JGSP **14** (2009) 1–11

JOURNAL OF
Geometry and Symmetry in Physics

ABOUT THE DENSITIES FOR STRAIGHT LINES IN SEMI-RIEMANNIAN SPACES

GRACIELA SILVIA BIRMAN

Communicated by Abraham A. Ungar

Abstract. We find some new expressions of density for straight lines and planes in euclidean spaces of dimension two and three. Also, we show the density for straight lines in surfaces which is expressed in terms of the coefficient of the first fundamental form. Finally, the density for tangent lines to a differentiable plane curve and for tangent plane of a differentiable surface are presented.

1. Introduction

Since the Buffon's needle problem in the XVIII century until now, through the Crofton's formulas, many papers were written about different expressions of the density for straight lines and its application to integral formulas related to a convex set and further to geometric probability.

The applications to tomography, [1] and stereology show us that we are far from considering that the interest in these differential forms obtained under different hypothesis is over.

Even in [2] we can find the technical way to obtain them and the corresponding supported theory but, for instance, the density for one and two dimensional linear spaces in $\mathbb{R}^3$ are not exhibit leaving many of integral formulas related to convex bodies as an open problems.

In this paper we want to show a most general expression of density for straight lines in semi-Riemannian spaces of dimension two and for lines and planes in $\mathbb{R}^3$. In particular, in the last sections we show density for tangent line to differential curves involving the curvature of the curve and for tangent planes to differential surfaces in terms of the Hessian of the surface.

2. Density for Planes in $\mathbb{R}^3$

It is well known that a straight line L in the Euclidean plane is determined by its distance to the origin and the angle that the direction perpendicular to L makes with a fixed direction as the x-axis, [2].

Analogously, a way to determine a plane π in $\mathbb{R}^3$ is by its distance to the origin and the angles that the direction perpendicular to π makes with the z-axis.

Calling π ($\circ$ L_2) to that plane, ρ its distance to the origin, $\overrightarrow{n}$ the normal vector to π and α the angle already described we get that the density for planes can be expressed by the following differential form

$$\mathrm{d}\pi = \mathrm{d}L_2 = \mathrm{d}L_2^3 = \mathrm{d}\rho\Lambda\mathrm{d}\overrightarrow{n}. \tag{1}$$

It is, also, well known that in a n-dimensional space, the density for r-spaces is a $(n-r)(r+1)-$differential form, thus, if $n=3$ and $r=2$ as in (1), we have that $(n-r)(r+1)=3$.

Now, we have to express ρ, $\overrightarrow{n}$ and α in terms of the others parameters which determine the plane in order to obtain an expression different from (1) but equivalent to.

If the plane is given by the equation

$$Ax + By + Cz = D \tag{2}$$

where $\overrightarrow{n} = (A, B, \sqrt{1-A^2-B^2})$ is the unitary normal vector to the plane.

If P_o is any point of the space, its distance to the plane given by (2) is

$$\mathrm{dist}(P_o, \mathrm{plane}) = \frac{(P_o - P)\times\overrightarrow{n}}{\shortparallel\overrightarrow{n}\shortparallel}$$

where $\times$ means scalar product.

If that point P_o is the origin, it results

$$\rho = \mathrm{dist}(O, \mathrm{plane}) = \mid D \mid$$

We assume that $D > 0$ and as $A^2 + B^2 + C^2 = 1$, computing $\mathrm{d}\rho$

$$\mathrm{d}\rho = \mathrm{d}D$$
$$\mathrm{d}C = \frac{-A\mathrm{d}A - B\mathrm{d}B}{\sqrt{1-A^2-B^2}}$$

i.e.,

$$\mathrm{d}\rho = \mathrm{d}D. \tag{3}$$

Finally, (1) becomes

$$\mathrm{d}L_2 = \mathrm{d}\rho\Lambda\mathrm{d}\overrightarrow{n} = \mathrm{d}D\Lambda\mathrm{d}\overrightarrow{n}. \tag{4}$$

