MAPPING BETWEEN NONLINEAR SCHRÖDINGER EQUATIONS WITH REAL AND COMPLEX POTENTIALS

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Abstract. A mapping between the stationary solutions of nonlinear Schrödinger equations with real and complex potentials is constructed and a set of exact solutions with real energies are obtained for a large class of complex potentials. As specific examples we consider the case of dissipative periodic soliton solutions of the nonlinear Schrödinger equation with complex potential.

1. Introduction

Nonlinear wave phenomena with time evolutions governed by non hermitian Hamiltonians are presently attracting a great interest both from the theoretical and the applicative point of view. The non hermiticity is in general due to the presence of a complex potential in the Hamiltonian accounting for typical dissipative and amplification effects met in classical and quantum contexts [5, 12]. In particular, dissipative solitons [4] of the nonlinear Schrödinger (NLS) equation with periodic complex potentials have been extensively investigated during the past years in connections with the propagation of light in nonlinear optical fibers with periodic modulations of the complex refractive index [13, 18]. Recently similar studies were done for matter wave solitons of Bose-Einstein condensates (BEC) trapped in absorbing optical lattices [1, 7] and in the presence of three body interatomic interactions [3]. In the linear context, the recent discovery [6] that the Schrödinger eigenvalue problem with complex potentials that are invariant under the combined parity and time reversal symmetry (so called $\mathcal{PT}$-potentials), may have fully real spectrum, has raised interest also in view of possible connection with the theory of quantum dissipative systems [10]. Complex potentials with $\mathcal{PT}$-symmetry are presently investigated in nonlinear optics [11] where it has been demonstrated that nonlinear media with *linear* damping and amplifications that are $\mathcal{PT}$-symmetric can support stable stationary localized and periodic states [14]. Also, quite recently, physical systems with $\mathcal{PT}$-symmetry have been successfully implemented in real experiments [9, 15, 17]. Solutions of the NLS equation with a complex potential which belong to the real part of the spectrum (real energies or real chemical

potentials) can exist, however, for generic complex potentials and it is therefore of interest to characterize them in general, independently from the $\mathcal{PT}$-symmetry.

The aim of the present paper is to show how one can systematically construct stationary solutions of the complex nonlinear Schrödinger equation via a mapping between real and complex NLS equations. The problem is formulated in terms of a nonlocal eigenvalue problem which involves only real potentials, whose eigenfunctions and eigenvalues fix amplitudes and energies of the stationary solutions of the complex NLS equation, respectively. The complex potentials and the phases of the solutions are also determined self-consistently through the mapping. To illustrate our approach we discuss the case of the NLS equation with different complex potentials for which we construct periodic dissipative solitons in the form of elliptic functions.

The paper is organized as follows. In Section 2 we introduce model equations and illustrate the mapping used to determine the solutions. In Section 3 we show how to construct exact solutions of the NLS with periodic complex potentials while in the last section the main results of the paper will be briefly summarized.

2. Model Equations and Mapping

The model equation we consider is the NLS equation with real and complex potentials both of linear and nonlinear types, e.g.

$$\mathrm{i}\psi_t = -\frac{1}{2}\psi_{xx} + (V_l(x) + \mathrm{i}W_l(x))\psi + (\sigma + V_{nl}(x) + \mathrm{i}W_{nl}(x))|\psi|^2\psi. \quad (1)$$

The case of the linear Schrodinger equation (e.g. $\sigma = V_{nl} = W_{nl} = 0$) can be used as an example of quantum dissipative system. In the nonlinear case the above equation can appear in connection with several interesting phenomena including light propagation in photonic crystals and Bose-Einstein condensates. Due to the possibility of different physical applications we shall keep equation (1) in normalized form, looking for stationary solutions of the type

$$\psi(x,t) = A(x)\mathrm{e}^{\mathrm{i}\theta(x)}\mathrm{e}^{-\mathrm{i}\omega t} \quad (2)$$

with the amplitude $A(x)$ and phase $\theta(x)$ as real functions. Substituting this expression into equation(1), we obtain the system of equations

$$\omega A + \frac{1}{2}A_{xx} - \sigma A^3 - \frac{A}{2}(\theta_x)^2 - V_l A - V_{nl}A^3 = 0 \quad (3)$$

