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# A NOTE ON THE CLASS OF SURFACES WITH CONSTANT SKEW CURVATURES

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**Abstract.** The goal of this paper is to analyze surfaces with constant skew curvature (CSkC), and show that the class of CSkC surfaces with non-constant principal curvatures does not contain any Bonnet surfaces.

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## 1. Introduction and Background

A CSkC (constant skew curvature) surface represents a smooth, immersed surface in a space form, whose difference of principal curvatures $k_1 - k_2 = a$ represents a positive constant. Note that this is equivalent with $H^2 - K = c^2 = \frac{a^2}{4}$ being constant, where $H$ is the mean curvature, and $K$ is the Gaussian curvature of the surface. This type of surface is a particular kind of *W-surface* (a surface that is characterized by a functional relationship between its principal curvatures, as stated by Chern in [2]). On the other hand, the characterization of a CSkC surface can be made in a more specific way. Surfaces in $\mathbb{R}^3$ whose principal curvatures satisfy a linear relation (i.e., $k_1 = pk_2 + q$, where $p$ and $q$ are real numbers) are called *linear Weingarten surfaces*. This class of surfaces has many relevant physical applications. For example, the Mylar balloon can be regarded as a specific example of a linear Weingarten surface (with $q = 0$) as it was done in [5], where a variational characterization was provided for linear Weingarten surfaces that generalize the Mylar balloon, in terms of beta functions.

Therefore, we may regard a CSkC surface as a linear Weingarten surface with $p = 1$ and $q$ non-zero, by excluding umbilic points.

Separately, we will recall the notion of Bonnet surface. The notations used in this article are the standard ones from most books on surface theory, such as [6], for example. Let us consider an oriented surface $\mathbb{M}^2$ in $\mathbb{R}^3$ of Riemannian metric $g$, characterized by a smooth mean curvature $H$. One of the famous questions that

Bonnet asked was: "When does there exist an isometric embedding $x : \mathbb{M}^2 \to \mathbb{R}^3$ such that the mean curvature function of the immersion is $H$?"

In 1867, Bonnet proved that any surface with constant mean curvature ($H =$ constant) in $\mathbb{R}^3$ (which is not locally umbilical) has the property that there is a non-trivial isometric deformation preserving the mean curvature. The past two decades reignited the researchers' interest in Bonnet surfaces, and several significant papers in on this topic have been published ever since.

The present work was born from the question: can a surface be CSkC and Bonnet at the same time, and, if that is the case, what does it represent?

Since $H^2 - K = \frac{(k_1-k_2)^2}{4} = c^2 \geq 0$ for any surface in three-space, it follows that $H^2 - K$ must be nonnegative. This notation assumes that $c$ is a smooth and positive function, which in general is not constant. Let us assume $c = \sqrt{H^2 - K} > 0$ (which is the case, away from umbilics). We fix an oriented, orthonormal coframing $(w_1, w_2)$, with dual frame field $(e_1, e_2)$. We know that there exists a unique one-form $w_2^1$ so that we can express the Cartan's structure equations as

$$\begin{aligned} \mathrm{d}w_1 + w_1^2 \wedge w_2 &= 0 \\ \mathrm{d}w_2 + w_2^1 \wedge w_1 &= 0 \\ \mathrm{d}w_2^1 + w_3^1 \wedge w_2^3 &= 0 \\ \mathrm{d}w_3^1 + w_2^1 \wedge w_3^2 &= 0 \\ \mathrm{d}w_3^2 + w_1^2 \wedge w_3^1 &= 0. \end{aligned} \tag{1}$$

The parameterization $x$ corresponds to a lifting $f$, so that the following correspondences take place

$$\begin{aligned} f^*\eta_1 &= w_1, \qquad & f^*\eta_3^1 &= w_3^1 = h_{11}w_1 + h_{12}w_2 \\ f^*\eta_2 &= w_2, \qquad & f^*\eta_3^2 &= w_3^2 = h_{12}w_1 + h_{22}w_2 \end{aligned} \tag{2}$$

where $h_{11} + h_{22} = 2H$ and $h_{11}h_{22} - h_{12}^2 = K$. We also know, by uniqueness of the Levi-Civita connection, that $f^*\eta_2^1 = w_2^1$.

