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SOME PROPERTIES OF VARIABLE BESOV-TYPE SPACES

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Abstract: In this article we introduce Besov-type spaces with variable smoothness and integrability, which unify and generalize the Besov-type spaces with fixed exponents. Under natural regularity assumptions on the exponent functions, we show that our spaces are well-defined, i.e., independent of the choice of basis functions and we establish some properties of these function spaces. Moreover the Sobolev embeddings for these function spaces are obtained.

Keywords: Besov spaces, embeddings, maximal function, variable exponent.

1. Introduction

It is well-known that function spaces have been a central topic in modern analysis, and are now of increasing applications in areas such as harmonic analysis and partial differential equations. Some examples of these spaces can be mentioned such as: Besov and Triebel-Lizorkin spaces, were introduced between 1959 and 1975; see, for example, [29]. They cover many well-known classical concrete function spaces such as Hölder-Zygmund spaces, Sobolev spaces, fractional Sobolev spaces. A comprehensive treatment of these function spaces and their history can be found in Triebel's monographs [30], [31] and [32].

In recent years, there has been increasing interest in a new family of function spaces, called $B_{p,q}^{s,\tau}$ spaces. These spaces unify and generalize many classical function spaces including Besov spaces, Besov-Morrey spaces (see, for example, [38, Corollary 3.3]). The theory of these function spaces has been considered by many researchers. When $s \in \mathbb{R}, \tau \in [0,\infty)$ and $1 \leqslant p,q < \infty$, the $B_{p,q}^{s,\tau}$ spaces were first introduced by El Baraka in [11] and [12]. In these papers, El Baraka investigated embeddings as well as Littlewood-Paley characterizations of Campanato spaces. El Baraka showed that the spaces $B_{p,q}^{s,\tau}$ cover certain Campanato spaces. Later on, the author gave in [7] a characterization for $B_{p,q}^{s,\tau}$ spaces by local means

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and maximal functions. For a complete treatment of $B_{p,q}^{s,\tau}$ spaces we refer the reader to the work of W. Yuan, W. Sickel and D. Yang, [38]. D. Yang and W. Yuan, in [35] introduced the scales of homogeneous Besov spaces $\dot{B}_{p,q}^{s,\tau}$, which generalize the homogeneous Besov spaces $\dot{B}_{p,q}^{s}$. See also [8], [10] and [26] for further results. All these results are obtained when s, τ, p, q are fixed.

In recent years, there has been growing interest in generalizing classical spaces such as Lebesgue, Sobolev spaces, Besov spaces, Triebel-Lizorkin spaces to the case with either variable integrability or variable smoothness. The motivation for the increasing interest in such spaces comes not only from theoretical purposes, but also from applications to fluid dynamics [21], image restoration [3] and PDE with non-standard growth conditions. The survey [20] provides information on existence and regularity results for PDE in the variable exponent setting. This includes also an extensive list of references on this subject. We only refer to the monograph [6] for further details and references on recent developments on this field.

The concept of function spaces with variable smoothness and the concept of variable integrability were first merged by Diening, Hästö and Roudenko in [5]. They defined Triebel-Lizorkin spaces $F_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}$ and proved a discretization by the so called φ-transform. Also atomic and molecular decomposition of these function spaces are obtained and used to derive trace results. The Sobolev embedding of these function spaces was proved by J. Vybíral, [33]. When α, p, q are constants they coincide with the usual function spaces $F_{p,q}^{s}$. Besov spaces of variable smoothness and integrability, $B_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}$, initially appeared in the paper of A. Almeida and P. Hästö [1]. Several basic properties were established, such as the Fourier analytical characterization. When p, q, α are constants they coincide with the usual function spaces $B_{p,q}^{s}$. Also Sobolev type embeddings and the characterization by approximations of these function spaces were obtained. Some properties of these function spaces such as local means characterizations, atomic decomposition and characterizations by ball means of differences can be found in [9] and [15].

The purpose of the present paper is to define and study a generalized scale of Besov-type spaces with variable smoothness and integrability. By setting some of the indices to appropriate values we recover all previously mentioned spaces as special cases.

The paper is organized as follows. First we give some preliminaries where we introduce notation and recall basic facts on function spaces with variable integrability. We also formulate here technical lemmas needed in the proofs of the main statements. For making the presentation clearer, we give their proofs later in Section 5. We then define the Besov-type space $B_{p(\cdot),q(\cdot)}^{\alpha(\cdot),\tau(\cdot)}$ in Section 3 and give several basic properties. We show that our spaces are well-defined, i.e., independent of the choice of basis functions. Moreover, we show that for some special parameters, the spaces $B_{p(\cdot),q(\cdot)}^{\alpha(\cdot),\tau(\cdot)}$ are just the Besov space $B_{\infty,\infty}^{\alpha(\cdot)+n(1/\tau(\cdot)-1/p(\cdot))}$. In Section 4 we prove elementary embeddings between these functions spaces, as well as Sobolev embeddings.

2. Preliminaries

As usual, we denote by $\mathbb{R}^n$ the n-dimensional real Euclidean space, $\mathbb{N}$ the collection of all natural numbers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. The letter $\mathbb{Z}$ stands for the set of all integer numbers. For a multi-index $\alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{N}_0^n$, we write $|\alpha| = \alpha_1 + ... + \alpha_n$. The Euclidean scalar product of $x = (x_1, ..., x_n)$ and $y = (y_1, ..., y_n)$ is given by $x \cdot y = x_1 y_1 + ... + x_n y_n$. The expression $f \lesssim g$ means that $f \leqslant c\, g$ for some independent constant c (and non-negative functions f and g), and $f \approx g$ means $f \lesssim g \lesssim f$. As usual for any $x \in \mathbb{R}$, $[x]$ stands for the largest integer smaller than or equal to x.

For $x \in \mathbb{R}^n$ and $r > 0$ we denote by $B(x, r)$ the open ball in $\mathbb{R}^n$ with center x and radius r. By $\mathrm{supp} f$ we denote the support of the function f , i.e., the closure of its non-zero set. If $E \subset \mathbb{R}^n$ is a measurable set, then $|E|$ stands for the (Lebesgue) measure of E and χ_E denotes its characteristic function.

The symbol $\mathcal{S}(\mathbb{R}^n)$ stands for the set of all Schwartz functions φ on $\mathbb{R}^n$ and we denote by $\mathcal{S}'(\mathbb{R}^n)$ the dual space of all tempered distributions on $\mathbb{R}^n$. We define the Fourier transform of a function $f \in \mathcal{S}(\mathbb{R}^n)$ by

$$\mathcal{F}(f)(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x) dx.$$

Its inverse is denoted by $\mathcal{F}^{-1} f$. Both $\mathcal{F}$ and $\mathcal{F}^{-1}$ are extended to the dual Schwartz space $\mathcal{S}'(\mathbb{R}^n)$ in the usual way.

The Hardy-Littlewood maximal operator $\mathcal{M}$ is defined on L^1_{loc} by

$$\mathcal{M} f(x) = \sup_{r>0} \frac{1}{|B(x, r)|} \int_{B(x,r)} |f(y)|\, dy.$$

2.1. Spaces $\ell^{q(\cdot)}(L^{p(\cdot)})$

The variable exponents that we consider are always measurable functions p on $\mathbb{R}^n$ with range in $[c, \infty[$ for some $c > 0$. We denote the set of such functions by $\mathcal{P}_0$. The subset of variable exponents with range $[1, \infty[$ is denoted by $\mathcal{P}$. We use the standard notation

$$p^- = \underset{x\in\mathbb{R}^n}{\text{ess-inf}}\, p(x), \qquad p^+ = \underset{x\in\mathbb{R}^n}{\text{ess-sup}}\, p(x).$$

We define $\varrho_p(t) = t^p$. The variable exponent modular is defined by

$$\varrho_{p(\cdot)}(f) = \int_{\mathbb{R}^n} \varrho_{p(x)}(|f(x)|) dx.$$

The variable exponent Lebesgue space $L^{p(\cdot)}$ consists of measurable functions f on $\mathbb{R}^n$ such that $\varrho_{p(\cdot)}(\lambda f) < \infty$ for some $\lambda > 0$. We define the Luxemburg (quasi)-norm on this space by the formula

$$\|f\|_{p(\cdot)} = \inf \Big\{ \lambda > 0 : \varrho_{p(\cdot)}\Big(\frac{f}{\lambda}\Big) \leqslant 1 \Big\}.$$

As is known, the following inequalities hold (see [6], Lemma 3.2.4)

$$\|f\|_{p(\cdot)} \leqslant 1 \Longleftrightarrow \varrho_{p(\cdot)}(f) \leqslant 1. \tag{2.1}$$

Moreover,

$$\min\Big(\varrho_{p(\cdot)}\Big(f\Big)^{\frac{1}{p^+}}, \varrho_{p(\cdot)}\Big(f\Big)^{\frac{1}{p^-}}\Big) \leqslant \|f\|_{p(\cdot)} \leqslant \max\Big(\varrho_{p(\cdot)}\Big(f\Big)^{\frac{1}{p^+}}, \varrho_{p(\cdot)}\Big(f\Big)^{\frac{1}{p^-}}\Big), \tag{2.2}$$

if $\varrho_{p(\cdot)}\Big(f\Big) > 0$ or $p^+ < \infty$. Let $p, q \in \mathcal{P}_0$. The mixed Lebesgue-sequence space $\ell^{q(\cdot)}(L^{p(\cdot)})$ is defined on sequences of $L^{p(\cdot)}$-functions by the modular

$$\varrho_{\ell^{q(\cdot)}(L^{p(\cdot)})}((f_v)_v) = \sum_v \inf\Big\{\lambda_v > 0 : \varrho_{p(\cdot)}\Big(\frac{f_v}{\lambda_v^{1/q(\cdot)}}\Big) \leqslant 1\Big\}.$$