$$\frac{1}{2}A\theta_{xx} + A_x\theta_x - W_l A - W_{nl}A^3 = 0. \quad (4)$$

These equations can be easily separated. In this respect notice that by multiplying equation (4) by A and integrating it twice one obtains

$$\frac{1}{2}\theta(x) = B_2 + \int_{-\infty}^{x} \frac{B_1 + F(y)}{A^2(y)}\mathrm{d}y \tag{5}$$

with

$$F(y) = \int_{-\infty}^{y} \left[W_l(z) + W_{nl}(z)A^2(z)\right] A^2(z)\mathrm{d}z \tag{6}$$

and B_1, B_2 integration constants. By substituting equation (5) into equation (3) we obtain the following nonlinear eigenvalue problem for the real amplitude A

$$\left\{ -\frac{1}{2}\frac{\partial}{\partial x^2} + V_l + (\sigma + V_{nl})A^2 + 2\left(\frac{F(x)}{A^2}\right)^2 \right\} A = \omega A \tag{7}$$

where the integration constants B_1, B_2, have been fixed to zero for simplicity. Note that for stationary solutions equation (7) is completely equivalent to equation (1) in the sense that any solution of (7) gives a stationary solution of (1) with the phase fixed by (5). Also note that the dependence on the complex potentials in the eigenvalue problems comes through the function F and for an arbitrary $F(x)$ (e.g. arbitrary complex potentials) the problem can become singular. It is possible, however, to construct potentials W_l and W_{nl} (e.g. functions F) so that the solutions of (7) are regular. This establishes a mapping between stationary solutions of the NLS equation with real potentials and stationary solution of equation (1) with the phase given by (5). In this respect, one can take F in general to be an analytical function of A^2 and derivatives e.g. $F(x) \equiv F(A^2, (A^2)_x, ...)$. In the simplest case F can be taken of the form

$$F(x) = \frac{1}{2}C_n A^{n+2}, \qquad n = 0, 1, 2... \tag{8}$$

with C_n constants to be determined. Equation (7) then reduces to the following NLS real eigenvalue problem

$$\left\{ -\frac{1}{2}\frac{\partial}{\partial x^2} + V_l + (\sigma + V_{nl})A^2 + \frac{C_n^2}{2}A^{2n} \right\} A = \omega A \tag{9}$$

which can be solved analytically for particular forms of the potentials V_l, V_{nl}, or numerically with high accuracy (using for example the self-consistent method discussed in [16]) for generic real potentials. In the following we therefore assume that the real amplitudes A and frequencies ω for given V_l and V_{nl} are exactly obtained from (9), either analytically or numerically.

On the other hand from equation (8) one can characterize the complex potentials which support such solutions. Using equation (6) we have indeed that equation (8) is satisfied if the amplitude A is related to W_l and W_{nl} by the relation

$$W_l + W_{nl}A^2 = C_n \left(\frac{n+2}{2n}\right)\frac{\mathrm{d}A^n}{\mathrm{d}x} \tag{10}$$

and from equations (5), (8), one gets that the phase is given by

$$\theta(x) = C_n \int_{-\infty}^{x} A^n \mathrm{d}y. \tag{11}$$

Note that in this case equation (10) allows to relate the constant C_n to the amplitude of the solution, A_0, and the amplitudes W_{0l}, W_{0nl}, of the linear and nonlinear complex potentials, respectively. In particular, for the case $W_{nl} = 0$ we have that

$$C_n = \frac{2}{n+2}\frac{W_{0l}}{A_0^n}, \qquad W_{0nl} = 0 \tag{12}$$

while for $W_l = 0$ one obtains

$$C_n = \frac{2}{n+2}\frac{W_{0nl}}{A_0^{n-2}}, \qquad W_{0l} = 0. \tag{13}$$

It is worth to note that while the case $n = 1$ leads to a pure cubic NLS eigenvalue problem, the case $n > 1$ introduces higher order nonlinearities in equation (9) which can however be eliminated by redefying the linear real potential as

$$V_l = \tilde{V}_l - \frac{C_n^2}{2}A^{2n} \tag{14}$$

or the nonlinear real potential as

$$V_{nl} = \tilde{V}_{nl} - \frac{C_n^2}{2}A^{2n-2} \tag{15}$$

(or a combination of both). Also notice that equations (2), (10) - (13) allow to map solutions of the real eigenvalue problem (9) into solutions of the NLS equation (1) with the corresponding complex potentials determined as in (10). It is clear that this approach can be extended to functions of the type

$$F(x) = \frac{1}{2}\sum_{n=0}^{k} C_n A^{n+2}, \qquad k = 0, 1, 2 \ldots \tag{16}$$

