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FINITE MORDELL-TORNHEIM MULTIPLE ZETA VALUES

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Abstract: We investigate a finite analogue of the Mordell-Tornheim multiple zeta values (the finite Mordell-Tornheim multiple zeta values). These values can be expressed by a linear combination of finite multiple zeta values, and its rules are described by the shuffle product. As a corollary, we give a certain relation among finite multiple zeta values.

Keywords: Mordell-Tornheim multiple zeta values; finite multiple zeta values.

1. Introduction

The multiple zeta values are defined by

$$\zeta(k_1, \ldots, k_r) := \sum_{\substack{m_1 > \cdots > m_r > 0 \\ m_i \in \mathbb{Z}}} \frac{1}{m_1^{k_1} \cdots m_r^{k_r}}$$

for positive integers $k_1, \ldots, k_r$ with $k_1 \geqslant 2$. The numbers $k := k_1 + \cdots + k_r$ and r are called the weight and the depth of $\zeta(k_1, \ldots, k_r)$, respectively. The case of depth 2 was studied by Euler, and general cases have been studied by many authors from 1990's. Many linear relations among multiple zeta values are known, and one of the goals of the study of multiple zeta values is to determine all such relations.

For positive integers $k_1, \ldots, k_{r+1}$, the Mordell-Tornheim multiple zeta values are defined by

$$\zeta^{MT}(k_1, \ldots, k_r; k_{r+1}) := \sum_{m_1, \ldots, m_r \geqslant 1} \frac{1}{m_1^{k_1} \cdots m_r^{k_r} (m_1 + \cdots + m_r)^{k_{r+1}}}. \quad (1)$$

These types of sums were first studied by Tornheim [10] and Mordell [6] and many relations are given. Tsumura proved that the value $\zeta^{MT}(k_1, \ldots, k_r; k_{r+1})$ can be expressed as a rational linear combination of products of the Mordell-Tornheim multiple zeta values of lower depth than r, when its depth r and weight

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$k_1+\cdots+k_{r+1}$ are of different parity. Note that Tornheim [10] proved this result for the case $r=2$. Matsumoto [7] considered the function (1) as an $(r+1)$-variable complex function and proved that this function can be meromorphically continued to the whole $\mathbb{C}^{r+1}$ space.

Recently, Kaneko-Zagier introduced a finite analogue of multiple zeta values, called *finite multiple zeta values* (cf. [4]). Let $\mathcal{A} := \prod_p \mathbb{Z}/p\mathbb{Z} / \bigoplus_p \mathbb{Z}/p\mathbb{Z}$ where p runs over all primes. The ring $\mathcal{A}$ naturally becomes $\mathbb{Q}$-algebra.

For positive integers $k_1,\ldots,k_r$, finite multiple zeta values are defined by

$$\zeta_{\mathcal{A}}(k_1,\ldots,k_r) := \left(\sum_{\substack{p>m_1>\cdots>m_r>0 \\ m_i\in\mathbb{Z}}} \frac{1}{m_1^{k_1}\cdots m_r^{k_r}} \right)_p \in \mathcal{A}.$$

From now on, we denote $(a_p)_p \in \mathcal{A}$ simply by a_p. Hence the definition above is written as

$$\zeta_{\mathcal{A}}(k_1,\ldots,k_r) = \sum_{\substack{p>m_1>\cdots>m_r>0 \\ m_i\in\mathbb{Z}}} \frac{1}{m_1^{k_1}\cdots m_r^{k_r}}.$$

Similar to the usual multiple zeta values, there are many relations among finite multiple zeta values. For example, the following identities are known.

Proposition 1.1 ([3]). *For any positive integers $k_1,\ldots,k_r$ and k, the following identities hold:*

1. $\zeta_{\mathcal{A}}(k,\ldots,k)=0$,
2. $\zeta_{\mathcal{A}}(k_1,\ldots,k_r) = (-1)^{k_1+\cdots+k_r}\zeta_{\mathcal{A}}(k_r,\ldots,k_1)$,
3. $\sum_{\sigma\in\mathfrak{S}_r} \zeta_{\mathcal{A}}(k_{\sigma(1)},\ldots,k_{\sigma(r)}) = 0$ *where $\mathfrak{S}_r$ is the symmetric group of degree r.*

For other relations, see [2], [3], [8] and [9].

