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ON THE GEOMETRY OF ORBITS OF CONFORMAL VECTOR FIELDS

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Abstract. Geometry of orbit is a subject of many investigations because it has important role in many branches of mathematics such as dynamical systems,control theory. In this paper it is studied geometry of orbits of conformal vector fields. It is shown that orbits of conformal vector fields are integral submanifolds of completely integrable distributions. Also for Euclidean space it is proven that if all orbits have the same dimension they are closed subsets.

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1. Introduction

The integral curves of vector fields and the orbits of an arbitrary set of vector fields on the smooth manifold have been studied in many investigations because of their importance in Control Theory, Dynamical systems, Foliation Theory and Physics [1, 4, 10, 14–16]. To the study of systems of vector fields from the point of view of Control Theory (the properties of accessibility and complete controllability) have been devoted numerous investigations [15, 16]. One of important class of vector fields is class of conformal vector fields which has wide applications. Geometry of conformal vector fields is subject of many papers [2–4, 12]. In this paper we study some properties of orbits of conformal vector fields.

In the paper, smoothness is understood as smoothness of the class C^{∞}.

2. Orbits of Vector Fields and Distributions

Let (M, g) be a smooth Riemannian manifold of dimension n with metric tensor g, $V(M)-$ a set of all smooth vector fields on a manifold M. The set $V(M)$ is a

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linear space over the field of real numbers and a Lie algebra with respect to the Lie bracket of vector fields.

Let D be a family of smooth vector fields defined on M. The family D can contain finitely or infinitely many smooth vector fields. For a vector field $X \in D$, by $X^t(x)$ we denote the integral curve of X passing through a point $x \in M$ for $t = 0$. The mapping $t \longrightarrow X^t(x)$ is defined in some domain $I(x)$, which in general depends not only on the field X but also on the initial point x. Throughout the following, we assume that $t \in I(x)$. If the domain $I(x)$ of the curve $t \longrightarrow X^t(x)$ coincides with the real line for each $x \in M$, then the vector field X is said to be complete.

Definition 1. *The orbit $L(x)$ of a family D of vector fields through a point x is the set of points y in M such that there exist real numbers $t_1, t_2, \dots, t_k$ and vector fields $X_1, X_2, \dots, X_k$ in D (where k is an arbitrary positive integer) such that*

$$y = X_k^{t_k}(X_{k-1}^{t_{k-1}}(...(X_1^{t_1})...)).$$

Obviously, if the family D consists of a single vector field, then the orbit is a smooth curve (a one-dimensional manifold).

The fundamental result in study of orbits is Sussmann theorem [14], which asserts that every orbit of smooth vector fields with Sussmann topology has differential structure with respect to which it is a immersed submanifold of M.

Recall that a mapping P that takes each point $x \in M$ to some subspace $P(x) \subset T_xM$ is called a distribution. If dim $P(x) = k$ for all $x \in M$, then P is called a k-dimensional distribution. A distribution P is said to be smooth if, for each point $x \in M$, there exists a neighborhood $U(x)$ of that point and smooth vector fields $X_1, X_2, \dots, X_m$ defined on $U(x)$ such that the vectors

$$X_1(y), X_2(y), \dots, X_m(y)$$

form a basis of the subspace $P(y)$ for each $y \in U(x)$. A family D of smooth vector fields naturally generates the smooth distribution that takes each point $x \in M$ to the subspace $P(x)$ of the tangent space T_xM spanned by the set

$$D(x) = \{X(x)\ ;\ X \in D\}.$$

Obviously, the dimension of the subspace $P(x)$ can vary from point to point.

A distribution P is said to be completely integrable if, through every point $x \in M$, there passes a connected submanifold N_x of the manifold M such that $T_yN_x = P(y)$ for all $y \in N_x$. Note we shall use T_pM to denote the tangent space to the manifold M at the point p. If N is a submanifold of M, and if $p \in N$, than T_pN

can be identified in a natural way with a subspace of T_pM. The submanifold N_x is called an integral submanifold of the distribution P.

For a vector field X, we write $X \in P$ if $X(x) \in P(x)$ for all $x \in M$.