The (quasi)-norm is defined as usual:

$$\|(f_v)_v\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} = \inf\Big\{\mu > 0 : \varrho_{\ell^{q(\cdot)}(L^{p(\cdot)})}\Big(\frac{1}{\mu}(f_v)_v\Big) \leqslant 1\Big\}. \tag{2.3}$$

If $q^+ < \infty$, then we can replace (2.3) by the simpler expression

$$\varrho_{\ell^{q(\cdot)}(L^{p(\cdot)})}((f_v)_v) = \sum_v \Big\||f_v|^{q(\cdot)}\Big\|_{\frac{p(\cdot)}{q(\cdot)}}.$$

Furthermore, if p and q are constants, then $\ell^{q(\cdot)}(L^{p(\cdot)}) = \ell^q(L^p)$. The case $p \equiv \infty$ can be included by replacing the last modular by

$$\varrho_{\ell^{q(\cdot)}(L^\infty)}((f_v)_v) = \sum_v \Big\||f_v|^{q(\cdot)}\Big\|_\infty.$$

It is known, cf. [1] and [14], that $\ell^{q(\cdot)}(L^{p(\cdot)})$ is a norm if $q(\cdot) \geqslant 1$ is constant almost everywhere (a.e.) on $\mathbb{R}^n$ and $p(\cdot) \geqslant 1$, or if $\frac{1}{p(x)} + \frac{1}{q(x)} \leqslant 1$ a.e. on $\mathbb{R}^n$, or if $1 \leqslant q(x) \leqslant p(x) < \infty$ a.e. on $\mathbb{R}^n$.

We say that $g : \mathbb{R}^n \to \mathbb{R}$ is *locally* log-*Hölder continuous*, abbreviated $g \in C^{\log}_{\mathrm{loc}}$, if there exists $c_{\log}(g) > 0$ such that

$$|g(x) - g(y)| \leqslant \frac{c_{\log}(g)}{\log(e + 1/|x-y|)} \tag{2.4}$$

for all $x, y \in \mathbb{R}^n$. We say that g satisfies the log-*Hölder decay condition*, if there exists $g_\infty \in \mathbb{R}$ and a constant $c_{\log} > 0$ such that

$$|g(x) - g_\infty| \leqslant \frac{c_{\log}}{\log(e + |x|)}$$

for all $x \in \mathbb{R}^n$. We say that g is *globally*-log-*Hölder continuous*, abbreviated $g \in C^{\log}$, if it is locally log-Hölder continuous and satisfies the log-Hölder decay condition. The constants $c_{\log}(g)$ and $c_{\log}$ are called the *locally* log-*Hölder constant* and the log-*Hölder decay constant*, respectively. We note that all functions $g \in C^{\log}_{\mathrm{loc}}$ always belong to L^∞.

We define the following class of variable exponents

$$\mathcal{P}^{\log} = \Big\{ p \in \mathcal{P} : \frac{1}{p} \text{ is globally-log-Hölder continuous} \Big\}.$$

We define $1/p_\infty := \lim_{|x|\to\infty} 1/p(x)$ and we use the convention $\frac{1}{\infty} = 0$. Note that although $\frac{1}{p}$ is bounded, the variable exponent p itself can be unbounded. It was shown in [6], Theorem 4.3.8, that $\mathcal{M} : L^{p(\cdot)} \to L^{p(\cdot)}$ is bounded if $p \in \mathcal{P}^{\log}$ and $p^- > 1$. Also, if $p \in \mathcal{P}^{\log}$, then convolution with a radially decreasing L^1-function is bounded on $L^{p(\cdot)}$:

$$\|\varphi * f\|_{p(\cdot)} \leqslant c\|\varphi\|_1 \|f\|_{p(\cdot)}.$$

We also refer to the papers [2] and [4], where various results on maximal function in variable Lebesgue spaces were obtained.

Very often we have to deal with the norm of characteristic functions on balls (or cubes) when studying the behavior of various operators in Harmonic Analysis. In classical L^p spaces the norm of such functions is easily calculated, but this is not the case when we consider variable exponents. Nevertheless, it is known that for $p \in \mathcal{P}^{\log}$ we have

$$\|\chi_B\|_{p(\cdot)} \|\chi_B\|_{p'(\cdot)} \approx |B|. \tag{2.5}$$

Also,

$$\|\chi_B\|_{p(\cdot)} \approx |B|^{\frac{1}{p(x)}}, \quad x \in B \tag{2.6}$$

for small balls $B \subset \mathbb{R}^n$ ($|B| \leqslant 2^n$), and

$$\|\chi_B\|_{p(\cdot)} \approx |B|^{\frac{1}{p_\infty}} \tag{2.7}$$

for large balls ($|B| \geqslant 1$), with constants only depending on the log-Hölder constant of p (see, for example, [6, Section 4.5]). Here p' denotes the conjugate exponent of p given by $1/p(\cdot) + 1/p'(\cdot) = 1$. These properties hold true if $p \in \mathcal{P}^{\log}_0$, since $\|\chi_B\|_{p(\cdot)} = \|\chi_B\|^{1/a}_{p(\cdot)/a}$ and $\frac{p}{a} \in \mathcal{P}^{\log}$ if $p^- \geqslant a$.

Recall that $\eta_{v,m}(x) = 2^{nv}(1 + 2^v |x|)^{-m}$, for any $x \in \mathbb{R}^n$, $v \in \mathbb{N}_0$ and $m > 0$. Note that $\eta_{v,m} \in L^1$ when $m > n$ and that $\|\eta_{v,m}\|_1 = c_m$ is independent of v. For $v \in \mathbb{Z}$ and $m = (m_1, ..., m_n) \in \mathbb{Z}^n$, let Q_{vm} be the dyadic cube in $\mathbb{R}^n$,

$$Q_{vm} = \{(x_1, ..., x_n) : m_i \leqslant 2^v x_i < m_i + 1, i = 1, 2, ..., n\}.$$

For the collection of all such cubes we use

$$\mathcal{Q} = \{Q_{vm} : v \in \mathbb{Z}, m \in \mathbb{Z}^n\}.$$

For each cube Q, we denote by x_Q the lower left-corner $2^{-v}m$ of $Q = Q_{vm}$, its side length by $l(Q)$ and for $r > 0$, we denote by rQ the cube concentric with Q having the side length $rl(Q)$. Furthermore, we put $v_Q = -\log_2 l(Q)$ and $v_Q^+ = \max(v_Q, 0)$.

By c we denote generic positive constants, which may have different values at different occurrences. Although the exact values of the constants are usually irrelevant for our purposes, sometimes we emphasize their dependence on certain parameters (e.g. $c(p)$ means that c depends on p, etc.). Further notation will be properly introduced whenever needed.

2.2. Some technical lemmas

In this subsection we present some results which are useful for us. The following lemma is proved [15, Lemma 19], see also [5, Lemma 6.1].

Lemma 2.8. *Let $\alpha \in C^{\log}_{\mathrm{loc}}$ and let $R \geqslant c_{\log}(\alpha)$, where $c_{\log}(\alpha)$ is the constant from (2.4) for α. Then*

$$2^{v\alpha(x)}\eta_{v,m+R}(x-y) \leqslant c\ 2^{v\alpha(y)}\eta_{v,m}(x-y)$$

with $c > 0$ independent of $x, y \in \mathbb{R}^n$ and $v, m \in \mathbb{N}_0$.

The previous lemma allows us to treat the variable smoothness in many cases as if it were not variable at all, namely we can move the term inside the convolution as follows:

$$2^{v\alpha(x)}\eta_{v,m+R} * f(x) \leqslant c\ \eta_{v,m} * (2^{v\alpha(\cdot)}f)(x).$$

The next lemma often allows us to deal with exponents which are smaller than 1, see [5, Lemma A.7].

Lemma 2.9. *Let $r > 0$, $v \in \mathbb{N}_0$ and $m > n$. Then there exists $c = c(r, m, n) > 0$ such that for all $g \in \mathcal{S}'(\mathbb{R}^n)$ with $\operatorname{supp} \mathcal{F}g \subset \{\xi \in \mathbb{R}^n : |\xi| \leqslant 2^{v+1}\}$, we have*

$$|g(x)| \leqslant c(\eta_{v,m} * |g|^r(x))^{1/r}, \qquad x \in \mathbb{R}^n.$$

The next lemma tells us that in most circumstances two convolutions are as good as one, see [5].

Lemma 2.10. *For $v_0, v_1 \in \mathbb{N}_0$ and $m > n$, we have*

$$\eta_{v_0,m} * \eta_{v_1,m} \approx \eta_{\min(v_0,v_1),m}$$

with the constant depending only on m and n.

