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NONCOMMUTATIVE GRASSMANNIAN $U(1)$ SIGMA-MODEL AND BARGMANN-FOCK SPACE

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Abstract. We consider the Grassmannian version of the noncommutative $U(1)$ sigma-model, which is given by the energy functional $E(P) = \|[a, P]\|^2_{HS}$, where P is an orthogonal projection on a Hilbert space H and the operator $a : H \to H$ is the standard annihilation operator. Using realization of H as the Bargmann-Fock space, we describe all solutions with one-dimensional image and prove that the operator $[a, P]$ is densely defined on H for some class of projections P with infinite-dimensional image and kernel.

1. Introduction

We consider the Grassmannian noncommutative $U(1)$ sigma-model, which is the noncommutative analogue of the classical $\mathbb{C}$-one-dimensional Grassmannian sigma-model. Firstly we describe the latter one. By $\mathrm{Gr}_k(\mathbb{C}^n)$ denote the complex Grassmannian (i.e., the manifold of k-dimensional complex planes in $\mathbb{C}^n$). We will consider its points as orthogonal projections on $\mathbb{C}^n$ with k-dimensional image (and $(n-k)$-dimensional kernel). Then the energy of any map $f : \mathbb{C}P^1 \to \mathrm{Gr}_k(\mathbb{C}^n)$ (i.e., for every z, $f(z)$ is a matrix of k-dimensional orthogonal projection on $\mathbb{C}^n$) is

$$E(f) := \int\limits_{\mathbb{C}P^1} \|\partial_{\bar{z}} f\|^2_{HS} \mathrm{d}x\mathrm{d}y = \int\limits_{\mathbb{C}P^1} \mathrm{tr}\,(\partial_{\bar{z}} f)^* \,\partial_{\bar{z}} f \mathrm{d}x\mathrm{d}y. \tag{1}$$

Extremals of $E(f)$ (solutions of this model) are called harmonic maps. (For details see [7].)

Under the studying of static $D0$-branes in $D2$-branes (see [3]) there appears the noncommutative analogue of the model above. (This analogue is also considered in [5] and [2].) To describe it, we regard the noncommutative plane $\mathbb{R}^2_\theta$. The transfer is based on the rules of the Weyl calculus of pseudodifferential operators and the result is the following. Instead of maps $f(\cdot) : \mathbb{C}P^1 \to \mathrm{Gr}_k(\mathbb{C}^n)$ we consider orthogonal projections on a Hilbert space H. Moreover, $\partial_{\bar{z}}(\cdot)$ is replaced by $\frac{1}{\sqrt{2\theta}}[a, \cdot]$, $\partial_z(\cdot)$ by $-\frac{1}{\sqrt{2\theta}}[a^*, \cdot]$, and $\int \mathrm{tr}$ by $2\theta\mathrm{tr}$. Here a and a^* are the standard annihilation and creation operators respectively (for their definition see Section 2). Therefore the classical functional $E(f)$ transfers to

$$E(P) = \|[a, P]\|_{HS}^2. \tag{2}$$

Extremals of $E(P)$ (which are analogues for harmonic maps) will be the main subject of our study.

The aim of the paper is to show how some mathematical difficulties that are connected with the description of the configuration space and solutions of the noncommutative sigma-model can be overcomed with the help of the realization of H as the Bargmann-Fock space. In Section 4 we describe all solutions with one-dimensional image (and therefore with one-dimensional kernel, since $E(P) = E(I - P)$). If we consider a projection P with infinite-dimensional image and kernel, then the questions whether the domain of $[a, P]$ is dense in H (in Section 2 this property is called admissibility) and whether the energy of P is finite become non-evident and non-equivalent. In Section 5 we prove admissibility for some class of such projections.

2. Basic Notions and Definitions

Let H be a separable Hilbert space with an orthonormal basis $\{e_0, e_1, \dots\}$. By a we denote the standard annihilation operator on H: $\operatorname{dom} a = \{x \in H;\ \sum_{j=0}^{\infty} j|x_j|^2 < \infty\}$, where $x_j := (x, e_j)$, and $a(e_j) = \sqrt{j}e_{j-1}$ for $j = 0, 1, 2, \dots$. The basis $\{e_0, e_1, \dots\}$ is called *canonical*. Note that $\operatorname{dom} a = \operatorname{dom} a^* =: D$ and $a^*(e_j) = \sqrt{j+1}e_{j+1}$. Besides that for any $\alpha \in \mathbb{C}$ there exists a unique (up to multiplying by θ, where $|\theta| = 1$) normed eigenvector c_α corresponding to the eigenvalue α. Namely, $(c_\alpha, e_j) = \mathrm{e}^{-\frac{|\alpha|^2}{2}} \frac{\alpha^j}{\sqrt{j!}}$ and these c_α are called *coherent states*. Note also that $[a, a^*] = I$ and the operator a is irreducible.