Being α=angle $(\overrightarrow{n}, z-\text{axis})$ we identify $\mathrm{d}\overrightarrow{n} \equiv \mathrm{d}\alpha$ given by $\cos\alpha = \sqrt{1-A^2-B^2}$ and $\sin\alpha\mathrm{d}\alpha = \dfrac{A\mathrm{d}A + B\mathrm{d}B}{\sqrt{1-A^2-B^2}}$, then

$$\mathrm{d}\alpha = \frac{A\mathrm{d}a + B\mathrm{d}B}{\sqrt{1-A^2-B^2}(A^2+B^2)^{1/2}}.$$

Substituting in (4), we obtain

$$\mathrm{d}L_2 = \mathrm{d}L_2^3 = \frac{\mathrm{d}D\Lambda(A\mathrm{d}A + B\mathrm{d}B)}{\sqrt{1-A^2-B^2}\sqrt{A^2+B^2}}. \tag{5}$$

3. Density for Lines in $\mathbb{R}^3$

As mentioned above, the density for straight lines in the Euclidean tridimensional space is a differential form of order four.

If we now look for an expression analogous to those (1) but in a tridimensional space, we have to consider the distance of such a straight line to the origin and its normal plane.

Calling ρ to the distance we can write

$$\mathrm{d}L_1^3 = \mathrm{d}\rho\Lambda\mathrm{d}L_2^3. \tag{6}$$

We leave out of consideration the case for straight lines through the origin ([2], page 202).

The distance from a line G to the origin is given by

$$\rho = \mathrm{dist}(G, O) = \frac{|\overrightarrow{OP}\Lambda\overrightarrow{u}|}{|\overrightarrow{u}|}$$

where $P \in G$, $O \notin G$ and $\overrightarrow{u}$ is a vector parallel to G.

If G is given by the equation

$$\frac{x-x_o}{A} = \frac{y-y_o}{B} = \frac{z-z_o}{C}$$

we have $\overrightarrow{u} = (A, B, C), P = (x_o, y_o, z_o)$ and

$$\rho = \mathrm{dist}(G, O) = \frac{\mid (x_o, y_o, z_o)\Lambda(A, B, C) \mid}{\mid \sqrt{A^2 + B^2 + C^2} \mid}.$$

By straightforward computation we have

$$\rho^2 = \frac{(y_o C - z_o B)^2 + (z_o A - x_o C)^2 + (x_o B - y_o A)^2}{A^2 + B^2 + C^2}.$$

We now compute $\mathrm{d}\rho$.

$$\begin{aligned}\rho \mathrm{d}\rho =& \frac{[A^2 + B^2 + C^2]\{(y_o C - z_o B)(C\mathrm{d}y_o + y_o\mathrm{d}C - B\mathrm{d}z_o - z_o\mathrm{d}B)}{(A^2 + B^2 + C^2)^2} \\ &+ \frac{(z_o A - x_o C)(A\mathrm{d}z_o + z_o\mathrm{d}A - C\mathrm{d}x_o - x_o\mathrm{d}C)}{(A^2 + B^2 + C^2)^2} \\ &+ \frac{(x_o B - y_o A)(B\mathrm{d}x_o + x_o\mathrm{d}B - A\mathrm{d}y_o - y_o\mathrm{d}A)}{(A^2 + B^2 + C^2)^2}\} \\ &- \frac{\{(y_o C - z_o B)^2 + (z_o A - x_o C)^2 + (x_o B - y_o A)^2}{(A^2 + B^2 + C^2)^2} \\ &\times [A\mathrm{d}A + B\mathrm{d}B + C\mathrm{d}C].\end{aligned}$$

