JGSP **12** (2008) 1–13

JOURNAL OF
Geometry and Symmetry in Physics

RESONANCES AS EIGENVALUES IN THE GEL'FAND TRIPLET APPROACH FOR FINITE-DIMENSIONAL FRIEDRICHS MODELS ON THE POSITIVE HALF LINE

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Communicated by Arno Bohm

Abstract. The eigenvalue problem for the resonances of finite-dimensional Friedrichs models on the positive half line is solved using an appropriate Gel'fand triplet. The associated Gamow vectors are uniquely determined by the corresponding eigenantilinear forms. They turn out to be the restriction of the eigenantilinear forms to the Hardy space part of the Gel'fand space. Conditions are presented such that there are only finitely many resonances and all resonances are simple poles of the scattering matrix.

1. Introduction

In quantum scattering systems bumps in cross sections often can be described by expressions like $\lambda \to c((\lambda - \lambda_0)^2 + (\frac{\Gamma}{2})^2)^{-1}$, where λ_0 is the *resonance energy*, $\Gamma/2$ the *halfwidth*, called Breit-Wigner formulas. Sometimes, if the scattering matrix is analytically continuable into the lower half plane, these bumps can be connected with complex poles $\lambda_0 - \mathrm{i}\frac{\Gamma}{2}$ of the scattering matrix in $\mathbb{C}_-$. Then $c((\lambda - \lambda_0) - \mathrm{i}\frac{\Gamma}{2})^{-1}$ is called the Breit-Wigner amplitude (see e.g. [1, pp. 428 - 429]). These poles are called *resonances* (see e.g. [2]).

The investigation of these poles requires knowledge of the connection between the Hamiltonian H and its scattering matrix w.r.t. the "free" Hamiltonian H_0 which is difficult to obtain, in general. Therefore, some authors work with the resolvent of H and systems of its matrix elements to define resonances by poles of their meromorphic continuations. Then the aim is to associate to these poles eigenvalues of certain non-selfadjoint operators, associated with H. This approach is called the Aigular-Balslev-Combes-Simon theory (see e.g. Hislop&Sigal [10]). The problem is then to identify these poles with poles of the meromorphic continuation of the scattering matrix.

In this paper for finite-dimensional Friedrichs models on the positive half line the resonances are characterized by their spectral properties w.r.t. H directly, i.e., it is shown that they are exactly the eigenvalues of an appropriate extension of H by a Gel'fand triplet. Further the corresponding eigenantilinear forms are used to derive the Gamow vectors of the resonance, which are eigenvectors of the decay semigroup connected with the Hamiltonian H. For this purpose the Hardy spaces play an essential role. For the introduction of this semigroup of the Toeplitz type and further details see Eisenberg *et al.* [4] and Strauss [5], see also [9]. Finally, the analytic structure of the scattering matrix is characterized and sufficient conditions are presented such that there are only simple poles and its Laurent main part turns out to be a linear combination of Gamow vectors, associated to the resonances. The paper is related to [8, Section 5] and [11] where the results are briefly described.

2. The Friedrichs Model on the Positive Half Axis

In the following we collect the concepts and the notations for the finite-dimensional Friedrichs model on $\mathbb{R}_+ := (0,\infty)$. Let $\mathcal{H}_{0,+} := L^2(\mathbb{R}_+,\mathcal{K},\mathrm{d}\lambda)$, where $\mathcal{K}$ denotes a multiplicity Hilbert space, $\dim\mathcal{K} < \infty$. We put $\mathcal{H}_0 := L^2(\mathbb{R},\mathcal{K},d\lambda)$ such that with $P_+f(\lambda) := \chi_{\mathbb{R}_+}(\lambda)f(\lambda)$, $f \in \mathcal{H}_0$, the Hilbert spaces $\mathcal{H}_{0,+}$ and $P_+\mathcal{H}_0$ can be canonically identified. Further let $\mathcal{E}$ be a finite-dimensional Hilbert space with $\dim\mathcal{E} = \dim\mathcal{K}$ and put $\mathcal{H} := \mathcal{H}_{0,+} \oplus \mathcal{E}$. The projection onto $\mathcal{E}$ is denoted by $P_\mathcal{E}$. H_0 is assumed to be the multiplication operator on $\mathcal{H}_{0,+}$. A is a selfadjoint operator on $\mathcal{E}$ with only positive eigenvalues. The selfadjoint operator H on $\mathcal{H}$ is given by a perturbation of $H_0 \oplus A$ as