As a finite analogue of the Mordell-Tornheim multiple zeta values, we define the *finite Mordell-Tornheim multiple zeta values* by

$$\zeta_{\mathcal{A}}^{MT}(k_1,\ldots,k_r;k_{r+1}) := \sum_{\substack{m_1,\ldots,m_r>0 \\ m_1+\cdots+m_r<p}} \frac{1}{m_1^{k_1}\cdots m_r^{k_r}(m_1+\cdots+m_r)^{k_{r+1}}} \in \mathcal{A}$$

for positive integers $k_1,\ldots,k_{r+1}$. It is clear that $\zeta_{\mathcal{A}}^{MT}(k_1,0;k_3)=\zeta_{\mathcal{A}}(k_3,k_1)$.

Following Hoffman [2], we introduce the algebraic setup of finite multiple zeta values. Let $\mathfrak{H} := \mathbb{Q}\langle x,y\rangle$ be the non-commutative polynomial ring over $\mathbb{Q}$ in two indeterminates x and y, and $\mathfrak{H}^1$ its subring $\mathbb{Q}+\mathfrak{H}y$. The shuffle product ш on $\mathfrak{H}$ is a $\mathbb{Q}$-bilinear map $\mathfrak{H}\times\mathfrak{H}\to\mathfrak{H}$ satisfying

$$w ш 1 = 1 ш w = w, \qquad (u_1w_1) ш (u_2w_2) = u_1(w_1 ш (u_2w_2)) + u_2((u_1w_1) ш w_2) \quad (2)$$

for $w, w_i \in \mathfrak{H}$ and $u_i = x$ or y $(i=1,2)$. We denote $x^{k-1}y$ by z_k for $k \geqslant 1$, and define the $\mathbb{Q}$-linear map $Z_{\mathcal{A}} : \mathfrak{H}^1 \to \mathcal{A}$ satisfying $Z_{\mathcal{A}}(z_{k_1}z_{k_2}\cdots z_{k_r}) = \zeta_{\mathcal{A}}(k_1,k_2,\ldots,k_r)$. For example, $Z_{\mathcal{A}}(x^2yxy) = Z_{\mathcal{A}}(z_3z_2) = \zeta_{\mathcal{A}}(3,2)$.

The finite Mordell-Tornheim multiple zeta values have the following expression.

Theorem 1.2. *For integers $k_1, \ldots, k_r \geqslant 1$ and $l \geqslant 0$, the following identity holds:*

$$\zeta_{\mathcal{A}}^{MT}(k_1, \ldots, k_r; l) = Z_{\mathcal{A}}(x^l(z_{k_1} ш \cdots ш z_{k_r})). \tag{3}$$

As a main result of the paper, we will prove this type of relation in a more general setting (see Theorem 2.1) and Theorem 1.2 is its special case. The right-hand side of (3) can be expressed by a linear combination of finite multiple zeta values, hence the finite Mordell-Tornheim multiple zeta values can be expressed by a linear combination of finite multiple zeta values.

2. Finite Mordell-Tornheim multiple zeta values and their generalization

For non-negative integers $k_1, \ldots, k_i, l_i, \ldots, l_r$ with $1 \leqslant i \leqslant r$, we define the function

$$T_{\mathcal{A}}((k_1, \ldots, k_i); (l_i, \ldots, l_r)) := \sum_{\substack{m_1, \ldots, m_r \geqslant 1 \\ m_1 + \cdots + m_r \leqslant p-1}} \frac{1}{m_1^{k_1} \cdots m_i^{k_i} N_i^{l_i} N_{i+1}^{l_{i+1}} \cdots N_r^{l_r}} \in \mathcal{A},$$

where $N_k := m_1 + \cdots + m_k$ for $1 \leqslant k \leqslant r$. This function contains finite multiple zeta values and the finite Mordell-Tornheim multiple zeta values. Indeed, $T_{\mathcal{A}}((k_1); (0, l_2, \ldots, l_r)) = \zeta_{\mathcal{A}}(l_r, \ldots, l_2, k_1)$ and $T_{\mathcal{A}}((k_1, \ldots, k_r); (l_r)) = \zeta_{\mathcal{A}}^{MT}(k_1, \ldots, k_r; l_r)$.