Also

$$\begin{aligned}\rho \mathrm{d}\rho =& \frac{(y_o C - z_o B)(C\mathrm{d}y_o + y_o\mathrm{d}C - B\mathrm{d}z_o - z_o\mathrm{d}B)}{(A^2 + B^2 + C^2)} \\ &+ \frac{(z_o A - x_o C)(A\mathrm{d}z_o + z_o\mathrm{d}A - C\mathrm{d}x_o - x_o\mathrm{d}C)}{(A^2 + B^2 + C^2)} \\ &+ \frac{(x_o B - y_o A)(B\mathrm{d}x_o + x_o\mathrm{d}B - A\mathrm{d}y_o - y_o\mathrm{d}A)}{(A^2 + B^2 + C^2)} \\ &- \frac{\{(y_o C - z_o B)^2 + (z_o A - x_o C)^2 + (x_o B - y_o A)^2}{(A^2 + B^2 + C^2)^2} \\ &\times [A\mathrm{d}A + B\mathrm{d}B + C\mathrm{d}C]\end{aligned}$$

$$\begin{aligned}\mathrm{d}\rho =& \sqrt{\frac{A^2 + B^2 + C^2}{(y_o C - z_o B)^2 + (z_o A - x_o C)^2 + (x_o B - y_o A)^2}}\Big\{\frac{(C^2 y_o - BCz_o)\mathrm{d}y_o}{(A^2 + B^2 + C^2)} \\ &+ \frac{(y_o^2 C - y_o z_o B)\mathrm{d}C - (BCy_o - Bz_o^2)\mathrm{d}z_o - (z_o y_o C - z_o^2 B)\mathrm{d}B}{(A^2 + B^2 + C^2)} \\ &+ \frac{(A^2 z_o - x_o AC)\mathrm{d}z_o + (z_o^2 A - x_o z_o C)\mathrm{d}A - (CAz_o - x_o C^2)\mathrm{d}x_o}{(A^2 + B^2 + C^2)}\end{aligned}$$

$$+ \frac{-(z_o x_o A - x_o^2 C)\mathrm{d}C + (B^2 x_o - ABy_o)\mathrm{d}x_o - (Bx_o^2 - x_o y_o A)\mathrm{d}B}{(A^2+B^2+C^2)}$$
$$+ \frac{-(ABx_o - A^2 y_o)\mathrm{d}y_o - (x_o y_o B - y_o^2 A)\mathrm{d}A}{(A^2+B^2+C^2)}$$
$$- \frac{A[(y_oC - z_oB)^2 + (z_oA - x_oC)^2 + (x_oB - y_oA)^2]\mathrm{d}A}{(A^2+B^2+C^2)^2}$$
$$- \frac{B[(y_oC - z_oB)^2 + (z_oA - x_oC)^2 + (x_oB - y_oA)^2]\mathrm{d}B}{(A^2+B^2+C^2)^2}$$
$$- \frac{C[(y_oC - z_oB)^2 + (z_oA - x_oC)^2 + (x_oB - y_oA)^2]\mathrm{d}C}{(A^2+B^2+C^2)^2}\Big\}.$$

Calling

$$\lambda = (A^2 + B^2 + C^2), \qquad \mu = (y_oC - z_oB)^2 + (z_oA - x_oC)^2 + (x_oB - y_oA)^2$$

we can rewrite

$$\begin{aligned}
\mathrm{d}\rho = & \frac{1}{\sqrt{\lambda\mu}}\{[(C^2 - B^2)x_o - ABy_o - CAz_o]\mathrm{d}x_o \\
& + [(C^2 + A^2)y_o - BCz_o - ABx_o)\mathrm{d}y_o \\
& + [(A^2 + B^2)z_o - BCy_o - ACx_o]\mathrm{d}z_o\} \\
& + \left[(z_o^2 A - x_o z_o C) - (x_o y_o B - y_o^2 A) - \frac{A\mu}{\lambda^2}\right]\mathrm{d}A \\
& - \left[(z_o y_o C - z_o^2 B) + (x_o^2 - x_o y_o A) + \frac{B\mu}{\lambda^2}\right]\mathrm{d}B \\
& + \left[(y_o^2 C - y_o z_o B) - (z_o x_o A - x_o^2 C) - \frac{C\mu}{\lambda^2}\right]\mathrm{d}C
\end{aligned} \tag{7}$$

