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STRAIGHT AND CIRCULAR MOTIONS OF A PARTICLE IN THE FIELD OF TWO FIXED ATTRACTORS

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Abstract. The two centre problem is studied both in the phase plane and in spacetime, assuming first a trajectory collinear, and, as a second case, a circular one through the newtonian attractors, finding a saddle equilibrium. For the latter problem a probably new differential equation is met and solved. Time is then obtained in both cases through elliptic integrals of all kinds and Jacobian functions.

1. Introduction

This problem consists of computing the motion of a test particle in the field of two fixed centres of newtonian attraction. It was first considered by Euler in 1760, who showed its integrability (see [7] for the early history of this problem). Nowadays, the system plays an important role both in macro and microphysics. In the past it represents a body moving under the attraction of two fixed stars. Passing to relativistic implications, Contopoulos *et al.*, [3], [4], have discovered, through numerical experiments, that in contrast with the classic two-centre problem, whose dynamics is completely integrable, relativistic motion of two black-holes in spacetime exhibits *chaotic* behavior. In the latter, the system is the simplest model of a diatomic molecule, since Pauli had applied it to the hydrogen molecular ion H_2^+ in his doctoral thesis, 1922, well before the birth of wave mechanics. Anyway, the assumption that their nuclei are fixed is known as "Born-Oppenheimer approximation" whose paper *Zur Quantentheorie der Molekeln*, 1927, describes the separation of electronic motion, nuclear vibrations, and molecular rotation. Such approximation is ubiquitous in quantum chemical calculations, the test particles being electrons which are assumed to "feel" the Coulomb attractive potential V of the nuclei clamped at certain space positions. Generalizing the attraction law to $V = ar^{2n}$, where r is a distance and n a real number, it has recently proved, [6], that a two fixed attractors problem is integrable when:

1. $n = 0$, trivial case of free motion
2. $n = -1$, the classic two-centre problem which is separable (Jacobi) in elliptic coordinates
3. $n = 1$, case of two uncoupled harmonic oscillators
4. $n = 2$, the case of Hamiltonian system with a non-homogeneous fourth degree potential is integrable and separable in elliptic coordinates [5].

The classic bicentral problem can be thought as a "soft" version of the "hard" planar three-body problem whenever the third mass is so small that it can not appreciably affect the remaining ones which can properly be assumed as fixed. Several approaches to it are possible, according to whether the third body trajectory is imposed or not, and which one, and/or if we there is either a 1-D motion or a 2-D one depending on the initial conditions nature. E. g. in [7] we assumed the body to move along the normal to the joint of two attractors, so that oscillating and non oscillating behaviors were detected. In the present work, we analyze the simplest conceivable three masses set, the collinear one. Going on with the 1-D task, we supplement this article with our research on the half circular path whose diameter is drawn through the attractors. After a phase portrait analysis, time equation – for both cases, thanks to their symmetries is integrated in closed form, through the Jacobi elliptic functions.

2. The Collinear Problem

For $a \in \mathbb{R}^+$, we consider, see for instance [1] page 418, the two-centre collinear motion (Figure 1), which leads to the following differential equation:

$$\begin{aligned} &\ddot{x} = \frac{B^2}{(a-x)^2} - \frac{A^2}{x^2} := f(x), \qquad A, B > 0 \\ &x(0) = x_0 \in (0, a)\,, \qquad \dot{x}(0) = v_0 \in \mathbb{R} \end{aligned} \tag{1}$$

First we assume $A \neq B$, so (1) has a Weierstraß function given by

$$\Phi(x) := 2\int_{x_0}^{x} f(\xi)\mathrm{d}\xi + v_0^2 = 2\left(\frac{B^2}{a-x} + \frac{A^2}{x}\right) - 2\left(\frac{B^2}{a-x_0} + \frac{A^2}{x_0}\right) + v_0^2$$

so that the equation for time reads

$$t = \operatorname{sign}\,(v_0)\int_{x_0}^{x} \frac{\mathrm{d}u}{\sqrt{\Phi(u)}}. \tag{2}$$

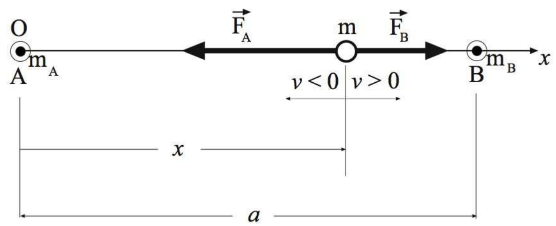

Figure 1. The two-centre collinear problem.

When $v_0 = 0$, the sign shall be chosen according to that of $f(x_0)$.

