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EXPLICIT BOUNDS ON THE LOGARITHMIC DERIVATIVE AND THE RECIPROCAL OF THE RIEMANN ZETA-FUNCTION

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Abstract: The purpose of this article is consider $|\zeta'(\sigma+it)/\zeta(\sigma+it)|$ and $|\zeta(\sigma+it)|^{-1}$ when σ is close to unity. We prove that $|\zeta'(\sigma+it)/\zeta(\sigma+it)| \leqslant 87\log t$ and $|\zeta(\sigma+it)|^{-1} \leqslant 6.9\times 10^6 \log t$ for $\sigma \geqslant 1-1/(8\log t)$ and $t \geqslant 45$.

Keywords: Riemann zeta-function, prime number theorem, zero-free region.

1. Introduction

Consider $\mu(n)$ the Möbius function, $M(x)=\sum_{n\leqslant x}\mu(n)$ and $m(x)=\sum_{n\leqslant x}\mu(n)/n$. It is known that $M(x)/x$ and $m(x)$ both tend to zero as x tends to infinity. Schoenfeld [10] showed that $|M(x)|/x \leqslant 2.9/(\log x)$ for $x>1$; this was improved by Ramaré [9] who showed that $|M(x)|/x \leqslant 0.013/(\log x)$ for $x \geqslant 1.1\times 10^6$. Ramaré [*op. cit.*] also proved that $|m(x)| \leqslant 0.026/(\log x)$ for $x \geqslant 61000$.

One can produce explicit bounds of the form

$$|m(x)| \leqslant C_1 \log^3 x \exp(-C_2\sqrt{\log x}), \tag{1.1}$$

where $C_1, C_2 > 0$, by following the arguments in §3.13 in [13]. Indeed, since $\sum_{n=1}^{\infty}\mu(n)/n^s = \zeta(s)^{-1}$ for all $\Re s = \sigma > 1$, one can use Perron's formula to show that

$$\sum_{n<x}\frac{\mu(n)}{n^{1+it}} = \frac{1}{2\pi i}\int_{c-iT}^{c+iT}\frac{1}{\zeta(1+it+w)}\frac{x^w}{w}\,dw + E(c,x,T), \tag{1.2}$$

where $c>0$ and $E(c,x,T)$ is an error term that can be estimated explicitly. If one has an explicit zero-free region for $\zeta(s)$, and an explicit bound for $|\zeta(s)|^{-1}$ in $\sigma \geqslant 1-1/(W\log t)$, then one may apply Cauchy's theorem to the integral in (1.2) and prove (1.1) with $C_2 = 1/W$. One can recover explicit bounds for $M(x)$ using (1.1) and partial summation.

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Similarly, if one has a bound for $|\zeta'(s)/\zeta(s)|$ one may follow §3.14 in [13] to bound $\psi(x) = \sum_{n\leqslant x}\Lambda(n)$, where $\Lambda(n)$ is the von Mangoldt function. Finally, one can consider $L(x) = \sum_{n<x}\lambda(n)$, where $\lambda(n)$ is Liouville's function, which defines the Dirichlet series $\zeta(2s)/\zeta(s) = \sum_{n=1}^{\infty}\lambda(n)n^{-s}$, for $\sigma > 1$. Provided that we have an explicit bound for $|\zeta(s)|^{-1}$, we may apply Perron's formula to obtain an explicit bound for $L(x)$.

Given the applications to $M(x), m(x), \psi(x)$ and $L(x)$, it seems natural to try to obtain an explicit bound for $|\zeta(s)|^{-1}$ and for $|\zeta'(s)/\zeta(s)|$. The point of this article is to prove

Theorem 1. *For $t \geqslant 45$ and for $\sigma \geqslant 1 - 1/(8\log t)$ we have*

$$\left|\frac{\zeta'(s)}{\zeta(s)}\right| \leqslant 87\log t, \qquad \frac{1}{|\zeta(s)|} \leqslant 6.9\times 10^6\log t. \tag{1.3}$$

Moreover, for s in the region $t \geqslant t_0$ and $\sigma \geqslant 1 - 1/(W\log t)$, bounds of the sort $|\zeta'(s)/\zeta(s)| \leqslant R_1\log t$ and $|\zeta(s)|^{-1} \leqslant R_2\log t$ are given in Table 1.