The normal plane to $\overrightarrow{u}$ has the equation (2) then using the formula (5) and substituting (7) in (6) we have

$$\mathrm{d}L_1^3 = \mathrm{d}\rho\Lambda\frac{\mathrm{d}D\Lambda(A\mathrm{d}A + B\mathrm{d}B)}{\sqrt{1 - A^2 - B^2}\sqrt{(A^2 + B^2)}}$$

and

$$\begin{aligned}
\mathrm{d}L_1^3 = & \Big\{\frac{1}{\sqrt{\lambda\mu}}\{[(C^2 - B^2)x_o - ABy_o - CAz_o]\mathrm{d}x_o \\
& + [(C^2 + A^2)y_o - BCz_o - ABx_o)\mathrm{d}y_o \\
& + [(A^2 + B^2)z_o - BCy_o - ACx_o]\mathrm{d}z_o\} \\
& + [(z_o^2 A - x_o z_o C) - (x_o y_o B - y_o^2 A) - \frac{A\mu}{\lambda^2}]\mathrm{d}A
\end{aligned}$$

$$- [(z_o y_o C - z_o^2 B) + (x_o^2 - x_o y_o A) + \frac{B\mu}{\lambda^2}]\mathrm{d}B$$
$$+ [(y_o^2 C - y_o z_o B) - (z_o x_o A - x_o^2 C) - \frac{C\mu]}{\lambda^2}]\Big\}\mathrm{d}C\Lambda\mathrm{d}L_2^3.$$

4. Density for Lines in Semi-Riemannian Spaces of Dimension Two

We consider the bidimensional space given by $(\mathbb{R}^2, \mathrm{d}s^2 = E\mathrm{d}u^2 + 2F\mathrm{d}u\mathrm{d}v + G\mathrm{d}v^2)$ and note with $<, >$ and $\shortparallel\,\shortparallel$, respectively, to the corresponding inner product and norm. In this space, a line $ax + by + c = 0$, is given by

$$L_1 \equiv \langle (a, b), (x, y) \rangle = -c.$$

It is known that the expression for the density for straight lines, $\mathrm{d}L_1$or $\mathrm{d}L_1^2$, in $\mathbb{R}^2$ is, [2]

$$\mathrm{d}L_1 = \mathrm{d}\rho\Lambda\mathrm{d}\theta$$

where ρ represents the distance from the line to the origin and θ is the angle between the direction perpendicular to line and the x-axis. Then

$$L_1 \equiv aEx + bGy + 2abFxy = -c.$$

In order to determine θ we know that

$$\cos\theta = \frac{\langle (a, b), (1, 0) \rangle}{\shortparallel (a, b) \shortparallel} = \frac{aE}{\sqrt{a^2E + b^2G + 2abF}}.$$

Noting that the derivative of $\arccos u$ is $\left(-\frac{u'}{\sqrt{1-u^2}}\right)$ we obtain

$$\mathrm{d}u = \frac{Eb(bG + aF)\mathrm{d}a - aE(bG + aF)\mathrm{d}b}{(a^2E + b^2G + 2abF)^{\frac{3}{2}}}$$

and then

$$\mathrm{d}\theta = -\frac{\frac{Eb(bG+aF)\mathrm{d}a - aE(bG+aF)\mathrm{d}b}{(a^2E+b^2G+2abF)^{\frac{3}{2}}}}{\frac{\sqrt{a^2E(1-E)+b^2G+2abF}}{\sqrt{a^2E+b^2G+2abF}}}$$

i.e.,

$$\mathrm{d}\theta = -\frac{E\sqrt{a^2E(1-E) + b^2G + 2abF}[bG + aF][b\mathrm{d}a - a\mathrm{d}b]}{a^2E + b^2G + 2abF}. \tag{8}$$

