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CASSINI OVALS IN HARMONIC MOTION ORBITS*

KHRISTO N. BOYADZHIEV AND IRINA A. BOYADZHIEV

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Abstract. We discover the appearance of interesting Cassinian ovals in the motion of a two-dimensional harmonic oscillator. The trajectories of the oscillating points are ellipses depending on a parameter. The locus of the foci of these ellipses is a Cassini oval. The form of this oval depends on the magnitude of the initial velocity.

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1. Introduction

In this note we point out an interesting geometric phenomenon. We consider mechanical vibrations on the plane where the vibrating point traces various ellipses. We show that the foci of these ellipses trace different Cassini ovals. The forms of these ovals depend only on the initial velocity.

The simple free non-damped harmonic oscillator is governed by the differential equation

$$x'' + \omega^2 x = 0 \tag{1}$$

where $x(t)$ is the position function, $t \geq 0$ is time, and ω is the angular frequency. This equation results from Newton's law $F = m\,a$ by using the force $F = -k\,x$, where $k > 0$ is a constant. Then we have $m\,x'' = -k\,x$ or (1) with $\omega^2 = k/m$.

Suppose the initial position is the point $(\mathring{x}, 0)$ in the xy-plane and the initial velocity is $\mathring{v} = x'(0)$. The law of motion is

$$x(t) = \mathring{x}\cos(\omega t) + \frac{\mathring{v}}{\omega}\sin(\omega t). \tag{2}$$

The point $M(x(t), 0)$ oscillates over the interval $[-c, c]$, where $c = \sqrt{\mathring{x}^2 + \frac{\mathring{v}^2}{\omega^2}}$.

Now we extend this motion to two dimensions by assuming that the initial velocity is a vector $\mathring{\mathbf{v}} = (\mathring{v}\cos\alpha,\ \mathring{v}\sin\alpha)$, cutting angle α with the x-axis. The motion

*Dedicated to the memory of Professor Vasil V. Tsanov 1948-2017.

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is described by the vector function $\mathbf{r}(t) = (x\,(t),\, y\,(t))$ whose coordinates satisfy the equations

$$x'' + \omega^2 x = 0, \qquad y'' + \omega^2 y = 0 \tag{3}$$

with initial position $(\mathring{x}, 0)$ and initial velocity $\mathring{\mathbf{v}} = (\mathring{v}\cos\alpha,\, \mathring{v}\sin\alpha)$. Solving for x and y separately we find the parametric equations

$$x = \mathring{x}\cos(\omega t) + \frac{\mathring{v}}{\omega}\cos\alpha\sin(\omega t), \qquad y = \frac{\mathring{v}}{\omega}\sin\alpha\sin(\omega t) \tag{4}$$

and the trajectory is an ellipse. Setting for convenience $p = \mathring{v}/\omega$ and replacing this in the second equation in (4) we write

$$\sin(\omega t) = \frac{y}{p\sin\alpha}\cdot \tag{5}$$

From the first equation in (4) $x = \mathring{x}\cos(\omega t) + y\cot\alpha$ we can solve for $\cos(\omega t)$

$$\cos(\omega t) = \frac{x - y\cot\alpha}{\mathring{x}}\cdot \tag{6}$$

Now from (5) and (6) we find the equation of the elliptical trajectory in Cartesian coordinates

$$\left(\frac{x - y\cot\alpha}{\mathring{x}}\right)^2 + \left(\frac{y}{p\sin\alpha}\right)^2 = 1. \tag{7}$$

When α changes we have a family of ellipses with center $(0\,,0)$ (notice the symmetry with respect to $(0, 0)$). One simple representative of this family is the ellipse corresponding to $\alpha = \dfrac{\pi}{2}$

$$\frac{x^2}{\mathring{x}^2} + \frac{y^2}{p^2} = 1$$

with semi axes $\mathring{x}$ and $p = \mathring{v}/\omega$.

For a dynamic illustration see the applet at `https://ggbm.at/AhqHxxBF`.

Now we shall try to identify the region G filled by all these ellipses when we change α in $(0\,,\pi)$. For this purpose we rewrite equation (7) as a quadratic equation for $\cot\alpha$.

