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A REMARK ON COMPACT MINIMAL SURFACES IN S^5 WITH NON-NEGATIVE GAUSSIAN CURVATURE

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Abstract. The purpose of this paper is to show that a generalized Clifford immersion with non-negative Gaussian curvature has constant contact angle, thus extending previous results.

1. Introduction

In [4] we introduced the notion of contact angle, which can be considered as a new geometric invariant useful for investigating the geometry of immersed surfaces in S^3. Geometrically, the contact angle β is the complementary angle between the contact distribution and the tangent space of the surface. Also in [4], we derived formulae for the Gaussian curvature and the Laplacian of an immersed minimal surface in S^3, and we gave a characterization of the Clifford Torus as the only minimal surface in S^3 with constant contact angle.

Recently, in [5], we constructed a family of minimal tori in S^5 with constant contact and holomorphic angles. These tori are parametrized by the following circle equation

$$a^2 + \left(b - \frac{\cos\beta}{1+\sin^2\beta}\right)^2 = 2\frac{\sin^4\beta}{(1+\sin^2\beta)^2} \tag{1}$$

where a and b are given in Section 3 (equation (9)). In particular, when $a = 0$, we recover the examples found by Kenmotsu [3]. These examples are defined for $0 < \beta < \frac{\pi}{2}$. Also, when $b = 0$, we find a new family of minimal tori in S^5, and these tori are defined for $\frac{\pi}{4} < \beta < \frac{\pi}{2}$. Also, in [5], when $\beta = \frac{\pi}{2}$, we give an alternative proof of this classification of a Theorem proved by Blair in [1], and Yamaguchi, Kon and Miyahara in [6] for Legendrian minimal surfaces in S^5 with constant Gaussian curvature.

The immersions that we investigate in this paper are those that satisfy the following conditions:

1) S is compact
2) $\imath$ is a minimal immersion
3) α is constant on S, and
4) The principal curvatures of the immersion in the direction of e_3 are constant and correspond to the directions e_1 and e_2.

We will call **generalized Clifford immersion** as the immersions $\imath$ of S into S^5 that verifies the conditions from 1) until 4).

As a consequence of the Gauss equation and using the above notation, supposing that S has non-negative Gaussian curvature, we have proved the main result:

Theorem 1. *Suppose that S is a generalized Clifford immersion with non-negative Gaussian curvature* $(K \geq 0)$*, then the contact angle* β *must be constant.*

2. Contact Angle For Immersed Surfaces In S^5

Consider in $\mathbb{C}^3$ the following objects:

- The Hermitian product: $(z, w) = \sum_{j=0}^{2} z^j \bar{w}^j$.
- The Inner product: $\langle z, w \rangle = \mathrm{Re}(z, w)$.
- The Unit sphere: $S^5 = \{z \in \mathbb{C}^3 \,;\, (z, z) = 1\}$.
- The *Reeb* vector field in S^5, given by: $\xi(z) = \mathrm{i}z$.
- The Contact distribution in S^5, which is orthogonal to ξ

$$\Delta_z = \{v \in T_z S^5 \,;\, \langle \xi, v \rangle = 0\}.$$

Note that Δ is invariant by the complex structure of $\mathbb{C}^3$.

Let now S be an orientable immersed surface in S^5.

Definition 2. *The* contact angle β *is the complementary angle between the contact distribution* Δ *and the tangent space* TS *of the surface.*

Let (e_1, e_2) be a local orthonormal frame of TS, where $e_1 \in TS \cap \Delta$. Then $\cos\beta = \langle \xi, e_2 \rangle$. Finally, let v be the unit vector in the direction of the orthogonal projection of e_2 on Δ, defined by the following relation

$$e_2 = \sin\beta v + \cos\beta\xi \tag{2}$$

Definition 3. *The holomorphic angle α is the angle given by* $\cos\alpha = \langle \mathrm{i}e_1, v\rangle$. *The holomorphic angle α is the analogue of the Kähler angle introduced by Chern and Wolfson in* [2].

3. Equations for Gaussian Curvature and Laplacian of a Minimal Surface in S^5 with Constant Holomorphic Angle α

In this section, we derive the equations for the Gaussian curvature and for the Laplacian of a minimal surface in S^5 in terms of the contact angle and the holomorphic angle.

