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SMALL FRACTIONAL PARTS OF POLYNOMIALS

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Abstract: Let $k \geqslant 6$. Using the recent result of Bourgain, Demeter, and Guth [5] on the Vinogradov mean value, we obtain new bounds for small fractional parts of polynomials $\alpha_k n^k + \dots + \alpha_1 n$ and additive forms $\beta_1 n_1^k + \dots + \beta_s n_s^k$. Our results improve earlier theorems of Danicic (1957), Cook (1972), Baker (1982, 2000), Vaughan and Wooley (2000), and Wooley (2013).

Keywords: Weyl sums, Vinogradov mean value, fractional parts of polynomials.

1. Introduction

Let $J_{s,k}(N)$ be the Vinogradov mean value,

$$J_{s,k}(N) := \int_{[0,1)^k} \left| \sum_{n=1}^{N} e(x_k n^k + \dots + x_1 n) \right|^{2s} dx_1 \dots dx_k.$$

Here s and k are natural numbers. Recently Wooley [12] (for $k = 3$) and Bourgain, Demeter, and Guth [5] (for $k \geqslant 4$) have established the main conjecture for $J_{s,k}(N)$, namely

$$J_{s,k}(N) \ll_{k,\varepsilon} N^{s+\varepsilon} + N^{2s-k(k+1)/2+\varepsilon}. \tag{1.1}$$

Here ε is an arbitrary positive number. In the present note we combine (1.1) with techniques from two earlier publications [3, 4] to obtain new bounds of the form

$$\min_{1 \leqslant n \leqslant N} \|\alpha_k n^k + \dots + \alpha_1 n\| \ll_{k,\varepsilon} N^{-\mu_k+\varepsilon} \qquad (k = 8, 9, \dots) \tag{i}$$

(with arbitrary real numbers $\alpha_1, \dots, \alpha_k$, $\beta_1, \dots, \beta_s$ here and below);

$$\min_{1 \leqslant n \leqslant N} \|\alpha_k n^k + \alpha_1 n\| \ll_{k,\varepsilon} N^{-\rho_k+\varepsilon} \qquad (k = 6, 7, \dots) \tag{ii}$$

$$\min_{\substack{0 \leqslant n_1, \dots, n_s \leqslant N \\ (n_1, \dots, n_s) \neq \mathbf{0}}} \|\beta_1 n_1^k + \dots + \beta_s n_s^k\| \ll N^{-\sigma_{s,k}+\varepsilon} \qquad (k = 6, 7, \dots, s \geqslant 1). \tag{iii}$$

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Theorem 1. *Let $k \geqslant 8$. Then (i) holds with $\mu_k = 1/2k(k-1)$.*

Theorem 2.

(a) *Let $k \geqslant 6$. Then (ii) holds with $\rho_k = 1/k(k-1)$.*
(b) *Let $k \geqslant 6$. For a certain positive absolute constant B, (ii) holds with $\rho_k = 1/k(2\log k + B\log\log k)$.*

Theorem 3.

(a) *Let $k \geqslant 6$, $1 \leqslant s \leqslant k(k-1)$. Then (iii) holds with $\sigma_{s,k} = s/k(k-1)$.*
(b) *Let*

$$F(J,s,k) = \min\left(\frac{s}{J}, \max_{J+1\leqslant h\leqslant s} \min\left(\frac{(2h-2)(s-k)+4k-4}{h(s-k)+4h-4}, \frac{s-h+J+1}{J}\right)\right)$$

Then (iii) holds for $k \geqslant 6$, $s > k(k-1)$ with

$$\sigma_{s,k} = F(k(k-1), s, k).$$

In particular,

$$\min_{\substack{0\leqslant n_1,\ldots,n_s\leqslant N\\(n_1,\ldots,n_s)\neq\mathbf{0}}} \|\beta_1 n_1^6 + \cdots + \beta_s n_s^6\| \ll N^{-s/30+\varepsilon} (1 \leqslant s \leqslant 56).$$

We note here the existing results in each case. Let $K = 2^{k-1}$.