In classical Lebesgue spaces $\frac{\|\chi_Q\|_\tau}{\|\chi_P\|_\tau} = \left(\frac{|Q|}{|P|}\right)^{1/\tau}$ for any cubes P and Q. We would like to generalize this to the case of a variable exponent τ. It is not clear how to replace $1/\tau$ in the formula by something in terms of τ. However, it turns out that if $\tau \in \mathcal{P}^{\log}_0$, then this is possible in some particular cases.

Lemma 2.11. *Let $\tau \in \mathcal{P}_0^{\log}$ and $k \in \mathbb{Z}^n$.*

(i) *For any cubes P and Q, we have*

$$\frac{\left\|\chi_{P+kl(Q)}\right\|_{\tau(\cdot)}}{\left\|\chi_P\right\|_{\tau(\cdot)}} \leqslant c\left(1+\frac{l(Q)}{l(P)}|k|\right)^{c_{\log}(\frac{1}{\tau})}$$

with $c>0$ independent of $l(Q)$, $l(P)$ and k.

(ii) *For any cubes P and Q, such that $P \subset Q$, we have*

$$C\left(\frac{|Q|}{|P|}\right)^{1/\tau^+} \leqslant \frac{\left\|\chi_Q\right\|_{\tau(\cdot)}}{\left\|\chi_P\right\|_{\tau(\cdot)}} \leqslant c\left(\frac{|Q|}{|P|}\right)^{1/\tau^-}$$

with $c, C>0$ are independent of $|Q|$ and $|P|$.

We introduce the abbreviations

$$\left\|(f_v)_v\right\|_{\ell^{\tau(\cdot),q(\cdot)}(L^{p(\cdot)})} = \sup_{P\in\mathcal{Q}}\left\|\left(\frac{f_v}{\left\|\chi_P\right\|_{\tau(\cdot)}}\chi_P\right)_{v\geqslant v_P^+}\right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})}.$$

Since the maximal operator is in general not bounded on $\ell^{q(\cdot)}(L^{p(\cdot)})$, see [1, Example 4.1] we need a replacement for it. It turned out that a convolution with radial decreasing functions fits very well into the scheme. The following lemma is the $\ell^{\tau(\cdot),q(\cdot)}(L^{p(\cdot)})$-version of Lemma 4.7 from A. Almeida and P. Hästö [1].

Lemma 2.12. *Let $p \in \mathcal{P}^{\log}$ with $1<p^- \leqslant p^+ \leqslant \infty$ and $q, \tau \in \mathcal{P}_0^{\log}$ with $0<q^- \leqslant q^+<\infty$. For $m>2n+c_{\log}(1/\tau)+c_{\log}(1/q)$, there exists $c>0$ such that*

$$\left\|(\eta_{v,m} * f_v)_v\right\|_{\ell^{\tau(\cdot),q(\cdot)}(L^{p(\cdot)})} \leqslant c\left\|(f_v)_v\right\|_{\ell^{\tau(\cdot),q(\cdot)}(L^{p(\cdot)})}.$$

The proof of Lemmas 2.11 and 2.12 is postponed to the Appendix.

3. The spaces $\boldsymbol{B}_{\boldsymbol{p}(\cdot),\boldsymbol{q}(\cdot)}^{\boldsymbol{\alpha}(\cdot),\boldsymbol{\tau}(\cdot)}$

In this section we present the Fourier analytical definition of Besov-type spaces of variable smoothness and integrability and we prove the basic properties in analogy to the Besov-type spaces with fixed exponents. We first need the concept of a resolution of unity. Let Ψ be a function in $\mathcal{S}(\mathbb{R}^n)$ satisfying $\Psi(x)=1$ for $|x| \leqslant 1$ and $\Psi(x)=0$ for $|x| \geqslant 2$. We put $\mathcal{F}\varphi_0(x)=\Psi(x)$, $\mathcal{F}\varphi(x)=\Psi(\frac{x}{2})-\Psi(x)$ and $\mathcal{F}\varphi_v(x)=\mathcal{F}\varphi(2^{-v+1}x)$ for $v=1,2,3,\dots$. Then $\{\mathcal{F}\varphi_v\}_{v\in\mathbb{N}_0}$ is a resolution of unity, $\sum_{v=0}^{\infty}\mathcal{F}\varphi_v(x)=1$ for all $x \in \mathbb{R}^n$. Thus we obtain the Littlewood-Paley decomposition

$$f=\sum_{v=0}^{\infty}\varphi_v * f$$

of all $f \in \mathcal{S}'(\mathbb{R}^n)$ (convergence in $\mathcal{S}'(\mathbb{R}^n)$).

Using the resolution of unity $\{\mathcal{F}\varphi_v\}_{v\in\mathbb{N}_0}$ we can define the norm

$$\|f\|_{B^{\alpha,\tau}_{p,q}} = \sup_{P\in\mathcal{Q}} \frac{1}{|P|^{\tau}} \left(\sum_{v=v_P^+}^{\infty} 2^{v\alpha q} \|(\varphi_v * f)\chi_P\|_p^q\right)^{1/q}$$

for constants α,τ and $p,q\in(0,\infty]$. The Besov-type space $B^{\alpha,\tau}_{p,q}$ consist of all distributions $f\in\mathcal{S}'(\mathbb{R}^n)$ for which $\|f\|_{B^{\alpha,\tau}_{p,q}}<\infty$. It is well-known that these spaces do not depend on the choice of the initial system $\{\mathcal{F}\varphi_v\}_{v\in\mathbb{N}_0}$ (up to equivalence of quasi-norms). More information on the classical theory of these spaces can be found in [7], [8], [27], [35], [36] and [38]; see also [10] for recent developments. Further results about Triebel-Lizorkin-type spaces can be found in [34].

Now, we define corresponding spaces.

Definition 3.1. Let $\{\mathcal{F}\varphi_v\}_{v\in\mathbb{N}_0}$ be a resolution of unity, $\alpha:\mathbb{R}^n\to\mathbb{R}$ and $p,q,\tau\in\mathcal{P}_0$. The Besov-type space $B^{\alpha(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)}$ is the collection of all $f\in\mathcal{S}'(\mathbb{R}^n)$ such that

$$\|f\|^{\varphi}_{B^{\alpha(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)}} = \sup_{P\in\mathcal{Q}} \left\| \left(\frac{2^{v\alpha(\cdot)}\varphi_v * f}{\|\chi_P\|_{\tau(\cdot)}}\chi_P\right)_{v\geqslant v_P^+}\right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} < \infty. \tag{3.2}$$

It is easy to see immediately that if α, τ, p and q are constants, then $B^{\alpha(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)} = B^{\alpha,1/\tau}_{p,q}$. When, $q\equiv\infty$ the Besov-type space $B^{\alpha(\cdot),\tau(\cdot)}_{p(\cdot),\infty}$ consist of all distributions $f\in\mathcal{S}'(\mathbb{R}^n)$ such that

$$\sup_{P\in\mathcal{Q},v\geqslant v_P^+} \left\|\frac{2^{v\alpha(\cdot)}\varphi_v * f}{\|\chi_P\|_{\tau(\cdot)}}\chi_P\right\|_{p(\cdot)} < \infty.$$

Now, we are ready to show that the definition of the spaces $B^{\alpha(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)}$ is independent of the chosen resolution of unity $\{\mathcal{F}\varphi_v\}_{v\in\mathbb{N}_0}$. This justifies our omission of the subscript φ in the sequel.

Theorem 3.3. *Let $\{\mathcal{F}\varphi_v\}_{v\in\mathbb{N}_0}$, $\{\mathcal{F}\psi_v\}_{v\in\mathbb{N}_0}$ be two resolutions of unity and let $\alpha\in C^{\log}_{\mathrm{loc}}$ and $p,q,\tau\in\mathcal{P}_0^{\log}$ with $0<q^+<\infty$. Then*

$$\|f\|^{\varphi}_{B^{\alpha(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)}} \approx \|f\|^{\psi}_{B^{\alpha(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)}}$$

for all $f\in\mathcal{S}'(\mathbb{R}^n)$.