The method of proof follows that in Titchmarsh [13, pp. 56-60]. In §2 explicit versions of Titchmarsh's Lemmas α and γ are given. These were first annunciated by Landau [5]. Bounds similar to those in (1.3), but without explicit constants, were proved by Gronwall [2, p. 96].

Landau's method contains two steps. First, one uses good bounds for $\zeta(s)$ near $\sigma = 1$ to deduce a zero-free region near $\sigma = 1$. Second, the bound on $\zeta(s)$ and the zero-free region are used to bound $|\zeta'(s)/\zeta(s)|$ and $|\zeta(s)|^{-1}$. We break into this argument after the first step. Instead of using the zero-free region obtained by Landau's method we use the one obtained by Kadiri [4]. This sharper zero-free region enables us to obtain relatively good bounds on $|\zeta'(s)/\zeta(s)|$ and $|\zeta(s)|^{-1}$.

It should be remarked that Ford's [1] theorem, that

$$|\zeta(\sigma+it)| \leqslant 76.2t^{4.45(1-\sigma)^{3/2}}\log^{2/3}t, \qquad \left(t\geqslant 3,\ \tfrac{1}{2}\leqslant\sigma\leqslant 1\right),$$

could be used to obtain results of the form

$$\frac{1}{|\zeta(s)|} \leqslant A(\log t)^{2/3}(\log\log t)^{1/3},$$

for some constant A, as well as a similar result for $|\zeta'(s)/\zeta(s)|$. One could burn the extra candle and estimate the size of the constant A. However it is likely that such results would improve on those in Theorem 1 only when t is extremely large.

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2. Preparatory lemmas

Lemma 1. *Let $f(s)$ be regular and let $\left|\frac{f(s)}{f(s_0)}\right| \leqslant A_1$ in $|s-s_0| \leqslant r$. Then*

$$\left|\frac{f'(s)}{f(s)} - \sum_{\rho} \frac{1}{s-\rho}\right| \leqslant \frac{4\log A_1}{r(1-2\alpha)^2}, \qquad (|s-s_0| \leqslant \alpha r), \tag{2.1}$$

where ρ runs through the zeroes of $f(s)$ for which $|s-\rho| \leqslant \frac{1}{2}r$, and where $\alpha < \frac{1}{2}$.

Proof. See [11, p. 151]. ■

Whereas Titchmarsh [13, Lemma α] proves Lemma 1 by applying the Borel–Carathéodory theorem and then Cauchy's theorem for derivatives, Tenenbaum in [11] proves Lemma 1 'in one go'. This diminishes the right hand side of (2.1). For example, when $\alpha = \frac{1}{4}$ the proof in [13] gives $48\log A_1/r$, whereas Lemma 1 gives $16\log A_1/r$.

Lemma 2. *Let $f(s)$ satisfy the conditions of Lemma 1, and let $\left|\frac{f'(s_0)}{f(s_0)}\right| \leqslant \frac{A_2}{r}$. Suppose also that $f(s) \neq 0$ when $|s-s_0| \leqslant r$ and $\sigma \geqslant \sigma_0 - \eta r'$, where $\eta > 1$ and $\eta r' \leqslant \alpha r$. Then*

$$\left|\frac{f'(s)}{f(s)}\right| \leqslant \frac{8\alpha\log A_1}{r(\eta-1)(1-2\alpha)^2} + \frac{\eta+1}{\eta-1}\frac{A_2}{r}, \qquad (|s-s_0| \leqslant r').$$

Proof. In the region $|s-s_0| \leqslant \alpha r$, bound the real part of $f'(s)/f(s)$ using Lemma 1 and note that, for $\sigma \geqslant \sigma_0 - \eta r'$, we have $\Re(s-\rho) > 0$. Now apply the Borel–Carathéodory theorem (see, e.g., [12, §5.5]) to the function $-f'(s)/f(s)$ on the circles $|s-s_0| = \eta r'$ and $|s-s_0| = r'$. ■

We shall also require the following bound on $\zeta(s)$ which we shall borrow from [14].