Let S be a minimal immersed Riemann surface in S^5 with constant holomorphic angle. Consider the normal vector fields

$$\begin{aligned} e_3 &= \mathrm{i}\csc\alpha e_1 - \cot\alpha v \\ e_4 &= \cot\alpha e_1 + \mathrm{i}\csc\alpha v \\ e_5 &= \csc\beta\xi - \cot\beta e_2 \end{aligned} \tag{3}$$

where $\beta \neq 0, \pi$ and $\alpha \neq 0, \pi$. Let $(e_j)_{1\leq\, j\leq 5}$ be an *adapted frame*.

Using (2) and (3), we get

$$v = \sin\beta e_2 - \cos\beta e_5, \quad \mathrm{i}v = \sin\alpha e_4 - \cos\alpha e_1, \quad \xi = \cos\beta e_2 + \sin\beta e_5. \tag{4}$$

It follows from (3) and (4) that

$$\begin{aligned} \mathrm{i}e_1 &= \cos\alpha\sin\beta e_2 + \sin\alpha e_3 - \cos\alpha\cos\beta e_5 \\ \mathrm{i}e_2 &= -\cos\beta z - \cos\alpha\sin\beta e_1 + \sin\alpha\sin\beta e_4. \end{aligned} \tag{5}$$

Let (θ^j) be the coframe of (e_j). Connection forms (θ^j_k) are given by

$$De_j = \theta^k_j e_k$$

and the second fundamental form with respect to this frame is given by

$$II^j \;=\; \theta^j_1\theta^1 + \theta^j_2\theta^2, \qquad j = 3, ..., 5.$$

Using (5) and differentiating v and ξ on the surface S, we get

$$\begin{aligned} D\xi \;=\;& -\cos\alpha\sin\beta\theta^2 e_1 + \cos\alpha\sin\beta\theta^1 e_2 + \sin\alpha\theta^1 e_3 + \sin\alpha\sin\beta\theta^2 e_4 \\ & -\cos\alpha\cos\beta\theta^1 e_5 \\ Dv \;=\;& (\sin\beta\theta^1_2 - \cos\beta\theta^1_5)e_1 + \cos\beta(\mathrm{d}\beta - \theta^2_5)e_2 + (\sin\beta\theta^3_2 - \cos\beta\theta^3_5)e_3 \\ & +(\sin\beta\theta^2_4 - \cos\beta\theta^4_5)e_4 + \sin\beta(\mathrm{d}\beta + \theta^5_2)e_5. \end{aligned} \tag{6}$$

Differentiating e_3, e_4 and e_5, we have

$$\begin{aligned}
\theta_3^1 &= -\theta_1^3\\
\theta_3^2 &= \sin\beta\theta_4^1 - \cos\beta\sin\alpha\theta^1\\
\theta_3^4 &= \csc\beta\theta_1^2 - \cot\alpha(\theta_1^3 + \csc\beta\theta_2^4)\\
\theta_3^5 &= \cot\beta\theta_2^3 - \csc\beta\sin\alpha\theta^1\\
\theta_4^1 &= -\csc\beta\theta_2^3 + \sin\alpha\cot\beta\theta^1\\
\theta_4^2 &= -\theta_2^4\\
\theta_4^3 &= \csc\beta\theta_2^1 + \cot\alpha(\theta_1^3 + \csc\beta\theta_2^4)\\
\theta_4^5 &= \cot\beta\theta_2^4 - \sin\alpha\theta^2\\
\theta_5^1 &= -\cos\alpha\theta^2 - \cot\beta\theta_2^1\\
\theta_5^2 &= \mathrm{d}\beta + \cos\alpha\theta^1\\
\theta_5^3 &= -\cot\beta\theta_2^3 + \csc\beta\sin\alpha\theta^1\\
\theta_5^4 &= -\cot\beta\theta_2^4 + \sin\alpha\theta^2.
\end{aligned} \tag{7}$$