$$H := (H_0 \oplus A) + \Gamma + \Gamma^*$$

where Γ denotes a partial isometry on $\mathcal{H}$ with the properties $\Gamma^*\Gamma = P_\mathcal{E}$, $\Gamma\Gamma^* < P_\mathcal{E}^\perp := \mathbb{1} - P_\mathcal{E}$. One has $\operatorname{dom} H = \operatorname{dom} H_0 \oplus \mathcal{E}$. In the following $\Gamma\lceil\mathcal{E}$ is identified with Γ without confusion. For $\mathcal{H} \ni x := f + e$, $f \in \mathcal{H}_{0,+}$, $e \in \mathcal{E}$ one has $\Gamma x = \Gamma e$, $\Gamma^* x = \Gamma^* f$. Therefore $(\Gamma e)(\lambda) = M(\lambda)e$, $\mathcal{E} \ni \Gamma^* f = \int_0^\infty M(\lambda)^* f(\lambda)\mathrm{d}\lambda$, where $\lambda \to M(\lambda) \in \mathcal{L}(\mathcal{E} \to \mathcal{K})$ is a.e. defined on $\mathbb{R}_+$. Note that $\int_0^\infty \|M(\lambda)\|_2^2\mathrm{d}\lambda < \infty$, where $\|\cdot\|_2$ denotes the Hilbert-Schmidt norm on $\mathcal{E}$. On this model we impose the following conditions:

i) $\Gamma\mathcal{E}$ is generating for $\mathcal{H}_{0,+}$ and $\mathcal{E}$ is generating for $\mathcal{H}$, i.e.,

$$\mathcal{H}_{0,+} = \overline{\{E_0(\Delta)f;\ f \in \Gamma\mathcal{E}\}}, \qquad \mathcal{H} = \overline{\{E(\Delta)e;\ e \in \mathcal{E}\}}.$$

where $E_0(\cdot),\ E(\cdot)$ denote the spectral measures of H_0, H, respectively.

ii) H has no real eigenvalues.

iii) $\int\limits_0^\infty \lambda^2 \|M(\lambda)\|_2^2 \mathrm{d}\lambda < \infty$. This is equivalent with $\Gamma\mathcal{E} \subset \mathrm{dom}\, H_0$.

iv) $M(\cdot)$ is holomorphically continuable into $\mathbb{C}_{<0} := \mathbb{C} \setminus (-\infty, 0]$, including the border $\mathbb{R}_- \pm \mathrm{i}0$ of the cut $(-\infty, 0]$, where $\mathbb{R}_-$ denotes the negative half line, $\mathbb{R}_- := (-\infty, 0)$ and $z \to M(z) \in \mathcal{L}(\mathcal{E} \to \mathcal{K})$ is invertible there, i.e., $z \to M(z)^{-1}$ is also holomorphic. Further

$$\sup_{z \in \mathbb{C}_{<0} \setminus K_\epsilon} \|M(z)\| < \infty, \qquad \sup_{z \in \mathbb{C}_{<0} \setminus K_\epsilon} \|M'(z)\| < \infty$$

where $K_\epsilon := \{z;\ |z| \le \epsilon\}$, $\epsilon > 0$, is a small circle around 0. Note that $z \to M(\overline{z})^*$ is also holomorphic on $\mathbb{C}_{<0}$ including the border $\mathbb{R}_- \pm \mathrm{i}0$.

An example for $M(\cdot)$ in the case $\mathcal{E} := \mathbb{C}$ is given by

$$M(z) := c \cdot \left(\frac{\log z}{z-1}\right)^2.$$

We put

$$\Phi(z) := \Gamma^*(z - H_0)^{-1}\Gamma\restriction\mathcal{E} = \int\limits_0^\infty \frac{M(\lambda)^* M(\lambda)}{z - \lambda}\mathrm{d}\lambda, \qquad z \in \mathbb{C}_{>0} \tag{1}$$

where $\mathbb{C}_{>0} := \mathbb{C} \setminus [0, \infty)$. $\Phi(\cdot)$ is holomorphic there. We use the the notation $\Phi_\pm(z) := \Phi(z),\ z \in \mathbb{C}_\pm$. Obviously $\Phi_\pm(\cdot)$ is holomorphic continuable into $\mathbb{C}_{<0}$ across $\mathbb{R}_+$, it is holomorphic there and one has the relation

$$\Phi_-(z) - \Phi_+(z) = 2\pi\mathrm{i} M(\overline{z})^* M(z), \qquad z \in \mathbb{C}_{<0}. \tag{2}$$

Note that $z \to M(\overline{z})^*$ is also holomorphic in $\mathbb{C}_{<0}$. From (1) we obtain

$$\frac{\Gamma^* E_0(\mathrm{d}\lambda)\Gamma}{\mathrm{d}\lambda}\restriction\mathcal{E} = M(\lambda)^* M(\lambda), \qquad \lambda \in \mathbb{R}_+$$

and from (1) and (2)

$$\sup_{z \in \mathbb{C}_{<0} \setminus K_\epsilon} \|\Phi_\pm(z)\| < \infty. \tag{3}$$