2.1. No Periodic Behavior Occurs

The motion is possible if $\Phi(x) \geq 0$. Owing to $\Phi(x) \to \infty$ as $x \to 0^+$ and $x \to a^-$, and in such a way the absolute minimum of $\Phi(x)$ for any $x \in (0, a)$ is

$$\Phi\left(a\,\frac{A}{A+B}\right) = v_0^2 - \frac{2\,(aA - (A+B)\,x_0)^2}{a\,(a-x_0)\,x_0}.$$

In order to establish if the motion ruled by (1) is periodic or not, it is necessary to examine the roots of its Weierstraß function $\Phi(x)$ for $x \in (0, a)$. A possible oscillation would take place between the $\Phi(x)$'s real roots

$$x_{1,2} = \frac{1}{2K}\left[aK + 2B^2 - 2A^2 \pm \sqrt{(aK + 2B^2 - 2A^2)^2 + 8KaA^2}\right]$$

$$K = v_0^2 - 2\left(\frac{B^2}{a - x_0} + \frac{A^2}{x_0}\right).$$

Since $\Phi(x)$ is convex for each $x \in (0, a)$, either $\Phi(x)$ has no real root in $(0, a)$ and then the motion turns out to be aperiodic, or it has two real roots x_1 and x_2, but its convexity implies that $\Phi(x) < 0$ for $x \in (x_1, x_2)$, so that no motion at all occurs between the roots. In any case, periodic motions between the fixed stars shall be excluded.

2.2. Phase Portrait Analysis

Before the exact integration of (2), let us discuss the features of the dynamic system. By setting

$$\dot{x} = g_1(x, y) = y, \qquad \dot{y} = g_2(x, y) = \frac{B^2}{(a-x)^2} - \frac{A^2}{x^2} \tag{3}$$

the equilibria turn out to be just at the $\Phi(x)$ stationary points, namely

$$(\overline{x}_1, 0) = \left(\frac{aA}{A-B},\, 0\right), \qquad (\overline{x}_2, 0) = \left(\frac{aA}{A+B}, 0\right)$$

$\overline{x}_1 \notin (0, a)$, since it contradicts also $\overline{x}_1 < a$ if $A > B$, and contradicts $\overline{x}_1 > 0$ if $A < B$. Hence, the only feasible critical point is $E \equiv (\overline{x}_2, 0)$, one of the Lagrangian points, which is well-known in Celestial Mechanics. Their construction is shown at Figure 2. The eigenvalues of the Jacobian matrix of (3) evaluated at E

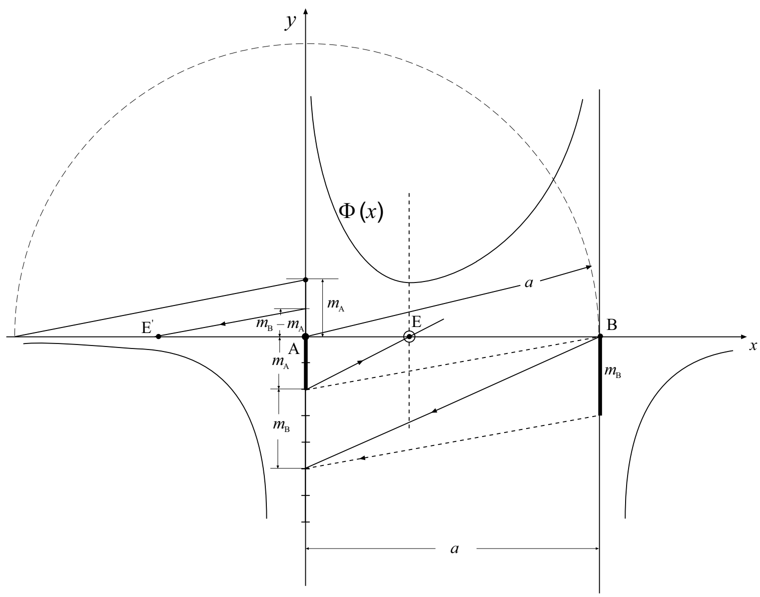

Figure 2. How to construct points E and $E' = (\overline{x}_1, 0)$ on the phase plane.

have opposite signs: E is then a saddle point for the system for any choice of the parameters a, A and B, namely regardless of distances and masses the attractors can have. The phase plane orbit can be detected by eliminating time from (3) and in such a way one gets

$$y(x) = \mp\sqrt{K + 2\left(\frac{B^2}{a-x} + \frac{A^2}{x}\right)}$$

where K was introduced before. By taking E instead of $(x_0,\, y_0)$ as the initial point, then we will get $y = y(a, A, B, K_E; x)$, i.e., the cartesian equation of the separatrix σ, which has four different branches (see Figure 3). Looking at σ, the strip $(0, a) \times \mathbb{R}$ can be divided in three regions

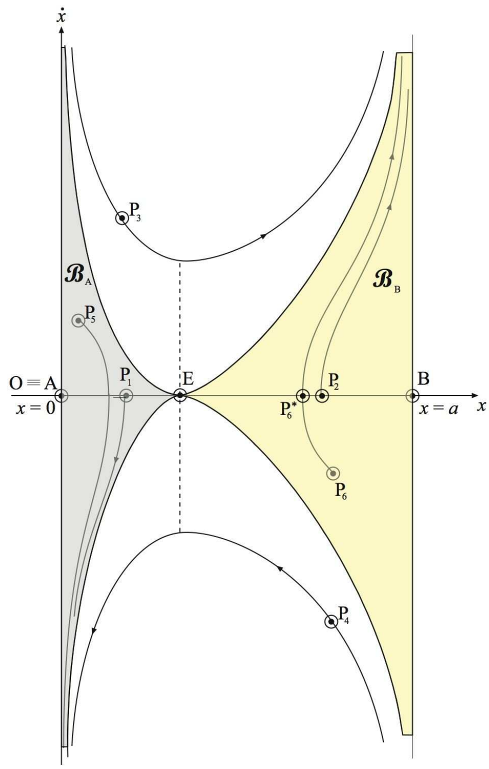

Figure 3. A phase portrait of the collinear motion with the separatrix and the basins $\mathfrak{B}_A$ and $\mathfrak{B}_B$ relevant to each centre A, B.