Lemma 3 (Cor. 1 [14]). *Let δ be a positive real number and let*

$$a_0(\sigma, Q_0, t) = \frac{\sigma+Q_0}{2t^2\log t} + \frac{\pi}{2\log t} + \frac{\pi(\sigma+Q_0)^2}{4t\log^2 t}, \qquad a_1(\sigma, Q_0, t) = \frac{\sigma+Q_0}{t}.$$

Then, for $\sigma \in [\frac{1}{2}, 1+\delta]$ and $t \geqslant t_0$ we have

$$|\zeta(s)| \leqslant 0.732(1+a_1(1+\delta,5,t_0))^{7/6}(1+a_0(1+\delta,5,t_0))^2 t^{1/6}\log t, \tag{2.2}$$

provided that

$$t \geqslant \max\{1.16, \exp[4\zeta(1+\delta)/3]\}. \tag{2.3}$$

3. Estimating $|\zeta'(s)/\zeta(s)|$

First consider $t_0 \geqslant H$, where $H = 3.06 \times 10^{10}$ is the height to which the Riemann hypothesis has been verified — see [7]. Let $s_0 = \sigma_0 + it_0 = 1 + \frac{c}{\log t_0} + it_0$, where c is a positive constant to be determined later. We aim at applying Lemma 2 with $r = \frac{1}{2}$. In the region $|s - s_0| \leqslant \frac{1}{2}$ we have

$$\frac{1}{2} \leqslant \sigma \leqslant 1 + \frac{1}{2} + \frac{c}{\log H}, \qquad t \leqslant t_0\left(1 + \frac{1}{2H}\right).$$

We shall apply Lemma 3 with $\delta = \frac{1}{2} + \frac{c}{\log H}$; the condition in (2.3) is certainly met for all $t \geqslant 34$. This shows that

$$|\zeta(s)| \leqslant 0.732\alpha_1 t_0^{\frac{1}{6}} \log t_0, \qquad (|s - s_0| \leqslant \tfrac{1}{2}),$$

where

$$\alpha_1 = (1 + a_1(\tfrac{3}{2} + \tfrac{c}{\log H}, 5, H - \tfrac{1}{2}))^{\frac{7}{6}} (1 + a_0(\tfrac{3}{2} + \tfrac{c}{\log H}, 5, H - \tfrac{1}{2}))^2 \left(1 + \tfrac{1}{2H}\right)^{\frac{7}{6}}. \quad (3.1)$$

We now bound $|\zeta(s_0)|$ trivially using the estimate $|\zeta(s_0)| \geqslant \zeta(2\sigma_0)/\zeta(\sigma_0)$. This, together with (3.1), shows that

$$\left|\frac{\zeta(s)}{\zeta(s_0)}\right| \leqslant A_1 := 0.732\alpha_1 t_0^{\frac{1}{6}} (\log t_0)^2 \frac{X(1 + \frac{c}{\log H})}{c} \quad (3.2)$$

where

$$X(t) = \frac{\zeta(t)(t-1)}{\zeta(2t)}. \quad (3.3)$$

Note that $X(t)$ is increasing and that $\lim_{t \to 1} X(t) = 6\pi^{-2}$.

To bound $|\zeta'(s)/\zeta(s)|$ we use the trivial bound $|\zeta'(s)/\zeta(s)| \leqslant -\zeta'(\sigma)/\zeta(\sigma)$ and Lemma 70.1 in [3], which shows that $-\zeta'(x)/\zeta(x) < 1/(x-1)$ for any real $x > 1$. We therefore have

$$\left|\frac{\zeta'(s_0)}{\zeta(s_0)}\right| \leqslant \frac{A_2}{r}, \qquad \text{where } A_2 = \frac{r \log t_0}{c}, \quad r = \frac{1}{2}. \quad (3.4)$$

3.1. Using the zero-free region

Suppose

$$\zeta(s) \neq 0, \qquad \text{for } \sigma \geqslant 1 - \frac{1}{R \log t}, \quad (t \geqslant 3).$$

Kadiri [4] has shown that one may take $R = 5.69693$. We keep the parameter R in the equations that follow. Let t' be a real number for which

$$\frac{c}{\log t_0} + \frac{1}{R \log(t_0 + t')} < t'.$$

It follows that there are no zeroes of $\zeta(s)$ in the region $|s-s_0| \leqslant 1+\frac{c}{\log t_0}-\frac{1}{R\log(t_0+t')}$. We may convert this into a slightly easier form to show that there are no zeroes of $\zeta(s)$ in the region

$$|s-s_0| \leqslant \frac{c+\frac{1}{\alpha_2 R}}{\log t_0},$$

where

$$\alpha_2 = 1+\frac{t'}{H\log H}.$$

To apply Lemma 2 we choose

$$\eta r' = \alpha r = \frac{1}{2}\alpha = \frac{c+\frac{1}{\alpha_2 R}}{\log t_0},$$

whence

$$r' = \frac{c+\frac{1}{\alpha_2 R}}{\eta\log t_0}, \qquad \alpha \leqslant \frac{2(c+\frac{1}{\alpha_2 R})}{\log H}.$$