In this case coefficient C_n are self-consistently determined from the real eigenvalue problem

$$\left\{ -\frac{1}{2}\frac{\partial}{\partial x^2} + V_l + (\sigma + V_{nl})A^2 + \frac{1}{2}\left(\sum_n C_n A^n\right)^2 \right\} A = \omega A. \quad (17)$$

and complex potentials and phase are given by

$$W_l + W_{nl}A^2 = \frac{1}{A^2}\frac{\mathrm{d}F}{\mathrm{d}x} \quad (18)$$

$$\theta(x) = \sum_n C_n \int_{-\infty}^{x} A^n \mathrm{d}y. \quad (19)$$

Note that the sum in equation (16) can include infinite terms and to have a map between real and complex NLS equations it is necessary to subtract higher order nonlinearities from the real linear and nonlinear potentials as done in equations (14)-(15). Finally we remark that if the functions A_x/A, A_{xx}/A, ... are bounded, the expression (16) can be further generalized as

$$F(x) = \sum_{n,m}^{k} C_{n,m}\frac{\mathrm{d}^m A^{n+2}}{\mathrm{d}x^m} \quad (20)$$

with $C_{n,m}$ suitable constants and with the complex potentials determined as (18). In all these cases a map between solutions of the real eigenvalue problem (17) and solutions of the NLS equation (1) is constructed.

The mapping guarantees that the constructed solutions always have real energies and may be therefore of physical interest. We finally remark that a similar approach based on a priori fixing of the solution and a posteriori determination of the complex potential, has been considered also in [2, 8], although not in terms of a mapping between stationary solutions of NLS equations. In the following we illustrate how the mapping works on some specific example.

3. Nonlinear Schrödinger Equation with Complex Potentials

3.1. Case $n = 1$

Let us consider first the simplest ansatz (8) with $n = 1$. We fix the nonlinearity to be attractive ($\sigma < 0$) and restrict to linear complex potentials (i.e., $W_{nl} = V_{nl} = 0$)

and with linear potential of the form $V_l = V_{0l} \operatorname{cn}^2(x, k)$. In this case the real eigenvalue problem (9)

$$\left[-\frac{1}{2} \frac{\partial}{\partial x^2} + V_{0l} \operatorname{cn}(x, k)^2 + \left(\sigma + \frac{C_1^2}{2} \right) A^2 \right] A = \omega A \qquad (21)$$

admits the following exact solutions in terms of elliptic functions

a) $A(x) = A_0 \operatorname{cn}(x, k)$

$$A_0 = \pm \sqrt{\frac{2(k^2 + V_{0l})}{2|\sigma| - C_1^2}}, \qquad \omega = \frac{1 - 2k^2}{2} \qquad (22)$$

b) $A(x) = A_0 \operatorname{sn}(x, k)$

$$A_0 = \pm \sqrt{\frac{2(k^2 + V_{0l})}{C_1^2 - 2|\sigma|}}, \qquad \omega = \frac{1 + k^2}{2} + V_{0l} \qquad (23)$$

c) $A(x) = A_0 \operatorname{dn}(x, k)$

$$A_0 = \pm \frac{1}{k} \sqrt{\frac{2(k^2 + V_{0l})}{2|\sigma| - C_1^2}}, \qquad \omega = \frac{k^2}{2} - 1 + V_{0l} \left(1 - \frac{1}{k^2} \right). \qquad (24)$$