With the help of the identity $\dfrac{1}{\sin^2\alpha} = 1 + \cot^2\alpha$ the equation becomes

$(x - y\cot\alpha)^2 p^2 + \mathring{x}^2 y^2 (1 + \cot^2\alpha) - \mathring{x}^2 p^2 = 0 \quad$ or

$$y^2(p^2 + \mathring{x}^2)\cot^2\alpha - 2xyp^2\cot\alpha + (x^2 - \mathring{x}^2)p^2 + \mathring{x}^2 y^2 = 0. \tag{8}$$

We reason that a point (x, y) is on the boundary of G if it can be reached by only one ellipse in the above set of ellipses. This means that for points (x, y) on the

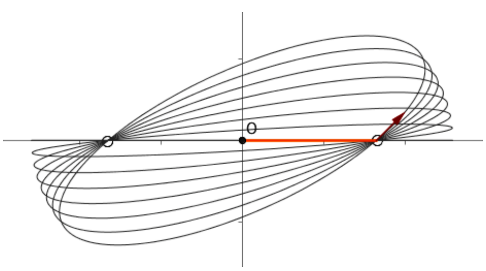

Figure 1. Orbits with the same speed, $\mathring{v} < \mathring{x}$.

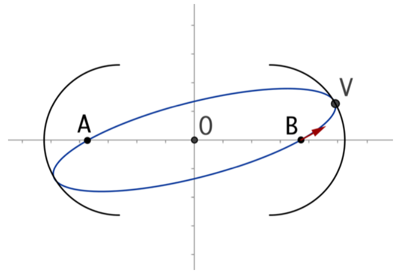

Figure 2. Trajectories of the vertices.

boundary, equation (8) has only one solution for $\cot\alpha$. In this case the discriminant is zero, i.e.,

$$4x^2y^2p^4 - 4y^2(p^2+\mathring{x}^2)[(x^2-\mathring{x}^2)p^2+\mathring{x}^2y^2] = 0.$$

Simplifying this we obtain

$$\frac{x^2}{\mathring{x}^2+p^2} + \frac{y^2}{p^2} = 1 \tag{9}$$

which is an ellipse with center at the origin, foci at $(-\mathring{x}, 0)$and $(\mathring{x}, 0)$, big axis $\sqrt{\mathring{x}^2+p^2}$ and small axis p, i.e., the axes are $a = \sqrt{\mathring{x}^2+\frac{\mathring{v}^2}{\omega^2}} = \sqrt{\mathring{x}^2+\frac{m\mathring{v}^2}{k}}$ (which is the maximal amplitude) and $p = \dfrac{\mathring{v}}{\omega} = \mathring{v}\sqrt{\dfrac{m}{k}}\cdot$ The enveloping ellipse can be seen in Figs. 3, 5 and 6.

The ellipse (9) is called ellipse of safety. All points in the plane beyond this ellipse cannot be hit by the oscillating particle.

If we rotate this ellipse about the x-axis, it will generate the ellipsoid of safety, with the same center and same foci. Beyond this ellipsoid is the "safe" space, where the oscillating point $M$with initial speed $|\mathring{\mathbf{v}}|$ cannot reach.

For ellipses of safety and other similar results see [3].

2. Cassini Ovals

In this section we present an interesting fact: when the angle α varies, the foci of the orbits (4) trace a remarkable curve, a Cassini oval. The Cassinian oval is defined as the locus of all points (u, v) whose distances to two fixed points (foci) $(-\lambda, 0)$ and $(\lambda, 0)$ have a constant product μ^2, i.e.,

$$[u+\lambda)^2+v^2]\,[(u-\lambda)^2+v^2] = \mu^4.$$

We prove the following theorem. Without loss of generality we assume that the frequency $\omega = 1$.

Theorem 1. *The foci of the elliptical trajectories*

$$x = \mathring{x}\cos(t) + \mathring{v}\cos\alpha\sin(t), \qquad y = \mathring{v}\sin\alpha\sin(t), \qquad 0 \leq\ t\ \leq 2\pi$$

with initial position $(\mathring{x}, 0)$ *and initial velocity* $\mathring{\mathbf{v}} = (\mathring{v}\cos\alpha,\, \mathring{v}\sin\alpha)$ *when* α *changes from* $\alpha = -\pi$ *to* $\alpha = \pi$ *trace a Cassini oval with Cartesian equation*

$$[(x+\mathring{x})^2 + y^2]\,[(x-\mathring{x})^2 + y^2] = \mathring{v}^4 \tag{10}$$

or equivalently

$$(x^2+y^2)^2 - 2\mathring{x}^2(x^2-y^2) = \mathring{v}^4 - \mathring{x}^4 \tag{11}$$

and polar equation

$$r^4 - 2\mathring{x}^2\, r^2\cos 2\theta = \mathring{v}^4 - \mathring{x}^4. \tag{12}$$