Proof. It is sufficient to show that there is $c>0$ such that for all $f\in\mathcal{S}'(\mathbb{R}^n)$ we have $\|f\|^{\psi}_{B^{\alpha(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)}} \leqslant c\,\|f\|^{\varphi}_{B^{\alpha(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)}}$. Interchanging the roles of ψ and φ we obtain the desired result. Putting $\varphi_{-1}=0$ we see $\mathcal{F}\psi_v = \mathcal{F}\psi_v\sum_{k=-1}^{k=1}\mathcal{F}\varphi_{v+k}$ for all $v\in\mathbb{N}_0$ and by the properties of the Fourier transform $\psi_v * f = \sum_{k=-1}^{k=1}\psi_v * \varphi_{v+k} * f$. Fix

$0 < r < \frac{1}{2}\min(p^-, q^-, 2)$ and $m > n + 2c_{\log}(\alpha)$ large. Since $|\psi_v| \leqslant c\, \eta_{v,2m/r}$, with $c > 0$ independent of v, we obtain

$$\begin{aligned}|\psi_v * \varphi_{v+k} * f| &\leqslant c\, \eta_{v,m/r} * |\varphi_{v+k} * f| \\ &\leqslant c\, \eta_{v,m/r} * (\eta_{v+k,m} * |\varphi_{v+k} * f|^r)^{1/r},\end{aligned}$$

where in the second inequality we used Lemma 2.9. By Minkowski's integral inequality the left-hand side is bounded by

$$\begin{aligned}&c\Big((\eta_{v,m/r} * \eta^{1/r}_{v+k,m})^r * |\varphi_{v+k} * f|^r\Big)^{1/r} \\ &\qquad = c\, 2^{n(v+k)(1/r-1)}\Big(\big(\eta_{v,m/r} * \eta_{v+k,m/r}\big)^r * |\varphi_{v+k} * f|^r\Big)^{1/r}.\end{aligned}$$

By Lemma 2.10 we have $\eta_{v,m/r} * \eta_{v+k,m/r} \approx \eta_{v+k,m/r}$. Then the last expression is bounded by

$$c\,(\eta_{v+k,m} * |\varphi_{v+k} * f|^r)^{1/r}.$$

This, together with Lemma 2.8, gives for any dyadic cube P of $\mathbb{R}^n$

$$\begin{aligned}&\left\|\left(\frac{2^{v\alpha(\cdot)}\psi_v * f}{\|\chi_P\|_{\tau(\cdot)}}\chi_P\right)_{v\geqslant v_P^+}\right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} \\ &\quad = \left\|\left(\frac{2^{v\alpha(\cdot)r}|\psi_v * f|^r}{\|\chi_P\|^r_{\tau(\cdot)}}\chi_P\right)_{v\geqslant v_P^+}\right\|^{1/r}_{\ell^{q(\cdot)/r}(L^{p(\cdot)/r})} \\ &\quad \lesssim \sum_{k=-1}^{k=1}\left\|\left(\frac{2^{v\alpha(\cdot)r}\eta_{v+k,m} * |\varphi_{v+k} * f|^r}{\|\chi_P\|^r_{\tau(\cdot)}}\chi_P\right)_{v\geqslant v_P^+}\right\|^{1/r}_{\ell^{q(\cdot)/r}(L^{p(\cdot)/r})} \\ &\quad \lesssim \sum_{k=-1}^{k=1}\left\|\left(\frac{\eta_{v+k,m-c_{\log}(\alpha)} * 2^{(v+k)\alpha(\cdot)r}|\varphi_{v+k} * f|^r}{\|\chi_P\|^r_{\tau(\cdot)}}\chi_P\right)_{v\geqslant v_P^+}\right\|^{1/r}_{\ell^{q(\cdot)/r}(L^{p(\cdot)/r})}.\end{aligned}$$

By the change of variable $v + k = i$, this expression is bounded by

$$c\sum_{k=-1}^{k=1}\left\|\left(\frac{\eta_{i,m-c_{\log}(\alpha)} * 2^{i\alpha(\cdot)r}|\varphi_i * f|^r}{\|\chi_P\|^r_{\tau(\cdot)}}\chi_P\right)_{i\geqslant v_P^+ + k}\right\|^{1/r}_{\ell^{q(\cdot)/r}(L^{p(\cdot)/r})}.$$

The factor $\left\|(\cdots)_{i\geqslant v_P^+ +k}\right\|_{\ell^{q(\cdot)/r}(L^{p(\cdot)/r})}^{1/r}$ with $k=1$ is bounded by

$$\left\|\left(\frac{\eta_{i,m-c_{\log}(\alpha)} * 2^{i\alpha(\cdot)r}|\varphi_i * f|^r}{\|\chi_P\|_{\tau(\cdot)}^r}\chi_P\right)_{i\geqslant v_P^+}\right\|_{\ell^{q(\cdot)/r}(L^{p(\cdot)/r})}^{1/r}$$

$$\lesssim \left\|(2^{v\alpha(\cdot)}\varphi_v * f)_v\right\|_{\ell^{\tau(\cdot),q(\cdot)}(L^{p(\cdot)})} = \|f\|^{\varphi}_{B^{\alpha(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)}},$$

where we have used Lemma 2.12. When $k=-1$, we use the fact that $\chi_P \leqslant \chi_{2P}$, $v_P - 1 = v_{2P}$ and $\|\chi_P\|_{\tau(\cdot)} \approx \|\chi_{2P}\|_{\tau(\cdot)}$, where this result can be proved by the properties (2.6) and (2.7) (in view of the fact that $|2P| = 2^n|P|$), which completes the proof. ■

Proposition 3.4. *Let B_J be any ball of $\mathbb{R}^n$ with radius 2^{-J}, $J\in\mathbb{Z}$. In the definition of the spaces $B^{\alpha(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)}$ if we replace the dyadic cubes P by the balls B_J, then we obtain equivalent quasi-norms.*

Proof. Let B_J be any ball of $\mathbb{R}^n$ with radius 2^{-J}. We set

$$\|f\|^{B}_{B^{\alpha(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)}} = \sup_{B_J}\left\|\left(\frac{2^{v\alpha(\cdot)}\varphi_v * f}{\|\chi_{B_J}\|_{\tau(\cdot)}}\chi_{B_J}\right)_{v\geqslant J^+}\right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})},$$

where $J^+ = \max(J,0)$ and the supremum is taken over all $J\in\mathbb{Z}$ and all balls B_J of $\mathbb{R}^n$ with radius 2^{-J}. First let us prove that

$$\|f\|_{B^{\alpha(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)}} \leqslant c\,\|f\|^{B}_{B^{\alpha(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)}}. \tag{3.5}$$

Let P be a dyadic cube in $\mathbb{R}^n$ and $r<\tau^-$. Since $\|\chi_P\|_{\tau(\cdot)} = \|\chi_P\|^{1/r}_{\tau(\cdot)/r}$ and $\frac{\tau}{r}\in\mathcal{P}^{\log}$, then (2.5) can be used to obtain $\|\chi_P\|_{\tau(\cdot)/r}\|\chi_P\|_{(\tau(\cdot)/r)'} \approx |P|$, where $(\frac{\tau(\cdot)}{r})'$ is the conjugate exponent of $\frac{\tau(\cdot)}{r}$. Hence

$$\begin{aligned}\frac{1}{\|\chi_P\|_{\tau(\cdot)/r}} &\lesssim \frac{\|\chi_P\|_{(\tau(\cdot)/r)'}}{|P|}\\ &\lesssim \frac{\left\|\chi_{B(x_P,\sqrt{n}l(P))}\right\|_{(\tau(\cdot)/r)'}}{|B(x_P,\sqrt{n}l(P))|}\\ &\lesssim \frac{1}{\left\|\chi_{B(x_P,\sqrt{n}l(P))}\right\|_{\tau(\cdot)/r}},\end{aligned}$$

since $P \subset B(x_P, \sqrt{n}l(P))$ and $|P| \approx |B(x_P, \sqrt{n}l(P))|$. Moreover

$$\begin{aligned}
&\left\| \left(\frac{2^{v\alpha(\cdot)}\varphi_v * f}{\|\chi_P\|_{\tau(\cdot)}} \chi_P \right)_{v \geqslant v_P^+} \right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} \\
&\qquad \lesssim \left\| \left(\frac{2^{v\alpha(\cdot)}\varphi_v * f}{\left\|\chi_{B(x_P,\sqrt{n}l(P))}\right\|_{\tau(\cdot)/r}} \chi_{B(x_P,\sqrt{n}l(P))} \right)_{v \geqslant v_P^+} \right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} \\
&\qquad \lesssim \|f\|^{B}_{B^{\alpha(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)}},
\end{aligned}$$

which proves estimate (3.5). By the properties of the dyadic cubes, there exists $k \in \mathbb{N}$ not depending on J such that

$$B_J \subset \cup_{l=1}^k Q_J^l, \qquad B_J \cap Q_J^l \neq \emptyset, \forall l = 1,..,k.$$

Here, Q_J^l is a dyadic cube of side length 2^{-J}. We claim that $\frac{\left\|\chi_{Q_J^l}\right\|_{\tau(\cdot)}}{\left\|\chi_{B_J}\right\|_{\tau(\cdot)}} \leqslant c$, where the constant $c > 0$ not depending on J. Then

$$\begin{aligned}
\left\| \left(\frac{2^{v\alpha(\cdot)}\varphi_v * f}{\|\chi_{B_J}\|_{\tau(\cdot)}} \chi_{B_J} \right)_{v \geqslant J^+} \right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} &\leqslant c \sum_{l=1}^k \left\| \left(\frac{2^{v\alpha(\cdot)}\varphi_v * f}{\left\|\chi_{Q_J^l}\right\|_{\tau(\cdot)}} \chi_{Q_J^l} \right)_{v \geqslant v_{Q_J^l}^+} \right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} \\
&\leqslant c \|f\|_{B^{\alpha(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)}}.
\end{aligned}$$

Let us prove the claim. We use (2.5) to obtain

$$\begin{aligned}
\frac{1}{\|\chi_{B_J}\|_{\tau(\cdot)/r}} \lesssim \frac{\|\chi_{B_J}\|_{(\tau(\cdot)/r)'}}{|B_J|} &\lesssim \frac{\left\|\chi_{(2\sqrt{n}+1)B_J}\right\|_{(\tau(\cdot)/r)'}}{|(2\sqrt{n}+1)B_J|} \\
&\lesssim \frac{1}{\left\|\chi_{(2\sqrt{n}+1)B_J}\right\|_{\tau(\cdot)/r}},
\end{aligned}$$

where $r < \tau^-$. Now the claim follows from the fact that $Q_J^l \subset (2\sqrt{n}+1)B_J$ for any $l = 1, ..., k$. ■

If we replace dyadic cubes P in Definition 3.1 by arbitrary cubes P, we obtain equivalent quasi-norms.