We use Lemma 2 and (3.2), (3.3), and (3.4) to prove

Lemma 4. *For $t_0 \geqslant H$*

$$\left|\frac{\zeta'(s)}{\zeta(s)}\right| \leqslant A\log t_0 + B\log\log t_0 + C, \qquad \left(|s-s_0| \leqslant \frac{c+\frac{1}{\alpha_2 R}}{\eta\log t_0}\right), \tag{3.5}$$

where

$$A = \frac{8}{3(\eta-1)(1-2\alpha)^2} + \left(\frac{\eta+1}{\eta-1}\right)\frac{1}{2c},$$
$$B = \frac{32}{(\eta-1)(1-2\alpha)^2},$$
$$C = \frac{16(\log\alpha_1 + \log(0.732X/c))}{(\eta-1)(1-2\alpha)^2}.$$

The bound in (3.5) holds whenever

$$\sigma_0 - \frac{c+\frac{1}{\alpha_2 R}}{\eta\log t_0} \leqslant \sigma \leqslant \sigma_0 + \frac{c+\frac{1}{\alpha_2 R}}{\eta\log t_0}.$$

For larger values of σ_0 we use the trivial bound on $|\zeta'(s)/\zeta(s)|$. Making the substitution $t_0 \mapsto t$ we obtain a bound on $|\zeta'(s)/\zeta(s)|$ for all $\sigma > 1-1/(W\log t)$ for some constant W. The result is summarised in

Theorem 2. *Let*

$$W = \frac{\eta\alpha_2 R}{1+(1-\eta)\alpha_2 Rc}, \qquad (\alpha_2 Rc(\eta-1) < 1). \tag{3.6}$$

Then, for all $t \geqslant H$ *and for* $\sigma \geqslant 1 - 1/(W \log t)$ *we have*

$$\left| \frac{\zeta'(\sigma + it)}{\zeta(\sigma + it)} \right| \leqslant R_1 \log t,$$

where

$$R_1 = \max \left\{ \frac{\eta}{\eta c + c + \frac{1}{\alpha_2 R}}, A + B\frac{\log\log H}{\log H} + \frac{C}{\log H} \right\}. \tag{3.7}$$

For a given W, we solve (3.6) for η and evaluate R_1 in (3.7) by varying $c \in [10^{-4}, 1]$ in increments of 10^{-4}. For example, when $R = 5.69693$, given $W = 8$, choosing $c = 0.1369$ gives $R_1 \leqslant 86.23$.

We now turn to the case when $0 < T_0 < t < H$. In this case there are no zeroes for $\sigma_0 - \eta r' > \frac{1}{2}$. We therefore choose

$$\eta r' = \alpha r = \frac{1}{2}\alpha, \qquad \text{whence} \;\; r' = \frac{\alpha}{2\eta},$$

where we require that α be less than $\frac{1}{2}$ to ensure that the conditions of Lemma 1 are satisfied. We now follow the argument leading to Lemma 4, noting the change of α and of r'. We arrive at a bound for $|\zeta'(s)/\zeta(s)|$ in the region $\sigma \geqslant \sigma_0 - \alpha/(2\eta)$. This region will be at least as wide as that in Theorem 2 if

$$\frac{\frac{\alpha}{2}\log T_0 - c\eta}{\eta} \geqslant \frac{1}{W}. \tag{3.8}$$

We use (3.8) to solve for T_0. We then optimise by varying $\alpha \in [10^{-2}, 1]$ in increments of 10^{-2}, $\eta \in [1.001, 3]$ in increments of 10^{-3}, and $c \in [0.001, 1]$ in increments of 10^{-3}. We compare the value of R_1 thus obtained with that obtained when $t \geqslant H$. For example, when $W = 8$ we have already shown that $R_1 \leqslant 86.23$ for all $t \geqslant H$. Choosing $\alpha = 0.23, c = 0.041, \eta = 2.631$ we have $R_1 \leqslant 86.11$ with $W = 8$ and $t \geqslant 44.61$. We continue in this way for other values of W: the results on R_1 are presented in Table 1.

