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SOLUTIONS FOR THE CONSTANT QUANTUM YANG-BAXTER EQUATION FROM LIE (SUPER)ALGEBRAS

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Abstract. We present a systematic procedure to obtain singular solutions of the constant quantum Yang-Baxter equation in arbitrary dimension. This approach, inspired by the Lie (super)algebra structure, is explicitly applied to the particular case of (graded) contractions of the orthogonal real algebra $\mathfrak{so}(N+1)$. In this way we show that "classical" contraction parameters which appear in the commutation relations of the contracted Lie algebras, become quantum deformation parameters, arising as entries of the resulting quantum R-matrices.

1. Introduction

Quantum R-matrices are solutions of the constant quantum Yang-Baxter equation (cQYBE)

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12} \tag{1}$$

where $R = \sum_i a_i \otimes b_i$ is a linear operator acting on a D^2-dimensional space and

$$R_{12} \equiv \sum_i a_i \otimes b_i \otimes 1,\ R_{13} \equiv \sum_i a_i \otimes 1 \otimes b_i,\ R_{23} \equiv \sum_i 1 \otimes a_i \otimes b_i. \tag{2}$$

The cQYBE can be considered as a limiting case of the QYBE with spectral parameters, which constitutes the algebraic keystone for the integrability properties of $(1+1)$ solvable models [1, 2]. Constant quantum R-matrices have been shown to be relevant in quantum group theory and non-commutative geometry [3], since constant quantum R-matrices can be used to get the defining relations for non-commutative spaces such as the ones obtained under different generalizations/deformations of the special relativity theory (see [4] and references therein). Several classifications for the solutions of the cQYBE, mainly concerning low dimensions, can be found in [5–9]. However, few constructive procedures for solutions in arbitrary dimensions D are available. The aim of this contribution is to present a systematic construction of multiparametric solutions of the cQYBE by means of the structure constants of any Lie (super)algebra. In Section 2 the generic R-matrix is constructed and in Section 3 this approach is used to obtain explicitly the solutions generated by a family of contractions of the Lie algebra $\mathfrak{so}(N+1)$. In this way, by restoring to the quantum group interpretation of quantum R-matrices, we show that the contraction parameters (which in this case are endowed with a precise geometrical and physical meaning) can be interpreted as quantum deformation parameters in some non-commutative framework.

2. Solutions for the Constant Quantum Yang-Baxter Equation

The main result of this contribution can be stated as follows.

Theorem 1. *Let $X_1, \ldots, X_d, X_D$ $(D = d+1)$ span a vector space endowed with a bilinear law*

$$X_i * X_j = C_{ij}^k X_k,\ i, j, k = 1, \ldots, D \ \textit{ such that } \ C_{ij}^k = 0 \ \textit{ if } \ i, j \textit{ or } k = D \quad (3)$$

while the remaining C_{ij}^k are completely arbitrary coefficients. Consider now the D^2-dimensional square R-matrix with entries given by

$$R_{(i,j),(k,\ell)} = C_{ij}^k \delta_\ell^D + C_{ij}^\ell \delta_k^D, \quad (i,j),(k,\ell) \in \{(1,1), \ldots, (D,D)\}. \quad (4)$$

Then R provides a D-state solution of the cQYBE.

We stress that each non-zero coefficient C_{ij}^k (3) is promoted into a quantum deformation one through the R-matrix (4). This, in turn, means that our approach affords the construction of multiparametric R-matrices by simply considering different coefficients and obviously, one can take all the C_{ij}^k equal to a single coefficient. Furthermore, the C_{ij}^k can be taken as (real or complex) constants as well as functions depending on some other parameters, without any restriction.

The composition law (3) is, in fact, a Lie (super)bracket inspired law, since the latter can be recovered as a particular case, once the C_{ij}^k are identified with some structure constants. Thus the mechanism (4) provides a way of making a connection between a Lie (super)algebra of dimension d and a $(d+1)$-state solution of the cQYBE. This is achieved by adding a central charge X_D (an explicit application is performed in Section 3). Note however that the connection is only in one way, by starting from a Lie (super) algebra one can obtain a corresponding solution for the cQYBE, but the reciprocal assertion is not compulsory true.