Definition 1. *An orthogonal projection P (that is, P is a bounded operator such that $P = P^2 = P^*$) is called admissible if the subspace $D_P := \{x \in D; Px \in D\}$ is dense in H. In this case the operator $A := [a, P]$ is densely defined.*

Proposition 2. *There is a simple criterion of admissibility: an orthogonal projection $P : H \to H$ is admissible if and only if the spaces $D \cap \mathrm{im}P$ and $D \cap \ker P$ are dense in $\mathrm{im}P$ and in $\ker P$ respectively.*

Therefore projections P and $I - P$ are admissible together. Also a projection P with $\dim(\mathrm{im}P) < \infty$ (or $\dim(\ker P) < \infty$) is admissible if and only if $\mathrm{im}P \subset D$ (respectively $\ker P \subset D$).

Definition 3. *Let a projection P be admissible. Then P is said to be a projection of finite energy if the operator $A = [a, P]$ is extended from D_P to the whole H by continuity and this extension is a Hilbert-Schmidt operator. In this case we define the energy of P by the formula*

$$E(P) := \|A\|_{HS}^2 = \|[a, P]\|_{HS}^2 \tag{3}$$

where $\|\cdot\|_{HS}$ is the Hilbert-Schmidt norm. Otherwise an admissible P is called a projection of infinite energy and $E(P) := \infty$.

Note that $E(P) = E(I - P)$.

Definition 4. *We say that a projection P_0 of finite energy is a solution if for any smooth curve $P(t)$ $(-\varepsilon < t < \varepsilon)$ of projections of finite energy such that $P(0) = P_0$, we have $\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} E(P(t)) = 0$. (The smoothness is understood in the sense of the operator norm.)*

Definition 5. *BPS-solution is an admissible projection P such that there exists a system of vectors $\{g_1, g_2, \dots\}$ in $\mathrm{im}P$ with two properties: $\overline{\mathrm{Span}}\{g_1, g_2, \dots\} = \mathrm{im}P$ ($\overline{\mathrm{Span}}$ denotes the closure of linear hull) and $ag_i \in \mathrm{im}P$ for any $i = 1, 2, \dots$.*

It is clear that if $\dim(\mathrm{im}P) < \infty$, then P is a BPS-solution if and only if $a(\mathrm{im}P) \subset \mathrm{im}P$ (i.e., $[a, P]P = 0$).[1]

In [2] it is proved that if P is admissible and $\dim(\mathrm{im}P) < \infty$, then

$$E(P) = \dim(\mathrm{im}P) + 2\|[a, P]P\|_{HS}^2. \tag{4}$$

Therefore a BPS-solution P with $\dim(\mathrm{im}P) = n < \infty$ is a minimum of the energy functional on the class of admissible projections with n-dimensional image. Since projections with different dimension of images are far from each other, BPS-solutions with finite-dimensional image are local minima of the energy functional on the space of all admissible projections. (In particular, they are solutions).

[1] In the noncommutative Grassmannian sigma-model the term "BPS-solution" was introduced in [5] for projections P such that $a(\mathrm{im}P) \subset \mathrm{im}P$. Since a has no infinite-dimensional invariant spaces, our definition generalizes the latter one.

Example 6. *The projection P with* $\mathrm{im}P = \mathrm{Span}\,\langle e_0, e_1, \ldots, e_n\rangle$ *is a BPS-solution and* $E(P) = n+1$. (*Recall that* $\{e_j\}$ *is the canonical basis.*)

3. Bargmann-Fock Space

Definition 7. *The Bargmann-Fock space is the functional space*

$$\mathcal{F}_z := \{f(z) \in \mathcal{O}(\mathbb{C});\ \int\limits_{\mathbb{C}} |f(z)|^2 \mathrm{e}^{-|z|^2} \mathrm{d}x\mathrm{d}y < \infty\} \tag{5}$$

where $\mathcal{O}(\mathbb{C})$ is the space of entire functions.