Sometimes it is useful/important to restrict $\sup_{P \in \mathcal{Q}}$ in the definition to a supremum taken with respect to dyadic cubes with side length $\leqslant 1$.

Lemma 3.6. *Let $\alpha \in C^{\log}_{\mathrm{loc}}$ and $p,q,\tau \in \mathcal{P}^{\log}_0$ with $\tau_\infty \in (0,p^-]$ and $0<q^+<\infty$. A tempered distribution f belongs to $B^{\alpha(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)}$ if and only if,*

$$\|f\|^{\#}_{B^{\alpha(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)}} = \sup_{\{P\in\mathcal{Q},|P|\leqslant 1\}} \left\| \left(\frac{2^{v\alpha(\cdot)}\varphi_v * f}{\|\chi_P\|_{\tau(\cdot)}} \chi_P \right)_{v\geqslant v_P} \right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} < \infty.$$

Furthermore, the quasi-norms $\|f\|_{B^{\alpha(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)}}$ and $\|f\|^{\#}_{B^{\alpha(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)}}$ are equivalent.

Proof. Let P be a dyadic cube such that $|P| = 2^{-Jn}$, for some $-J \in \mathbb{N}$. Let $\{Q_m : m = 1,...,2^{-Jn}\}$ be the collection of all dyadic cubes with volume 1 and such that $P = \cup_{m=1}^{2^{-Jn}} Q_m$. In view of the definition of $B^{\alpha(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)}$, it suffices to show that $\|f\|_{B^{\alpha(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)}} \lesssim \|f\|^{\#}_{B^{\alpha(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)}}$. By the scaling argument, it suffices to consider the case $\|f\|^{\#}_{B^{\alpha(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)}} = 1$ and show that the modular of f on the left-hand side is bounded. In particular, we will show that

$$\sum_{v=v_P^+}^{\infty} \left\| \left| \frac{2^{v\alpha(\cdot)}\varphi_v * f}{\|\chi_P\|_{\tau(\cdot)}} \right|^{q(\cdot)} \chi_P \right\|_{\frac{p(\cdot)}{q(\cdot)}} \leqslant 1$$

for any dyadic cube P, with $|P|>1$. We set

$$\max_{m=1,...,2^{-Jn}} \left\| \left| \frac{2^{v\alpha(\cdot)}\varphi_v * f}{\|\chi_{Q_m}\|_{\tau(\cdot)}} \right|^{q(\cdot)} \chi_{Q_m} \right\|_{\frac{p(\cdot)}{q(\cdot)}} = \left\| \left| \frac{2^{v\alpha(\cdot)}\varphi_v * f}{\|\chi_{Q_{m_0}}\|_{\tau(\cdot)}} \right|^{q(\cdot)} \chi_{Q_{m_0}} \right\|_{\frac{p(\cdot)}{q(\cdot)}} = \delta.$$

Thus it remains only to prove that $\left\| \left| \frac{2^{v\alpha(\cdot)}\varphi_v * f}{\|\chi_P\|_{\tau(\cdot)}} \right|^{q(\cdot)} \chi_P \right\|_{\frac{p(\cdot)}{q(\cdot)}} \leqslant \delta$. This claim can be reformulated as showing that $\left\| \delta^{-\frac{1}{q(\cdot)}} \frac{2^{v\alpha(\cdot)}\varphi_v * f}{\|\chi_P\|_{\tau(\cdot)}} \chi_P \right\|_{p(\cdot)} \leqslant 1$, which is equivalent to

$$\int_P \delta^{-\frac{p(x)}{q(x)}} \frac{\left|2^{v\alpha(x)}\varphi_v * f(x)\right|^{p(x)}}{\|\chi_P\|^{p(x)}_{\tau(\cdot)}} dx \leqslant 1.$$

This integral can be rewritten us

$$\sum_{m=1}^{2^{-Jn}} \int_{Q_m} \delta^{-\frac{p(x)}{q(x)}} \frac{\left|2^{v\alpha(x)}\varphi_v * f(x)\right|^{p(x)}}{\|\chi_P\|^{p(x)}_{\tau(\cdot)}} dx$$
$$\lesssim \sum_{m=1}^{2^{-Jn}} 2^{Jn} \int_{Q_m} \delta^{-\frac{p(x)}{q(x)}} \frac{\left|2^{v\alpha(x)}\varphi_v * f(x)\right|^{p(x)}}{\|\chi_{Q_m}\|^{p(x)}_{\tau(\cdot)}} dx \lesssim 1$$

where the second inequality is obtained from the estimate $\|\chi_P\|_{\tau(\cdot)}^{p(x)} \approx |P|^{p(x)/\tau_\infty} \geqslant 2^{-Jnp^-/\tau_\infty} \geqslant 2^{-Jn}$ which is a consequence of (2.7) since $|P| > 1$ and $\tau_\infty \in (0, p^-]$. The last inequality follows from the fact that

$$\left\| \delta^{-\frac{1}{q(\cdot)}} \frac{2^{v\alpha(\cdot)}\varphi_v * f}{\|\chi_{Q_m}\|_{\tau(\cdot)}} \chi_{Q_m} \right\|_{p(\cdot)} \leqslant 1$$

for any $m = 1, ..., 2^{-Jn}$, which follows immediately from the definition of δ. (combined with, if $\|g\|_{p(\cdot)} \leqslant 1$, then $\varrho_{p(\cdot)}(g) \leqslant \|g\|_{p(\cdot)}^t$, where $t \leqslant p^-$, see [6, Lemma 3.2.4], which completes the proof. ■

Remark 3.7. This result with fixed exponents is given in [38, Lemma 2.2] with $1/\tau$ in place of τ.

The next theorem gives conditions where the spaces $B_{p(\cdot),q(\cdot)}^{\alpha(\cdot),\tau(\cdot)}$ are just the Besov spaces $B_{\infty,\infty}^{\alpha(\cdot)+n(1/\tau(\cdot)-1/p(\cdot))}$. This result with fixed exponents is given in [37].

Theorem 3.8. *Let $\alpha \in C_{\mathrm{loc}}^{\log}$ and $p, q, \tau \in \mathcal{P}_0^{\log}$ with $\tau_\infty \in (0, p^-]$ and $0 < q^+ < \infty$. If $(1/\tau - 1/p)^- > 0$ or $(1/\tau - 1/p)^- \geqslant 0$ and $q \equiv \infty$, then*

$$B_{p(\cdot),q(\cdot)}^{\alpha(\cdot),\tau(\cdot)} = B_{\infty,\infty}^{\alpha(\cdot)+n(1/\tau(\cdot)-1/p(\cdot))},$$

with equivalent norms.

Proof. We consider only $(1/\tau - 1/p)^- > 0$. The case $(1/\tau - 1/p)^- \geqslant 0$ and $q = \infty$ can be proved analogously with the necessary modifications. Since $\tau_\infty \in (0, p^-]$, then we use the equivalent norm given in the previous lemma. First let us prove the following estimate

$$\|f\|_{B_{p(\cdot),q(\cdot)}^{\alpha(\cdot),\tau(\cdot)}} \lesssim \|f\|_{B_{\infty,\infty}^{\alpha(\cdot)+n(1/\tau(\cdot)-1/p(\cdot))}}$$

for any $f \in B_{\infty,\infty}^{\alpha(\cdot)+n(1/\tau(\cdot)-1/p(\cdot))}$. By the scaling argument, we see that it suffices to consider only the case $\|f\|_{B_{\infty,\infty}^{\alpha(\cdot)+n(1/\tau(\cdot)-1/p(\cdot))}} = 1$ and show that the modular of f on the left-hand side is bounded. Let P be a dyadic cube with volume 2^{-nv_P}, $v_P \in \mathbb{N}$. By (2.6), we obtain

$$\begin{aligned}
&\frac{2^{v\alpha(x)}|\varphi_v * f(x)|}{\|\chi_P\|_{\tau(\cdot)}} \\
&\qquad \leqslant c\, 2^{v(\alpha(x)+n(1/\tau(x)-1/p(x))+n(v_P-v)(1/\tau(x)-1/p(x))+nv_P/p(x)}|\varphi_v * f(x)| \\
&\qquad \leqslant c\, 2^{n(v_P-v)(1/\tau(x)-1/p(x))+nv_P/p(x)} \, \|f\|_{B_{\infty,\infty}^{\alpha(\cdot)+n(1/\tau(\cdot)-1/p(\cdot))}} \\
&\qquad = c\, 2^{n(v_P-v)(1/\tau(x)-1/p(x))+nv_P/p(x)}
\end{aligned}$$

for any $x \in P$. Then for any $v \geqslant v_P$

$$\begin{aligned}\Big\| \Big| \frac{2^{v\alpha(\cdot)}\varphi_v * f}{\|\chi_P\|_{\tau(\cdot)}} \Big|^{q(\cdot)} \chi_P \Big\|_{\frac{p(\cdot)}{q(\cdot)}} &\leqslant \Big\| \Big| c\, 2^{n(v_P - v)(1/\tau(\cdot) - 1/p(\cdot)) + nv_P/p(\cdot)} \Big|^{q(\cdot)} \chi_P \Big\|_{\frac{p(\cdot)}{q(\cdot)}} \\ &\leqslant c\, 2^{n(v_P - v)(1/\tau - 1/p)^- q^-} \Big\| \Big| 2^{nv_P/p(\cdot)} \Big|^{q(\cdot)} \chi_P \Big\|_{\frac{p(\cdot)}{q(\cdot)}}.\end{aligned}$$