(i) This is known with $\mu_k = 1/K$ $(2 \leqslant k \leqslant 8)$ (Baker [1]) and $\mu_k = 1/4k(k-2)$ for $k \geqslant 9$ (Wooley [11]).
(ii) Only the special case $\alpha_1 = 0$ has been considered separately from (i). Here the result is known with $\rho_2 = 4/7$ (Zaharescu [14]); $\rho_k = 1/K$ $(3 \leqslant k \leqslant 6)$ (Danicic [7]), while there are the values $\rho_7 = 1/57.23$, $\rho_8 = 1/69.66$, $\rho_9 = 1/82.08$, $\rho_{10} = 1/94.62$, $\rho_{11} = 1/107.27, \ldots, \rho_{20} = 1/222.16$, given by Vaughan and Wooley [9], which are better than the present method gives (in the monomial case) for $k \geqslant 11$. There is an absolute positive constant C such that, for $k \geqslant 6$,
$$\min_{1\leqslant n\leqslant N} \|\alpha n^k\| \ll_{k,\varepsilon} N^{-1/k(\log k + C\log\log k)} \tag{1.2}$$
(Wooley [10]).
(iii) This is known with $\sigma_{s,k} = s/K$ for $k \geqslant 2$, $1 \leqslant s \leqslant K$ (Cook [6]), and
$$\sigma_{s,k} = F(K,s,k) \quad (k \geqslant 4, s > K)$$
(Baker [4]). For $k = 2, 3$ and $s > K$, see Baker [1, 4]; for example, $\sigma_{3,2} = 9/8$ and $\sigma_{5,3} = 5/4$.

We refer the reader to Heath-Brown [8], Wooley [10], and Vaughan and Wooley [9] for results of the kind: for irrational α, we have

$$\|\alpha n^k\| < n^{-\tau_k}$$

for infinitely many k. For example, one may take $\tau_k = 1/9.028k$ for every k [10].

2. Bounds for Weyl sums

We suppose throughout (as we may) that ε is sufficiently small and N is sufficiently large in terms of k, ε; we write $\eta = \varepsilon^2$.

Theorem 4. *Let $k \geqslant 3$ and $\varepsilon > 0$. Suppose that the Weyl sum*

$$g_k(\boldsymbol{\alpha}; N) := \sum_{n=1}^{N} e(\alpha_k n^k + \cdots + \alpha_1 n)$$

satisfies

$$|g_k(\boldsymbol{\alpha}; N)| \geqslant A > N^{1-1/2k(k-1)+\varepsilon}. \tag{2.1}$$

Then there exist integers q, $a_1, \ldots, a_k$ such that

$$1 \leqslant q \leqslant N^{\varepsilon}(NA^{-1})^k \tag{2.2}$$

and

$$|q\,\alpha_j - a_j| \leqslant N^{-j+\varepsilon}(NA^{-1})^k \qquad (1 \leqslant j \leqslant k). \tag{2.3}$$

If $\alpha_{k-1} = \cdots = \alpha_2 = 0$, then the same conclusion holds with the weaker lower bound.

$$|g_k(\boldsymbol{\alpha}; N)| \geqslant A > N^{1-1/k(k-1)+\varepsilon} \tag{2.4}$$

in place of (2.1)*.*

Proof. We initially proceed exactly as in the proof of [3, Theorem 4.3] with θ replaced by 0 and ℓ replaced by $(k-1)/2$. This is permissible since we have

$$J_{s,k-1}(N) \ll N^{s+\varepsilon}$$

with $s = k(k-1)/2$, in place of the bound for $J_{s,k-1}(N)$ used in [3]. We find that for $j = 2, \ldots, k$ there are coprime pairs of integers q_j, b_j with

$$1 \leqslant q_j \ll (NA^{-1})^{k(k-1)}(\log N)^C, \qquad |q\alpha_j - b_j| \leqslant N^{-j+\varepsilon}(NA^{-1})^{k(k-1)}$$

where we shall use C for an unspecified positive constant depending on k. Let q_0 be the l.c.m of $q_2, \ldots, q_k$. We now follow the argument of [3, pp. 41–42] to obtain

$$q_0 \ll (\log N)^C (NA^{-1})^{k(k-1)}. \tag{2.5}$$

It follows that, with $a_j = q_0 b_j / q_j$, we have

$$|q_0\alpha_j - \alpha_j| \leqslant N^{-j+2\varepsilon}(NA^{-1})^{2k(k-1)} \qquad (j = 2, \ldots, k). \tag{2.6}$$

We now appeal to Lemma 4.6 of [3], which we restate here for clarity as Lemma 1.