By considering definition (4), one has several lines (the lines (D,j) and (i,D), $i,j = 1, \ldots, D$, that is $2D-1$ lines) and columns (the columns (k,ℓ) for which both k and ℓ are different from D, that is $(D-1)^2+1$ columns) which are identically zero. Hence $\det R = 0$, whatever the rest of the entries of the R-matrix with free coefficients C_{ij}^k are, so that we are always dealing with singular (non-invertible) solutions of the cQYBE.

2.1. Proof of Theorem 1

Let $\mathcal{E}^{(i,j),(k,\ell)}$ be the D^2-dimensional square matrix with only zero entries except for the $((i,j),(k,\ell))$ entry, which is equal to one. The set $(\mathcal{E}^{(i,j),(k,\ell)}$, $(i,j),(k,\ell) \in \{(1,1)\dots,(D,D)\})$ forms a basis of the square matrices $\mathcal{M}_{D^2}(\mathbb{K})$ over the field $\mathbb{K}$. Note that

$$\mathcal{E}^{(i,j),(k,\ell)} = \varepsilon^{i,k} \otimes \varepsilon^{j,\ell} \tag{5}$$

where $\varepsilon^{i,k}$ is the D-dimensional square matrix with only zero entries except for the (i,k) entry, which is equal to one. Thus $\varepsilon^{i,k} \otimes \varepsilon^{j,\ell}$, (with $i,j,k,\ell = 1,\dots,D$) is also a basis of $\mathcal{M}_{D^2}(\mathbb{K})$. Hence one can write the R-matrix with entries (4) as

$$R = R_{(i,j),(k,\ell)}\mathcal{E}^{(i,j),(k,\ell)} = R_{(i,j),(k,\ell)}\varepsilon^{i,k} \otimes \varepsilon^{j,\ell} \tag{6}$$

where hereafter we assume sum over repeated indices. Then the three-sites tensor product R-matrices (2), which belong to $\mathcal{M}_{D^3}(\mathbb{K})$, read

$$\begin{aligned} R_{12} &= R_{(i_1,j_1),(k_1,\ell_1)}\varepsilon^{(i_1,k_1)} \otimes \varepsilon^{(j_1,\ell_1)} \otimes \mathrm{Id} \\ R_{13} &= R_{(i_2,j_2),(k_2,\ell_2)}\varepsilon^{(i_2,k_2)} \otimes \mathrm{Id} \otimes \varepsilon^{(j_2,\ell_2)} \\ R_{23} &= R_{(i_3,j_3),(k_3,\ell_3)}\mathrm{Id} \otimes \varepsilon^{(i_3,k_3)} \otimes \varepsilon^{(j_3,\ell_3)} \end{aligned} \tag{7}$$

where Id is the D-dimensional unit matrix.

The strategy we adopt is to explicitly calculate the tensorial products in the LHS and RHS of the cQYBE (1) showing that both of them are identically equal to zero. For this, one needs the following formulas (which can be directly checked)

$$\begin{aligned} \varepsilon^{(i,k)} \otimes \varepsilon^{(j,\ell)} &= E^{(i-1)D+j,(k-1)D+\ell} \\ E^{I,J} \otimes \mathrm{Id} &= \sum_{m=1}^{D} F^{(I-1)D+m,(J-1)D+m} \\ \varepsilon^{(i,k)} \otimes \mathrm{Id} &= \sum_{m=1}^{D} E^{(i-1)D+m,(k-1)D+m} \\ E^{I,J} \otimes \varepsilon^{j,\ell} &= F^{(I-1)D+j,(J-1)D+\ell} \\ \mathrm{Id} \otimes \varepsilon^{i,k} &= \sum_{m=1}^{D} E^{(m-1)D+i,(m-1)D+k} \end{aligned} \tag{8}$$

for any $i, k, j, \ell = 1, \ldots, D$; $I, J = 1, \ldots, D^2$; and where $E^{I,J}$ is the D^2-dimensional square matrix with only zero entries except for the (I, J) entry, which is equal to one, while $F^{A,B}$ $(A, B = 1, \ldots, D^3)$ is the D^3-dimensional square matrix with only zero entries except for the (A, B) entry, which is equal to one. From these expressions, the matrices (7) can be rewritten as