Furthermore, since $H^2 - K = c^2 > 0$, these can be solved in terms of an extra parameter in the form

$$h_{11} = H + c\ \cos\varphi, \qquad h_{12} = c\ \sin\varphi, \qquad h_{22} = H - c\ \cos\varphi. \tag{3}$$

**Note 1.** (The case of isothermal coordinates.) If $e^i$ is an orthonormal frame, then we have $\mathrm{d}w_3 = 0$, $w_1 = \sqrt{g_{11}}\mathrm{d}u$, $w_2 = \sqrt{g_{22}}\mathrm{d}v$.

The following definitions and terminology are following the referenced works of Bryant [1], as well as Corales-Kenmotsu [3].

**Definition 1.** **a.** *We introduce the scalar functions $c_i$ and $H_i$ for $i = 1, 2$ as being defined by the equalities*

$$\mathrm{d}c = c_1 w_1 + c_2 w_2 \tag{4}$$

*and, respectively*

$$\mathrm{d}H = H_1 w_1 + H_2 w_2. \tag{5}$$

Remark that these functions are well defined, considering the given basis $w_1, w_2$ of the tangent bundle of the surface.

**b.** We introduce the functions $C$, $S$ and $T$ by the following expressions:

$$C = 2c_1 H_1 - 2c_2 H_2 - cH_{11} + cH_{22} \tag{6}$$

$$S = 2c_2 H_1 + 2c_1 H_2 - 2cH_{12} \tag{7}$$

$$T = 2c^4 - 2H^2 c^2 + c(c_{11} + c_{22}) - c_1^2 - c_2^2 - H_1^2 - H_2^2 \tag{8}$$

where $H_{ij}$ and $c_{ij}$ are defined by the following equalities:

$$\begin{aligned} \mathrm{d}H_1 &= -H_2 w_1^2 + H_{11} w_1 + H_{12} w_2, \qquad & \mathrm{d}c_1 &= -c_2 w_1^2 + c_{11} w_1 + c_{12} w_2 \\ \mathrm{d}H_2 &= H_1 w_1^2 + H_{12} w_1 + H_{22} w_2, \qquad & \mathrm{d}c_2 &= c_1 w_1^2 + c_{12} w_1 + c_{22} w_2. \end{aligned} \tag{9}$$

**Remark 2.** Theorem 1 (in [3]) *states*: *Let $M$ be a piece of an oriented surface in $\mathbb{R}^3$ such that it has no umbilic points. Then, $M$ admits a non-trivial isometric deformation preserving the mean curvature function if and only if one of the following two (equivalent) conditions holds*

$$\nabla\left(\frac{\nabla H}{H^2 - K}\right)(Z, Z) = 0 \qquad \textit{for} \qquad Z = \frac{e_1 - \mathrm{i}e_2}{2} \tag{10}$$

*and, respectively*

$$(H^2 - K)(\Delta \log(\sqrt{H^2 - K}) - 2K) - |\mathrm{grad} H|^2 = 0, \qquad \Delta\phi = 0. \tag{11}$$

*In this source, $\nabla$ represents the covariant derivative and $\Delta$ represents the Laplace-Beltrami operator with respect to the induced Riemannian metric of the surface. Again, for more information, please see* [3, p 75] *and* [1, p 49].

**Remark 3.** *In is straightforward to see that the above-stated* Theorem 1 *from* [3] *can be rephrased as follows*: *Let $M$ be a piece of an oriented surface in $\mathbb{R}^3$ such that it has no umbilic points. Then, $M$ admits a non-trivial isometric deformation preserving the mean curvature function if and only if one of the following conditions holds*: $C = S = 0$, *respectively*, $T = \Delta\phi = 0$.