The operator function

$$L_\pm(z) := (z\mathbb{1}_\mathcal{E} - A) - \Phi_\pm(z), \qquad z \in \mathbb{C}_{<0}$$

the so-called Livšic-matrix, is decisive in the following. It is holomorphic in $\mathbb{C}_{<0}$. One has $L_+(z)^* = L_-(\overline{z})$. The so-called partial resolvent $P_\mathcal{E}(z-H)^{-1}P_\mathcal{E}$ satisfies the equation

$$L_\pm(z) \cdot P_\mathcal{E}(z-H)^{-1}P_\mathcal{E}\restriction\mathcal{E} = P_\mathcal{E}(z-H))^{-1}P_\mathcal{E}\restriction\mathcal{E} \cdot L_\pm(z) = \mathbb{1}_\mathcal{E}, \qquad z \in \mathbb{C}_\pm$$

(see e.g. [6]), that is

$$P_\mathcal{E}(z-H)^{-1}P_\mathcal{E}\restriction\mathcal{E} = L_\pm(z)^{-1}, \qquad z \in \mathbb{C}_\pm$$

and this equation shows that $z \to L_\pm(z)^{-1} \in \mathcal{L}(\mathcal{E})$ is holomorphic on $\mathbb{C}_\pm$. Furthermore, according to ii), $\lambda \to L_\pm(\lambda \pm \mathrm{i}0)^{-1} =: L_\pm(\lambda)$ is holomorphic on $\mathbb{R}_+$ (see e.g. [6]). On $\mathbb{C}_{<0} \cup \{\mathbb{R}_- \pm \mathrm{i}0\}$ it is meromorphic. From (3) it follows that

$$\sup_{z\in\mathbb{C}_{<0}\setminus K_R} \|L_\pm(z)^{-1}\| < \infty, \qquad K_R := \{z;\, |z| \le R\} \tag{4}$$

where R is sufficiently large. Therefore, $L_+(\cdot)^{-1}$ has at most denumerably many poles, where at most 0 can be an accumulation point. They are contained in $\mathbb{C}_-$ including the border $\mathbb{R}_- - \mathrm{i}0$. The nonreal poles are called *resonances*. The set of all resonances is denoted by $\mathcal{R}$. A point $\zeta_0 \in \mathbb{C}_-$ is a resonance iff $\det L_+(\zeta_0) = 0$. Note further that $P_\mathcal{E}(\lambda \pm \mathrm{i}0 - H)^{-1}P_\mathcal{E}\restriction\mathcal{E} = L_\pm(\lambda)^{-1}$, $\lambda \in \mathbb{R}_+$. This implies

$$\begin{aligned} \frac{P_\mathcal{E}E(\mathrm{d}\lambda)P_\mathcal{E}}{\mathrm{d}\lambda}\restriction\mathcal{E} &= \frac{1}{2\pi\mathrm{i}}(L_-(\lambda)^{-1} - L_+(\lambda)^{-1}) \\ &= L_\pm(\lambda)^{-1}M(\lambda)^*M(\lambda)L_\mp(\lambda)^{-1}, \qquad \lambda > 0. \end{aligned} \tag{5}$$

The left hand side vanishes for $\lambda < 0$. Therefore spec $H = [0,\infty)$ and it follows that H is pure absolutely continuous.

3. Spectral Representations, Wave Operators, Wave Matrices and Scattering Matrix

Since $\Gamma + \Gamma^*$ is a finite-dimensional perturbation the wave operators $W_\pm = W_\pm(H, H_0) :=$s-$\lim_{t\to\pm\infty} \mathrm{e}^{\mathrm{i}tH}\mathrm{e}^{-\mathrm{i}tH_0}P_\mathcal{E}^\perp$ exist, they are isometric from $\mathcal{H}_0$ onto $\mathcal{H}$. Furthermore, $W_\pm^* = W_\pm(H_0, H) =$s-$\lim_{t\to\pm\infty} \mathrm{e}^{\mathrm{i}tH_0}\mathrm{e}^{-\mathrm{i}tH}$.

The Hilbert spaces $\mathcal{H}_{0,+}$ and $\mathcal{H}$ have natural spectral representations w.r.t. $E_0(\cdot)$, $E(\cdot)$, respectively, which are explicitly given by spectral integrals (see e.g. [7])

$$\mathcal{H}_{0,+} \ni x = \int_0^\infty E_0(\mathrm{d}\mu)\Gamma f(\mu) \tag{6}$$

$$\mathcal{H} \ni y = \int_0^\infty E(\mathrm{d}\lambda) g(\lambda) \tag{7}$$

where $\mu \to f(\mu) \in \mathcal{E}$, $\lambda \to g(\lambda) \in \mathcal{E}$ are vector functions with values in $\mathcal{E}$ such that the integrals (6), (7) exist. Note that (6) exists iff $\int_0^\infty \|M(\mu)f(\mu)\|^2_{\mathcal{K}}\mathrm{d}\mu < \infty$, i.e., iff the function $\mu \to M(\mu)f(\mu)$ is an element of $\mathcal{H}_{0,+}$. The integral (7) exists iff $\int_0^\infty \|M(\lambda)L_+(\lambda)^{-1}g(\lambda)\|^2_{\mathcal{K}}\mathrm{d}\lambda < \infty$, i.e., iff the function $\lambda \to M(\lambda)L_+(\lambda)^{-1}g(\lambda)$ is an element of $\mathcal{H}_{0,+}$. The function $f(\cdot)$ is called the E_0-*representer* of x and $g(\cdot)$ the E-*representer* of y .