$$\begin{aligned} R_{12} &= R_{(i_1,j_1),(k_1,\ell_1)} F^{A_1,B_1} \\ R_{13} &= R_{(i_2,j_2),(k_2,\ell_2)} F^{A_2,B_2} \\ R_{23} &= R_{(i_3,j_3),(k_3,\ell_3)} F^{A_3,B_3} \end{aligned} \tag{9}$$

where

$$\begin{aligned} A_1 &= (i_1 - 1)D^2 + (j_1 - 1)D + m_1 \\ B_1 &= (k_1 - 1)D^2 + (\ell_1 - 1)D + m_1 \\ A_2 &= (i_2 - 1)D^2 + (m_2 - 1)D + j_2 \\ B_2 &= (k_2 - 1)D^2 + (m_2 - 1)D + \ell_2 \\ A_3 &= (m_3 - 1)D^2 + (i_3 - 1)D + j_3 \\ B_3 &= (m_3 - 1)D^2 + (k_3 - 1)D + \ell_3. \end{aligned} \tag{10}$$

Notice that in each matrix (9) one has five summations from 1 to D. Four of them correspond to the repeated indices i_a, j_a, k_a, ℓ_a and the other to m_a ($a = 1, 2, 3$). By taking into account the property

$$F^{A_1,B_1} F^{A_2,B_2} = \delta^{A_2,B_1} F^{A_1,B_2} \tag{11}$$

we obtain that the LHS of the cQYBE (1) contains the matrix product

$$F^{A_1,B_1} F^{A_2,B_2} F^{A_3,B_3} = \delta^{A_2,B_1} \delta^{A_3,B_2} F^{A_1,B_3}. \tag{12}$$

Since the dimension D is arbitrary, equations (10) imply that

$$\begin{aligned} \delta^{A_2,B_1} &= \delta^{k_1,i_2} \delta^{\ell_1,m_2} \delta^{j_2,m_1} \\ \delta^{A_3,B_2} &= \delta^{k_2,m_3} \delta^{i_3,m_2} \delta^{\ell_2,j_3}. \end{aligned} \tag{13}$$

By firstly inserting (12), (13) and the matrix elements $R_{(i,j),(k,\ell)}$ (4) in the LHS of the cQYBE (1), and secondly expanding the terms one finally obtains that

$$R_{12} R_{13} R_{23} = 0 \tag{14}$$

since $C^{k_a}_{Dj_a} = C^{k_a}_{i_a D} = C^{D}_{i_a j_a} = 0$. Similarly, one also find that $R_{23} R_{13} R_{12} = 0$, so that the R-matrix defined by Theorem 1 is a D-state solution of the cQYBE.

QED

3. R-matrices from Contractions of Orthogonal Lie Algebras

Theorem 1 provides some general results to construct singular solutions of the cQYBE. Nevertheless, as commented above, this can be specially applied to Lie (super)algebras by simply introducing their structure constants in the definition of the R-matrix entries (4). In this Section we construct explicitly the R-matrices corresponding to a particular family of contracted algebras which are obtained from $\mathfrak{so}(N+1)$. As a byproduct, we find that the graded contraction parameters are promoted into quantum deformation ones within such R-matrices.

Let us consider the real Lie algebra $\mathfrak{so}(N+1)$ whose $\frac{1}{2}N(N+1)$ generators J_{ab} $(a,b=0,1,\ldots,N,\, a<b)$ satisfy the non-vanishing Lie brackets given by

$$[J_{ab},J_{ac}]=J_{bc},\qquad [J_{ab},J_{bc}]=-J_{ac},\qquad [J_{ac},J_{bc}]=J_{ab} \tag{15}$$

where $a<b<c$. The $\mathbb{Z}_2^{\otimes N}$-graded contractions of $\mathfrak{so}(N+1)$ contain the so called Cayley-Klein (CK) orthogonal Lie algebras [10]. This family, denoted collectively $\mathfrak{so}_\kappa(N+1)$, depends on N real contraction parameters $\kappa=(\kappa_1,\ldots,\kappa_N)$. The non-zero commutators turn out to be [10]