1. the *free zones*, namely that one above σ, with $\dot{x} > 0$ and that one below σ, with $\dot{x} < 0$. No dominant massive effect is present over there.
2. the *attraction basin* $\mathfrak{B}_A$ of the mass A, as a part of the plane bounded by the $\sigma-$branches with $0 < x < \overline{x}_2$ and the vertical axis $x = 0$, their asymptote.
3. the *attraction basin* $\mathfrak{B}_B$ of the mass B, as a part of the plane bounded by the $\sigma-$branches with $\overline{x}_2 < x < a$, and the vertical axis $x = a$, their asymptote.

Notice that each trajectory cuts the x-axis, as shown in Figure 3, providing a complete phase portrait of (3). We will restrict our considerations to the following points.

1. The point P_1 belongs to the attraction basin $\mathfrak{B}_A$ of the mass A which attracts it. P_1 starts from the rest position so that x_{P_1} decreases, all the motion takes place in the down side of $\mathfrak{B}_A$, tending asymptotically to $x = 0$.

2. The phase portrait P_2 motion is symmetric to that of P_1 and its movement takes place in the upper side of $\mathfrak{B}_B$, aiming asymptotically to $x = a$.
3. At $t = 0$ the point P_6 is already moving, the sign of its velocity tells us that it is pointing towards A. But A can be never caught because $P_6 \in \mathfrak{B}_B$, then it is attracted by B in two steps: first it is deviated to A, reducing its speed till the stop P_6. Subsequently, the speed will start increasing, but pointing towards B and $x = a$ will be the asymptotic destination which the state representative point is aiming to, with a theoretically infinite body speed.
4. The P_5 motion description is symmetric to that of P_6.
5. P_3: at $t = 0$ the point is moving, and its velocity sign tells us it is pointing towards B, but $P_3 \notin \mathfrak{B}_A$, $P_3 \notin \mathfrak{B}_B$. Then it is free to catch B directly, without any imposed deviation. Its velocity will keep its sign unchanged, but in approaching B, first it will decrease till its minimum for $x = x_E$ (the saddle), then it will increase monotonically far from it, aiming to its asymptotic destination with infinite speed.
6. The P_4 motion description is symmetric to that of P_3.

2.3. Integration

On the above phase portrait analysis, the motion main features can be sketched as follows. If the starting point is different from E, the motion will take place directly towards the predominant mass if its velocity is pointing out towards it. Alternatively it will deviate from it to a stop, after which it will go back and definitely, towards the larger mass. If the starting point is E, if $v_0 \neq 0$, the point will impact that of two attractors it meets according to the sense of its initial velocity. Finally, with a starting speed v_0 large enough, namely beyond the threshold $v_0^2 > 2\left[B^2/(a - x_0) - A^2/(x_0)\right]$, then $\Phi(x)$ is strictly positive and the initial (great) impulse succeeds to push the bead towards the fixed star following its sense of motion, *regardless to which star has a prevailing mass*. The meaningful cases are then two: the zero velocity case, including very low velocities, causing an evolution towards the predominant star and the high velocity case, when the initial impulsion pushes the bead to move towards one of the attractors, apart from their mass ratio.

2.4. Integration if $v_0 = 0$

When the body starts at zero speed, then $\Phi(x^*) < 0$, so x_1 and x_2 are two zeros to Φ in $(0, a)$, then integration may have sense only in $(0, x_1) \cup (x_2, a)$. Let us see the effect of the possible x_0 values: $\Phi(x^*) < 0 \iff x_0 \in (0, \xi_1) \cup (\xi_2, a)$ where

$$\xi_{1,2} = \frac{4aA^2 + 4aAB + a^2v_0^2 \pm av_0\sqrt{a^2v_0^2 + 8aAB}}{4(A+B)^2 + 2av_0^2}.$$

When the motion starts from the quiet $(v_0 = 0)$, then x_0 cannot be whatsoever, but shall lie outside of (ξ_1, ξ_2). On the other hand, if $x_0 \in (\xi_1, \xi_2)$, $\Phi(x)$ is nonnegative and the motion will either occur directly towards the predominant mass, if that is the direction of the velocity vector, or away from that mass, till it stops and goes back to the most attractive center. Notice that in special case $x_0 = x^*$, an unstable equilibrium solution occurs, since if $v_0 \neq 0$ the particle will point towards one of the two attractors according to the initial velocity direction. When $v_0 = 0$, the Weierstraß function zeros are

$$x_1 = x_0, \qquad x_2 = \frac{(a - x_0)aA^2}{aA^2 - (A^2 - B^2)x_0}.$$

The case of their coincidence leads to two possibilities

- $x_1 = x_2 = x_0 = \overline{x}_1$, not acceptable
- $x_1 = x_2 = x_0 = \overline{x}_2$, in which $f(x)$ vanishes and no motion occurs.