*Both* [3] *and* [1] *proved that these conditions are equivalent, more precisely*

$$C = S = 0 \Longleftrightarrow T = \Delta\phi = 0.$$

Next, we will use the result stated in Remark 2 (Theorem 1 of [3]) in order to prove the following

**Theorem 4.** *Let $\phi : U \to \mathbb{R}^3$ represent an isometric immersion of the surface $S$ that contains no umbilics. Let $H$ and $K$ represent its mean curvature and Gaussian curvature.*

*Next, assume that the surface $S$ is at the same time Bonnet and CSkC* (*constant skew curvature*).

*Then, $S$ must have curvature $K = 0$ and constant mean curvature $H = c$, that is, $S$ must be a* (*patch of*) *a circular cylinder.*

**Proof:** Let us consider an immersion of $S$ which is CSkC and a Bonnet surface, at the same time.

As stated in the second remark, according to Theorem 1 of [3], the property of the surface $S$ being a Bonnet surface is equivalent to one of the following conditions

$$\nabla\left(\frac{\nabla H}{H^2 - K}\right)(Z, Z) = 0 \qquad \text{for} \qquad Z = \frac{e_1 - \mathrm{i}e_2}{2} \tag{12}$$

and, respectively

$$(H^2 - K)(\Delta\log(\sqrt{H^2 - K}) - 2K) - |\mathrm{grad}H|^2 = \Delta\phi = 0. \tag{13}$$

These represent the equivalent equations (13) and (14) from [3].

Hence, a CSkC surface that is also a Bonnet surface must simultaneously satisfy

$$H^2 - K = c^2 = \text{constant}, \qquad |\mathrm{grad}H|^2 = -2c^2K. \tag{14}$$

On the other hand, a straightforward computation shows that the condition $T = 0$, that must be satisfied by each Bonnet surface, can be rewritten as

$$(H^2 - K)\Delta H - 2H|\mathrm{grad}H|^2 = \frac{2K(H^2 - K)^2}{H} \tag{15}$$

(see [3, p 76] for details).

Combining the previous four conditions in the latter equation, we immediately obtain the following two possibilities, which must hold at all points of $S$

i) $K = 0$, and hence $H = c$.

ii) $H^2 = -K = \frac{c^2}{2} = \mathrm{constant} > 0$.

Hence, $K = 0$ remains the only possibility. This implies that $H = c$ (modulo a possible change of orientation, by virtue of $H^2 - K = c^2$). Therefore, the surface is a (patch of) a circular cylinder. ■

*An Alternative Proof*

**Proof:** We hereby provide yet another proof, based on the works of Bryant [1]. This proof is based on a more sophisticated setting, which nevertheless is worth showing, due to the elegance and significance of the results used.

We shall restrict our analysis to the open set $U \subset M$ where $\mathrm{d}H \neq 0$, i.e., where $H_1^2 + H_2^2 > 0$.

We take the coframing $(w_1, w_2)$ so that the dual frame field $(e_1, e_2)$ has the property that $e_1$ points in the direction of steepest increase for $H$, i.e., in the direction of the gradient of $H$. Note that, since we are away from umbilics, a steepest ascent direction always exists. This means that, for this coframing, we have $H_2 = 0$ and $H_1 > 0$.

In this case, equations $C = S = 0$ simplify as: $H_{12} = (\dfrac{c_2}{c})H_1$ and $H_{11} - H_{22} = (\dfrac{2c_1}{c})H_1$.

Moreover, looking back at the structure equations found so far, this implies that $\mathrm{d}H = H_1 w_1$ and that there is a function $P$ so that

$$H_1^{-1}\mathrm{d}H_1 = (cP + \frac{c_1}{c})w_1 + (\frac{c_2}{c})w_2, \qquad -w_1^2 = (\frac{c_2}{c})w_1 + (cP - \frac{c_1}{c})w_2.$$

Following the arguments presented by Bryant in [1], it is straightforward to verify

$$
\begin{aligned}
&\mathrm{d}w_1 = 0, \qquad \mathrm{d}w_2 = (cP - \frac{c_1}{c})w_1 \wedge w_2 \\
&\mathrm{d}c = c_1 w_1, \qquad \mathrm{d}H = H_1 w_1 \\
&\mathrm{d}c_1 = (2c^3 - 2H^2 c + c_1 cP - \frac{2c_1^2}{c} - \frac{H_1^2}{c})w_1 \\
&\mathrm{d}H_1 = H_1(cP + \frac{c_1}{c})w_1, \qquad \mathrm{d}P = (c^2H^2 + H_1^2 - c^4 - c^4P^2)w_1.
\end{aligned} \tag{16}
$$

From the first equation, note that $w_1 = \mathrm{d}x$ for some function $x$, uniquely defined up to an additive constant.