A distribution P is said to be involutive if the inclusion $X, Y \in P$ implies that $[X, Y] \in P$, where $[X, Y]$ is the Lie bracket of the fields X and Y.

The Frobenius theorem [8, p 10] provides a necessary and sufficient condition for the completely integrability of a distribution of constant dimension.

Theorem 2. *A distribution P on a manifold M is completely integrable if and only if it is involutive.*

Let $A(D)$ be the smallest Lie algebra containing the set D. By setting $A_x(D) = \{X(x)\ ;\ X \in A(D)\}$, we obtain an involutive distribution $P_D : x \to A_x(D)$. If the dimension $\dim A_x(D)$ is independent of x, then the distribution $P_D : x \to A_x(D)$ is completely integrable by the Frobenius theorem.

If the dimension $\dim A_x(D)$ depends on x, then, as following example from [9] shows, the distribution $P_D : x \to A_x(D)$ is not necessarily completely integrable. Let $M = \mathbb{R}^2$ with Cartesian coordinate system (x, y) and the set D consists from vector field $\dfrac{\partial}{\partial x}$ and vector fields $f(x)\dfrac{\partial}{\partial y}$, where $f(x)$ is any smooth function such that all derivatives $f^k(x)$, $k = 0, 1, 2, \ldots$ vanishes at the point $x = 0$. If a point p lies on $y-$ axis, then we have $\dim A_p(D) = 1$ and $A_p(D))$ is parallel to the $x-$ axis, while we have $\dim A_p(D) = 2$ otherwise. Therefore there does not exists an integral manifold through a point p.

The Frobenius theorem generalized by Hermann to distributions of variable dimension provides a necessary and sufficient condition for the complete integrability of distributions which is finitely generated [5].

Definition 3. *A system of vector fields $D = \{X_1, X_2, ..., X_k\}$ on M is in involution if there exist smooth real-valued functions $f_{ij}^l(x), x \in M,\ i, j, l = 1, ..., k$ such that for each (i, j) it takes*

$$[X_i, X_j] = \sum_{l=1}^{k} f_{ij}^l(x) X_l.$$

Theorem 4 (Herman) . *The system $D = \{X_1, X_2, ..., X_k\}$ of smooth vector fields on M generates completely integrable distribution if and only if it is in involution.*

3. Geometry of Orbits of Conformal Vector Fields

In this section we study geometry of orbits of conformal vector fields and in particularly we show that if the set D consists of conformal vector fields, then the distribution $P_D : x \to A_x(D)$ is completely integrable.

Definition 5. *A vector field X is conformal if $L_X g = \sigma g$, where σ is a function on (M, g), $L_X g$ denotes Lie derivative of the metric g with respect to X.*

It is known that a vector field X on (M, g) is conformal if and only if the local one-parameter group of local transformations generated by vector field X consists of conformal transformations. A local one-parameter group of local transformations generated by a conformal vector field consists of homotheties if σ is a constant, and consists of isometries if $\sigma = 0$.

We recall that a diffeomorphism $\phi : M \to M$ is called a conformal transformation if $\mathrm{d}\phi(g) = \lambda g$, where $\mathrm{d}\phi(g)(u, v)$ =$g(\mathrm{d}\varphi(u), \mathrm{d}\varphi(v))$, λ is a positive function on (M, g), u, v -tangent vectors. If λ constant, then ϕ is a transformation of homothety. If λ is identically equal to 1, then ϕ is isometry. Examples of conformal vector fields are Killing vector fields. Recall that a vector field on (M, g) is called a Killing field if its flow consists of isometries of a Riemannian manifold (M, g), that is $L_X g = 0$. Geometry of orbits of Killing vector fields is studied in [10].

Numerous studies have been devoted to the study of geometry of conformal vector fields [2–4, 7, 10, 12, 13], in particular in [3] it was proved that if the manifold is compact, then the set of fixed points of the conformal vector field is a submanifold of even codimension.