Lemma 1. *Suppose that there are integers r, $v_2, \ldots, v_k$ such that* $\gcd(r, v_2, \ldots, v_k) = 1$,

$$|q_j r - v_j| \leqslant N^{1-j}/4k^4 \qquad (j = 2, \ldots, k), \tag{2.7}$$

and that

$$|g_k(\boldsymbol{\alpha}; N)| \geqslant H > r^{1-1/k} N^{\varepsilon}. \tag{2.8}$$

There is a natural number $t \leqslant 2k^2$ such that

$$tr \leqslant (NH^{-1})^k N^{\varepsilon}, \tag{2.9}$$

$$t|\alpha_j r - v_j| \leqslant (NH^{-1})^k N^{-j+\varepsilon} \qquad (j = 2, \ldots, k) \tag{2.10}$$

$$\|tr\, \alpha_1\| \leqslant (NH^{-1}) N^{-1+\varepsilon}. \tag{2.11}$$

We now apply the lemma with $A = H$, $r = q_0 d^{-1}$, $v_j = a_j d^{-1}$ where $d = \gcd(q_0, a_2, \ldots, a_k)$. From (2.5) and (2.6),

$$|\alpha_j r - v_j| \leqslant N^{-j+2\varepsilon}(NA^{-1})^{2k(k-1)} \leqslant N^{-j+1}(4k^4)^{-1}$$

since

$$(NA^{-1})^{2k(k-1)} \leqslant N^{1-12\varepsilon}$$

and $r \leqslant N^{1-5\varepsilon}$,

$$Ar^{-1+1/k} N^{-2\varepsilon} \geqslant N^{1-1/k(k-1)-1+1/k-C\varepsilon} \gg 1.$$

The inequalities (2.9)–(2.11) now yield the first assertion of the theorem with $q = tr$. For the second assertion, since $\alpha_2, \ldots, \alpha_{k-1}$ are 0, we may take $r = q_k$, $v_k = b_k$, $v_2 = \cdots = v_{k-1} = 0$, $H = A$ in the application of Lemma 1. (The inequality (2.4) suffices in the earlier part of the argument.) We know that

$$|r\alpha_k - a_k| \leqslant N^{-k+\varepsilon}(NA^{-1})^{k(k-1)}$$

rather than the weaker bound (2.6). We may now complete the proof in the same way as before. ∎

3. Proof of Theorems 1, 2, and 3

Proof of Theorem 1. Suppose there is no solution of

$$1 \leqslant n \leqslant N, \qquad \|\alpha_k n^k + \cdots + \alpha_1 n\| \leqslant N^{-1/J+\varepsilon} \tag{3.1}$$

where J denotes $2k(k-1)$. By [3, Theorem 2.2] we have

$$\sum_{m=1}^{M} |g_k(m\boldsymbol{\alpha}; N)| > N/6,$$

where $M = [N^{1/J-\varepsilon}]$. There is an integer m, $1 \leqslant m \leqslant M$ such that

$$|g_k(m\alpha; N)| > A = N/6M.$$

We have

$$(NA^{-1})^{2k(k-1)} \ll M^{2k(k-1)} \ll N^{1-2k(k-1)\varepsilon}.$$

By Theorem 4 there is a natural number $q = tr$ such that

$$q \ll N^{\varepsilon}(NA^{-1})^k \ll M^k, \tag{3.2}$$

$$\|qm\alpha_j\| \ll (NA^{-1})^k N^{-j+\varepsilon} \ll M^k N^{-j+\varepsilon} \qquad (j = 1, \ldots, k). \tag{3.3}$$

Now let $n = qm$. Then

$$n \ll M^{k+1} \ll N^{(k+1)/J} \ll N^{1-\varepsilon},$$
$$\|n^j\alpha_j\| \leqslant n^{j-1}\|n\alpha_j\| \ll M^{(k+1)(j-1)+k}N^{-j+\varepsilon} \ll M^{-1}N^{-\varepsilon}$$

since $M^{(k+1)j} \ll N^{(k+1)j/J-(k+1)\varepsilon} \ll N^{j-2\varepsilon}$. It follows that n satisfies (3.1), which is a contradiction. This completes the proof of Theorem 1. ■