Time equation has the same sign of $f(x_0)$

$$f(x_0) = \frac{B^2}{(a-x_0)^2} - \frac{A^2}{x_0^2} \begin{cases} < 0 & \text{if} \quad 0 < x_0 < \overline{x}_2 < a \\ > 0 & \text{if} \quad 0 < \overline{x}_2 < x_0 < a \end{cases}$$

in such a way we find

$$t = \operatorname{sign}\left(f(x_0)\right) \sqrt{\frac{x_0(a-x_0)}{2\left[aA^2 - (A^2 - B^2)x_0\right]}} \int_{x_0}^{x} \sqrt{\frac{\xi(a-\xi)}{(\xi - x_0)(\xi - x_2)}}\, \mathrm{d}\xi \qquad (4)$$

Lemma 1. *The sign of* (4) *does not depend on the mass coefficients.*

Proof: If $A > B$: $x_2 < x_0 \iff 0 < x_2 < \overline{x}_2 < x_0 < a < \overline{x}_1 \iff f(x_0) > 0$. On the other hand, $x_2 > x_0 \iff 0 < x_0 < \overline{x}_2 < x_2 < a < \overline{x}_1 \iff f(x_0) < 0$.

If $A < B$, by repeating the previous procedure: $x_2 < x_0 \iff \overline{x}_1 < 0 < x_2 < \overline{x}_2 < x_0 < a \iff f(x_0) > 0$. Vice versa, $x_2 > x_0 \iff \overline{x}_1 < 0 < x_0 < \overline{x}_2 < x_2 < a \iff f(x_0) < 0$. ■

So we can state

Theorem 2. *Integral* (4) *can be evaluated*

$$t = \frac{a - x_0}{B}\sqrt{\frac{x_0(a - x_0)}{2a}}\, I_1(\phi, \alpha_1, k_1), \quad \textit{if } x_2 < x_0$$

$$t = \frac{x_0}{A}\sqrt{\frac{x_0(a - x_0)}{2a}}\, I_2(\psi, \alpha_2, k_2), \qquad \textit{if } x_2 > x_0$$

where $I_1(\phi(x), \alpha_1, k_1)$ denotes some combination between incomplete elliptic integrals of all three kinds, being ϕ a function of x, see below and $k_{1,}\alpha_1$ belong to the problem data set. A fourth and last term within I_1 is a purely goniometric function of ϕ and then of x. The same for $I_2(\psi(x), \alpha_2, k_2)$ with some adjustments

$$k_1^2 k_2^2 = 1, \qquad \frac{\alpha_1^2}{\alpha_2^2} = \frac{(a - x_0)^2 A^2}{B^2 x_0^2} = k_2^2 = \frac{1}{k_1^2} \tag{5}$$

Proof: Case $x_2 < x_0$: (4) has the unique singularity x_0. This integral can be computed (see [2], page 122 formula 256.19 with $b = 0$)

$$t = \frac{(a - x_0)x_0}{B}\sqrt{\frac{a - x_0}{2ax_0}} \int_0^{F(\phi,k_1)} \frac{\mathrm{cn}^2 u\, \mathrm{dn}^2 u\, \mathrm{d}u}{\left(1 - \alpha_1^2\, \mathrm{sn}^2 u\right)^2}$$

where

$$\phi(x) = \arcsin\sqrt{\frac{(a - x_2)(x - x_0)}{(a - x_0)(x - x_2)}}, \qquad k_1^2 = \frac{(a - x_0)^2 A^2}{B^2 x_0^2}$$

$$\alpha_1^2 = \frac{[aA^2 - (A^2 - B^2)x_0](a - x_0)}{aB^2 x_0}$$

and $F(\phi, k_1)$ is the first kind incomplete elliptic integral of amplitude ϕ and modulus k_1 while sn u, cn u and dn u denote the Jacobi elliptic functions of amplitude $\phi = \mathrm{am}(u, k_1)$. Consequently, (4) turns out to be

$$t = \frac{a - x_0}{B}\sqrt{\frac{x_0(a - x_0)}{2a}}\left[-\frac{E(\phi, k_1)}{\alpha_1^2} + \left(\frac{\alpha_1^2 + k_1^2}{\alpha_1^4}\right) F(\phi, k_1)\right.$$

$$\left. + \left(1 - \frac{k_1^2}{\alpha_1^4}\right)\Pi(\phi, \alpha_1, k_1) + \frac{\sin(2\phi)\sqrt{1 - k_1^2 \sin^2\phi}}{2(1 - \alpha_1^2 \sin^2\phi)}\right]$$

where $E(\phi, k_1)$ and $\Pi(\phi, \alpha_1, k_1)$ denote the second and the third kind incomplete elliptic integrals of amplitude ϕ, modulus k_1 and parameter α_1.

Case $x_2 > x_0$: by formula 253.20, $b = 0$ (see [2], page 109) the integral in (4) can be represented as

$$
\begin{aligned}
t =& \frac{x_0}{A}\sqrt{\frac{(a-x_0)x_0}{2a}} \int_0^{F(\psi,k_2)} \frac{\mathrm{cn}^2 u\, \mathrm{dn}^2 u\, \mathrm{d}u}{\left(1-\alpha_2^2\, \mathrm{sn}^2 u\right)^2} \\
=& \frac{x_0}{A}\sqrt{\frac{x_0(a-x_0)}{2a}} \left[-\frac{E(\psi,k_2)}{\alpha_2^2} + \left(\frac{\alpha_2^2+k_2^2}{\alpha_2^4}\right) F(\psi,k_2) \right. \\
& \left. + \left(\frac{\alpha_2^4-k_2^2}{\alpha_2^4}\right) \Pi(\psi,\alpha_2,k_2) + \frac{\sin(2\psi)\sqrt{1-k_2^2\sin^2\psi}}{2(1-\alpha_2^2\, \sin^2\psi)} \right]
\end{aligned}
$$