The norm on the right-hand side is bounded by 1. To show that, we investigate the corresponding modular:

$$\varrho_{\frac{p(\cdot)}{q(\cdot)}}\Big(\Big| 2^{nv_P/p(\cdot)} \Big|^{q(\cdot)} \chi_P \Big) = \int_P \Big| 2^{nv_P/p(x)} \Big|^{p(x)} dx = 2^{nv_P} \int_P dx = 1.$$

Hence

$$\sum_{v=v_P^+}^{\infty} \Big\| \Big| \frac{2^{v\alpha(\cdot)}\varphi_v * f}{\|\chi_P\|_{\tau(\cdot)}} \Big|^{q(\cdot)} \chi_P \Big\|_{\frac{p(\cdot)}{q(\cdot)}} \leqslant c \sum_{v=v_P^+}^{\infty} 2^{n(v_P - v)(1/\tau - 1/p)^- q^-} \leqslant c.$$

Let $f \in B^{\alpha(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)}$, with $\|f\|_{B^{\alpha(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)}} = 1$. By Lemma 2.9 we have for any $x \in \mathbb{R}^n, m > n$

$$\begin{aligned}&2^{v(\alpha(x)+n(1/\tau(x)-1/p(x))} |\varphi_v * f(x)| \\ &\qquad \leqslant c\, 2^{v(\alpha(x)+n(1/\tau(x)-1/p(x))} (\eta_{v,m} * |\varphi_v * f|^{p^-}(x))^{1/p^-} \\ &\qquad \leqslant c \Big\| 2^{v(\alpha(x)+n/\tau(x))} \varphi_v * f(\cdot)(1 + 2^v|x - \cdot|)^{-m/2p^-} \Big\|_{p(\cdot)} \\ &\qquad\quad \times \Big\| 2^{vn/t(x)} (1 + 2^v|x - \cdot|)^{-m/2p^-} \Big\|_{t(\cdot)},\end{aligned}$$

by Hölder's inequality, with $\frac{1}{p^-} = \frac{1}{p(\cdot)} + \frac{1}{t(\cdot)}$. The second norm on the right-hand side is bounded if $m > \frac{2np^-}{t^-} + 2c_{\log}(t)$ (this is possible since m can be taken large enough). To show that, we investigate the corresponding modular:

$$\begin{aligned}\varrho_{t(\cdot)}(2^{vn/t(x)}(1 + 2^v|x - \cdot|)^{-m/2p^-}) &= \int_{\mathbb{R}^n} 2^{vnt(y)/t(x)} (1 + 2^v|x - y|)^{-mt(y)/2p^-} dy \\ &\leqslant 2^{vn} \int_{\mathbb{R}^n} (1 + 2^v|x - y|)^{-(m - 2c_{\log}(1/t))t^-/2p^-} dy \\ &< \infty,\end{aligned}$$

where we used Lemma 2.8. Again by the same lemma the first norm is bounded by

$$\Big\| 2^{v(\alpha(\cdot)+n/\tau(x))} \varphi_v * f(\cdot)(1 + 2^v|x - \cdot|)^{-h} \Big\|_{p(\cdot)},$$

where $h = \frac{m}{2p^-} - c_{\log}(\alpha)$. Let now prove that this expression is bounded. We investigate the corresponding modular:

$$\begin{aligned}
&\varrho_{p(\cdot)}(2^{v(\alpha(\cdot)+n/\tau(x))}\varphi_v * f(\cdot)(1+2^v|x-\cdot|)^{-h}) \\
&\qquad = \int_{\mathbb{R}^n} 2^{v(\alpha(y)+n/\tau(x))p(y)}|\varphi_v * f(y)|^{p(y)}(1+2^v|x-y|)^{-hp(y)}dy \\
&\qquad = \int_{|y-x|<2^{-v}}(\cdots)dy + \sum_{i=0}^{\infty}\int_{2^{i-v}\leqslant|y-x|<2^{i-v+1}}(\cdots)dy \\
&\qquad \leqslant \sum_{i=0}^{\infty} 2^{-ihp^-}\int_{|y-x|<2^{i-v+1}} 2^{v(\alpha(y)+n/\tau(x))p(y)}|\varphi_v * f(y)|^{p(y)}dy. \qquad (3.9)
\end{aligned}$$

By Lemma 2.11 (ii), we have

$$\frac{\left\|\chi_{B(x,2^{i-v+1})}\right\|_{\tau(\cdot)}}{\left\|\chi_{B(x,2^{-v})}\right\|_{\tau(\cdot)}} \leqslant c\ 2^{ni/\tau^-}, \qquad x \in \mathbb{R}^n, \tag{3.10}$$

and by the property (2.6), we have $\left\|\chi_{B(x,2^{-v})}\right\|_{\tau(\cdot)} 2^{vn/\tau(x)} \approx 1$. Then the right-hand side of (3.9) is bounded by

$$\begin{aligned}
&\sum_{i=0}^{\infty} 2^{(n/\tau^- - hp^-)i}\int_{B(x,2^{i-v+1})} \frac{2^{v\alpha(y)p(y)}|\varphi_v * f(y)|^{p(y)}}{\left\|\chi_{B(x,2^{i-v+1})}\right\|_{\tau(\cdot)}^{p(y)}}dy \\
&\quad \leqslant \sum_{i=0}^{\infty} 2^{(n/\tau^- - hp^-)i} \\
&\qquad \times \left(\sum_{j=(v-i-1)^+}^{\infty}\left(\int_{B(x,2^{i-v+1})} \frac{2^{j\alpha(y)p(y)}|\varphi_j * f(y)|^{p(y)}}{\left\|\chi_{B(x,2^{i-v+1})}\right\|_{\tau(\cdot)}^{p(y)}}dy\right)^{1/(p/q)^-}\right)^{(p/q)^-}
\end{aligned}$$

and since $\|f\|_{B^{\alpha(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)}} = 1$, we obtain

$$\sum_{j=J^+}^{\infty}\left\|\left|\frac{2^{j\alpha(\cdot)}\varphi_j * f}{\|\chi_{B_J}\|_{\tau(\cdot)}}\right|^{q(\cdot)}\chi_{B_J}\right\|_{\frac{p(\cdot)}{q(\cdot)}} \leqslant 1$$

and

$$\int_{B_J} 2^{j\alpha(y)p(y)}\frac{|\varphi_j * f(y)|^{p(y)}}{\|\chi_{B_J}\|_{\tau(\cdot)}^{p(y)}}dy \leqslant \left\|\left|\frac{2^{j\alpha(\cdot)}\varphi_j * f}{\|\chi_{B_J}\|_{\tau(\cdot)}}\right|^{q(\cdot)}\chi_{B_J}\right\|_{\frac{p(\cdot)}{q(\cdot)}}^{(p/q)^-}$$

for any ball B_J and $j \geqslant J^+$. Therefore,

$$\varrho_{p(\cdot)}(2^{v(\alpha(\cdot)+n/\tau(x))}\varphi_v * f(\cdot)(1+2^v|x-\cdot|)^{-h}) \leqslant c\sum_{i=0}^{\infty} 2^{(n/\tau^- - hp^-)i} < \infty$$

for any $h > n/p^-\tau^-$. The proof is completed by the scaling argument. ■

Remark 3.11. Under the hypothesis of the previous theorem we have $\|\cdot\|_{B^{\alpha(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)}}$ is an equivalent quasi-norm on $B^{\alpha(\cdot)+n(1/\tau(\cdot)-1/p(\cdot))}_{\infty,\infty}$. We further conclude that

$$2^{v(\alpha(x)+n(1/\tau(x)-1/p(x)))}|\varphi_v * f(x)| \leqslant c\,\|f\|_{B^{\alpha(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)}}$$

for any $x \in \mathbb{R}^n$, $\alpha \in C^{\log}_{\mathrm{loc}}$ and $p, q, \tau \in \mathcal{P}^{\log}_0$.

In the next theorem we prove that it is possible to define these spaces by replacing $v \geqslant v_P^+$ by $v \geqslant 0$, in Definition 3.1. For fixed exponents, see [27].

Theorem 3.12. *Let $\alpha \in C^{\log}_{\mathrm{loc}}$ and $p, q, \tau \in \mathcal{P}^{\log}_0$ with $0 < q^+ < \infty$. If $(1/\tau - 1/p)^+ < 0$ or $(1/\tau - 1/p)^+ \leqslant 0$ and $q \equiv \infty$, then*

$$\|f\|_{B^{\alpha(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)}}^{\blacktriangle} = \sup_{P \in \mathcal{Q}} \left\| \left(\frac{2^{v\alpha(\cdot)}\varphi_v * f}{\|\chi_P\|_{\tau(\cdot)}} \chi_P \right)_{v \geqslant 0} \right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})},$$

is an equivalent quasi-norm in $B^{\alpha(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)}$.