Similar solutions can be constructed for the case of a repulsive nonlinearity $\sigma > 0$ with linear potentials of the form $V_l = V_{0l} \operatorname{sn}^2(x, k)$. In this case we have

d) $A(x) = A_0 \operatorname{cn}(x, k)$

$$A_0 = \pm \sqrt{\frac{2(V_{0l} - k^2)}{C_1^2 + 2\sigma}}, \qquad \omega = \frac{1 - 2k^2}{2} + V_{0l} \qquad (25)$$

e) $A(x) = A_0 \operatorname{sn}(x, k)$

$$A_0 = \pm \sqrt{\frac{2(k^2 - V_{0l})}{C_1^2 + 2\sigma}}, \qquad \omega = \frac{1 + k^2}{2} \qquad (26)$$

f) $A(x) = A_0 \operatorname{dn}(x, k)$

$$A_0 = \pm \frac{1}{k} \sqrt{\frac{V_{0l} - k^2}{C_1^2 + 2\sigma}}, \qquad \omega = \frac{k^2}{2} - 1 + \frac{V_{0l}}{k^2}. \qquad (27)$$

Using the above mapping we can readily construct the stationary solutions of the corresponding complex NLS with linear complex potentials given by

$$W_l = \frac{3}{2} C_1 A_x, \qquad C_1 = \frac{2}{3} \frac{W_{0l}}{A_0} \qquad (28)$$

and with the phase given by $\theta(x) = C_1 \int_{-\infty}^{x} A(y)\mathrm{d}y$. Thus, for example, from the solution a) we get

$$A = A_0 \mathrm{cn}(x, k), \qquad\qquad V_l(x) = V_{0l}\, \mathrm{cn}^2(x, k)$$

$$A_0 = \frac{\sqrt{9\,(k^2 + V_{0l}) + 2W_{0l}^{\ 2}}}{3\sqrt{|\sigma|}}, \qquad\qquad \omega = \frac{1 - 2k^2}{2} \qquad\qquad (29)$$

$$W_l = -W_{0l}\, \mathrm{sn}(x)\mathrm{dn}(x), \qquad\qquad \theta(x) = \frac{2W_{0l}}{3k}\, \mathrm{arccos}(\mathrm{dn}(x)).$$

In similar manner one proceeds with the other solutions above. It is also clear that exact solutions of this type can be constructed also for other types of linear elliptic potentials (we omit them for brevity).

3.2. Case $n = 2$

As a further application of the ansatz (8) we consider the case $n = 2$ for which the mappings involves higher order nonlinearities. We assume as before that $V_{nl} = W_{nl} = 0$. In order to balance the quintic nonlinearity in equation (9), the potential V_l must be taken as in equation (14). We take $\tilde{V}_l = V_{0l}\, \mathrm{cn}^{\,2}(x, k)$ and consider a solution of the form $A(x) = A_0\, \mathrm{cn}(x, k)$. One can then check that this is a solution of (9) with

$$V_l(x) = V_{0l}\mathrm{cn}^2(x, k) - \frac{C_2^2}{2} A_0^4\, \mathrm{cn}^4(x, k) \qquad\qquad (30)$$

if $A_0^2 = V_{0l} + k^2$ and $\omega = \frac{1-2k^2}{2}$. From the mapping we then have that

$$C_2 = \frac{W_{0l}}{2A_0^2} = \frac{W_{0l}}{2(V_0 + k^2)}$$

$$W_l(x) = 2C_2 A A_x = -W_{0l}\, \mathrm{cn}(x)\mathrm{sn}(x)\mathrm{dn}(x) \qquad\qquad (31)$$

and the phase is

$$\theta(x) = x - \frac{x}{k^2} + \frac{1 - k^2 + k^2\mathrm{cn}^2(x, k)}{k^2\mathrm{dn}^2(x, k)} E(\mathrm{am}(x, k), k). \qquad\qquad (32)$$