where

$$
\psi(x) = \arcsin\sqrt{\frac{x_2(x_0-x)}{x_0(x_2-x)}}, \qquad k_2^2 = \frac{B^2 x_0^2}{(a-x_0)^2 A^2}
$$

$$
\alpha_2^2 = \frac{[aA^2 - (A^2-B^2)x_0]x_0}{(a-x_0)aA^2}.
$$

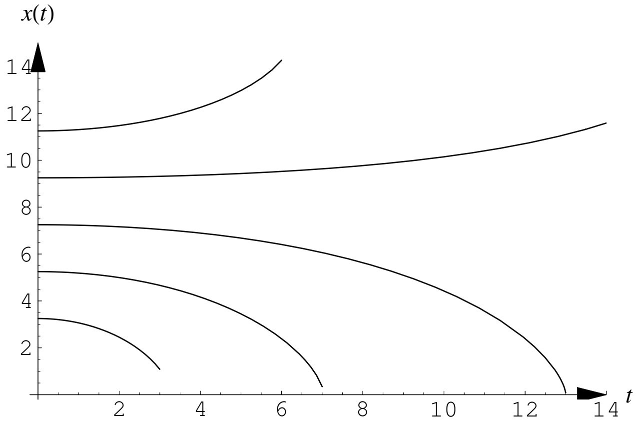

Figure 4. Some solutions $x(t)$ coming from inverting $t = I_2(\psi(x))$ and $t = I_1(\phi(x))$, $a = 15,\ A > B$, where the bead starts from a state of rest.

■

2.5. Integration if $v_0 > 0$

Passing to a high initial speed, $\Phi(x)$ is seen to remain strictly positive. The initial impulsion is strong enough to push the body towards the star according to its velocity. The stationary unstable case always leads to some of the attractors without

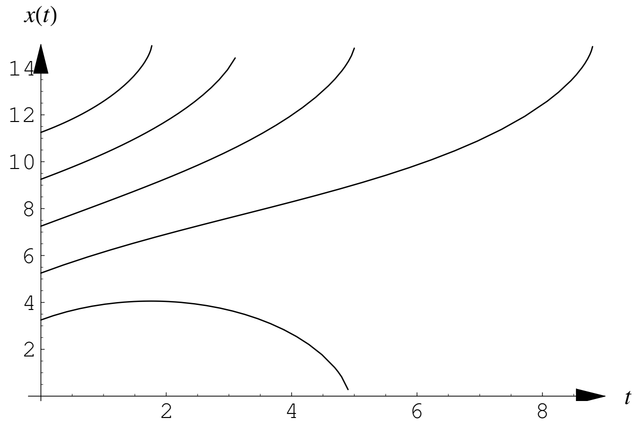

Figure 5. A sketch of some different solutions when $a = 15$, $v_0 = 1$, $A > B$.

any periodicity. The relevant formulæ are not much different. In the $\Phi(x)$ expression there will be a further numerical term $v_0 > 0$ and $x_1 \neq x_0$. Figure 5 displays the solutions when the initial velocity is equal to 1 and the predominant mass is A. The first curve of Figure 5 from the bottom describes a behavior like that of point P_5 of Figure 3, while the second one could be referred to P_3.

2.6. The Case $A = B$

Should the mass coefficients coincide, the problem gets simpler

$$\ddot{x} = \frac{a\,A^2(2x-a)}{(a-x)^2x^2} := \overline{f}(x), \qquad x(0) = x_0 \in (0, a), \qquad \dot{x}(0) = v_0 \in \mathbb{R}$$

$Q \equiv (a/2, 0)$ is the unique equilibrium (saddle) point. When $v_0 = 0$, the analysis carried out in the case $A \neq B$ can be repeated: $x_2 > x_0 \iff x_0 \in (0, a/2)$ so that

$$\begin{aligned} t =& \frac{x_0}{A}\sqrt{\frac{x_0(a-x_0)}{2a}}\left[-\frac{E(\psi,k_2)}{\alpha_2^2}+\left(\frac{\alpha_2^2+k_2^2}{\alpha_2^4}\right)F(\psi,k_2)\right. \\ &\left.+\left(\frac{\alpha_2^4-k_2^2}{\alpha_2^4}\right)\Pi(\psi,\alpha_2,k_2)+\frac{\sin(2\psi)\sqrt{1-k_2^2\sin^2\psi}}{2(1-\alpha_2^2\,\sin^2\psi)}\right]. \end{aligned}$$

When $x_2 < x_0 \iff x_0 \in (a/2, a)$

$$\begin{aligned} t =& -\frac{a-x_0}{A}\sqrt{\frac{x_0(a-x_0)}{2a}}\left[-\frac{E(\phi,k_1)}{\alpha_1^2}+\left(\frac{\alpha_1^2+k_1^2}{\alpha_1^4}\right)F(\phi,k_1)\right. \\ &\left.+\left(1-\frac{k_1^2}{\alpha_1^4}\right)\Pi(\phi,\alpha_1,k_1)+\frac{\sin(2\phi)\sqrt{1-k_1^2\sin^2\phi}}{2(1-\alpha_1^2\sin^2\phi)}\right]. \end{aligned}$$