$$[J_{ab},J_{ac}]=\kappa_{ab}J_{bc},\quad [J_{ab},J_{bc}]=-J_{ac},\quad [J_{ac},J_{bc}]=\kappa_{bc}J_{ab} \tag{16}$$

without sum over repeated indices, and where the two-index parameters κ_{ab} are expressed in terms of the N basic ones through

$$\kappa_{ab}=\kappa_{a+1}\kappa_{a+2}\cdots\kappa_b,\quad a,b=0,1,\ldots,N,\quad a<b. \tag{17}$$

Each contraction parameter κ_μ can take a positive, negative or zero value, so that $\mathfrak{so}_\kappa(N+1)$ comprises 3^N Lie algebras (some of them are isomorphic). For instance [11], when $\kappa_\mu\neq 0$ for any μ, $\mathfrak{so}_\kappa(N+1)$ is a simple pseudo-orthogonal algebra $\mathfrak{so}(p,q)$ $(p+q=N+1)$ (the B_l and D_l Cartan series). When $\kappa_1=0$ we recover the inhomogeneous algebras $\mathfrak{iso}(p',q')$ $(p'+q'=N)$ and when all $\kappa_\mu=0$, we find the flag algebra $\mathfrak{i}\ldots\mathfrak{iso}(1)$. We recall that kinematical algebras, such us Poincaré, Galilei, (anti-)de Sitter, etc., associated to different models of spacetimes of constant curvature also belong to this CK family of algebras [11].

Now, in order to apply Theorem 1 we enlarge the CK algebra $\mathfrak{so}_\kappa(N+1)$ with an additional central generator Ξ, that is $[\Xi,J_{ab}]=0$, for all ab. The vector space corresponding to $\mathfrak{so}_\kappa(N+1)\oplus\mathbb{R}$ is spanned by $D=\frac{1}{2}N(N+1)+1$ elements. We label the D generic generators $\{X_1,,\ldots,X_{\frac{1}{2}N(N+1)},X_D\}$ as $\{J_{01},\ldots,J_{N-1\,N},\Xi\equiv X_D\}$ according to the increasing order of indices ab with $a<b$. Here the indices of the entries (4) run as $i,j,k,\ell=\{01,02,\ldots,0N,12,\ldots,N-$

$1\, N, D\}$. Then by taking into account the structure constants of (16) we obtain that the D-state solution of the cQYBE associated to $\mathfrak{so}_\kappa(N+1)$ is the R-matrix with the following non-zero entries

$$\begin{array}{ll} R_{(ab,ac),(bc,D)} = \kappa_{ab}, & R_{(ab,ac),(D,bc)} = \kappa_{ab} \\ R_{(ac,ab),(bc,D)} = -\kappa_{ab}, & R_{(ac,ab),(D,bc)} = -\kappa_{ab} \\ R_{(ab,bc),(ac,D)} = -1, & R_{(ab,bc),(D,ac)} = -1 \\ R_{(bc,ab),(ac,D)} = 1, & R_{(bc,ab),(D,ac)} = 1 \\ R_{(ac,bc),(ab,D)} = \kappa_{bc}, & R_{(ac,bc),(D,ab)} = \kappa_{bc} \\ R_{(bc,ac),(ab,D)} = -\kappa_{bc}, & R_{(bc,ac),(D,ab)} = -\kappa_{bc} \end{array} \tag{18}$$

where $a, b, c = 0, 1, \ldots, N$ and $a < b < c$. Thus we have obtained a multiparametric solution of the cQYBE, which holds simultaneously for the 3^N particular Lie algebras contained in the CK family. The maximum number of quantum deformation parameters is N $(\kappa_1, \ldots, \kappa_N)$, that corresponds to $\mathfrak{so}(p,q)$ but through the contractions $\kappa_\mu = 0$ this number is subsequently reduced up to reach the flag algebra, for which there is no quantum parameters other than the constants ± 1.