It was shown in [13] that if a Riemannian manifold (M, g) is different from a Euclidean space or a sphere, then on a manifold (M, g) there exists a Riemannian metric $\tilde{g}$ conformally equivalent to the Riemannian metric g such that the group of conformal transformations of a manifold (M, g) is a group of isometries of $(M, \tilde{g})$ (see also [4]). This fact shows that all conformal vector fields on manifolds are Killing vector fields with respect to the Riemannian metric $\tilde{g}$. It follows that on manifolds that are different from the Euclidean space and from the sphere, the study of the geometry of conformal vector fields reduces to studying Killing vector fields [10].

Note that the Lie bracket of two conformal fields and a linear combination of conformal fields over the field of real numbers are conformal fields as well. Therefore, the set $\mathrm{Conf}(M)$ of all conformal vector fields on the manifold M is a Lie algebra over the field of real numbers. In addition, it is well known that the dimension of the Lie algebra $\mathrm{Conf}(M)$ of conformal vector fields on a connected Riemannian manifold M does not exceed $(n+1)(n+2)/2$, where $n = \dim M$, $n \geq 3$ [8, p.310, Theorem 1].

Now by $A(D)$ we denote the smallest Lie subalgebra of Lie algebra $\mathrm{Conf}(M)$ containing the set D. Since the algebra $\mathrm{Conf}(M)$ is finite-dimensional, it follows that there exists finite number vector fields $X_1, X_2, \ldots, X_m$ in $A(D)$ such that the vectors $X_1(x), X_2(x), \ldots, X_m(x)$ form a basis of the subspace $A_x(D)$ for each $x \in M$.

Therefore, Theorem 5 implies the following assertion for the case in which the family D consists of conformal vector fields.

Theorem 6. *Let $n \geq 3$, $D-$ be family of conformal vector fields. Then every orbit of D is a integral manifold of completely integrable distribution $P_D : x \to A_x(D)$.*

Proof: In this case, the distribution $P_D : x \longrightarrow A_x(D)$ finitely generated and by the Hermann theorem, it is completely integrable.

Let N_x be an integral submanifold of the distribution $P_D : x \to A_x(D)$ passing through the point x, $L(x)-$ the orbit passing through the point x. Then, by the definition of an orbit $L(x) \subset N_x$, and in addition it is known that $\dim A_x(D) \leq \dim L(x)$ [14]. Since $\dim A_y(D) = \dim N_x$ for every $y \in N_x$ if that $\dim A_x(D) = k$, integral submanifold N_x and the orbit $L(x)$ are k dimensional manifold. It follows from here orbit $L(x)$ is open subset of N_x. Since different orbits do not intersect, by virtue of the connectedness of N_x we get that $L(x) = N_x$. ■

In the case $n \geq 3$ this theorem permits one to study the geometry and topology of the orbit $L(x_0)$ of the conformal vector fields with the use of the mapping

$$\varphi : (t_1, t_2, \ldots, t_m) \to X_m^{t_m}(X_{m-1}^{t_{m-1}}(...(X_1^{t_1}(x_0)...))) \qquad (1)$$

where $(t_1, t_2, \ldots, t_m) \in U \subset \mathbb{R}^m$ and U is a neighborhood of the origin in $\mathbb{R}^m$. The following theorem shows that each point in the orbit $L(x_0)$ can be reached from x_0 by finitely many "switches" using of the vector fields $X_1, X_2, \ldots, X_m$ in a certain order.

Theorem 7. *The set of points of the form*

$$y = X_m^{t_m}(X_{m-1}^{t_{m-1}}(...(X_1^{t_1}(x_0)...))) \qquad (2)$$

where $(t_1, t_2, \ldots, t_m) \in U$ coincides with the orbit $L(x_0)$.