Moduli and parameters are

$$k_1^2=\frac{(a-x_0)^2}{x_0^2},\qquad k_2^2=\frac{x_0^2}{(a-x_0)^2},\qquad \alpha_1^2=a-x_0,\qquad \alpha_2^2=\frac{x_0}{a-x_0}.$$

3. Circular Path with Two Attractors on a Chord

When the system is a planar one, for the mobile point P(x, y) of mass m it is compulsory to choose bi-polar coordinates, such as ρ_A and ρ_B, Figure 6. We deal with the special case of the circular trajectory. The attractors A and B are fixed at the ends of the chord whose length is a, being C the center and R the radius of the circle, with the plane through P, A, B normal to the (m, P) weight direction.

As before, no resistance will withdraw the P-motion which is entirely due to the forces $\vec{F}_A$ and $\vec{F}_B$ exerted by masses M_A and M_B. We assume as $x-$reference axis the AB direction and put the origin O in A. By $\psi(t)$ and $\theta(t)$ we denote now the variable angles of PA and PB with Ox, while α_0 is the constant inclination of the chord with respect to AC. The mobile point has to be thought on a *horizontal* smooth plane so that its weight is balanced by plane's normal reaction, a plane where a round track $\mathfrak{C}$ is cut so that P moves along it under the attractions. Minding Figure 6, if ρ_A and ρ_B are the P-distances from the centres A and B, we have

$$\rho_{_A}^2=4R^2\cos^2(\psi+\alpha_0)\,,\qquad \rho_{_B}^2=(a-x)^2+y^2$$

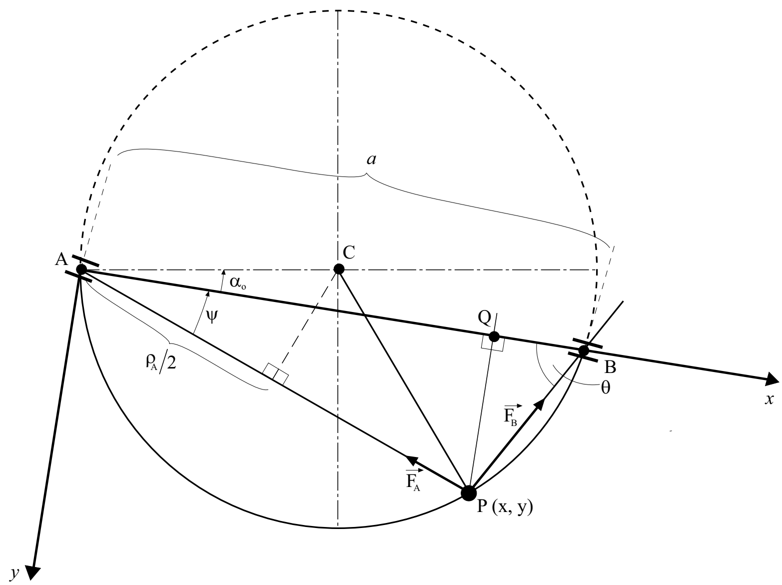

Figure 6. Circular path of a bead under two attractors on a chord.

so that the scalar $x-$equation is

$$\ddot{x}(t) = -\frac{GM_A}{4R^2}\frac{\cos\psi}{\cos^2(\psi+\alpha_0)} + \frac{GM_B}{\rho_B^2}\cos\theta.$$

Then knowing that $\cos\theta = a - x/\rho_B$ with

$$x = 2R\cos(\psi+\alpha_0)\cos\psi, \qquad y = 2R\cos(\psi+\alpha_0)\sin\psi$$

we will eliminate all the remaining state variables but ψ, obtaining the autonomous ψ-equation

$$\begin{aligned} 2R\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left[\cos(\psi+\alpha_0)\cos\psi\right] &= -\frac{GM_A}{4R^2}\frac{\cos\psi}{\cos^2(\psi+\alpha_0)} \\ &\quad + GM_B\frac{a-2R\cos(\psi+\alpha_0)\cos\psi}{\left[a^2+4R\cos^2(\psi+\alpha_0)-4aR\cos(\psi+\alpha_0)\cos\psi\right]^{3/2}} \end{aligned}$$

which we do not deem having any hope of being solved in closed form.

3.1. Attractors at the Ends of a Diameter

Let the chord is AB rotated counterclockwise by α_0. The attractors will be at the end contrivances of a diameter $|AB| = a$, as in Figure 7. Doing in the $\psi-$equation

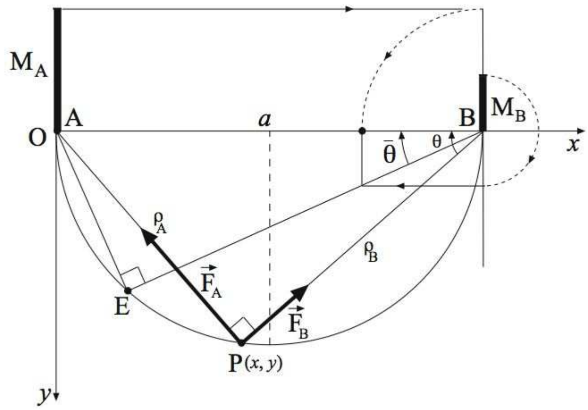

Figure 7. Sketch of the half circular path bicentral motion, and its equilibrium point.