The following is the main result of the present paper. Note that this theorem includes Theorem 1.2 because $T_{\mathcal{A}}((k_1, \ldots, k_r); (l_r)) = \zeta_{\mathcal{A}}^{MT}(k_1, \ldots, k_r; l_r)$ in the case $i = r$.

Theorem 2.1. *Let i and r be integers with $1 \leqslant i \leqslant r$. For integers $k_1, \ldots, k_i \geqslant 1$, $l_i \geqslant 0$ and $l_{i+1}, \ldots, l_r \geqslant 1$, we have*

$$T_{\mathcal{A}}((k_1, \ldots, k_i); (l_i, \ldots, l_r)) = Z_{\mathcal{A}}(z_{l_r} \cdots z_{l_{i+1}} x^{l_i}(z_{k_1} ш \cdots ш z_{k_i})).$$

In particular, the value $T_{\mathcal{A}}((k_1, \ldots, k_i); (l_i, \ldots, l_r))$ can be expressed by a linear combination of finite multiple zeta values of weight $\sum_{s=1}^{i} k_s + \sum_{t=i}^{r} l_t$.

Proof. We prove the theorem by induction on $k_1 + \cdots + k_i$. We first consider the case $(k_1, \ldots, k_i) = (1, \ldots, 1)$. By using the partial fraction decomposition

$$\frac{1}{m_1 \cdots m_i} = \frac{1}{m_1 + \cdots + m_i} \sum_{1 \leqslant c \leqslant i} \underbrace{\frac{1}{m_1 \cdots m_i}}_{\text{remove } c\text{-th}}, \tag{4}$$

we have

$$T_{\mathcal{A}}((1,\dots,1);(l_i,\dots,l_r)) = \sum_{\substack{m_1,\dots,m_r\geqslant 1\\ m_1+\cdots+m_r\leqslant p-1}} \frac{1}{m_1\cdots m_i N_i^{l_i}\cdots N_r^{l_r}}$$

$$= \sum_{\substack{m_1,\dots,m_r\geqslant 1\\ m_1+\cdots+m_r\leqslant p-1}} \sum_{1\leqslant c\leqslant i} \frac{1}{\underbrace{m_1\cdots m_i}_{\text{1531:remove } c\text{-th}}} \frac{1}{N_i^{l_i+1}N_{i+1}^{l_{i+1}}\cdots N_r^{l_r}}$$

$$= \sum_{\substack{m_1,\dots,m_r\geqslant 1\\ m_1+\cdots+m_r\leqslant p-1}} \frac{i}{m_1\cdots m_{i-1}} \frac{1}{N_i^{l_i+1}N_{i+1}^{l_{i+1}}\cdots N_r^{l_r}}.$$

By repeating this procedure, we have

$$T_{\mathcal{A}}((1,\dots,1);(l_i,\dots,l_r))$$
$$= i! \sum_{\substack{m_1,\dots,m_r\geqslant 1\\ m_1+\cdots+m_r\leqslant p-1}} \frac{1}{m_1(m_1+m_2)\cdots(m_1+\cdots+m_{i-1})} \frac{1}{N_i^{l_i+1}N_{i+1}^{l_{i+1}}\cdots N_r^{l_r}}$$
$$= i!\zeta_{\mathcal{A}}(l_r,\cdots,l_{i+1},l_i+1,\underbrace{1,\cdots,1}_{i-1}).$$

On the other hand, since $\underbrace{z_1 ш \cdots ш z_1}_{i} = i!\, z_1^i$, we have

$$Z_{\mathcal{A}}(z_{l_r}\cdots z_{l_{i+1}} x^{l_i}(\underbrace{z_1 ш \cdots ш z_1}_{i})) = i! Z_{\mathcal{A}}(z_{l_r}\cdots z_{l_{i+1}} z_{l_i+1}\underbrace{z_1\cdots z_1}_{i-1}))$$
$$= i!\zeta_{\mathcal{A}}(l_r,\cdots,l_{i+1},l_i+1,\underbrace{1,\cdots,1}_{i-1}).$$

Hence the assertion holds for the case $(k_1,\dots,k_i)=(1,\dots,1)$.