As a further example of $n = 2$ we consider the case of pure nonlinear optical lattices, i.e., $V_l = W_l = 0$. Fixing $\tilde{V}_{nl} = 0$ and looking for solutions of the type $A(x) = A_0\mathrm{cn}(x, k)$, we have from equation (15) that that

$$V_{nl}(x) = -\frac{C_2^2}{2} A_0^2\, \mathrm{cn}^2(x, k). \qquad\qquad (33)$$

with C_2 fixed according to equation (13) as $C_2 = W_{0nl}/2$. One can easily check that this is indeed a solution of the eigenvalue problem (9) if

$$\omega = \frac{1}{2} - k^2, \qquad A_0 = \frac{k}{\sqrt{|\sigma|}} \tag{34}$$

(we consider $\sigma < 0$). From the mapping we have that this is also a solution of the NLS with the complex part of the nonlinear potential fixed according to equation (10) as

$$W_{nl}(x) = 2C_2\frac{A_x}{A} = -W_{0nl}\frac{\mathrm{sn}(x,k)\mathrm{dn}(x,k)}{\mathrm{cn}(x,k)}. \tag{35}$$

For the cases $n > 2$ one can proceed in similar manner.

3.3. General Case

Let us now consider an example with the more general ansatz (16). To this regard we take $F(x) = \frac{1}{2}(C_0 + C_2A^2)A^2$ and look for solutions of the form $A(x) = A_0\mathrm{dn}(x,k)$. Let us fix the linear potentials as $V_l = W_l = 0$ and the real nonlinear potential as $V_{nl} = V_{0nl} - \alpha_2^2A^2$ with V_{0nl} a constant and with $\alpha_n = \frac{C_n}{\sqrt{2}}, n = 0, 2$ (notice that we fixed all coefficients for $n \neq 0, 2$, equal to zero). By substituting these expressions into the real eigenvalue problem we find that $A(x)$ is indeed a solution if

$$\omega = \alpha_0^2 + \frac{k^2}{2} - 1, \qquad \alpha_0 = -\frac{1 + A_0^2(\sigma + V_{0nl})}{2A_0^2\alpha_2}. \tag{36}$$

Thus, for example, if we fix $V_{0nl} = 2/k^2$, $\alpha_2 = -1/k$ and consider $\sigma = 1$ (repulsive interactions), we have

$$V_{nl} = \frac{2 - \mathrm{dn}^2(x,k)}{k^2} = \frac{1}{k^2} + \mathrm{sn}^2(x,k)). \tag{37}$$

From equation (36) we have

$$\omega = \sigma - 1 + \frac{1}{A_0^2} + \frac{k^2\left[(1 + 2\sigma A_0^2) + (\sigma^2 + 2)A_0^4\right]}{4A_0^4} + \frac{1}{k^2}$$

$$\alpha_0 = \frac{1}{k} + \frac{k(1 + \sigma A_0^2)}{2A_0^2} \tag{38}$$

and from (16) we get the function F as

$$F(x) = \frac{1}{\sqrt{2}}(\alpha_0 + \alpha_2A^2)A^2$$

$$= \frac{A_0^2 \mathrm{dn}^2(x,k)}{2\sqrt{2}k} \left[2 + \frac{k^2}{A_0^2}(1 + \sigma A_0^2) - 2A_0^2 \mathrm{dn}^2(x,k) \right].$$

Substituting into equations (18) we finally get the complex potential as

$$W_{nl} = \sqrt{2}k \frac{\mathrm{sn}(x,k)\mathrm{cn}(x,k)}{\mathrm{dn}^3(x,k)} \times \left[2 - \frac{1}{A_0^2} - \frac{k^2}{2A_0^4}(1 + \sigma A_0^2) - 2k^2 \mathrm{sn}^2(x,k) \right]. \quad (39)$$

The phase of the solution can be readily obtained from equation (19). Notice that in the case $\sigma = 1, A_0 = 1$, this solution coincides with the one derived in [2] with a slightly different approach. We remark that the above solutions of the complex NLS equations not only have real energies but are also stable (not shown for brevity) under time evolution.

4. Conclusions

In conclusion we have demonstrated the possibility to construct stationary solutions of the linear and nonlinear Schrodinger equation with complex potentials via a mapping with stationary solutions of the NLS equation with suitable real potentials. In particular we showed that by means of this mapping it is possible to construct sets of exact solutions with real energies for different types of complex potentials. The presented approach can be applied to other types equations, including the linear Schrödinger equation describing quantum dissipative oscillators, and the NLS equation with arbitrary higher order nonlinearities, as it will be discussed elsewhere.

Acknowledgements

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