This Cassini oval sits symmetrically inside the ellipse of safety. In particular, when $\mathring{x} = \mathring{v}$ *the locus of the foci is Bernoulli's lemniscate with Cartesian equation*

$$(x^2+y^2)^2 = 2\mathring{x}^2(x^2-y^2) \tag{13}$$

and polar equation

$$r^2 = 2\mathring{x}^2\,\cos 2\theta, \qquad -\frac{\pi}{4} \leq \theta \leq \frac{\pi}{4}\cdot \tag{14}$$

Moreover, in this case $\alpha = 2\theta$*, where* θ *is the polar angle. Thus equation* (13) *becomes*

$$r^2 = 2\mathring{x}^2\,\cos\alpha, \qquad -\frac{\pi}{2} \leq \alpha \leq \frac{\pi}{2}\cdot \tag{15}$$

Remark 2. *This result seems to be new. The Cassini ovals were discovered experimentally by the second author while studying the ellipses of safety.*

Remark 3. *Although in equation* (15) $\cos\alpha$ *is only non-negative, allowing the polar radius in this equation to be negative we obtain the entire lemniscate.*

Remark 4. *As stated in the theorem, when* $\mathring{v} = \mathring{x}$ *the Cassini oval is a lemniscate. The other two cases* $\mathring{v} < \mathring{x}$ *and* $\mathring{v} > \mathring{x}$ *provide two-pieces and one-piece ovals correspondingly. All three cases of Cassini ovals are presented in* Figs. 3, 5 *and* 6 *below. In these figures* $A = (-\mathring{x}, 0)$ *and* $B = (\mathring{x}, 0)$ *are the foci of the enveloping ellipse and also the foci of the Cassini ovals. The points* F_1 *and* F_2 *are the foci and also the endpoints of the degenerated ellipse-segment* $[-a, a],\ a = \sqrt{\mathring{x}^2 + \mathring{v}^2}$ *occurring for* $\alpha = 0$*. These are the only two common points for the Cassini ovals and the enveloping ellipse.*

Remark 5. *Because of the symmetry with respect to the x-axis, rotation around this axis will generate Cassini surfaces of revolution. For example, the three-dimensional locus of foci in the case of lemniscates will be the surface in* Fig.4.

Notice that equation (10) is a bipolar equation expressing the fact that if $F(x, y)$ is a point on the Cassini oval, the product of its distances to A and B is $\mathring{v}^2$.

Good references for Cassini ovals are Lawrence [1], Mladenov [2], Teixeira [4] and Yates [5].

Proof of the theorem. We first give a proof for the most interesting case $\mathring{x} = \mathring{v}$. For this case it is easy to write explicit parametric equations for the trajectory of the foci. The proof of the general case will be different.

When $\mathring{x} = \mathring{v}$ and $\omega = 1$ equations (4) become

$$x = \mathring{x}(\cos t + \cos\alpha \sin t), \qquad y = \mathring{x} \sin\alpha \sin t. \tag{16}$$

From this we compute

$$r^2 = x^2 + y^2 = \mathring{x}^2 \, (\cos^2 t + \sin^2 t \cos^2\alpha + \sin^2 t \;\; \sin^2\alpha + \cos\alpha \sin 2t).$$

That is

$$r^2 = \mathring{x}^2 \, (1 + \cos\alpha \sin 2t). \tag{17}$$

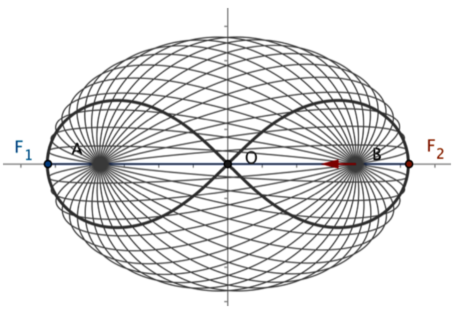

Figure 3. Lemniscate, $\mathring{v} = \mathring{x}$.

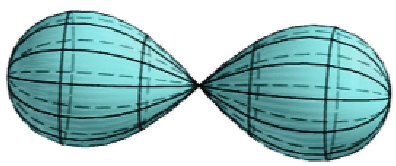

Figure 4. Surface of revolution for the lemniscate.