The distance from any point (x_o, y_o) to L_1 is given by

$$\operatorname{dist}(L_1,(x_o,y_o)) = \frac{aEx_o + bGy_o + 2abFx_oy_o + c}{\sqrt{a^2E + b^2G + 2abF}}$$

then

$$\operatorname{dist}(L_1,(0,0)) = \frac{\mid c \mid}{\sqrt{a^2E + b^2G + 2abF}} = \rho$$

and

$$\mathrm{d}\rho = \frac{(\sqrt{a^2E + b^2G + 2abF})\mathrm{d}c - c\mathrm{d}(\sqrt{a^2E + b^2G + 2abF})}{a^2E + b^2G + 2abF}.$$

Again,

$$\mathrm{d}(\sqrt{a^2E + b^2G + 2abF}) = \frac{(aE + bF)\mathrm{d}a + (bG + aF)\mathrm{d}b}{(\sqrt{a^2E + b^2G + 2abF})}$$

and substituting, we have

$$\mathrm{d}\rho = \frac{(\sqrt{a^2E + b^2G + 2abF})\mathrm{d}c - c\left[\frac{(aE+bF)\mathrm{d}a+(bG+aF)\mathrm{d}b}{(\sqrt{a^2E+b^2G+2abF})}\right]}{a^2E + b^2G + 2abF}$$

i.e.,

$$\mathrm{d}\rho = \frac{(a^2E + b^2G + 2abF)\mathrm{d}c - c[(aE + bF)\mathrm{d}a + (bG + aF)\mathrm{d}b]}{(a^2E + b^2G + 2abF)^{\frac{3}{2}}} \tag{9}$$

Now joining (8) and (9) we obtain

$$\begin{aligned}\mathrm{d}\rho\Lambda\mathrm{d}\theta =& \frac{Eb[bG + aF]\mathrm{d}a + aE[bG + aF)\mathrm{d}b}{(a^2E + b^2G + 2abF)^{\frac{3}{2}}} \\ &\Lambda \frac{(a^2E + b^2G + 2abF)\mathrm{d}c - c[(aE + bF)\mathrm{d}a + (bG + aF)\mathrm{d}b]}{(a^2E + b^2G + 2abF)^{\frac{3}{2}}}\end{aligned}$$

or

$$\begin{aligned}\mathrm{d}\rho\Lambda\mathrm{d}\theta =& \frac{E[bG + aF]}{(a^2E + b^2G + 2abF)^3}\{b(a^2E + b^2G + 2abF)\mathrm{d}a\Lambda\mathrm{d}c \\ &+ a(a^2E + b^2G + 2abF)\mathrm{d}b\Lambda\mathrm{d}c - cb^2G + ca^2E)\mathrm{d}a\Lambda\mathrm{d}b\}.\end{aligned}$$

Finally, we have

$$\begin{aligned}\mathrm{d}L_1 =& \frac{E[bG + aF]}{(a^2E + b^2G + 2abF)^3}\{b(a^2E + b^2G + 2abF)\mathrm{d}a\Lambda\mathrm{d}c \\ &+ a(a^2E + b^2G + 2abF)\mathrm{d}b\Lambda\mathrm{d}c - cb^2G + ca^2E)\mathrm{d}a\Lambda\mathrm{d}b\}.\end{aligned} \tag{10}$$

Taking $E = G = 1$, $F = 0$, the result is

$$\mathrm{d}L_1 = \frac{b}{(a^2+b^2)^3}\{ba^2+b^3)\mathrm{d}a\Lambda\mathrm{d}c+(a^3+ab^2)\mathrm{d}b\Lambda\mathrm{d}c+c(a^2-b^2)\mathrm{d}a\Lambda\mathrm{d}b\} \quad (11)$$

and for $E = 1$, $G = -1$, $F = 0$, the result is

$$\mathrm{d}L_1 = \frac{-b}{(a^2-b^2)^3}\{ba^2-b^3)\mathrm{d}a\Lambda\mathrm{d}c+(a^3-ab^2)\mathrm{d}b\Lambda\mathrm{d}c+c(a^2+b^2)\mathrm{d}a\Lambda\mathrm{d}b\} \quad (12)$$

which are new expressions for density for straight lines in the Euclidean and Lorentzian plane, respectively.