Proof: Let $\mathrm{rank}\{X_1, X_2, \ldots, X_m\} = k$ at the point x_0, and let $x \in L(x_0)$. Then there exist vector fields $Y_1, Y_2, \ldots, Y_p$ in D such that

$$x = Y_p^{t_p}(y_{p-1}^{t_{p-1}}(...(y_1^{t_1}(x_0)...))).$$

The mapping

$$\psi : x_0 \to x = Y_p^{t_p}(y_{p-1}^{t_{p-1}}(...(y_1^{t_1}(x_0)...)))$$

is a conformal transformation of the Riemannian manifold M. Consider the vector fields $Z_i = \mathrm{d}\psi(X_i)$, where $\mathrm{d}\psi$ is the differential of the mapping ψ. The one-parameter local transformation group of the vector field Z_i has the form $\psi X_i^t \psi^{-1}$, where X_i^t is the flow of the vector field X_i [8, Proposition 1.7, p.14]. Since ψ is a conformal transformation, it follows that the vector field Z_i is a conformal vector field as well. Therefore, the vector fields Z_i can be linearly expressed via the vector fields $X_1, X_2, \ldots, X_m$. Since $\mathrm{rank}\{Z_1, Z_2, \ldots, Z_m\} = k$ at the point $x \in L(x_0)$, we find that $\mathrm{rank}\{X_1, X_2, \ldots, X_m\}$ is also equal to k at the point $x \in L(x_0)$; i.e., $\mathrm{dim}A_x(D) = k$ for all $x \in L(x_0)$. This, together with the Frobenius theorem, implies that $\mathrm{dim}L(x_0) = k$.

Consider the mapping (1) of U into the orbit $L(x_0)$. Since the rank of the mapping (1) is equal to the rank of the family $\{X_1, X_2, \ldots, X_m\}$ and $m \geq k$, it follows that it is a mapping of maximum rank. Therefore, the set of points of the form

$$y = X_m^{t_m}(X_{m-1}^{t_{m-1}}(...(X_1^{t_1}(x_0)...)))$$

is an open subset of $L(x_0)$.

Indeed, if $y_0 = X_m^{t_m^0}(X_{m-1}^{t_{m-1}^0}(...(X_1^{t_1^0}(x_0)...)))$, then, by virtue of the rank theorem [8, Proposition 1.1, p 8], there exists a neighborhood V of the point $(t_1^0, t_2^0, \ldots, t_m^0)$ and a neighborhood U of the point y_0 in $L(x_0)$ such that

$$\{X_m^{t_m}(X_{m-1}^{t_{m-1}}(...(X_1^{t_1}(x_0)...))); (t_1, t_2, \ldots, t_m) \in V\} = U.$$

It follows that $y_0 \in U \subset L(x_0)$.

Now let us show that the set of points of the form (2) is a closed subset of $L(x_0)$.

Let the points $y_i = X_m^{t_m}(X_{m-1}^{t_{m-1}}(...(X_1^{t_1}(x_0)...)))$ converge to a point $y_0 \in L(x_0)$. Since the rank of the vectors $X_1, X_2, \ldots, X_m$ at the point $y_0 \in L(x_0)$ is equal to k, it follows from the rank theorem that there exists a neighborhood U of y_0 in $L(x_0)$ that consists of points of the form

$$X_m^{t_m^0}(X_{m-1}^{t_{m-1}^0}(...(X_1^{t_1^0}(x_0)...))).$$

Therefore, if $y_i \in U$, then $y_0 = X_m^{\tau_m}(X_{m-1}^{\tau_{m-1}}(...(X_1^{\tau_1}(x_0)...)))$, where $\tau_j = t_j - t_j^0$. The proof of the Theorem 7 is complete. ■

Theorem 8. *Let $M = \mathbb{R}^n$, the set D consists of conformal vector fields and $\dim A_x(D) = k$ for any $x \in M$, where $0 < k \leq n$. Then each orbit of the family D is closed subset.*

Proof: In case $k = n$ every orbit L is connected open subset of $\mathbb{R}^n$ (n-dimensional manifold) and since different orbits do not intersect, by virtue of the connectedness of $\mathbb{R}^n$ we get that $L = \mathbb{R}^n$ [14].

We will suppose $0 < k < n$. By Frobenius theorem all orbits are $k-$ dimensional manifolds. We consider two cases: $n \geq 3$ and $n = 2$.