The conditions of minimality and of symmetry are equivalent to the following equations

$$\theta_1^\lambda \wedge \theta^1 + \theta_2^\lambda \wedge \theta^2 = \theta_1^\lambda \wedge \theta^2 - \theta_2^\lambda \wedge \theta^1 = 0. \tag{8}$$

On the surface S, we consider

$$\theta_1^3 \quad = \quad a\theta^1 + b\theta^2.$$

It follows from (8) that

$$\begin{aligned}
\theta_1^3 =& a\theta^1 + b\theta^2\\
\theta_2^3 =& b\theta^1 - a\theta^2\\
\theta_1^4 =& (b\csc\beta - \sin\alpha\cot\beta)\theta^1 - a\csc\beta\theta^2\\
\theta_2^4 =& -a\csc\beta\theta^1 - (b\csc\beta - \sin\alpha\cot\beta)\theta^2\\
\theta_1^5 =& \mathrm{d}\beta\circ J - \cos\alpha\theta^2\\
\theta_2^5 =& -\mathrm{d}\beta - \cos\alpha\theta^1\\
\theta_3^4 =& -\sec\beta\mathrm{d}\beta\circ J + a\cot\alpha\cot^2\beta\theta^1\\
&+ (b\cot\alpha\cot^2\beta - \cos\alpha\cot\beta\csc\beta + 2\sec\beta\cos\alpha)\theta^2\\
\theta_3^5 =& (b\cot\beta - \csc\beta\sin\alpha)\theta^1 - a\cot\beta\theta^2\\
\theta_4^5 =& -a\cot\beta\csc\beta\theta^1 + (\sin\alpha(\cot^2\beta - 1) - b\csc\beta\cot\beta)\theta^2.
\end{aligned} \tag{9}$$

We suppose that the second fundamental forms in the direction e_3 are constant. The purpose of this paper is to study the case $b = 0$. Therefore, we have

$$\theta_1^3 \quad = \quad a\theta^1.$$

It follows from (9) that

$$\begin{aligned} \theta_1^3 &= a\theta^1 \\ \theta_2^3 &= -a\theta^2 \\ \theta_1^4 &= -\sin\alpha\cot\beta\theta^1 - a\csc\beta\theta^2 \\ \theta_2^4 &= -a\csc\beta\theta^1 + \sin\alpha\cot\beta\theta^2 \\ \theta_1^5 &= \mathrm{d}\beta\circ J - \cos\alpha\theta^2 \\ \theta_2^5 &= -\mathrm{d}\beta - \cos\alpha\theta^1 \end{aligned} \tag{10}$$

where J is the complex structure of S is given by $Je_1 = e_2$ and $Je_2 = -e_1$. Moreover, normal connection forms are given by

$$\begin{aligned} \theta_3^4 =& -\sec\beta\mathrm{d}\beta\circ J + a\cot\alpha\cot^2\beta\theta^1 \\ &+ (2\sec\beta\cos\alpha - \cos\alpha\cot\beta\csc\beta)\theta^2 \\ \theta_3^5 =& -\csc\beta\sin\alpha\theta^1 - a\cot\beta\theta^2 \\ \theta_4^5 =& -a\cot\beta\csc\beta\theta^1 + \sin\alpha(\cot^2\beta - 1)\theta^2 \end{aligned} \tag{11}$$

while the Gauss equation is equivalent to the equation

$$\mathrm{d}\theta_2^1 + \theta_k^1 \wedge \theta_2^k \quad = \quad \theta^1 \wedge \theta^2. \tag{12}$$

Therefore, using equations (10) and (12), we have

$$K \quad = \quad 1 - (1 + csc^2\beta)a^2 - |\nabla\beta + \cos\alpha e_1|^2 - \sin^2\alpha\cot^2\beta$$

where β_1 and β_2 are defined by $\beta_1 = \mathrm{d}\beta(e_1)$ and $\beta_2 = \mathrm{d}\beta(e_2)$.
Using (7) and the complex structure of S, we get

$$\theta_2^1 \quad = \quad \tan\beta(\mathrm{d}\beta\circ J - 2\cos\alpha\theta^2). \tag{13}$$

