-order spatial curve, 9 conditions (9) must be satisfied, that is, out of 18 coordinates of a spatial curve, only 9 can be set. As initial conditions, we take the ordinates and abscissas of the four vertices of the characteristic polygon $\mathbf{r}_j = [x_{1j}, x_{2j}], j = 0 \ldots 3$ (10).

Consider the first isotropy condition for the fifth-order Bezier curve (9), namely, the isotropy of the side of the characteristic polygon, which coincides in direction with the tangent at the point $\mathbf{r}_0$ in form

$$\sum_{k=1}^{3}(x_{k1} - x_{k0})^2 = 0.$$

According to the initial data, only the coordinate x_{31} remains unknown in such an expression. Considering that for a flat curve

$$(x_{11} - x_{10})^2 + (x_{21} - x_{20})^2 = 0 \quad (11)$$

the first condition will be

$$x_{31} - x_{30} = 0. \quad (12)$$

We will have the required value $x_{31} = x_{30}$.

We consider the second condition in the form

$$\sum_{k=1}^{3}(x_{k1} - x_{k0})(x_{k2} - x_{k1}) = 0.$$

Substituting relations (11) and (12) into this expression, we obtain the identity. So, the second condition allows you to specify an arbitrary coordinate x_{32}, which is determined and depends on the order of the curve.

In the third condition

$$\sum_{k=1}^{3}[16(x_{k2} - x_{k1})^2 + 12(x_{k1} - x_{k0})(x_{k3} - x_{k2})] = 0$$

we substitute (12) and the isotropic conditions of the links of the polygon $\mathbf{r}_0\mathbf{r}_1$, $\mathbf{r}_1\mathbf{r}_2$, $\mathbf{r}_2\mathbf{r}_3$. From condition (10) for the plane we get the ratio $x_{32} = x_{31}$, which means that three points $\mathbf{r}_0, \mathbf{r}_1, \mathbf{r}_2$ lie in a plane in a three-dimensional complex space.

Similarly, substituting conditions (10) and the values found in condition (9), we find all other unknowns

$$\begin{aligned}
x_{14} &= \mathrm{i}(x_{24} - x_{23}) + x_{13} \\
x_{24} &= \frac{x_{22} - x_{21}}{(x_{21} - x_{20})^2}\left(3(x_{21} - x_{20})(x_{23} - x_{22}) - 2(x_{22} - x_{21})^2\right) + x_{23} \\
x_{34} &= \frac{3(x_{22} - x_{21})}{x_{21} - x_{20}}(x_{33} - x_{32}) + x_{33} \\
x_{15} &= -\frac{1}{\mathrm{i}}\left(3(x_{23} - x_{22}) - 2\frac{(x_{22} - x_{21})^2}{x_{21} - x_{20}}\right)^2 + \frac{9}{\mathrm{i}}(x_{33} - x_{32})^2 + x_{14} \\
x_{25} &= -\frac{18(x_{33} - x_{32})^2}{x_{21} - x_{20}} - \mathrm{i}(x_{15} - x_{14}) + x_{24} \\
x_{35} &= x_{33} + 3(x_{33} - x_{32}) \\
&\quad\times\left(\frac{(x_{22} - x_{21})(x_{21} - x_{20}) - 4(x_{22} - x_{21})^2 + 6(x_{23} - x_{22})(x_{21} - x_{20})}{(x_{21} - x_{20})^2}\right).
\end{aligned} \tag{13}$$

Thus, we obtain the coordinates for the isotropic Bezier curve without solving the system of nonlinear equations (9). For visualization of the obtained curve, the real and imaginary parts are singled out separately.

3. Modeling of the Minimum Surface

Weierstrass way [5], that is, the surface will be constructed on the basis of an isotropic guide curve with a conformal change of the parameter $t = u + v\mathrm{i}$. The equation of the surface will be obtained on the basis of the selection of the real or imaginary parts of the resulting expression coordinates of a spatial parametric isotropic curve

$$x_k(u,v) = x_k(t). \tag{14}$$

If we take the fifth-order Bezier curve as a guide curve, we will have

$$\begin{aligned}\mathbf{r}(u+\mathrm{i}v) = \Re[&\mathbf{r}_0(1-u-\mathrm{i}v)^5 + 5\mathbf{r}_1(1-u-\mathrm{i}v)^4(u+\mathrm{i}v)\\ &+ 10\mathbf{r}_2(1-u-\mathrm{i}v)^3(u+\mathrm{i}v)^2 + 10\mathbf{r}_3(1-u-\mathrm{i}v)^2(u+\mathrm{i}v)^3\\ &+ 5\mathbf{r}_4(1-u-\mathrm{i}v)(u+\mathrm{i}v)^4 + \mathbf{r}_5(u+\mathrm{i}v)^5].\end{aligned} \tag{15}$$