The space $\mathcal{F}_z$ is a Hilbert space with respect to the scalar product

$$(f, g) := \frac{1}{\pi} \int\limits_{\mathbb{C}} f(z)\overline{g(z)} \mathrm{e}^{-|z|^2} \mathrm{d}x\mathrm{d}y. \tag{6}$$

Besides that the functions $\left\{\frac{z^n}{\sqrt{n!}}\right\}_{n=0}^{\infty}$ form an orthonormal basis for $\mathcal{F}_z$.

The mapping of canonical basis for a in H to the basis $\left\{\frac{z^n}{\sqrt{n!}}\right\}_{n=0}^{\infty}$ gives us the isomorphism between H and $\mathcal{F}_z$. This isomorphism takes the operator a to $\frac{\mathrm{d}}{\mathrm{d}z}(\cdot)$, a^* to the operator of multiplication by z, the coherent state c_α to the function $\mathrm{e}^{-\frac{|\alpha|^2}{2}} \mathrm{e}^{\alpha z}$.

One can prove the following property of $\mathcal{F}_z$ (the proof is straightforward): for any $\alpha \in \mathbb{C}$ and for any $n = 0, 1, \ldots$ we have

$$(f, z^n \mathrm{e}^{\alpha z}) = f^{(n)}(\bar{\alpha}). \tag{7}$$

For more details on $\mathcal{F}_z$ see [1].

4. Description of Solutions with One-dimensional Image

Let P be an admissible projection on a normed vector $p \in H$. According to the criterion of admissibility (Proposition 2), $p \in D = \mathrm{dom}\, a$. Using (4), we get $E(P) = 1 + 2\|[a, P]P\|_{HS}^2$. Choosing an orthonormal basis $\{p_i\}_{i=0}^{\infty}$ in H such that $p_0 = p$, we easily obtain

$$\|[a, P]P\|_{HS}^2 = \|ap\|^2 - |(ap, p)|^2. \tag{8}$$

Let us consider the curve P_t passing through P that consists of the projections on the vectors $p_t = \frac{p+th}{\sqrt{1+t^2}}$, where $h \in D$ is a normalized and orthogonal to p vector, $t \in \mathbb{R}$, $|t| \ll 1$. Then P_t is admissible and $P_0 = P$. Now we write $E(P_t)$ up to the second order terms

$$E(P_t) = E(P) + 4\mathrm{Re}L_P(h)t + R_P(h)t^2 + O(t^3) \tag{9}$$

where

$$L_P(h) = (ah, ap) - (ap, p)(h, ap) - (p, ap)(ah, p) \tag{10}$$

$$\begin{aligned} R_P(h) =& 2[\|ah\|^2 + 2|(ap,p)|^2 - \|ap\|^2 - |(ap,h)|^2 - |(ah,p)|^2 \\ &- 2\mathrm{Re}\,((ap,p)(h,ah) + (ap,h)(p,ah))]. \end{aligned} \tag{11}$$

Now suppose that P is a solution. Then $\mathrm{Re}L_P(h)$ should be zero for any normed $h \in D \cap (\mathrm{Span}\, p)^\perp$. Since h can be replaced by $\mathrm{i}h$ and $L_P(h)$ is a linear function, we obtain that $L_P(h) = 0$ for these h. We know that $p \in D$ and $h \in D$. Therefore the condition $L_P(h) = 0$ is equivalent to

$$(ah, ap) = (ap, p)(h, ap) + (p, ap)(h, a^*p). \tag{12}$$

Note that the right-hand side of (12) is a bounded linear functional of h on the whole H. Since P is a solution, (12) holds true for all $h \in D \cap (\mathrm{Span}\, p)^\perp$. Therefore (ah, ap) is extended to a bounded linear functional of h on $(\mathrm{Span}\, p)^\perp$. Since $p \in D$, (ah, ap) is a linear functional on $\mathrm{Span}\, p$. Hence (ah, ap) is extended to a bounded linear functional of h on the whole H. Thus the operator a^* is defined at the vector ap and (12) can be rewritten as the following condition

$$(h, a^*ap) = (ap, p)(h, ap) + (p, ap)(h, a^*p), \qquad \text{for all } h \perp p. \tag{13}$$

So the vector $(a^*ap - (p, ap)ap - (ap, p)a^*p)$ should be collinear to p. Thus we get the necessary condition for P to be a solution

$$a^*ap - (p, ap)ap - (ap, p)a^*p = \mu p, \qquad \text{for some } \mu \in \mathbb{C}. \tag{14}$$