Next we assume that the assertion holds for all $(k_1,\dots,k_i)$ with $k_1+\cdots+k_i=k$. Let $(k_1,\dots,k_i)\in\mathbb{N}^i$ with $k_1+\cdots+k_i=k+1$. Without loss of generality, we may assume that $(k_1,\dots,k_i)=(k_1,\dots,k_j,1,\dots,1)$ where $k_1,\dots,k_j\geqslant 2$ for some j. Then, by (4) again,

$$T_{\mathcal{A}}((k_1,\dots,k_i);(l_i,\dots,l_r))$$
$$= \sum_{\substack{m_1,\dots,m_r\geqslant 1\\ m_1+\cdots+m_r\leqslant p-1}} \frac{1}{m_1^{k_1-1}\cdots m_i^{k_i} N_i^{l_i+1}N_{i+1}^{l_{i+1}}\cdots N_r^{l_r}} + \cdots$$
$$+ \sum_{\substack{m_1,\dots,m_r\geqslant 1\\ m_1+\cdots+m_r\leqslant p-1}} \frac{1}{m_1^{k_1}\cdots m_j^{k_j-1} m_{j+1}\cdots m_i N_i^{l_i+1}N_{i+1}^{l_{i+1}}\cdots N_r^{l_r}}$$
$$+ (i-j) \sum_{\substack{m_1,\dots,m_r\geqslant 1\\ m_1+\cdots+m_r\leqslant p-1}} \frac{1}{m_1^{k_1}\cdots m_j^{k_j} m_{j+1}\cdots m_{i-1} N_i^{l_i+1}N_{i+1}^{l_{i+1}}\cdots N_r^{l_r}}.$$

We use the notation $z_1^{\text{ш} r} = \underbrace{z_1 \text{ш} \cdots \text{ш} z_1}_{r}$. Then, by the inductive assumption, we have

$$\begin{aligned} &T_{\mathcal{A}}((k_1, \ldots, k_i); (l_i, \ldots, l_r)) \\ &\qquad = Z_{\mathcal{A}}(z_{l_r} \cdots z_{l_{i+1}} x^{l_i+1}(z_{k_1-1} \text{ш} \cdots \text{ш} z_{k_j} \text{ш} z_1^{\text{ш}(i-j)})) \\ &\qquad\quad + \cdots + Z_{\mathcal{A}}(z_{l_r} \cdots z_{l_{i+1}} x^{l_i+1}(z_{k_1} \text{ш} \cdots \text{ш} z_{k_j-1} \text{ш} z_1^{\text{ш}(i-j)})) \\ &\qquad\quad + (i-j) Z_{\mathcal{A}}(z_{l_r} \cdots z_{l_{i+1}} z_{l_i+1}(z_{k_1} \text{ш} \cdots \text{ш} z_{k_j} \text{ш} z_1^{\text{ш}(i-j-1)})). \end{aligned}$$

On the other hand, by (2), we have

$$\begin{aligned} &Z_{\mathcal{A}}(z_{l_r} \cdots z_{l_{i+1}} x^{l_i}(z_{k_1} \text{ш} \cdots \text{ш} z_{k_i})) \\ &\qquad = Z_{\mathcal{A}}(z_{l_r} \cdots z_{l_{i+1}} x^{l_i+1}(z_{k_1-1} \text{ш} \cdots \text{ш} z_{k_j} \text{ш} z_1^{\text{ш}(i-j)})) \\ &\qquad\quad + \cdots + Z_{\mathcal{A}}(z_{l_r} \cdots z_{l_{i+1}} x^{l_i+1}(z_{k_1} \text{ш} \cdots \text{ш} z_{k_j-1} \text{ш} z_1^{\text{ш}(i-j)})) \\ &\qquad\quad + (i-j) Z_{\mathcal{A}}(z_{l_r} \cdots z_{l_{i+1}} x^{l_i} y(z_{k_1} \text{ш} \cdots \text{ш} z_{k_j} \text{ш} z_1^{\text{ш}(i-j-1)})). \end{aligned}$$

Therefore the assertion holds for $(k_1, \ldots, k_i)$ and this completes the proof. ■

Remark 2.2. This method can be applied for the classical Mordell-Tornheim multiple zeta values. For example, one can prove the identity

$$\zeta^{MT}(k_1, \ldots, k_r; l) = Z(x^l(z_{k_1} \text{ш} \cdots \text{ш} z_{k_r})) \qquad (k_1, \ldots, k_r, l \geqslant 1). \tag{5}$$

Here $Z : \mathfrak{H}^0 \to \mathbb{R}$ is the $\mathbb{Q}$-linear map satisfying $Z(z_{k_1} \cdots z_{k_r}) = \zeta(k_1, \ldots, k_r)$ where $\mathfrak{H}^0 := \mathbb{Q} + x\mathfrak{H}y$. Bradley and Zhou [1, Theorem 1.1] proved that $\zeta^{MT}(k_1, \ldots, k_r; l)$ can be written in the form of a linear combination of the usual multiple zeta values. Equation (5) can be regarded as an explicit expression of their result.