Proof. In view of the definition of $B^{\alpha(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)}$, it suffices to prove that

$$\|f\|_{B^{\alpha(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)}}^{\blacktriangle} \lesssim \|f\|_{B^{\alpha(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)}}.$$

Let $f \in B^{\alpha(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)}$, with $\|f\|_{B^{\alpha(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)}} = 1$. In view of the proof of previous theorem, we have

$$2^{v(\alpha(x)+n(1/\tau(x)-1/p(x)))}|\varphi_v * f(x)| \leqslant c$$

for any $x \in \mathbb{R}^n$. Then for any $0 \leqslant v \leqslant v_P$

$$\begin{aligned}
\left\| \left| \frac{2^{v\alpha(\cdot)}\varphi_v * f}{\|\chi_P\|_{\tau(\cdot)}} \right|^{q(\cdot)} \chi_P \right\|_{\frac{p(\cdot)}{q(\cdot)}} &\leqslant c \left\| \left| 2^{(v_P - v)(1/\tau(\cdot)-1/p(\cdot)) + nv_P/p(\cdot)} \right|^{q(\cdot)} \chi_P \right\|_{\frac{p(\cdot)}{q(\cdot)}} \\
&\leqslant c\, 2^{(v_P - v)(1/\tau - 1/p)^+ q^-} \left\| \left| 2^{nv_P/p(\cdot)} \right|^{q(\cdot)} \chi_P \right\|_{\frac{p(\cdot)}{q(\cdot)}} \\
&\leqslant c\, 2^{(v_P - v)(1/\tau - 1/p)^+ q^-}.
\end{aligned}$$

Therefore,

$$\sum_{v=0}^{v_P} \left\| \left| \frac{2^{v\alpha(\cdot)}\varphi_v * f}{\|\chi_P\|_{\tau(\cdot)}} \right|^{q(\cdot)} \chi_P \right\|_{\frac{p(\cdot)}{q(\cdot)}} \leqslant c.$$

The proof is completed by the scaling argument. The case $q \equiv \infty$ can be easily proved. ■

4. Embeddings

For the spaces $B^{\alpha(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)}$ introduced above we want to show some embedding theorems. We say a quasi-normed space A_1 is continuously embedded in another quasi-normed space A_2, $A_1 \hookrightarrow A_2$, if $A_1 \subset A_2$ and there is a $c > 0$ such that $\|f\|_{A_2} \leqslant c\,\|f\|_{A_1}$ for all $f \in A_1$. We begin with the following elementary embeddings.

Theorem 4.1. *Let $\alpha \in C^{\log}_{\mathrm{loc}}$ and $p, q_0, q_1, \tau \in \mathcal{P}^{\log}_0$.*

(i) *If $q_0 \leqslant q_1$, then*

$$B^{\alpha(\cdot),\tau(\cdot)}_{p(\cdot),q_0(\cdot)} \hookrightarrow B^{\alpha(\cdot),\tau(\cdot)}_{p(\cdot),q_1(\cdot)}.$$

(ii) *If $(\alpha_0 - \alpha_1)^- > 0$, then*

$$B^{\alpha_0(\cdot),\tau(\cdot)}_{p(\cdot),q_0(\cdot)} \hookrightarrow B^{\alpha_1(\cdot),\tau(\cdot)}_{p(\cdot),q_1(\cdot)}.$$

The proof can be obtained by using the same method as in [1, Theorem 6.1]. We next consider embeddings of Sobolev-type. It is well-known that

$$B^{\alpha_0,\tau}_{p_0,q} \hookrightarrow B^{\alpha_1,\tau}_{p_1,q}.$$

if $\alpha_0 - n/p_0 = \alpha_1 - n/p_1$, where $0 < p_0 < p_1 \leqslant \infty, 0 \leqslant \tau < \infty$ and $0 < q \leqslant \infty$ (see e.g. [38, Corollary 2.2]). In the following theorem we generalize these embeddings to variable exponent case.

Theorem 4.2. *Let $\alpha_0, \alpha_1 \in C^{\log}_{\mathrm{loc}}$ and $p_0, p_1, q, \tau \in \mathcal{P}^{\log}_0$ with $0 < q^+ < \infty$. If $\alpha_0 > \alpha_1$ and $\alpha_0(x) - \frac{n}{p_0(x)} = \alpha_1(x) - \frac{n}{p_1(x)}$ with $\left(\frac{p_0}{p_1}\right)^- < 1$, then*

$$B^{\alpha_0(\cdot),\tau(\cdot)}_{p_0(\cdot),q(\cdot)} \hookrightarrow B^{\alpha_1(\cdot),\tau(\cdot)}_{p_1(\cdot),q(\cdot)}.$$

Proof. Let B_J be any ball of $\mathbb{R}^n$ with radius 2^{-J}, $J \in \mathbb{Z}$. By Lemma 2.9 we have for any $x \in B_J, m > n, d > 0$

$$\begin{aligned} |\varphi_v * f(x)| &\leqslant c(\eta_{v,2m} * |\varphi_v * f|^d(x))^{1/d} \\ &\leqslant c \sum_{i=0}^{\infty} 2^{-mi} (\eta_{v,m} * |\varphi_v * f\chi_{B_{J-i-1}}|^d(x))^{1/d}, \end{aligned}$$

where, $\varphi_v * f\chi_{B_{J-i-1}} = (\varphi_v * f)\chi_{B_{J-i-1}}$. Fix $0 < r < \frac{1}{2}\min(p^-, q^-, 2)$, then

$$\left\| \left(\frac{2^{v\alpha_1(\cdot)}\varphi_v * f}{\|\chi_{B_J}\|_{\tau(\cdot)}} \chi_{B_J} \right)_{v \geqslant J^+} \right\|^r_{\ell^{q(\cdot)}(L^{p_1(\cdot)})}$$

$$\lesssim \sum_{i=0}^{\infty} 2^{(n/\tau^- - m)ri} \left\| \left(\frac{2^{v\alpha_1(\cdot) - ni/\tau^-}}{\|\chi_{B_J}\|_{\tau(\cdot)}} \left(\eta_{v,m} * |\varphi_v * f\chi_{B_{J-i-1}}|^d\right)^{1/d} \right)_{v \geqslant J^+} \right\|^r_{\ell^{q(\cdot)}(L^{p_1(\cdot)})}.$$

Let us prove that the norm on the right-hand side is bounded by

$$\left\| \left(\frac{2^{v\alpha_0(\cdot)}\varphi_v * f}{\|\chi_{B_{J-i-1}}\|_{\tau(\cdot)}} \chi_{B_{J-i-1}} \right)_{v\geqslant J^+} \right\|_{\ell^{q(\cdot)}(L^{p_0(\cdot)})}. \tag{4.3}$$

By the scaling argument, we see that it suffices to consider the case when the last expression is less than or equal 1. Therefore we will prove

$$\sum_{v\geqslant J^+} \left\| \left| \frac{c\ 2^{v\alpha_1(\cdot)-ni/\tau^-}}{\|\chi_{B_J}\|_{\tau(\cdot)}} \left(\eta_{v,m} * |\varphi_v * f\chi_{B_{J-i-1}}|^d\right)^{1/d} \right|^{q(\cdot)} \right\|_{\frac{p_1(\cdot)}{q(\cdot)}} \leqslant C.$$

This clearly follows from the inequality

$$\begin{aligned} &\left\| \left| \frac{c\ 2^{v\alpha_1(\cdot)-ni/\tau^-}}{\|\chi_{B_J}\|_{\tau(\cdot)}} (\eta_{v,m} * |\varphi_v * f\chi_{B_{J-i-1}}|^d)^{1/d} \right|^{q(\cdot)} \right\|_{\frac{p_1(\cdot)}{q(\cdot)}} \\ &\qquad \leqslant \left\| \left| \frac{2^{v\alpha_0(\cdot)}\varphi_v * f}{\left\|\chi_{B_{J-i-1}}\right\|_{\tau(\cdot)}} \right|^{q(\cdot)} \chi_{B_{J-i-1}} \right\|_{\frac{p_0(\cdot)}{q(\cdot)}} + 2^{-v} = \delta. \end{aligned}$$

This claim can be reformulated as showing that

$$\left\| \left| \frac{c\ \delta^{-\frac{1}{q(\cdot)}} 2^{v\alpha_1(\cdot)-ni/\tau^-}}{\|\chi_{B_J}\|_{\tau(\cdot)}} (\eta_{v,m} * |\varphi_v * f\chi_{B_{J-i-1}}|^d)^{1/d} \right|^{q(\cdot)} \right\|_{\frac{p_1(\cdot)}{q(\cdot)}} \leqslant 1,$$

which is equivalent to

$$\int_{\mathbb{R}^n} c^{p_1(x)}\ \delta^{-\frac{p_1(x)}{q(x)}} \frac{2^{v(\alpha_1(x)-ni/\tau^-)p_1(x)}(\eta_{v,m} * |\varphi_v * f\chi_{B_{J-i-1}}|^d(x))^{p_1(x)/d}}{\|\chi_{B_J}\|_{\tau(\cdot)}^{p_1(x)}} dx \leqslant 1. \tag{4.4}$$