Note further that $(\left(\int_0^\infty E_0(\mathrm{d}\mu)\Gamma f(\mu)\right)(\lambda) = (\Gamma f(\lambda))(\lambda) = M(\lambda)f(\lambda)$. The wave operators $W_\pm$ on $\mathcal{H}_{0,+}$ and $W_\pm^*$ on $\mathcal{H}$ can be calculated explicitly.

Lemma 1. *The wave operators are given by the following expressions*

$$W_\pm\left(\int_0^\infty E_0(\mathrm{d}\mu)\Gamma f(\mu)\right) = \int_0^\infty E(\mathrm{d}\lambda)L_\pm(\lambda)f(\lambda)$$
$$W_\pm^*\left(\int_{-\infty}^\infty E(\mathrm{d}\lambda)g(\lambda)\right) = \int_{-\infty}^\infty E_0(\mathrm{d}\lambda)\Gamma L_\pm(\lambda)^{-1}g(\lambda).$$

See [8] for the Proof.

Lemma 1 says: if $\lambda \to f(\lambda)$ is the E_0-representer of $x \in \mathcal{H}_{0,+}$ then the E-representer of $W_\pm x \in \mathcal{H}$ is given by $\lambda \to L_\pm(\lambda)f(\lambda)$. Conversely, if $\lambda \to g(\lambda)$ is the E-representer of $y \in \mathcal{H}$ then the E_0-representer of $W_\pm^* y \in \mathcal{H}_{0,+}$ is given by $\lambda \to L_\pm(\lambda)^{-1}g(\lambda)$. That is, the elements $x \in \mathcal{H}_{0,+}$, $x = W_+^* y$, $y \in \mathcal{H}$, can be characterized by the functions $\lambda \to x(\lambda) := M(\lambda)L_+(\lambda)^{-1}g(\lambda)$, where $\|x\|^2 = \int_0^\infty \|M(\lambda)L_+(\lambda)^{-1}g(\lambda)\|^2_{\mathcal{K}}\mathrm{d}\lambda$. For example, the element $W_+^* e$, $e \in \mathcal{E}$

corresponds to the $\mathcal{K}$-valued function

$$\lambda \to M(\lambda)L_+(\lambda)^{-1}e \tag{8}$$

the element $W_+^*\Gamma e$ is characterized by

$$\lambda \to M(\lambda)L_+(\lambda)^{-1}(\lambda - A)e \tag{9}$$

because $(\lambda - A)e$ is the E-representer of Γe

$$\int_0^\infty E(\mathrm{d}\lambda)(\lambda - A)e = \int_0^\infty \lambda E(\mathrm{d}\lambda)e - \int_0^\infty E(\mathrm{d}\lambda)Ae = He - Ae = \Gamma e.$$

Note that $e \in \operatorname{dom} H$, because $He = Ae + \Gamma e$. Recall that $\operatorname{dom} H$ consists of all $g \in \mathcal{H}$ whose E-representers $g(\cdot)$ have the property that also $\lambda \to \lambda g(\lambda)$ is an E-representer. This is consistent with the fact that $\operatorname{dom} H$ is transformed into $\operatorname{dom} H_0$ by W_+^* (see equation (9)).

In general, operator functions with these properties are called the *wave matrices* of $W_\pm$, $W_\pm^*$ w.r.t. given fixed spectral representations (see [7, p. 177] for these concepts). Note that wave matrices are well-defined only if the spectral representations are fixed.

Lemma 2. *The wave matrices of* $W_\pm$, $W_\pm^*$ *wr.t. the natural spectral representations in* $\mathcal{H}_{0,+}$, $\mathcal{H}$ *are given by*

$$W_\pm(\lambda) = L_\pm(\lambda), \qquad W_\pm^*(\lambda) = L_\pm(\lambda)^{-1}, \qquad \lambda \in \mathbb{R}_+.$$

As is well-known (see e.g. [7, p. 398 ff.]) the explicit form of the scattering matrix $S_\mathcal{K}(\lambda) := (W_+^* W_-)(\lambda)$ in the usual $\mathcal{K}$-representation of $\mathcal{H}_{0,+}$ is given by the formula

$$S_\mathcal{K}(\lambda) = \mathbb{1}_\mathcal{K} - 2\pi \mathrm{i} M(\lambda)L_+(\lambda)^{-1}M(\lambda)^*, \qquad \lambda \in \mathbb{R}_+. \tag{10}$$

This expression leads immediately to

Proposition 3. *The scattering matrix* $S_\mathcal{K}(\cdot)$ *is meromorphically continuable into* $\mathbb{C}_{<0}$ *its poles coincide with the resonances and*

$$\sup_{z \in \mathbb{C}_{<0} \setminus K_R} \|S_\mathcal{K}(z)\| < \infty.$$

Proof: $L_+(\cdot)^{-1}$ is meromorphic on $\mathbb{C}_{<0}$, its poles are the resonances. Then from (iv) and (10) one obtains the assertion. ■