I. Assuming $n \geq 3$, which allows to use Theorem 7. In this case algebra $A(D)$ is finitely generated.

Let vector fields $X_1, X_2, ..., X_m$ from $A(D)$ form a basis of the Lie subalgebra $A(D)$. The condition $\dim A_x(D) = k$ for any $x \in M$ implies that rank of the map (1) is equal to k at $t_1 = 0, t_2 = 0, \ldots, t_m = 0$ for any point $x \in M$ because rank of (1) is equal the rank of the vectors $X_1, X_2, \ldots, X_m$ at the point x.

Let $O \in \mathbb{R}^n$ be the origin of the coordinate system.There are k vector fields $\{X_{i_1}, X_{i_2}, \ldots, X_{i_k}\}$ from the set $\{X_1, X_2, \ldots, X_m\}$ which are linearly independent at the point O. We put $Y_j = X_{i_j}$ for $j = 1, 2, ..., k$.

It is well known that a conformal transformation of Euclidean space is an affine transformation. Consequently, flows of conformal of the vector fields Y_j have the form $Y_j^t(x) = \lambda_j(t)A_j(t)x + b_jt$ for each $x \in \mathbb{R}^n$, where $A(t)$ is an orthogonal matrix, $\lambda_j(t)$ function with condition $\lambda_j(0) = 1$, b_j- are vectors.

Integral lines of vector fields Y_j passing through the point O at $t = 0$ have the form $Y_j^t(O) = b_jt$, for $j = 1, 2, ..., k$. From here follows that b_j- are linear independent vectors and orbit L_0 of D, passing through the point O, contains k straight lines parallel to vectors b_j. Since the conformal transformations translate straight lines to straight lines, we can conclude that the orbit L_0 contains k dimensional plane, and hence it is k dimensional plane.

Since $b_j \neq 0$, it follows from the equality $Y_j^{t+s}(x) = Y_j^t(Y_j^s(x))$ that $\lambda_j(t)A_j(t)b_j = b_j$ for every $x \in \mathbb{R}^n$, and therefore $\lambda_j(t) = 1$, $A_j(t)b_j = b_j$. It means that the vector b_j is parallel to the plane of fixed points of $A_j(t)$ or $A_j(t) = E$ for all t, where E is the identity matrix. It follows Y_j are complete Killing vector fields.

Now, if $x \in \mathbb{R}^n \setminus L_0$, since vectors b_j are linearly independent, vector fields Y_j linearly independent at point x. We consider

$$\varphi : (t_1, t_2, \ldots, t_k) \in \mathbb{R}^k \to Y_m^{t_k}(Y_{k-1}^{t_{k-1}}(...(Y_1^{t_1}(x)...))). \tag{3}$$

Rank of the map (3) is equal to k at any point and as follows from the Theorem 7 the set of points of the form

$$y = Y_k^{t_m}(Y_{k-1}^{t_{k-1}}(...(Y_1^{t_1}(x)...)))$$

where $(t_1, t_2, \ldots, t_k) \in \mathbb{R}^k$, coincides with the orbit $L(x)$, containing the point x. Now let $x_j \in L(x)$, $x_j \to y$ at $j \to \infty$, where $L(x)-$ the orbit containing the point x. The points x_j have the form

$$x_j = Y_k^{t_k^j} t(Y_{k-1}^{t_{k-1}^j}...(Y_1^{t_1^j}(x)...))$$

where $(t_1^j, t_2^j, \cdots, t_k^j) \in \mathbb{R}^k$. Conformal mappings

$$\varphi(t_1^j, t_2^j, ..., t_k^j)(x) = Y_k^{t_k^j} t(Y_{k-1}^{t_{k-1}^j}...(Y_1^{t_1^j}(x)...))$$

have the form

$$\varphi(t_1^j, t_2^j, ..., t_k^j)(x) = A(t_1^j, t_2^j, ..., t_k^j)x + b(t_1^j, t_2^j, ..., t_k^j)$$

where the orthogonal matrix $A(t_1^j, t_2^j, ..., t_k^j)$ is multiplication orthogonal matrices $A_p(t_p^j)$. The vector $b(t_1^j, t_2^j, ..., t_k^j)$ has the form

$$\sum_{l=1}^{k-1}\{\prod_{p=l+1}^{k} A_p(t_p^j)\}b_l t_l^j + b_k t_k^j. \tag{4}$$

From (4) follows that if $x_j \to y$ for $j \to \infty$, since vectors b_p are non-zero, the sequence t_p^j has a finite limit for every p: $t_p^j \to t_p^0$ for $j \to \infty$, where t_p^0 is a finite number. From here follows that $y = \varphi(t_1^0, t_2^0, ..., t_k^0)(x)$, i.e., $y \in L(x)$. Consequently, $L(x)$ is a closed set.