Now use the isomorphism between H and the Bargmann-Fock space $\mathcal{F}_z$ constructed in the Section 3. Denote (ap, p) by λ and let function $f(z) \in \mathcal{F}_z$ correspond to the vector p. Then (14) is equivalent to

$$zf'(z) - \bar{\lambda}f'(z) - \lambda z f(z) = \mu f(z), \qquad \text{for some } \mu \in \mathbb{C}.$$

Choosing only entire and normalized solutions (and denote $\mu + |\lambda|^2$ by m), we obtain that all solutions with one-dimensional image are contained among the projections on the vectors

$$f(z) = C\mathrm{e}^{\lambda z}(z - \bar{\lambda})^m \tag{15}$$

where $\lambda \in \mathbb{C}$, $m = 0, 1, \ldots$ and $C = (\mathrm{e}^{-|\lambda|^2}/m!)^{1/2}$. Further on we will show that all these projections are solutions.

Definition 8. *For any $\lambda \in \mathbb{C}$ we consider the operator of noncommutative translation*

$$U_\lambda : f(z) \mapsto \mathrm{e}^{-\frac{|\lambda|^2}{2}} \mathrm{e}^{\lambda z} f(z - \bar{\lambda}) \tag{16}$$

which is an automorphism of $\mathcal{F}_z$.

Proposition 9. *For any admissible projection P*

$$E(P) = E(U_\lambda P U_\lambda^{-1}). \tag{17}$$

Using U_λ and (15), we obtain that all solutions with one-dimensional image are among the projections on the vectors $U_\lambda(e_m)$, where $\lambda \in \mathbb{C}$, $m = 0, 1, 2, \ldots$ and $\{e_j\}$ is the canonical basis ($e_j = \frac{z^j}{\sqrt{j!}}$). Moreover, since U_λ is an automorphism and $E(P) = E(U_\lambda P U_\lambda^{-1})$, our problem is reduced to the question about the description of solutions among the projections on the basis vectors e_j, $j = 0, 1, 2, \ldots$.

We know that the projection on e_0 is a minimum of the energy (because it is a BPS-solution). Now prove that the projection P on e_j, where $j = 1, 2, \ldots$, is a solution but neither minimum nor maximum.

Let $P(t)$ be an arbitrary smooth curve such that $P(0) = P$. Then $P(t) = P + t\varphi + O(t^2)$, where $\varphi \in T_P Pr(H)$ ($T_P Pr(H)$ is the tangent space to the space of orthogonal projections at the point P). $T_P Pr(H)$ consists of bounded φ such that $P + \varphi = (P + \varphi)^2 = (P + \varphi)^*$ up to the first order term by φ, i.e.,

$$T_P Pr(H) = \{\text{bounded } \varphi;\ P\varphi + \varphi P = \varphi = \varphi^*\}. \tag{18}$$

Now we can write up to the first order terms

$$E(P(t)) = E(P + t\varphi + O(t^2)) = E(P) + 2t\mathrm{Re}([a, P], [a, \varphi])_{HS} + O(t^2). \tag{19}$$

Hence P is a solution if and only if the first variation

$$E_P^1(\varphi) := 2\mathrm{Re}([a, P], [a, \varphi])_{HS} \tag{20}$$

is equal to zero for any $\varphi \in T_P Pr(H)$. In our case ($\mathrm{im}P = \mathrm{Span}\, e_j$) this holds true. (It can be checked by straightforward calculation of $T_P Pr(H)$ and matrix elements of $[a, P]$ and $[a, \varphi]$ in the canonical basis.)

For proving that P with $\mathrm{im}P = \mathrm{Span}\, e_j$ for $j = 1, 2, \dots$ is neither minimum nor maximum it is sufficient to find two vectors h_1 and h_2 from $D \cap (\mathrm{Span}\, e_j)^{\perp}$ such that $R_P(h_1) > 0$ and $R_P(h_2) < 0$ ($R_P(h)$ was defined in (11)). We take $h_1 = e_{j+2}$ and $h_2 = e_0$.

So we obtain the following theorem.

Theorem 10. *A projection P with one-dimensional image is a solution if and only if*

$$\mathrm{im}P = \mathrm{Span}\,\langle U_\lambda e_j \rangle = \mathrm{Span}\,\langle (a^* - \bar{\lambda})^j c_\lambda \rangle, \qquad \lambda \in \mathbb{C},\ j = 0, 1, \dots. \tag{21}$$

Among them minima are projections on

$$\mathrm{Span}\,\langle U_\lambda e_0 \rangle = \mathrm{Span}\,\langle c_\lambda \rangle, \qquad j = 0 \tag{22}$$

i.e., all one-dimensional BPS-solutions. There are no maxima.