Concerning the scattering matrix in the natural spectral representation of $\mathcal{H}_{0,+}$ by the E_0-representers one has

Lemma 4. *W.r.t. the natural spectral representation of $\mathcal{H}_{0,+}$ by the E_0-representers the scattering matrix $S_{\mathcal{E}}(\cdot)$ is given by*

$$S_{\mathcal{E}}(\lambda) = L_+(\lambda)^{-1}L_-(\lambda) = L_+(\lambda)^{-1}L_+(\lambda)^*.$$

This means if $f \in \mathcal{H}_{0,+}$ and $\tilde{f}(\cdot)$ is its E_0-representer, i.e., $f(\lambda) = M(\lambda)\tilde{f}(\lambda)$ then $S_{\mathcal{E}}(\lambda)\tilde{f}(\lambda)$ is the E_0-representer of Sf, where $(Sf)(\lambda) = S_{\mathcal{K}}(\lambda)f(\lambda)$.

Proof: We have to prove that $S_{\mathcal{K}}(\lambda)M(\lambda)\tilde{f}(\lambda) = M(\lambda)S_{\mathcal{E}}(\lambda)\tilde{f}(\lambda)$. But this is obvious because of

$$M(\lambda)L_+(\lambda)^{-1}L_-(\lambda) = (\mathbb{1}_{\mathcal{K}} - 2\pi \mathrm{i} M(\lambda)L_+(\lambda)^{-1}M(\lambda)^*)M(\lambda) = S_{\mathcal{K}}(\lambda)M(\lambda)$$

■

4. The Gel'fand Triplet

The Gel'fand space $\mathcal{S} \subset \mathcal{H}_{0,+}$ is defined to be the manifold of all $s \in \mathcal{H}_{0,+}$ with

$$s(\lambda) := M(\lambda)L_+(\lambda)^{-1}g(\lambda)$$

such that $\lambda \to g(\lambda)$ has a holomorphic continuation into $\mathbb{C}_{<0}$ and $\lambda \to \lambda g(\lambda)$ is also an E-representer. Note that $g(\cdot)$ is the E-representer of $g := W_+s$.

Obviously $\mathcal{S}$ is dense in $\mathcal{H}_{0,+}$ w.r.t. the Hilbert topology. The Gel'fand topology in the Gel'fand space $\mathcal{S}$ is defined by the collection of norms

$$\mathcal{S} \ni s \to [s]_K := \|s\|_{\mathcal{H}_{0,+}} + \sup_{z\in K} \|g(z)\|_{\mathcal{K}}, \qquad K \subset \mathbb{C}_{<0}$$

where K is compact. $\mathcal{S}$ is closed w.r.t. this topology. Then $\mathcal{S} \subset \mathcal{H}_{0,+} \subset \mathcal{S}^\times$ is a *Gel'fand triplet* where $\mathcal{S}^\times$ denotes the set of all continuous linear forms (w.r.t. the Gel'fand topology) on $\mathcal{S}$. Note that $W_+^*\mathcal{E} \oplus W_+^*\Gamma\mathcal{E} \subset \mathcal{S}$. This follows from (8) and (9). The Gel'fand space $\mathcal{S}$ can be transformed into $\mathcal{H}$ by $\mathcal{G} := W_+\mathcal{S}$, i.e., $\mathcal{G}$ is the set of all $g \in \mathcal{H}$ such that $g(\cdot)$ is holomorphic in $\mathbb{C}_{<0}$ and $\lambda \to \lambda g(\lambda)$ is also an E-representer. The topology of $\mathcal{G}$ is defined by the injection of the topology of $\mathcal{S}$. Then $\mathcal{G}$ is a Gel'fand space in $\mathcal{H}$ and $\mathcal{G} \subset \mathcal{H} \subset \mathcal{G}^\times$ is the corresponding

Gel'fand triplet. One has $\mathcal{E} \oplus \Gamma\mathcal{E} \subset \mathcal{G}$ and $H\mathcal{G} \subseteq \mathcal{G}$. Further, $\mathcal{G} = \mathcal{E} \oplus \Phi$, where $\Phi := P_{\mathcal{E}}^{\perp}\mathcal{G} \subseteq \mathcal{H}_{0,+}$. Then $\Phi \subseteq \mathrm{dom}\, H_0$, hence $H_0\Phi \subseteq \Phi$ follows. Note that $\mathcal{G}^{\times} = \Phi^{\times} \times \mathcal{E}$ and for $d = \phi + e$, $d^{\times} = \{\phi^{\times}, e^{\times}\}$ one has

$$\langle d \mid d^{\times}\rangle = \langle \phi \mid \phi^{\times}\rangle + (e, e^{\times})_{\mathcal{E}}.$$