II. Let us consider the case $n = 2$. In this case we can not use Theorem 7, because algebra $A(D)$ is not finitely dimensional. If $\dim A_x(D) = 1$ for any $x \in \mathbb{R}^2$, then orbits generates one dimensional foliation F on $\mathbb{R}^2$. In this case as follows from results of [6, 11] every orbit is homeomorphic to $\mathbb{R}^1$ and is a closed subset of $\mathbb{R}^2$ (every orbit is level set of a continuous function [6]).

The case $n = 1$ is proved similarly. The proof of the Theorem 8 is complete. ■

Example 9. *Let us consider a set D fields which contains conform vector fields $X = \dfrac{\partial}{\partial x_1}$ and $Y = -x_2\dfrac{\partial}{\partial x_1} + x_1\dfrac{\partial}{\partial x_2}$ on Euclidean plane $\mathbb{R}^2$ with cartesian coordinates x_1, x_2. These vector fields generate the smooth distribution*: $(x_1, x_2) \to$

$P(x_1, x_2)$, where the subspace $P(x_1, x_2)$ is spanned by the set of vectors $\{X(x_1, x_2)\ ;\ X \in D\}$. We have $\dim P(x, y) = 2$ for every point (x, y) different from the points $(0, x_2)$, where $\dim P(x, y) = 1$. This smooth distribution is finite generated, but the D it is not in involution. In this case the smallest Lie subalgebra $A(D)$ of Lie algebra $\mathrm{Conf}(M)$ containing the set D is three dimensional. Vector fields X, Y and $Z = \dfrac{\partial}{\partial x_2}$ are basic fields of algebra $A(D)$. We can check that $\mathrm{dim} A_x(D) = 2$ for every point $x \in \mathbb{R}^2$. The distribution $x \to A_x(D)$ is completely integrable by Herman theorem, every orbit of D is $\mathbb{R}^2$.

Example 10. *If we consider a set D which contains only vector field $X = x_1 \dfrac{\partial}{\partial x_1} + x_2 \dfrac{\partial}{\partial x_2}$, than algebra $A(D)$ is one dimensional algebra. But $\dim A_x(D)$ is not constant. We have $\mathrm{dim} A_{(x,y)}(D)) = 1$ for every (x, y) different from $(0, 0)$, where* $\mathrm{dim} A_{(x,y)}(D) = 0$.

Example 11. *Let us consider the set D, which contains only one conformal vector field*

$$X = -x_2 \frac{\partial}{\partial x_1} + x_1 \frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_3}$$

in $M = \mathbb{R}^3$. In this case flow of X has following form $X^t(x) = A(t)x + bt$ for each $t \in \mathbb{R}$, where $b = \{0, 0, 1\}^T$

$$A(t) = \begin{pmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

In this case $\mathrm{dim} A_x(D) = 1$ for any $x \in M$. The orbit L_0, passing through the origin O of the coordinate system is axis OZ. Other orbits are helices.

Remark 12. *As simple examples show* Theorem 8 *is not true without supposition $\mathrm{dim} A_x(D) = k$ for any $x \in M$. Really, let $M = \mathbb{R}^n(x_1, x_2, ..., x_n)$ and the set D contains only the conformal vector field $X = \sum_{i=1}^n x_i \dfrac{\partial}{\partial x_i}\cdot$ In this case for any point $p(x_1, x_2, ..., x_n)$, with $\sum_{i=1}^n x_i^2 > 0$, orbit $L(p)$ is not closed subset.*

Remark 13. *As well known irrational winding of the two dimensional torus shows, in general* Theorem 8 *is not true for Riemannian manifolds different from Euclidean spaces.*

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