Assume first $-\frac{\pi}{2} \le \alpha \le \frac{\pi}{2}$, so that $\cos\alpha \ge 0$. The maximal and minimal values of r^2 are the squares of the big axis a and the small axis b of the ellipse (16). They are obtained when $\sin 2t = \pm 1$ ($t = \frac{\pi}{4}, \frac{3\pi}{4}, \ldots$). Thus

$$a^2 = \mathring{x}^2 \, (1 + \cos\alpha), \qquad b^2 = \mathring{x}^2 \, (1 - \cos\alpha). \tag{18}$$

For the focal distance c we have $c^2 = a^2 - b^2$, so that

$$c^2 = 2\mathring{x}^2 \cos\alpha. \tag{19}$$

The vertex of the ellipse in the first quadrant happens when $t = \frac{\pi}{4}$ and the coordinates of this vertex are (from (16))

$$V\left(\frac{\mathring{x}(1+\cos\alpha)}{\sqrt{2}},\ \frac{\mathring{x}\sin\alpha}{\sqrt{2}}\right). \tag{20}$$

Multiplying by c/a these coordinates we find the coordinates of the focus in the first quadrant as functions of α

$$x = \mathring{x}\sqrt{\cos\alpha\,(1+\cos\alpha)}, \qquad y = \frac{\mathring{x}\sin\alpha\sqrt{\cos\alpha}}{\sqrt{1+\cos\alpha}}. \tag{21}$$

From here $x^2 + y^2 = 2\mathring{x}^2\cos\alpha$ confirming (19) and also

$$x^2 - y^2 = 2\mathring{x}^2\cos^2\alpha. \tag{22}$$

Equation (13) follows immediately from here. The restriction $\cos\alpha \geq 0$ is not essential, because equation (13) extends by symmetry to all quadrants.

Let now θ be the polar angle and $x = r\cos\theta,\ y = r\sin\theta$ the standard polar relations. Then

$$x^2 - y^2 = r^2(\cos^2\theta - \sin^2\theta) = r^2\cos 2\theta$$

and comparing this to (22) and (20) for the coordinates of the focus we find that $\cos\alpha = \cos 2\theta$ and thus $\alpha = 2\theta$. The proof is completed.

For a dynamic illustration of this motion see the GeoGebra applet at

`https://ggbm.at/xQbfBxwY`

Remark 6. *Notice that the vertex V in* (20) *traces a semicircle for* $-\frac{\pi}{2} \leq \alpha \leq \frac{\pi}{2}$ *with radius* $\frac{\mathring{x}}{\sqrt{2}}$ *and center* $\left(\frac{\mathring{x}}{\sqrt{2}},\ 0\right)$. *The vertex cannot complete the whole circle when α moves beyond* $\pm\frac{\pi}{2}$, *because at* $\alpha = \pm\frac{\pi}{2}$ *the ellipse* (16) *becomes a circle with $a = b$ and the next moment the two axes a and b change places. The second half of the circle is centered at* $\left(-\frac{\mathring{x}}{\sqrt{2}},\ 0\right)$, *see* Fig.2.

Proof of the theorem in the case $\mathring{v} \neq \mathring{x}$. When $\mathring{v} < \mathring{x}$ the Cassini curve consists of two ovals, as shown in Fig.5. When $\mathring{v} > \mathring{x}$ the Cassini oval consists of one piece (see Fig. 6). We need to prove equation (10). When $\mathring{v} \neq \mathring{x}$ we compute from (4)

$$r^2 = x^2 + y^2 = \mathring{x}^2\cos^2 t + \mathring{v}^2\sin^2 t + 2\mathring{x}\mathring{v}\cos\alpha\sin t\cos t$$

which can be written as

$$r^2 = \frac{1}{2}(\mathring{x}^2 + \mathring{v}^2) + \frac{1}{2}(\mathring{x}^2 - \mathring{v}^2)\cos 2t + \mathring{x}\mathring{v}\cos\alpha \sin 2t. \tag{23}$$

Maximal and minimal values occur when

$$\tan 2t = \frac{2\mathring{x}\mathring{v}\cos\alpha}{\mathring{x}^2 - \mathring{v}^2}.$$