5. The Eigenvalue Problem for the Resonances

The Gel'fand triplet $\mathcal{G} \subset \mathcal{H} \subset \mathcal{G}^{\times}$ yields a unique extension $H^{\times}$ on $\mathcal{G}^{\times}$ given by

$$\langle d \mid H^{\times} d^{\times}\rangle := \langle Hd \mid d^{\times}\rangle, \qquad d \in \mathcal{G},\, d^{\times} \in \mathcal{G}^{\times}.$$

The eigenvalue equation for eigenvalues $\zeta_0 \in \mathbb{C}_-$ of $H^{\times}$ reads then

$$H^{\times} d_0^{\times} = \zeta_0 d_0^{\times}, \qquad d_0^{\times} := \{\phi_0^{\times}(\zeta_0, e_0)\}, \qquad e_0 \in \mathcal{E},\, \zeta_0 \in \mathbb{C}_-,\, \phi_0^{\times} \in \Phi^{\times}.$$

For the part $\phi_0^{\times}$ of a solution we impose a *Boundary condition*: $\phi_0^{\times}$ is required to be the analytic continuation into $\mathbb{C}_-$ across $\mathbb{R}_+$ of a holomorphic vector antilinear form $\phi_0^{\times}(z, e_0)$ on $\mathbb{C}_+$ such that the Φ-part of the eigenvalue equation is an identity on $\mathbb{C}_+$.

Theorem 5. *The antilinear form* $d_0^{\times}$ *is an eigensolution iff* e_0 *satisfies* $L_+(\zeta_0)e_0 = 0$*, i.e., if* ζ_0 *is a resonance and* $e_0 \in \ker L_+(\zeta_0)$*. The associated vector antilinear form on* $\mathbb{C}_+$ *is given by*

$$\langle \phi \mid \phi_0^{\times}(z, e)\rangle = (\phi, R_0(z)\Gamma e)_{\mathcal{H}_{0,+}}, \qquad z \in \mathbb{C}_+. \tag{11}$$

The dimension of the corresponding eigenspace is $\dim \ker L_+(\zeta_0)$*, the geometric multiplicity of the eigenvalue* 0 *of* $L_+(\zeta_0)$*.*

Proof: It follows the course of the similar proof of Theorem 1 of [8]. The crucial point in the proof is to show that (11) has an analytic continuation into $\mathbb{C}_-$ across $\mathbb{R}_+$. First one has

$$\langle \phi \mid \phi_0^{\times}(z, e)\rangle = (g, R_0(z)\Gamma e)_{\mathcal{H}_{0,+}}, \qquad z \in \mathbb{C}_+$$

where $g := W_+ s,\ s \in \mathcal{S},\ \phi = P_{\mathcal{E}}^{\perp} g$. Second, using the identity

$$R_0(z)\Gamma e + e = R(z)L_+(z)e, \qquad e \in \mathcal{E},\ z \in \mathbb{C}_+$$

one obtains

$$(g, R(z)L_+(z)e) = (\Psi_-(\overline{z}), L_+(z)e) \tag{12}$$

where

$$\Psi_\pm(z) := \int_0^\infty \frac{1}{z-\mu} L_-(\mu)^{-1} M(\mu)^* M(\mu) L_+(\mu)^{-1} g(\mu)\mathrm{d}\mu, \qquad z \in \mathbb{C}_\pm.$$

Then one obtains the assertion by inspection of (12). ■

The second result concerns the structure of the corresponding eigenantilinear form $s_0^\times$ of $H_0^\times$ w.r.t. the Gel'fand triplet $\mathcal{S} \subset \mathcal{H}_{0,+} \subset \mathcal{S}^\times$. This antilinear form is given by

$$s_0^\times(\zeta_0, e_0) = (W_+^*)^\times d_0^\times(\zeta_0, e_0). \tag{13}$$

Theorem 6. *The eigenantilinear form $s_0^\times$ of $H_0^\times$ w.r.t. the Gel'fand triplet $\mathcal{S} \subset \mathcal{H}_{0,+} \subset \mathcal{S}^\times$, associated to $d_0^\times$ by (13) is an antilinear form on $\mathcal{S}$ of pure Dirac type*

$$\langle s \mid s_0^\times(\zeta_0, e_0)\rangle = 2\pi\mathrm{i}(s(\overline{\zeta_0}), k_0)_\mathcal{K}, \qquad s \in \mathcal{G}$$

where $k_0 := M(\zeta_0)e_0$.

Proof: Since

$$\langle s \mid s_0^\times(z, e_0)\rangle = (g, R(z)L_+(z)e_0)$$

it is an easy implication of (12). ■

6. The Gamow Vector Associated to $s_0^\times(\zeta_0, e_0)$

First one has to construct an appropriate submanifold of $\mathcal{S}$. Let $\mathcal{H}_+^2 \subset \mathcal{H}_0$ be the Hardy space of the upper half plane. Then $P_+\mathcal{H}_+^2 \subset \mathcal{H}_{0,+}$ is a dense inclusion w.r.t. the Hilbert topology. Note that $P_+\mathcal{H}_+^2$ is a Hilbert space of its own w.r.t. te Hilbert norm $\langle u\rangle := \|P_+^{-1}u\|$, $u \in P_+\mathcal{H}_+^2$ because P_+ is injective between $\mathcal{H}_+^2$ and $P_+\mathcal{H}_+^2$.