Remark 2.3. Kuba [5] introduced a finite analogue T_N of the Mordell-Tornheim multiple zeta values as

$$T_N := \sum_{\substack{m_1+m_2 \leqq N \\ m_1, m_2 \geqslant 1}} \frac{1}{m_1^{k_1} m_2^{k_2} (m_1+m_2)^{k_3}} \qquad (k_1, k_2, k_3 \geqslant 1)$$

for any positive integer N. He proved an identity which expresses T_N in terms of finite analogue of multiple zeta values ([5, Theorem 5]). In our setting ($N = p-1$), his result can be written in the following form:

$$\begin{aligned} \zeta_{\mathcal{A}}^{MT}(k_1, k_2; l) = &\sum_{i=k_2}^{k_1+k_2-1} \binom{i-1}{k_2-1} \zeta_{\mathcal{A}}(l+i, k_1+k_2-i) \\ &+ \sum_{j=k_1}^{k_1+k_2-1} \binom{j-1}{k_1-1} \zeta_{\mathcal{A}}(l+j, k_1+k_2-j) \end{aligned} \tag{6}$$

for $k_1, k_2 \geqslant 1$ and $l \geqslant 0$. This equation also follows from our Theorem 1.2 because it hold that

$$z_{k_1} ш z_{k_2} = \sum_{i=k_2}^{k_1+k_2-1} \binom{i-1}{k_2-1} z_i z_{k_1+k_2-i} + \sum_{j=k_1}^{k_1+k_2-1} \binom{j-1}{k_1-1} z_j z_{k_1+k_2-j}.$$

3. Relations for finite multiple zeta values

In this section, we give some linear relations among finite multiple zeta values by using Theorem 2.1. First we give the following lemma, which is an analogue of Proposition 1.1 (ii).

Lemma 3.1. *For integers $k_1, \ldots, k_r, l \geqslant 0$, the following identity holds:*

$$\zeta_{\mathcal{A}}^{MT}(k_1, \ldots, k_r; l) = (-1)^{k_1+l} \zeta_{\mathcal{A}}^{MT}(l, k_2, \ldots, k_r; k_1). \tag{7}$$

Proof. By changing the variables as $m_1 \mapsto p - m_1 - \cdots - m_r$ in the summation, we have

$$\begin{aligned}
\zeta_{\mathcal{A}}^{MT}(k_1, \ldots, k_r; l) &= \sum_{\substack{m_1, \ldots, m_r \geqslant 1 \\ m_1 + \cdots + m_r \leqslant p-1}} \frac{1}{m_1^{k_1} \cdots m_r^{k_r} (m_1 + \cdots + m_r)^l} \\
&= \sum_{\substack{m_1, \ldots, m_r \geqslant 1 \\ m_1 + \cdots + m_r \leqslant p-1}} \frac{1}{(p - m_1 - \cdots - m_r)^{k_1} m_2^{k_2} \cdots m_r^{k_r} (p - m_1)^l} \\
&= \sum_{\substack{m_1, \ldots, m_r \geqslant 1 \\ m_1 + \cdots + m_r \leqslant p-1}} \frac{(-1)^{k_1+l}}{m_1^{l} m_2^{k_2} \cdots m_r^{k_r} (m_1 + \cdots + m_r)^{k_1}} \\
&= (-1)^{k_1+l} \zeta_{\mathcal{A}}^{MT}(l, k_2, \ldots, k_r; k_1).
\end{aligned}$$

■

Theorem 3.2. *Let r be a positive integer. For positive integers $k_1, \ldots, k_r$, the following identities hold:*

$$Z_{\mathcal{A}}(x^l (z_{k_1} ш \cdots ш z_{k_r})) = (-1)^{k_1+l} Z_{\mathcal{A}}(x^{k_1}(z_l ш z_{k_2} ш \cdots ш z_{k_r})) \qquad (l \geqslant 1). \tag{8}$$

$$Z_{\mathcal{A}}(z_{k_1} ш \cdots ш z_{k_r}) = (-1)^{k_1} Z_{\mathcal{A}}(z_{k_1}(z_{k_2} ш \cdots ш z_{k_r})). \tag{9}$$

Both sides of (8) and (9) can be expressed in terms of finite multiple zeta values, hence these equations give linear relations among finite multiple zeta values.