Equation (15) determines the minimum surface based on a fifth-order Bezier curve. Let us show by example that this equation determines the minimal surface. To do this, we will calculate the coefficients of the first quadratic form F

$$F = \sum_{k=1}^{3} x_{k,u}(u,v)x_{k,v}(u,v), \quad E = \sum_{k=1}^{3}(x_{k,u}(u,v))^2, \quad G = \sum_{k=1}^{3}(x_{k,v}(u,v))^2$$

where subscript after comma means differentiation according to the corresponding parameter, and also calculate the average curvature H for the surface of the form

$$H = \frac{1}{2}\frac{LG - 2MF + NE}{EG - F^2} \tag{16}$$

where

$$L = \frac{1}{\sqrt{EG-F^2}}\begin{vmatrix} x_{1,uu} & x_{2,uu} & x_{3,uu}\\ x_{1,u} & x_{2,u} & x_{3,u}\\ x_{1,v} & x_{2,v} & x_{3,v}\end{vmatrix}, \quad M = \frac{1}{\sqrt{EG-F^2}}\begin{vmatrix} x_{1,uv} & x_{2,uv} & x_{3,uv}\\ x_{1,u} & x_{2,u} & x_{3,u}\\ x_{1,v} & x_{2,v} & x_{3,v}\end{vmatrix}$$

$$N = \frac{1}{\sqrt{EG-F^2}}\begin{vmatrix} x_{1,vv} & x_{2,vv} & x_{3,vv}\\ x_{1,u} & x_{2,u} & x_{3,u}\\ x_{1,v} & x_{2,v} & x_{3,v}\end{vmatrix}.$$

Let us consider one example. We construct a minimal surface based on the deformation of the Bezier curve, if the following values are given

$$\begin{aligned}
x_{10} &= 1 + 2\mathrm{i}, & x_{20} &= 3\mathrm{i}, & x_{31} &= 3 + 6\mathrm{i} \\
\Re x_{11} &= 2, & \Re x_{21} &= 2, & x_{32} &= 8 + 5\mathrm{i} \\
\Re x_{12} &= 2, & \Re x_{22} &= 2, & x_{30} &= -3 - 2\mathrm{i} \\
\Re x_{13} &= 2, & \Re x_{23} &= 2, & x_{33} &= 4 + \mathrm{i}.
\end{aligned}$$

On the basis of the isotropy of the plane curve (10) we find

$$\begin{aligned}
\Im x_{11} &= 4, & \Im x_{21} &= 2, & \Im x_{12} &= 4 \\
\Im x_{22} &= 2, & \Im x_{13} &= 4, & \Im x_{23} &= 2.
\end{aligned}$$

Now we calculate the coordinates on the basis of deformation equations (11), (12), (13)

$$\begin{aligned}
&x_{31} = x_{30} = -3 - 2\mathrm{i}, && x_{24} = 2 + 4\mathrm{i}, && x_{14} = 2 + 4\mathrm{i}, \\
&x_{32} = x_{30} = -3 - 2\mathrm{i}, && x_{25} = -66.4 - 221.2\mathrm{i}, && x_{15} = -221.2 + 72.4\mathrm{i} \\
&x_{34} = 4 + \mathrm{i} \\
&x_{35} = 4 + \mathrm{i}.
\end{aligned}$$

We substitute the resulting values in the expression for the surface

$$x_1(u,v) = \frac{N_1}{D}, \qquad x_2(u,v) = \frac{N_2}{D}, \qquad x_3(u,v) = \frac{N_3}{D}$$

where

$$\begin{aligned}
N_1 = {} & (1.0 + 2.0\mathrm{i})(1 - u - \mathrm{i}v)^5 + (10.0 + 20.0\mathrm{i})(1 - u - \mathrm{i}v)^4(u + \mathrm{i}v) \\
& + (20.0 + 40.0\mathrm{i})(1 - u - \mathrm{i}v)^3(u + \mathrm{i}v)^2 \\
& + (20.0 + 40.0\mathrm{i})(1 - u - \mathrm{i}v)^2(u + \mathrm{i}v)^3 \\
& + (10.0 + 20.0\mathrm{i})(1 - u - \mathrm{i}v)(u + \mathrm{i}v)^4 + (-221.2 + 72.4\mathrm{i})(u + \mathrm{i}v)^5
\end{aligned}$$