Lemma 7. *The inclusion $\mathcal{S} \cap P_+\mathcal{H}_+^2 \subset P_+\mathcal{H}_+^2$ is dense w.r.t. the Hilbert topology.*

Proof: Let $\mathcal{U} \subset P_+\mathcal{H}_+^2$ be a dense submanifold of vectors u such that $u \in \operatorname{dom} H_0$ and $\lambda \to u(\lambda)$ is holomorphic continuable into $\mathbb{C}$ (for example use the parts on the positive half line of inverse Fourier transforms of Schwartz space functions on the positive half line with compact support). Then the E-representer of u is given by $g(\lambda) := L_+(\lambda)M(\lambda)^{-1}u(\lambda)$ which is holomorphic continuable into $\mathbb{C}_{<0}$ such that $\lambda \to u(\lambda) = M(\lambda)L_+(\lambda)^{-1}g(\lambda)$ is from $\mathcal{S}$. Then $\mathcal{U} \subset \mathcal{S} \cap P_+\mathcal{H}_+^2$ and this implies the assertion. ■

Theorem 8. *The restricted eigenantilinear form $s_0^\times \upharpoonright \mathcal{S} \cap P_+\mathcal{H}_+^2$*

$$\mathcal{S} \cap P_+\mathcal{H}_+^2 \ni s \to 2\pi \mathrm{i}(s(\overline{\zeta_0}), k_0)_\mathcal{K}$$

is even continuous w.r.t. the Hilbert space topology $\langle \cdot \rangle$ of $P_+\mathcal{H}_+^2$, i.e., it can be continuously extended onto $\mathrm{clo}_{\langle \cdot \rangle}(\mathcal{S} \cap P_+\mathcal{H}_+^2) = P_+\mathcal{H}_+^2$. That is, $s_0^\times \upharpoonright P_+\mathcal{H}_+^2$ is realized by the $P_+\mathcal{H}_+^2$-vector $\mathbb{R}_+ \ni \lambda \to k_0(\zeta_0 - \lambda)^{-1}$ via the relation

$$2\pi \mathrm{i}(s(\overline{\zeta_0}), k_0) = \int_{-\infty}^{\infty} \left(s(\lambda), \frac{k_0}{\zeta_0 - \lambda} \right)_\mathcal{K} \mathrm{d}\lambda. \tag{14}$$

where in (14) the (unique) ectensions of $s(\cdot)$ and $\lambda \to k_0(\zeta_0 - \lambda)^{-1}$ onto the whole real line have to be used.

Proof: (14) follows immediately from the Paley-Wiener theorem. ■

Note that the vector function $\lambda \to k_0/(\zeta_0 - \lambda)$ from $\mathcal{H}_{0,+}$ is called a Gamow vector of the resonance ζ_0 with parameter $k_0 := M(\zeta_0)e_0$. It is a special eigenvector of the so-called decay semigroup $t \to P_+Q_+\exp(-\mathrm{i}t\tilde{H}_0)P_+^{-1}$ on $P_+\mathcal{H}_+^2$, where $\tilde{H}_0$ denotes the multiplication operator on $\mathcal{H}_0$ as an extension of H_0 on $\mathcal{H}_{0,+}$ and Q_+ the projection onto $\mathcal{H}_+^2$.

7. Laurent Representation of the Scattering Matrix on $\mathbb{C}_{<0}$

In this section we introduce two further conditions

v) The point 0 is not an accumulation point of $\mathcal{R}$, i.e., $\mathcal{R}$ is finite.

vi) The operator function $\mathbb{R}_- \ni \lambda \to S_\mathcal{K}(\lambda - \mathrm{i}0)$ is holomorphic on $\mathbb{R}_-$ and bounded, i.e., $\sup_{\lambda \in \mathbb{R}_-} \|S_\mathcal{K}(\lambda - \mathrm{i}0)\| < \infty$ (note that $S_\mathcal{K}(\cdot)$ is for $\lambda > 0$ a priori bounded, because each $S_\mathcal{K}(\lambda)$ is unitary).

If

$$\inf_{|e|=1} \left| \int_0^\infty \frac{\|M(\lambda)e\|^2}{z - \lambda} \mathrm{d}\lambda \right| > N$$

(N arbitrary large) is satisfied for sufficiently small $|z|$ then v) is true. The example of Section 2 satisfies v). It satisfies vi) if A (which is a positive scalar in this case) is sufficiently large.