5. Density for Tangent Lines to Plane Curves in $\mathbb{R}^2$

Let C be a differentiable in $\mathbb{R}^2$ given by $y = f(x)$. The equation of the tangent line to that curve, T, at the point $(x_o, f(x_o))$ can be written

$$y - f(x_o) = f'(x_o)(x - x_o)$$

i.e.,

$$y = f'(x_o)x + (f(x_o) - f'(x_o)x_o). \quad (13)$$

The tangent line equation for T, also can be written

$$x\cos\phi + y\sin\phi = \rho \quad (14)$$

where ϕ is the angle that the perpendicular direction to T makes with the $x-$ axis. Calling τ the angle between the line T and the x-axis, we know that $\phi = \tau + \dfrac{\pi}{2}$ and $\mathrm{d}\phi = \mathrm{d}\tau$.

Comparing with (9) we get $y = -x\cot\phi + \dfrac{\rho}{\sin\phi}$ or $y = x\tan\tau - \dfrac{\rho}{\cos\tau}$ and

$$\rho = f'(x_o)x_o\cos\tau - \cos\tau f(x_o)$$

then

$$\mathrm{d}\rho = x_o f''(x_o)\cos\tau\mathrm{d}x_o + \sin\tau[f(x_o) - x_o f'(x_o)]\mathrm{d}\tau$$

and

$$\mathrm{d}T = \mathrm{d}\rho\Lambda\mathrm{d}\phi = \mathrm{d}\rho\Lambda\mathrm{d}\tau = x_o f''(x_o)\cos\tau\mathrm{d}x_o\Lambda\mathrm{d}\tau. \quad (15)$$

5.1. Other Expressions

Proposition 1. *In the plane, a C^2 curve C is represented by $f(s)$ where s is the arclength and its curvature $k(s)$, then the density for tangent straight lines to C, T, is*

$$\mathrm{d}T = \mathrm{d}\rho\Lambda\mathrm{d}\phi = s.k(s)\sin\phi\mathrm{d}s\Lambda\mathrm{d}\phi$$

or equivalently

$$\mathrm{d}T = \mathrm{d}\rho\Lambda\mathrm{d}\tau = s.k(s)\cos\tau\mathrm{d}s\Lambda\mathrm{d}\tau$$

where τ is the angle between T and the x-axis.

Proof: The proof follows immediately by taking the absolute value of (15). ■

In the following we forget the arc lenght parameter and assume that the curve C is given by the expression $(x, f(x))$, then the vector which generates the tangent line is $(1, f'(x)) = v$. Thus, the tangent line T, can be written as

$$T \equiv (1, f'(x))t + (x, f(x)) = (x + t, f'(x)t + f(x))$$

The distance from the line $r \equiv Ax + By + C = 0$ to any point $P_o = P_o(x_o, y_o)$ is given by

$$\mathrm{d}(P_o, \mathrm{d}r) = \frac{|C|(Ax_o + By_o + C)}{C\sqrt{A^2 + B^2}}.$$

In our case it would be $A = f'(x)$, $B = 1$, $C = -xf'(x) + f(x)$.
Applying it to the tangent line T and the point $P_o = (0, 0)$ results

$$\mathrm{d}\rho = \mathrm{d}(T, (0, 0)) = \frac{|-xf'(x) + f(x)|}{[1 + f'(x)]^{\frac{1}{2}}}$$

which send us to a former case.
By differentiation,

$$\mathrm{d}\rho = \left| \frac{\mathrm{d}x}{2(1 + f'(x))^{\frac{3}{2}}} \{2xf''(x) - xf'(x)f''(x) - f(x)f''(x)\} \right|.$$

Finally we obtain

$$\mathrm{d}T = \left| \frac{2xf''(x) - xf'(x)f''(x) - f(x)f''(x)}{2(1 + f'(x))^{\frac{3}{2}}} \right| \mathrm{d}x\Lambda\mathrm{d}\phi.$$