$\alpha_0 \mapsto 0^+$ and $\frac{a}{2} \mapsto R$, it becomes

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(\cos^2\psi\right) = -\frac{A}{\cos\psi} + \frac{B}{\sin\psi}$$

where $A = M_A G/a^3$, $B = M_B G/a^3$ are positive invariable quantities and $\psi =$ PÂB is our unknown function of time. If we look at Figure 7, where AB now is a diameter, we get: $\psi + \theta = \pi/2$. Then, passing to θ, the following differential equation to $\theta(t)$ is obtained

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(\sin^2\theta\right) = \frac{B}{\cos\theta} - \frac{A}{\sin\theta}, \qquad \theta(0) = \theta_0, \qquad \dot{\theta}(0) = \dot{\theta}_0.$$

Such a equation does not seem to have been met before. Anyway, putting $\sin^2\theta(t) = z(t)$ we get

$$\begin{aligned} &\dot{z} = v, \qquad \dot{v} = \frac{B}{\sqrt{1-z}} - \frac{A}{\sqrt{z}} \\ &z(0) = z_0 = \sin\theta_0, \qquad v(0) = \dot{z}_0 = 2\dot{\theta}_0\cos\theta_0\sin\theta_0. \end{aligned} \tag{6}$$

The equilibrium point detection in Figure 7, say E, leads to

$$E \equiv (\bar{v}, \bar{z}) \equiv \left(0, \frac{A^2}{\sqrt{B^2 + A^2}}\right)$$

and then to the construction of $\bar{\theta}$ such that

$$\sin\bar{\theta} = \frac{M_A}{\sqrt{M_B^2 + M_A^2}}.$$

The orbit equation $v = v(z)$ can be easy integrated, as the reader can check. The eigenvalues of the (6) Jacobian matrix at E are: $\lambda_{1,2} = \mp\sqrt[4]{B^2 + A^2}$, so that E has a saddle nature for every choice of B and A. The $z(t)$ behavior depends on the roots of the Weierstraß function $\widetilde{\Phi}(z)$

$$\widetilde{\Phi}(z) := 4\left[A\sqrt{z_0} + B\sqrt{1-z_0}\right] - 4\left[A\sqrt{z} + B\sqrt{1-z}\right] + v_0^2$$

so that, assuming $v_0 = \dot{z}_0 > 0$, the equation for time reads

$$\frac{1}{2}t = \int\limits_{z_0}^{z} \frac{\mathrm{d}u}{\sqrt{-A\sqrt{u} - B\sqrt{1-u} + \frac{1}{2}\left[\frac{\dot{z}_0^2}{2} + 2\left(A\sqrt{z_0} + B\sqrt{1-z_0}\right)\right]}}.$$

The expression within the square brackets is the unitary mass "energy excess", namely the difference between the initial kinetic energy and the potential binding start-up ones due to the gravitating masses M_B and M_A. If such an excess, say C_0, is positive, all the square root content is certainly positive so that all z - and then all θ - values will be allowed, and no periodic motion will occur, regardless to the initial conditions. On the contrary, if $C_0 < 0$, the initial kinetic energy is not enough, and the motion will have more restricted features. Namely, not all z-values will be free, so that the quadratic equation giving $\widetilde{\Phi}(z)$ zeros will provide real and complex roots, and then periodic and/or aperiodic behaviors, according to the initial position and speed. Coming back to variable θ, time equation will assume its final form:

$$t = -2 \int\limits_{\arcsin\sqrt{z}}^{\arcsin\sqrt{z_0}} \frac{\sin(2\theta)\,\mathrm{d}\theta}{\sqrt{A\cos\theta + B\sin\theta + C_0}} \tag{7}$$

where $C_0 := \frac{1}{2}[\dot{z}_0^2/2 - 2(A\sqrt{z_0} + B\sqrt{1-z_0})]$. Two possibilities can occur

1. $0 < |C_0| < \sqrt{B^2 + A^2}$. The integral (7) does not appear in [2], but it can be calculated through the contrivance:

$$\int \frac{\sin(2\theta)\,\mathrm{d}\theta}{\sqrt{B\cos\theta + A\sin\theta + C_0}} = -\frac{1}{BA}\left[\int \frac{(B\cos\theta - A\sin\theta)^2\,\mathrm{d}\theta}{\sqrt{A\cos\theta + B\sin\theta + C_0}}\right.$$
$$\left. - \int \frac{B^2\cos^2\theta\,\mathrm{d}\theta}{\sqrt{A\cos\theta + B\sin\theta + C_0}} - \int \frac{B^2\sin^2\theta\,\mathrm{d}\theta}{\sqrt{A\cos\theta + B\sin\theta + C_0}}\right].$$