Now we extend the scattering operator S, defined on $\mathcal{H}_{0,+}$, to a bounded operator on $\mathcal{H}_0$ using the boundary values $S_{\mathcal{K}}(\lambda \pm \mathrm{i}0)$ for $\lambda < 0$

$$S_{\pm} f(\lambda) := S_{\mathcal{K}}(\lambda \pm \mathrm{i}0) f(\lambda), \qquad \lambda \in \mathbb{R} \setminus \{0\}, \qquad f \in \mathcal{H}_0.$$

For brevity we write $S(\lambda \pm \mathrm{i}0) =: S_{\pm}(\lambda)$. Now it turns out that for vectors $g \in \mathcal{H}^2_-$ the projection of $S_- g$ onto the Hardy space $\mathcal{H}^2_+$ has a very simple form.

Theorem 9. *Let the conditions* i)–vi) *be satisfied and* $g \in \mathcal{H}^2_-$. *Then the projection* $Q_+ S_- g$ *of* $S_- g$ *onto the Hardy space* $\mathcal{H}^2_+$ *has the following form:*

$$(Q_+ S_- g)(z) = \sum_{\zeta \in \mathcal{R}} \frac{S_{-1,\zeta}\, g(\zeta)}{z - \zeta}, \qquad z \in \mathbb{C}_+ \tag{15}$$

where $S_{-1,\zeta}$ *denotes the residuum of* $S_{\mathcal{K}}(\cdot)$ *at* ζ.

Proof: It is given in [9, p. 20 f.]. See also Gadella [10] for related calculations. ■

Since the right hand side of (15) is also well-defined on $\mathbb{C}_-$ and even rational on $\mathbb{C}$ we obtain

Corollary 10. *The poles of the scattering matrix* $S_{\mathcal{K}}(\cdot)$ *are necessarily simple and the Laurent representation of* $S_{\mathcal{K}}(\cdot)$ *in* $\mathbb{C}_{<0}$ *is given by*

$$S_{\mathcal{K}}(z) = \sum_{\zeta \in \mathcal{R}} \frac{S_{-1,\zeta}}{z - \zeta} + H_{\mathcal{K}}(z), \qquad z \in \mathbb{C}_{<0}$$

where $z \to H_{\mathcal{K}}(z)$ is holomorphic on $\mathbb{C}_{<0}$.

Proof: The projection of $S_- g$ onto $\mathcal{H}^2_-$ is given by

$$(Q_- S_- g)(z) = -\frac{1}{2\pi \mathrm{i}} \int_{-\infty}^{\infty} \frac{S_-(\lambda) g(\lambda)}{\lambda - z} \mathrm{d}\lambda, \qquad z \in \mathbb{C}_-.$$

The right hand side of (15) is a rational function on $\mathbb{C}$, i.e., $Q_+ S_- g(\cdot)$ has an analytic continuation into $\mathbb{C}_-$ given by this expression. On the other hand $Q_+ S_- g +$

$Q_-S_-g = S_-g$. This means: the function $\mathbb{R}_+ \ni \lambda \to S_{\mathcal{K}}(\lambda)g(\lambda)$ has an analytic continuation into $\mathbb{C}_-$ given by

$$(S_-g)(z) = S_{\mathcal{K}}(z)g(z) = \sum_{\zeta\in\mathcal{R}} \frac{S_{-1,\zeta}\, g(\zeta)}{z-\zeta} - \frac{1}{2\pi \mathrm{i}} \int\limits_{-\infty}^{\infty} \frac{S_-(\lambda)g(\lambda)}{\lambda - z} \mathrm{d}\lambda, \qquad z \in \mathbb{C}_-.$$

This is true for all $g \in \mathcal{H}^2_-$. Putting, for example, $g(z) := k(z-\mathrm{i})^{-1}$, $k \in \mathcal{K}$, then one obtains

$$S_{\mathcal{K}}(z) = \sum_{\zeta\in\mathcal{R}} \frac{S_{-1,\zeta}}{z-\zeta} + H_{\mathcal{K}}(z), \qquad z \in \mathbb{C}_-$$

where the first term is the main part and

$$H_{\mathcal{K}}(z) = \sum_{\zeta\in\mathcal{R}} \frac{S_{-1,\zeta}}{\zeta - \mathrm{i}} - \frac{1}{2\pi\mathrm{i}} \int\limits_{-\infty}^{\infty} \frac{S_-(\lambda)}{(\lambda-\mathrm{i})(\lambda - z)} \mathrm{d}\lambda$$

is the holomorphic part of $S_{\mathcal{K}}(\cdot)$ in $\mathbb{C}_-$. However

$$S_{\mathcal{K}}(z) - \sum_{\zeta\in\mathcal{R}} \frac{S_{-1,\zeta}}{z-\zeta}$$

is holomorphic on $\mathbb{C}_{<0}$. This shows the assertion. ■

Note that the terms

$$z \to \frac{S_{-1,\zeta}\, g}{z-\zeta}, \qquad g \in \mathcal{K},\ \zeta \in \mathcal{R}$$

are Gamow vectors because for simple poles one has $\ker L_+(\zeta) = \mathrm{im}\, \mathrm{Res}_{z=\zeta} S_{\mathcal{K}}(z)$ $= \mathrm{im} S_{-1,\zeta}$.

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