4. Bounding $1/|\zeta(s)|$

We follow the argument on page 60 of [13]. If $1 - \frac{1}{W \log t} \leqslant \sigma \leqslant 1 + \frac{d}{\log t}$, for some $d > 0$, then, by Theorem 2, we have

$$\begin{aligned} \log \frac{1}{|\zeta(s)|} &\leqslant -\Re \log \zeta(s) \\ &= -\Re \log \zeta\left(1 + \frac{d}{\log t} + it\right) + \int_{\sigma}^{1+\frac{d}{\log t}} \Re \frac{\zeta'}{\zeta}(\xi + it)\, d\xi \\ &\leqslant \log \zeta\left(1 + \frac{d}{\log t}\right) - \log\zeta\left(2\left(1 + \frac{d}{\log t}\right)\right) + R_1\left(d + \frac{1}{W}\right), \end{aligned}$$

for $t \geqslant t_0$ where t_0, W and R_1 are in Table 1. Write

$$\zeta(\sigma) = \zeta(\sigma)(\sigma - 1)/(\sigma - 1) = Y(\sigma)/(\sigma - 1),$$

whence

$$|\zeta(s)|^{-1} \leqslant \frac{Y(1 + \frac{d}{\log t_0})e^{R_1(d+1/W)}}{d\zeta(2(1 + \frac{d}{\log t_0}))} \log t, \qquad (1 - 1/(W \log t) \leqslant \sigma \leqslant 1 + d/\log t). \tag{4.1}$$

If $\sigma_1 \geqslant \sigma \geqslant 1 + \frac{d}{\log t}$ we have

$$|\zeta(s)|^{-1} \leqslant \frac{X(\sigma_1)}{d} \log t. \tag{4.2}$$

Finally, for $\sigma \geqslant \sigma_1$ we have

$$|\zeta(s)|^{-1} \leqslant \frac{\zeta(\sigma_1)}{\zeta(2\sigma_1)} \leqslant \frac{\zeta(\sigma_1)}{\zeta(2\sigma_1) \log t_0} \log t. \tag{4.3}$$

We now optimise the maximum of (4.1), (4.2) and (4.3) by varying $d \in [10^{-4}, 1)$ in increments of 10^{-4}. The values of R_2 are presented in Table 1: this proves Theorem 1.

Table 1: Bounds for $|\zeta'(s)/\zeta(s)| \leqslant R_1 \log t$ and $|\zeta(s)|^{-1} \leqslant R_2 \log t$ and in $\sigma \geqslant 1 - 1/(W \log t)$ for $t \geqslant t_0$

W	R_1	R_2	t_0
6	548.53	7.8×10^{43}	34
7	140.03	1.3×10^{11}	34
8	86.23	6.9×10^6	44.61
9	64.98	1.5×10^5	63.91
10	53.60	1.9×10^4	79.35
11	46.50	5.3×10^3	95.45
12	41.64	2252	113.30

5. Conclusion

The dominant factor in (4.1) is $d^{-1} \exp(R_1(d + 1/W))$. It is the exponential dependence on R_1 that leads to such large values of R_2 in Table 1. Both R_1 and R_2 would be diminished were one in possession of any of the following: a higher height to which the Riemann hypothesis has been proved (a larger value of H), a wider zero-free region (a smaller value of R), or a better bound on $\zeta(s)$ across the critical strip (improving (2.2)). As noted in [8], the bound in (2.2) appears to be far from optimal. It is hoped that future researchers are able to improve on the methods of attacking this problem.

5.1. Note added in proof

Recently, in [6] it was announced that one could take $R = 5.573412$. Conditional on this bound of R one could refine the bounds in Table 1 as follows.

Table 2: Bounds for $|\zeta'(s)/\zeta(s)| \leqslant R_1 \log t$ and $|\zeta(s)|^{-1} \leqslant R_2 \log t$ and in $\sigma \geqslant 1 - 1/(W \log t)$ for $t \geqslant t_0$ — with $R = 5.573412$

W	R_1	R_2	t_0
6	382.58	3.2×10^{30}	34
7	125.60	1.3×10^{10}	34
8	80.38	3.1×10^{6}	50.28
9	61.54	9.6×10^{4}	70.59
10	51.19	1.5×10^{4}	90.87
11	44.65	4.4×10^{3}	111.12
12	40.14	1900	132.16

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