Let us illustrate explicitly this construction with the $N = 2$ case.

3.1. Solutions of the cQYBE Associated to $\mathfrak{so}_{\kappa_1,\kappa_2}(3)$

The CK algebra with $N = 2$ is $\mathfrak{so}_{\kappa_1,\kappa_2}(3)$, which depends on two real coefficients (κ_1, κ_2) and is spanned by three generators $\{J_{01}, J_{02}, J_{12}\}$ fulfilling

$$[J_{01}, J_{02}] = \kappa_1 J_{12}, \quad [J_{01}, J_{12}] = -J_{02}, \quad [J_{02}, J_{12}] = \kappa_2 J_{01}. \tag{19}$$

According to the pair (κ_1, κ_2) we find that $\mathfrak{so}_{\kappa_1,\kappa_2}(3)$ covers nine Lie algebras: $\mathfrak{so}(3)$ for $(+,+)$, $\mathfrak{so}(2,1)$ for $(+,-)$, $(-,+)$ and $(-,-)$, $\mathfrak{iso}(2)$ for $(+,0)$ and $(0,+)$, $\mathfrak{iso}(1,1)$ for $(-,0)$ and $(0,-)$, and $\mathfrak{iiso}(1)$ for $(0,0)$. By considering the Lie group $\mathrm{SO}_{\kappa_1,\kappa_2}(3)$, both contraction parameters, κ_1, κ_2, can be identified with the constant curvature of the two-dimensional homogeneous space of points $\mathrm{SO}_{\kappa_1,\kappa_2}(3)/\langle J_{12}\rangle$ and of lines $\mathrm{SO}_{\kappa_1,\kappa_2}(3)/\langle J_{01}\rangle$, respectively [12]. Furthermore, if $\{J_{01}, J_{02}, J_{12}\}$ are interpreted, in this order, as time-translation, spatial-translation and boost generators, then the six CK algebras with $\kappa_2 \leq 0$ are kinematical ones [12]. In this case, besides the geometrical interpretation, the contraction parameters have a physical meaning as well, since they can be expressed as $\kappa_1 = \pm 1/\tau^2$ where τ is the universe radius and $\kappa_2 = -1/c^2$ (c is the speed of light). Thus $\mathfrak{so}_{\kappa_1,\kappa_2}(3)$ and $\mathrm{SO}_{\kappa_1,\kappa_2}(3)/\langle J_{12}\rangle$ comprises the following kinematical algebras and (1+1) spacetimes [12]: the anti-de Sitter $(+1/\tau^2, -1/c^2)$, Minkowskian $(0,-1/c^2)$, de Sitter $(-1/\tau^2,-1/c^2)$, oscillating Newton-Hooke $(+1/\tau^2,0)$, Galilean $(0,0)$ and expanding Newton-Hooke ones $(-1/\tau^2,0)$.

Next we present the $D=4$ state solution of the cQYBE coming from $\mathfrak{so}_{\kappa_1,\kappa_2}(3)$. At this dimension the indices $i,j,k,\ell=\{01,02,12,D=4\}$, so that the entries (18) give rise to the following 16×16 R-matrix

$$R=\left[\begin{array}{cccc|cccc|cccc|cccc}
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&\kappa_1&0&0&\kappa_1&0\\
0&0&0&0&0&0&0&-1&0&0&0&0&0&-1&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
\hline
0&0&0&0&0&0&0&0&0&0&0&-\kappa_1&0&0&-\kappa_1&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&\kappa_2&0&0&0&0&0&0&0&0&\kappa_2&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
\hline
0&0&0&0&0&0&0&1&0&0&0&0&0&1&0&0\\
0&0&0&-\kappa_2&0&0&0&0&0&0&0&0&-\kappa_2&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
\hline
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0
\end{array}\right].$$

Now, if we consider this R-matrix as the structure constant matrix for a non-commutative space constructed by using the standard FRT approach [3], we get a direct relationship between "classical" contraction/curvature parameters and quantum deformation ones. Moreover, we find that physical classical quantities such as τ and c can also be promoted into quantum deformation parameters. The construction of such quantum spaces is currently in progress.

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