$$
\begin{aligned}
N_2 = &(0.0 + 3.0\mathrm{i})(1 - u - \mathrm{i}v)^5 + (10.0 + 10.0\mathrm{i})(1 - u - \mathrm{i}v)^4(u + \mathrm{i}v) \\
&+ (20.0 + 20.0\mathrm{i})(1 - u - \mathrm{i}v)^3(u + \mathrm{i}v)^2 \\
&+ (20.0 + 20.0\mathrm{i})(1 - u - \mathrm{i}v)^2(u + \mathrm{i}v)^3 \\
&+ (10.0 + 10.0\mathrm{i})(1 - u - \mathrm{i}v)(u + \mathrm{i}v)^4 + (-66.4 - 221.2\mathrm{i})(u + \mathrm{i}v)^5 \\
N_3 = &(-3.0 - 2.0\mathrm{i})(1 - u - \mathrm{i}v)^5 + (-15.0 - 10.0\mathrm{i})(1 - u - \mathrm{i}v)^4(u + \mathrm{i}v) \\
&+ (-30.0 - 20.0\mathrm{i})(1 - u - \mathrm{i}v)^3(u + \mathrm{i}v)^2 \\
&+ (40.0 + 10.0\mathrm{i})(1 - u - \mathrm{i}v)^2(u + \mathrm{i}v)^3 \\
&+ (20.0 + 5.0\mathrm{i})(1 - u - \mathrm{i}v)(u + \mathrm{i}v)^4 + (4.0 + 1.0\mathrm{i})(u + \mathrm{i}v)^5 \\
D = &1.0(1 - u - \mathrm{i}v)^5 + 5.0(1 - u - \mathrm{i}v)^4(u + \mathrm{i}v) \\
&+ 10.0(1 - u - \mathrm{i}v)^3(u + \mathrm{i}v)^2 + 10.0(1 - u - \mathrm{i}v)^2(u + \mathrm{i}v)^3 \\
&+ 5.0(1 - u - \mathrm{i}v)(u + \mathrm{i}v)^4 + 1.0(u + \mathrm{i}v)^5.
\end{aligned}
$$

Let's select the real part of the resulting equation (15) and display the real part (Fig. 1).

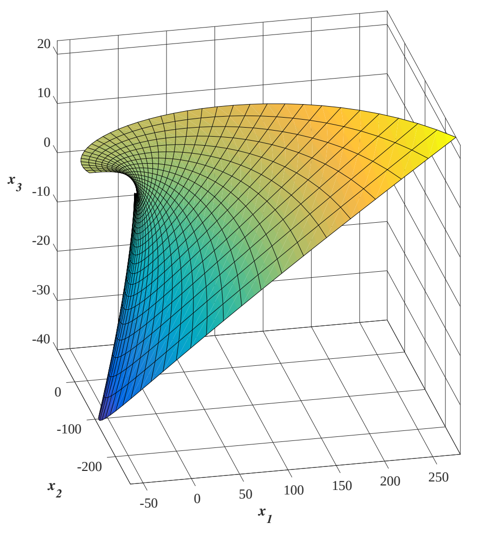

Figure 1. Minimal surface with isotropic guide Bezier curve, constructed on the basis of a flat curve deformation.

$$
\begin{aligned}
\Re x_1 = &\,5.0u - 10.0v - 5.0v^4 - 70.4v^5 + 10.0v^2 + 20.0v^3 - 30.0uv^2 - 60.0u^2v\\
&+ 30.0u^2v^2 + 40.0u^3v - 40.0uv^3 + 2222.0u^3v^2 - 1111.0uv^4 - 352.0u^4v\\
&+ 704.0u^2v^3 + 40.0uv - 222.2u^5 - 5.0u^4 + 10.0u^3 - 10.0u^2 + 1.0\\
\Re x_2 = &\,10.0u + 5.0v - 10.0v^4 + 224.2v^5 + 20.0v^2 - 10.0v^3 - 60.0uv^2 + 30.0u^2v\\
&+ 60.0u^2v^2 - 20.0u^3v + 20.0uv^3 + 664.0u^3v^2 - 332.0uv^4 + 1121.0u^4v\\
&- 2242.0u^2v^3 - 20.0uv - 66.4u^5 - 10.0u^4 + 20.0u^3 - 20.0u^2\\
\Re x_3 = &- 3.0 - 105.0v^4 - 18.0v^5 + 30.0v^3 - 210.0uv^2 - 90.0u^2v + 630.0u^2v^2\\
&+ 180.0u^3v - 180.0uv^3 - 420.0u^3v^2 + 210.0uv^4 - 90.0u^4v + 180.0u^2v^3\\
&+ 42.0u^5 - 105.0u^4 + 70.0u^3.
\end{aligned}
$$

In order to prove that the surface is minimal, we check the value of the average curvature, the coefficients of the second quadratic form and their symmetry. As we can see H and F are close to zero, that is, the surface is minimal and the coordinate lines are orthogonal. The equality $E = G$ shows that the grid is isothermal.

4. Conclusions

As a result of the studies, it was proposed to construct a surface on the basis of a conformal replacement of the parameter in the fifth-order isotropic Bezier curve equation. The curve was constructed on the basis of the deformation of a flat isotropic curve. Using the coefficients of the basic quadratic forms, it was shown that the surface will be minimal with an orthogonal and isothermal coordinate grid.

Further research may be related to the use of functions of a complex variable in the case of quasiconformal replacement of parameters.

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