The previously shown system can be rewritten as

$$
\begin{aligned}
&c' = c_1, \qquad H' = H_1 \\
&c_1' = 2c^3 - 2H^2 c + c_1 cP - 2\frac{c_1^2}{c} - \frac{H_1^2}{c} \\
&H_1' = H_1(cP + \frac{c_1}{c}), \qquad P' = c^2H^2 + H_1^2 - c^4 - c^4P^2.
\end{aligned} \tag{17}
$$

For the given immersion, assume that $H$ is non-constant and $c^2 = H^2 - K$ constant. Therefore, $c_1 = 0$. Next, recall that we have chosen the direction of steepest increase for $H$ (i.e. $H_1 > 0$ and $H_2 = 0$). Then, the previous system becomes

$$
\begin{aligned}
&H' = H_1, \qquad H_1' = H_1 cP \\
&P' = c^2H^2 + H_1^2 - c^4 - c^4P^2, \qquad H_1^2 = 2c^2(c^2 - H^2).
\end{aligned} \tag{18}
$$

From the second equation of the system, it is very important to remark that we have two possibilities:

i) either $H_1$ is identically zero – which means that the mean curvature is constant

or

ii) $H_1$ is an exponential function that will never vanish.

If we differentiate $H_1^2 = 2c^2(c^2 - H^2)$, and if we take into account the second equation of the previous system, we get

$$
2H_1H_1cP = -2c^2(2HH_1). \tag{19}
$$

From this equation, our study branches out in the following two cases:

*C*ase i) $H_1$ is everywhere zero, leading to $H$ = constant, everywhere (as a singular solution).

*C*ase ii) If $H_1$ is nowhere zero, it will represent an exponential solution of the second equation of the system.

Further, we obtain $P = -\frac{2Hc}{H_1}$, which can be rewritten as: $H_1 P + 2Hc = 0$, since in this case, $H_1$ never vanishes.

By differentiating, we obtain

$$P' = c^2H^2 + H_1^2 - c^4 - c^4P^2 = -2c + \frac{2Hc}{H_1^2}H_1cP = -2c + \frac{2Hc}{H_1^2}(-2Hc^2) \quad (20)$$

which can be rewritten as

$$c^2H^2 + 2c^2(c^2 - H^2) - c^4 - c^4(\frac{2Hc}{H_1})^2 = -2c + \frac{2Hc}{H_1^2}(-2Hc^2) \quad (21)$$

or, equivalently

$$c^4 - c^2H^2 - \frac{4c^6H^2}{H_1^2} = -2c - \frac{4H^2c^3}{H_1^2} \quad (22)$$

which in its turn is equivalent to

$$c^4 - c^2H^2 - \frac{4c^6H^2}{2c^2(c^2 - H^2)} = -2c - \frac{4H^2c^3}{2c^2(c^2 - H^2)}. \quad (23)$$

In its most compact form, this reduces to

$$H^2(H^2 - 4c^2) = -2c - c^4. \quad (24)$$

Note that this implies that the mean curvature $H$ must be constant, which Case ii) did not allow.

Therefore, the only possibility remains the first case, that $H$ is constant, while $K = 0$.

This ends the second, alternative proof.

■

**Corollary 5.** *The class of constant skew curvature surfaces with principal curvatures nonconstant cannot contain any Bonnet surfaces, in other words, for this class there is no non-trivial isometric deformation in* $\mathbb{R}^3$ *that preserves the mean curvature* $H$.

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