6. Density for Tangent Planes to Surfaces in $\mathbb{R}^3$

Let S be a differentiable surface in $\mathbb{R}^3$ given by $z = f(x,y)$, and be $p \in S$, $p = (x_o, y_o, f(x_o, y_o))$. The equation of the tangent plane to S at the point p is given by

$$z - z_o = f_x(x_o, y_o)(x - x_o) + f_y(x_o, y_o)(y - y_o).$$

Comparing with (2) we have

$$A = f_x(x_o, y_o), \qquad B = f_y(x_o, y_o), \qquad C = -1$$
$$D = x_o f_x(x_o, y_o) + y_o f_y(x_o, y_o) - z_o$$
$$\sqrt{1 - A^2 - B^2} = \sqrt{1 - f_x(x_o, y_o)^2 - f_y(x_o, y_o)^2}$$

and

$$\sqrt{A^2 + B^2} = \sqrt{f_x(x_o, y_o)^2 + f_y(x_o, y_o)^2}$$

In order to apply (5) we get that, for tangent planes to surfaces, is

$$\mathrm{d}L_2 = \frac{\mathrm{d}D\Lambda(A\mathrm{d}A + B\mathrm{d}B)}{\sqrt{1 - A^2 - B^2}\sqrt{A^2 + B^2}}.$$

For the above, we compute

$$\sqrt{1 - A^2 - B^2} = \sqrt{1 - f_x(x_o, y_o)^2 - f_y(x_o, y_o)^2}$$

and

$$\sqrt{A^2 + B^2} = \sqrt{f_x(x_o, y_o)^2 + f_y(x_o, y_o)^2}$$
$$\mathrm{d}A = f_{xx}\mathrm{d}x + f_{xy}\mathrm{d}y, \qquad A\mathrm{d}A = f_x(x_o, y_o)(f_{xx}\mathrm{d}x + f_{xy}\mathrm{d}y)$$
$$\mathrm{d}B = f_{xy}\mathrm{d}x + f_{yy}\mathrm{d}y, \qquad B\mathrm{d}B = f_y(x_o, y_o)(f_{xy}\mathrm{d}x + f_{yy}\mathrm{d}y)$$
$$A\mathrm{d}A + B\mathrm{d}B = f_x(x_o, y_o)(f_{xx}\mathrm{d}x + f_{xy}\mathrm{d}y) + f_y(x_o, y_o)(f_{xy}\mathrm{d}x + f_{yy}\mathrm{d}y)$$
$$\mathrm{d}D = f_x\mathrm{d}x + x f_{xx}\mathrm{d}x + x f_{xy}\mathrm{d}y + f_y\mathrm{d}y + y f_{yx}\mathrm{d}x + y f_{yy}\mathrm{d}y - \mathrm{d}z$$

multiplying $\mathrm{d}D\Lambda(A\mathrm{d}A + B\mathrm{d}B)$ and

$$\mathrm{d}L_2 = \mid (f_{xx}f_{yy} - f_{xy}^2) \mid \mathrm{d}x\Lambda\mathrm{d}y\Lambda\mathrm{d}z. \tag{16}$$

From (5) we find

Proposition 2. *The density for tangent plane of a differentiable surface in* $\mathbb{R}^3$ *given by* $z = f(x,y)$ *is the expression* (16) provided that the Hessian is non zero.

Acknowledgements

This work was partially supported by CONICET of Argentina.

References

[1] Grinberg E. and Quinto E. (Eds), *Integral Geometry and Tomography*, Contemporary Mathematics vol. 113, AMS, RI, 1990.

[2] Santaló L., *Integral Geometry and Geometric Probability*, Encyclopedia of Mathematics and Applications vol. 1, Addison-Wesley, Reading, MA, 1976.

Graciela Silvia Birman
Núcleo Consolidado Matemática
Pura y Aplicada (NuCOMPA)
Departamento de Matemática
Facultad de Ciencias Exactas
Universidad Nacional del Centro
de la Provincia de Buenos Aires
Pinto 399- (7000) Tandil
ARGENTINA
E-mail address: gbirman@exa.unicen.edu.ar