If maximum occurs for some t , then the next extremum, the minimum, will happen for $t + \frac{\pi}{2}$ as $\tan 2t$ is the same. Thus for the two semi axes a and b of the ellipse (4) we have

$$\begin{aligned} a^2 &= \tfrac{1}{2}(\mathring{x}^2 + \mathring{v}^2) + \tfrac{1}{2}(\mathring{x}^2 - \mathring{v}^2)\cos 2t + \mathring{x}\mathring{v}\cos\alpha \sin 2t \\ b^2 &= \tfrac{1}{2}(\mathring{x}^2 + \mathring{v}^2) - \tfrac{1}{2}(\mathring{x}^2 - \mathring{v}^2)\cos 2t - \mathring{x}\mathring{v}\cos\alpha \sin 2t \end{aligned}$$

and from here we obtain the remarkable equation

$$a^2 + b^2 = \mathring{x}^2 + \mathring{v}^2. \tag{24}$$

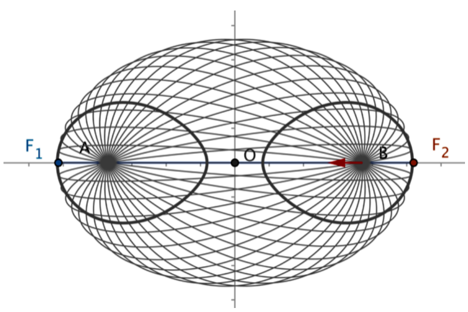

Figure 5. Two Cassini ovals, $\mathring{v} < \mathring{x}$.

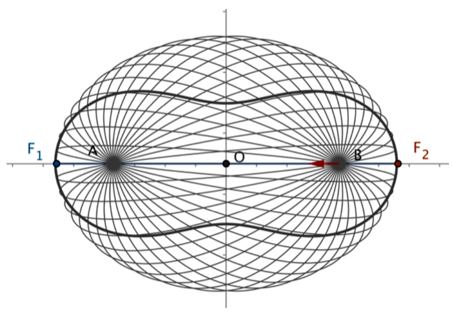

Figure 6. Cassini oval with $\mathring{v} > \mathring{x}$.

We use now the construction in Fig.2 with an arbitrary elliptical orbit.
As before, $A = (-\mathring{x}, 0)$, $B = (\mathring{x}, 0)$. The quadrilateral AF_2BF_1 is a parallelogram – note that the diagonals AB and F_1F_2 cut each other in half. Thus $AF_2 = BF_1$ and $AF_1 = BF_2$. From the parallelogram law we have

$$AB^2 + F_1F_2^2 = 2(F_1B^2 + F_2B^2).$$

Note that $AB = 2\mathring{x}$ and $F_1F_2^2 = 4(a^2 - b^2)$, so that

$$F_1B^2 + F_2B^2 = 2OF_1^2 + 2\mathring{x}^2. \tag{25}$$

At the same time, using the property of the ellipse

$$4a^2 = (F_1A + F_1B)^2 = F_1A^2 + F_1B^2 + 2F_1AF_1B.$$

Therefore, with the help of (25) and using that $F_1A = F_2B$

$$2F_1AF_1B = 4a^2 - F_2B^2 - F_1B^2 = 4a^2 - 2(a^2 - b^2) - 2\mathring{x}^2 = 2(a^2 + b^2) - 2\mathring{x}^2$$

and now using (24)

$$F_1AF_1B = \mathring{v}^2$$

which is equation (10).

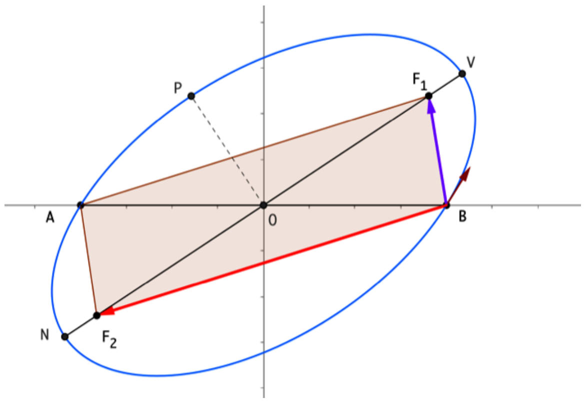

Figure 7. An important parallelogram.

Remark 7. *In* Fig.7*,* $a = OV$ *and* $b = OP$*. Equation* (24) *shows that the segment* PV *has constant length* $\sqrt{\mathring{x}^2 + \mathring{v}^2}$ *for all elliptical orbits. This quantity is exactly the large radius of the enveloping ellipse* (9) *when* $\omega = 1$*. Moreover, the enveloping ellipse and all Cassini ovals have the same foci* A *and* B*.*

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Khristo N. Boyadzhiev
Department of Mathematics and Statistics
Ohio Northern University
Ada, OH 45810, USA
E-mail address: k-boyadzhiev@onu.edu

Irina A. Boyadzhiev
Department of Mathematics
Ohio State University (Lima)
Lima, OH 45804, USA
E-mail address: boyadzhiev.1@osu.edu