The first of the right hand side integrals is given by formula 293.04 page 179 of [2]. The second and third can be solved by means of formulæ 293.06 and 293.07, page 180, which allow to integrate the product of

$$\frac{1}{\sqrt{A\cos\theta + B\sin\theta + C_0}}$$

to a rational function of $\cos\theta$ and $\sin\theta$ respectively. In such a way, we obtain

$$t = \frac{4}{(A^2+B^2)^2}\sqrt{\frac{2}{\sqrt{A^2+B^2}}}\int_0^{u_1}[BAH^2(C_0,B,A)\,\mathrm{cn}^4 u + BAC_0^2$$

$$-2\sqrt{B^2+A^2}H(C_0,B,A)BA\mathrm{sn}^2 u\,\mathrm{dn}^2 u - 2C_0H(C_0,B,A)\mathrm{cn}^2 u$$

$$-(\sqrt{2\sqrt{B^2+A^2}H(C_0,B,A)}(B^2-A^2)\mathrm{dn}u\,\mathrm{sn}u)(C_0+H(C_0,B,A)\,\mathrm{cn}^2 u)]\,\mathrm{d}u$$

where $H(C_0,B,A) = C_0 + \sqrt{B^2+A^2}$.
2. $C_0 > \sqrt{B^2+A^2} > 0$, see [2], page 180

$$t = \frac{8}{\sqrt{H(C_0,B,A)}(B^2+A^2)^2}\int_0^{u_1}\big[BAH^2(C_0,B,A)\mathrm{dn}^4 u + BAC_0^2$$
$$-4BA(B^2+A^2)\,\mathrm{sn}^2 u\,\mathrm{cn}^2 u + C_0\sqrt{B^2+A^2}(B^2+A^2)H(C_0,B,A)\,\mathrm{sn}u\,\mathrm{cn}u$$
$$-2BAC_0H(C_0,B,A)\,\mathrm{dn}^2 u + (B^4-A^4)H(C_0,B,A)\,\mathrm{sn}u\,\mathrm{cn}u\,\mathrm{dn}^2 u\big]\,\mathrm{d}u.$$

The above expressions of time require in both cases

i) $C_0 < \sqrt{B^2+A^2}$ ii) $C_0 > \sqrt{B^2+A^2}$

the evaluation of ten integrals of Jacobian elliptic functions between 0 and u_1, where

$$u_1 = u_1(z) = \arcsin\sqrt{\frac{\sqrt{B^2+A^2} + A\sqrt{z} - C_0\cos(\arcsin\sqrt{z})}{C_0+\sqrt{B^2+A^2}}}.$$

For convenience of the reader we detail the location of the required integrals available in [2].

i$_1$) $\int \mathrm{cn}^4 u\,\mathrm{d}u$, formula 312.04, page 193

i$_2$) $\int \mathrm{sn}^2 u\,\mathrm{dn}^2\,\mathrm{d}u$, formula 361.02, page 212 i$_3$) $\displaystyle\int \mathrm{cn}^2 u \, \mathrm{d}u$, formula 312.02, page 193

i$_4$) $\displaystyle\int \mathrm{dn} u \, \mathrm{sn} u \, \mathrm{d}u$, formula 360.02, page 211

i$_5$) $\displaystyle\int \mathrm{dn} u \, \mathrm{sn} u \, \mathrm{cn}^2 u \, \mathrm{d}u$, formula 360.13 (with $m = 2$), page 211

ii$_1$) $\displaystyle\int \mathrm{dn}^4 u \, \mathrm{d}u$, formula 314.04, page 194

ii$_2$) $\displaystyle\int \mathrm{sn}^2 u \, \mathrm{cn}^2 \, \mathrm{d}u$, formula 361.01, page 212

ii$_3$) $\displaystyle\int \mathrm{dn}^2 u \, \mathrm{d}u$, formula 314.02, page 194

ii$_4$) $\displaystyle\int \mathrm{cn} u \, \mathrm{sn} u \, \mathrm{d}u$, formula 360.03, page 211

ii$_5$) $\displaystyle\int \mathrm{sn} u \, \mathrm{cn} u \, \mathrm{dn}^2 u \, \mathrm{d}u$, formula 360.11 (with $n = 2$), page 211.

In such a way the problem has been solved completely.

4. Conclusions

The motion of a test particle in the field of two fixed newtonian attractors exhibits a very complex and rich dynamics. The collinear system is found to have an equilibrium point whose nature is a saddle and whose orbit can be easily integrated. The phase analysis highlights the existence of special attraction basins. Despite its apparent simplicity, the relevant time equation is much involved and our closed form integration provides time as addition (I_1or I_2) of elliptic integrals of I, II and III kind. Assuming as a trajectory an arc of a circle $\mathfrak{C}$, we obtain a second order autonomous $\psi-$equation, of intractable nature. But if not a chord anymore, but a diameter connects the fixed centres, then a different nonlinear $\theta-$ODE probably new, is met, tractable, even if quite heavy. In fact, we are led to the integral

$$\int \frac{\sin(2\theta) \, \mathrm{d}\theta}{\sqrt{A \cos\theta + B \sin\theta + C_0}}$$

which can not be found in the classic literature on elliptic functions. A way has been found for reducing it (for each of the two possibilities) to the sum of five known integrals of Jacobian elliptic functions. The motion, namely time equation $t = f(\theta)$ is then computable through a long expression (we do not provide) involving elliptic integrals of I and II kind, and the Jacobian functions cn, sn, dn whose arguments hold u_1 and then θ, being $u_1 = u_1(z)$ with $z = \sin^2\theta$. Let us observe finally that integrability of the circular system is founded upon the special symmetry of the $\mathfrak{C}$-diameter passing through the attractors so that the $\mathfrak{C}$-centre falls exactly into the middle of the attractors. The special symmetry made the problem solvable.

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