Proof of Theorem 2(a). We follow the above proof; this time, J denotes $k(k-1)$. The second assertion of Theorem 4 provides an integer $q = tr$ satisfying (3.2), and (3.3) for the relevant values $j = 1$, k. Now we complete the proof as before. ■

Proof of Theorem 2(b). This is a simple consequence of Wooley's bound (1.2). Let $\nu = \nu(k)$ have the property that

$$\min_{1 \leqslant n \leqslant N} \|\alpha n^k\| \ll_k N^{-\nu}$$

for $N \geqslant 1$ and real α. Let $a = \frac{1}{2+\nu}$, $b = 1 - a$. By Dirichlet's theorem there is a natural number $\ell \leqslant N^b$ with

$$\|\alpha_1\ell\| \leqslant N^{-b}.$$

We now choose another natural number $m \leqslant N^a$ with

$$\|\alpha_k\ell^k m^k\| \ll N^{-a\nu} = N^{-\nu/(2+\nu)}.$$

Note that

$$\|\alpha_1\ell m\| \leqslant N^{a-b} = N^{2a-1}.$$

Since $2a - 1 = -\frac{\nu}{2+\nu}$, we have, with $n = \ell m$,

$$1 \leqslant n \leqslant N, \qquad \|\alpha_k n^k + \alpha_1 n\| \ll N^{-\nu/(2+\nu)}.$$

Taking $\nu = 1/k(\log k + C\log\log k)$, we obtain

$$\frac{\nu}{2+\nu} = \frac{1}{2k\log k + 2C\log\log k + 1},$$

so that Theorem 2(b) holds with a suitable choice of B. ■

Example. If we take $k = 20$, $\nu = 1/222.16$ from [9], we obtain the value $1/445.32$ for ρ_{20}, which is not as good as Theorem 2(a). The proof of Theorem 2(b) is relatively crude, so it may be possible to do better using ideas from [9], [10].

Proof of Theorem 3(b). We can follow the proof of Theorem 1.8 of [4] (in the case $k \geqslant 4$) verbatim, replacing K by $J := k(k-1)$. The role of Lemma 5.2 of [4] is played by Theorem 4 in conjunction with [3, Lemma 8.6]. ■

Proof of Theorem 3(a). Write $J = k(k-1)$ again. We assume that there is no solution of

$$\|\beta_1 n_1^k + \cdots + \beta_s n_s^k\| \leqslant N^{-s/J+\varepsilon} \tag{3.4}$$

with $0 \leqslant n_1, \ldots, n_s \leqslant N$, $(n_1, \ldots, n_s) \neq \mathbf{0}$. Let

$$S_i(m) = \sum_{n=1}^{N} e(m\beta_i n^k), \qquad L = [N^{s/J-\varepsilon}].$$

Following [4], Lemma 5.1, we find that there is a set $\mathcal{B}$ of natural numbers, $\mathcal{B} \subset [1, L]$, and there are positive numbers $B_1 \geqslant \cdots \geqslant B_s$ such that

$$B_i < |S_i(m)| \leqslant 2B_i \qquad (i = 1, \ldots, s)$$

and

$$B_1 \ldots B_s \, |\mathcal{B}| \gg N^{s-\eta}.$$

(This may require a reordering of $\beta_1, \ldots, \beta_s$.) We can now follow the proof of Lemma 5.4 of [4], with K replaced by J, to obtain the inequality

$$|\mathcal{B}| \ll LN^{-1+2k\eta} |\mathcal{B}|^{k/s}.$$

Suppose first that $s > k$. Then

$$LN^{-1+2k\eta} \gg |\mathcal{B}|^{1-k/s} \gg 1,$$

contrary to the definition of L.

Suppose now that $s \leqslant k$. Then

$$L^{\frac{k}{s}-1} \geqslant |\mathcal{B}|^{\frac{k}{s}-1} \gg L^{-1} N^{1-2k\eta},$$

$$L \gg N^{\frac{s}{k}-2s\eta}.$$

This is again contrary to the definition of L, and we conclude that there is a solution of (3.4). ■

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