5. Infinite-dimensional BPS-solutions

Here we construct the class K of projections that are BPS-solutions with infinite-dimensional kernel (their admissibility will be proved). The most part of them will have also infinite-dimensional image and it will be clear which of them have this property. Here we will work only in the Bargmann-Fock space $\mathcal{F}_z$ (see Section 3).

Description of the class K. Let $\phi(z)$ be an exponential polynomial (i.e., $\phi(z) = \sum_\lambda p_\lambda(z) \mathrm{e}^{\lambda z}$, where $p_\lambda(z)$ are polynomials and $\sum_\lambda$ is finite). Define the projection P_ϕ. Let $\{\alpha_0, \alpha_1, \dots\}$ be the set of zeroes of $\phi(z)$ (possibly finite). Let n_i be the multiplicity of zero α_i. Then

$$\mathrm{im}P_\phi := \overline{\mathrm{Span}}\,\langle \{\mathrm{e}^{\bar{\alpha}_i z}, z\mathrm{e}^{\bar{\alpha}_i z}, \dots, z^{n_i - 1}\mathrm{e}^{\bar{\alpha}_i z}\}_{i=0}^{\infty} \rangle. \tag{23}$$

A projection P belongs to K if and only if there exists an exponential polynomial ϕ such that $P = P_\phi$.

Using the property of $\mathcal{F}_z$ (7), we see that

$$\ker P_\phi = \{g(z) \in \mathcal{F}_z\, ;\ g(z) = f(z)\phi(z),\ \mathrm{where} f(z) \in \mathcal{O}(\mathbb{C})\}. \tag{24}$$

So this kernel is infinite-dimensional. It is evident that P_ϕ has infinite-dimensional image if and only if $\phi(z)$ has infinite number of zeroes. For proving admissibility of P_ϕ we use the criterion of admissibility (Proposition 2). It is evident that $D \cap \mathrm{im}P$ is dense in $\mathrm{im}P$ (because $z^n \mathrm{e}^{\alpha z} \in D$). To prove that $D \cap \ker P$ is dense in $\ker P$, it is sufficient to check that the system of functions $\{\mathrm{e}^{\alpha z}\phi(z)\}_{\alpha \in \mathbb{C}}$ is dense in $\ker P_\phi$ (because $\mathrm{e}^{\alpha z}\phi(z) \in D$). This fact was proved in [6]. So P_ϕ is admissible and looking at formula (23) for $\mathrm{im}P_\phi$, we see that P_ϕ is a BPS-solution.

Example 11. *ϕ is a usual polynomial. Then* $\dim(\mathrm{im}P_\phi) < \infty$. *E.g., if* $\phi(z) = z^2(z + 1 - \mathrm{i})$, *then*

$$\mathrm{im}P_\phi = \mathrm{Span}\,\langle 1, z, \mathrm{e}^{-(1+\mathrm{i})z}\rangle. \tag{25}$$

Example 12. $\phi(z) = \sin \pi z$. *Then*

$$\mathrm{im}P_\phi = \overline{\mathrm{Span}}\,\langle \{\mathrm{e}^{nz}\}_{n\in\mathbb{Z}}\rangle = \overline{\mathrm{Span}}\,\langle \{c_n\}_{n\in\mathbb{Z}}\rangle. \tag{26}$$

Example 13. $\phi(z) = z(z - \mathrm{i})\sin^2 z$. *Then*

$$\mathrm{im}P_\phi = \overline{\mathrm{Span}}\,\langle \{\mathrm{e}^{\pi nz}\}_{n\in\mathbb{Z}}, \{z\mathrm{e}^{\pi nz}\}_{n\in\mathbb{Z}}, z^2, \mathrm{e}^{-\mathrm{i}z}\rangle. \tag{27}$$

Remark 14. *By now it is unknown whether any projection of finite energy with infinite-dimensional image and kernel exists. We try to find such projection among BPS-solutions, in particular, among of projections from the class* K. *Unfortunately, all projections for that we managed to compute the energy turned out to have infinite energy.* (*E.g.,* $P_{\sin \pi z}$ *from Example 12 is such projection.*)

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