Proof of Theorem 3.2. By Lemma 3.1, we have

$$\zeta_{\mathcal{A}}^{MT}(k_1, \ldots, k_r; l) = (-1)^{k_1+l} \zeta_{\mathcal{A}}^{MT}(l, k_2, \ldots, k_r; k_1)$$

for positive integers $k_1, \ldots, k_r, l$. Equation (8) is obtained by Theorem 1.2.

To prove (9), we consider

$$\begin{aligned}
\zeta_{\mathcal{A}}^{MT}(k_1, \ldots, k_r; 0) &= (-1)^{k_1} \zeta_{\mathcal{A}}^{MT}(0, k_2, \ldots, k_r; k_1) \\
&= (-1)^{k_1} T_{\mathcal{A}}((k_2, \ldots, k_r); (0, k_1)),
\end{aligned}$$

which is also obtained by Lemma 3.1. Then (9) is obtained by Theorem 2.1. ■

Remark 3.3. Equation (9) can be immediately follows from the equation (2.3) in [4].

In the last of this paper, we give some examples.

For $(k_1,\ldots,k_r)=(k_1,k_2,\ldots,k_{r-t},\overbrace{1,\ldots,1}^{t})$ $(k_{r-t}\geqslant 2)$, we define $u_{k_1,\ldots,k_r}=t+1$. For example, $u_{2,3}=1$, $u_{3,1}=2$ and $u_{1,1,1}=4$. Then the following identity holds:

Proposition 3.4. *For positive integers k, l and r, it holds that*

$$(-1)^l\sum_{m_1+\cdots+m_r=k+r}u_{m_1,\ldots,m_r}\zeta_{\mathcal{A}}(l+1,m_1,\ldots,m_r)\\
=(-1)^k\sum_{m_1+\cdots+m_r=l+r}u_{m_1,\ldots,m_r}\zeta_{\mathcal{A}}(k+1,m_1,\ldots,m_r).$$

Proof. Let $Y_r(k,l):=Z_{\mathcal{A}}(x^l(z_k ш y^{r-1}))$. Since

$$\begin{aligned}z_k ш y^{r-1}&=\sum_{i_1+\cdots+i_r=k+r-1}u_{i_2,\ldots,i_r}x^{i_1-1}yx^{i_2-1}y\cdots x^{i_r-1}y\\
&=\sum_{i_1+\cdots+i_r=k+r-1}u_{i_2,\ldots,i_r}z_{i_1}z_{i_2}\cdots z_{i_r},\end{aligned}$$

we have

$$\begin{aligned}Y_r(k,l)&=\sum_{i_1+\cdots+i_r=k+r-1}u_{i_2,\ldots,i_r}\zeta_{\mathcal{A}}(l+i_1,i_2,\ldots,i_r)\\
&=\sum_{\substack{i_1+\cdots+i_r=k+l+r-1\\ i_1\geqslant l+1}}u_{i_2,\ldots,i_r}\zeta_{\mathcal{A}}(i_1,i_2,\ldots,i_r).\end{aligned}$$

Hence it holds that

$$Y_r(k+1,l)-Y_r(k,l+1)=\sum_{i_2+\cdots+i_r=k+r-1}u_{i_2,\ldots,i_r}\zeta_{\mathcal{A}}(l+1,i_2,\ldots,i_r).$$

By putting $k_1=k$ and $k_2=\cdots=k_r=1$ in (8), we have $Y_r(k,l)=(-1)^{k+l}Y_r(l,k)$. Therefore

$$\sum_{i_2+\cdots+i_r=k+r-1}u_{i_2,\ldots,i_r}\zeta_{\mathcal{A}}(l+1,i_2,\ldots,i_r)\\
=(-1)^{k+l}\sum_{i_2+\cdots+i_r=l+r-1}u_{i_2,\ldots,i_r}\zeta_{\mathcal{A}}(k+1,i_2,\ldots,i_r).$$

By replacing $r-1$ by r, we obtain the result. ■

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