Differentiating (13), we conclude that

$$\begin{aligned} \mathrm{d}\theta_2^1 \quad = \quad & -((1 + \tan^2\beta)|\nabla\beta|^2 + \tan\beta\Delta\beta + 2\cos\alpha(1 + 2\tan^2\beta)\beta_1 \\ & +4\tan^2\beta\cos^2\alpha)\theta^1 \wedge \theta^2 \end{aligned}$$

where $\Delta = \mathrm{tr}\nabla^2$ is the Laplacian of S. The Gaussian curvature is therefore given by

$$\begin{aligned} K =& -(1+\tan^2\beta)|\nabla\beta|^2 - \tan\beta\Delta\beta - 2\cos\alpha(1+2\tan^2\beta)\beta_1 \\ & - 4\tan^2\beta\cos^2\alpha. \end{aligned} \tag{14}$$

From (13) and (14), we obtain the following formula for the Laplacian of S

$$\begin{aligned} \tan\beta\Delta\beta =& (1+\csc^2\beta)a^2 + \sin^2\alpha(1-\tan^2\beta) \\ & - \tan^2\beta(|\nabla\beta + 2\cos\alpha e_1|^2 - |\sin\alpha(1-\cot^2\beta)|^2). \end{aligned} \tag{15}$$

4. Proof of Theorem 1

In this section, in order to compute Gauss-Codazzi-Ricci equations, we consider that the holomorphic angle α is constant, and suppose that the principal curvature in the direction of e_3 is constant, that is, a is constant. The following Codazzi-Ricci equations

$$\begin{aligned} d\theta_1^3 + \theta_2^3\wedge\theta_1^2 + \theta_4^3\wedge\theta_1^4 + \theta_5^3\wedge\theta_1^5 &= 0 \\ d\theta_2^4 + \theta_1^4\wedge\theta_2^1 + \theta_3^4\wedge\theta_2^3 + \theta_5^4\wedge\theta_2^5 &= 0 \\ d\theta_4^5 + \theta_1^5\wedge\theta_4^1 + \theta_2^5\wedge\theta_4^2 + \theta_3^5\wedge\theta_4^3 &= 0 \end{aligned}$$

simplify to

$$\begin{aligned} \beta_2 =& \frac{(3-\cos^2\beta)a}{\sin\beta\cos\beta}(-2\sin\alpha\csc\beta\beta_1 - \sin\alpha\cos\alpha\csc\beta(3-\cot^2\beta) \\ & + a^2\cot\alpha\csc\beta\cot^2\beta). \end{aligned} \tag{16}$$

Moreover, the system of Codazzi-Ricci equations

$$\begin{aligned} d\theta_2^3 + \theta_1^3\wedge\theta_2^1 + \theta_4^3\wedge\theta_2^4 + \theta_5^3\wedge\theta_2^5 &= 0 \\ d\theta_1^4 + \theta_2^4\wedge\theta_1^2 + \theta_3^4\wedge\theta_1^3 + \theta_5^4\wedge\theta_1^5 &= 0 \\ d\theta_3^5 + \theta_1^5\wedge\theta_3^1 + \theta_2^5\wedge\theta_3^2 + \theta_4^5\wedge\theta_3^4 &= 0 \\ d\theta_2^5 + \theta_1^5\wedge\theta_2^1 + \theta_3^5\wedge\theta_2^3 + \theta_4^5\wedge\theta_2^4 &= 0 \end{aligned}$$

reduces to

$$\beta_1 = -2\cos\alpha. \tag{17}$$

Also using (17) in equation (14), we have

$$K = -(1+\tan^2\beta)\beta_2^2 - \tan\beta\Delta\beta. \tag{18}$$

Therefore

$$\tan\beta\Delta\beta = -K - (1 + \tan^2\beta)\beta_2^2. \tag{19}$$

Now using the condition that $K \geq 0$ and the Hopf's Lemma (for $0 < \beta < \pi/2$), we get that the contact angle β is constant, which prove Theorem 1. □

Theorem 1 of [5] states that any generalized Clifford immersion of constant contact and holomorphic angles is a flat torus. Combining this with Theorem 1 of this paper, we have the following

Corollary 4. *Any generalized Clifford immersion of a compact Riemann surface with non-negative Gaussian curvature is a flat torus.*

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