Since α_1, p_1 and q are log-Hölder continuous and $\delta \in [2^{-v}, 1+2^{-v}]$, we can move $2^{v(\alpha_1(x)-\frac{n}{p_1(x)})}$ and $\delta^{-\frac{1}{q(x)}}$ inside the convolution by Lemma 2.8:

$$\begin{aligned} &\delta^{-\frac{1}{q(x)}} \frac{2^{v(\alpha_1(x)-\frac{n}{p_1(x)})-ni/\tau^-} \left(\eta_{v,m} * |\varphi_v * f\chi_{B_{J-i-1}}|^d(x)\right)^{1/d}}{\|\chi_{B_J}\|_{\tau(\cdot)}} \\ &\qquad \leqslant c \left(\eta_{v,h} * \left(\frac{\delta^{-\frac{d}{q(\cdot)}} 2^{v(\alpha_1(\cdot)-\frac{n}{p_1(\cdot)})d} |\varphi_v * f\chi_{B_{J-i-1}}|^d}{\left\|\chi_{B_{J-i-1}}\right\|_{\tau(\cdot)}} \right)(x) \right)^{1/d} \\ &\qquad \leqslant c \left\| \frac{\delta^{-\frac{1}{q(\cdot)}} 2^{v\alpha_0(\cdot)}\varphi_v * f}{\left\|\chi_{B_{J-i-1}}\right\|_{\tau(\cdot)}} \chi_{B_{J-i-1}} \right\|_{p_0(\cdot)} \left\| 2^{vn/t(\cdot)}(1+2^v|x-\cdot|)^{-h/d} \right\|_{t(\cdot)}, \end{aligned}$$

by (3.10) and Hölder's inequality, with $\frac{1}{d} = \frac{1}{p_0(\cdot)} + \frac{1}{t(\cdot)}$. Here $h = m - c_{\log}(\alpha_1 - \frac{n}{p_1}) - c_{\log}(\frac{1}{q})$. The first norm on the right-hand side is bounded by 1 due to the choice of δ and the second norm is bounded if $h > nd/t^-$ (this is possible since m can be taken large enough). Now with the appropriate choice of $c > 0$, we find that the left-hand side of (4.4) can be rewritten as

$$\int_{\mathbb{R}^n} c^{p_0(x)} \left(\frac{c\, \delta^{-\frac{1}{q(x)}} 2^{v(\alpha_1(x) - \frac{n}{p_1(x)}) - ni/\tau^-}}{\|\chi_{B_J}\|_{\tau(\cdot)}} (\eta_{v,m} * |\varphi_v * f\chi_{B_{J-i-1}}|^d(x))^{1/d} \right)^{p_1(x) - p_0(x)}$$

$$\times \left(\frac{\delta^{-\frac{1}{q(x)}} 2^{v(\alpha_1(x) + \frac{n}{p_0(x)} - \frac{n}{p_1(x)}) - ni/\tau^-}}{\|\chi_{B_J}\|_{\tau(\cdot)}} (\eta_{v,m} * |\varphi_v * f\chi_{B_{J-i-1}}|^d(x))^{1/d} \right)^{p_0(x)} dx$$

$$\leqslant \int_{\mathbb{R}^n} \left(\frac{c}{\left\|\chi_{B_{J-i-1}}\right\|_{\tau(\cdot)}} (\eta_{v,h} * \delta^{-\frac{d}{q(\cdot)}} 2^{v\alpha_0(\cdot)d} |\varphi_v * f\chi_{B_{J-i-1}}|^d(x))^{1/d} \right)^{p_0(x)} dx,$$

where $h = m - c_{\log}(\alpha_0) - c_{\log}(\frac{1}{q})$. The last expression is bounded by 1 if and only if

$$\left\| \frac{c(\eta_{v,h} * \delta^{-\frac{d}{q(\cdot)}} 2^{v\alpha_0(\cdot)d} |\varphi_v * f\chi_{B_{J-i-1}}|^d)^{1/d}}{\left\|\chi_{B_{J-i-1}}\right\|_{\tau(\cdot)}} \right\|_{p_0(\cdot)} \leqslant 1.$$

Since convolution is bounded in $L^{p(\cdot)}$ when $p \in \mathcal{P}^{\log}$ and m can be taken large enough, again with an appropriate choice of $c > 0$ the left-hand side of this expression is bounded by

$$\left\| \frac{\delta^{-\frac{1}{q(\cdot)}} 2^{v\alpha_0(\cdot)} \varphi_v * f}{\left\|\chi_{B_{J-i-1}}\right\|_{\tau(\cdot)}} \chi_{B_{J-i-1}} \right\|_{p_0(\cdot)} \leqslant 1,$$

due to the choice of δ. We obtain the desired estimate in view that (4.3) is bounded by

$$\left\| \left(\frac{2^{v\alpha_0(\cdot)} \varphi_v * f}{\left\|\chi_{B_{J-i-1}}\right\|_{\tau(\cdot)}} \chi_{B_{J-i-1}} \right)_{v \geqslant (J-i-1)^+} \right\|_{\ell^{q(\cdot)}(L^{p_0(\cdot)})} \leqslant \|f\|_{B^{\alpha_0(\cdot),\tau(\cdot)}_{p_0(\cdot),q(\cdot)}}$$

for any $f \in B^{\alpha_0(\cdot),\tau(\cdot)}_{p_0(\cdot),q(\cdot)}$. ∎

Remark 4.5. We would like to mention that similar arguments give

$$B^{\alpha_0(\cdot),\tau(\cdot)}_{p_0(\cdot),q(\cdot)} \hookrightarrow B^{\alpha_0(\cdot) - \frac{n}{p_0(\cdot)},\tau(\cdot)}_{\infty,q(\cdot)}$$

if $\alpha_0 \in C^{\log}_{\mathrm{loc}}$ and $p_0, q, \tau \in \mathcal{P}^{\log}_0$, with $0 < q^+ < \infty$.

Let $\alpha \in C^{\log}_{\mathrm{loc}}, p, q, \tau \in \mathcal{P}^{\log}_0$ and $\alpha_0 = (\alpha(\cdot) - \frac{n}{p(\cdot)})^-$. We obtain

$$B^{\alpha(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)} \hookrightarrow B^{\alpha(\cdot),\tau(\cdot)}_{p(\cdot),\infty} \hookrightarrow B^{\alpha(\cdot) - \frac{n}{p(\cdot)},\tau(\cdot)}_{\infty,\infty} \hookrightarrow B^{\alpha_0,\tau(\cdot)}_{\infty,\infty} \hookrightarrow \mathcal{S}'(\mathbb{R}^n),$$

where the last inclusion is just the well known constant exponent case [38, Proposition 2.3]. Let $0 < p_1 < \infty$ and $\alpha_1 = (\alpha(\cdot) - \frac{n}{p(\cdot)} + \frac{n}{p_1})^+$. We obtain

$$\mathcal{S}(\mathbb{R}^n) \hookrightarrow B^{\alpha_1,\tau(\cdot)}_{p_1,q^-} \hookrightarrow B^{\alpha(\cdot)-\frac{n}{p(\cdot)}+\frac{n}{p_1},\tau(\cdot)}_{p_1,q^-} \hookrightarrow B^{\alpha(\cdot),\tau(\cdot)}_{p(\cdot),q^-} \hookrightarrow B^{\alpha(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)},$$

where the first inclusion follows again by [38, Proposition 2.3]. Thus we obtain:

Theorem 4.6. *Let $\alpha \in C^{\log}_{\mathrm{loc}}$ and $p, q, \tau \in \mathcal{P}^{\log}_0$ with $0 < q^+ < \infty$. Then*

$$\mathcal{S}(\mathbb{R}^n) \hookrightarrow B^{\alpha(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)} \hookrightarrow \mathcal{S}'(\mathbb{R}^n).$$

Recall that the Besov space of variable smoothness and integrability is the collection of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\|f\|_{B^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} = \left\| \left(2^{v\alpha(\cdot)} \varphi_v * f \right)_{v \geqslant 0} \right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} < \infty,$$

which introduced and investigated in [1], see [9] and [15] for further results. Now we establish further embedding results for the spaces $B^{\alpha(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)}$.

Theorem 4.7. *Let $\alpha \in C^{\log}_{\mathrm{loc}}$ and $p, q, \tau \in \mathcal{P}^{\log}_0$ with $0 < q^- \leqslant q^+ < \infty$. If $(p_2 - p_1)^+ \leqslant 0$, then*

$$B^{\alpha(\cdot)+n/\tau(\cdot)+n/p_2(\cdot)-n/p_1(\cdot)}_{p_2(\cdot),q(\cdot)} \hookrightarrow B^{\alpha(\cdot),\tau(\cdot)}_{p_1(\cdot),q(\cdot)}.$$

Proof. Using the Sobolev embeddings

$$B^{\alpha(\cdot)+n/\tau(\cdot)+n/p_2(\cdot)-n/p_1(\cdot)}_{p_2(\cdot),q(\cdot)} \hookrightarrow B^{\alpha(\cdot)+n/\tau(\cdot)}_{p_1(\cdot),q(\cdot)},$$

see [1, Theorem 6.4], it is sufficient to prove that $B^{\alpha(\cdot)+n/\tau(\cdot)}_{p_1(\cdot),q(\cdot)} \hookrightarrow B^{\alpha(\cdot),\tau(\cdot)}_{p_1(\cdot),q(\cdot)}$. The property (2.7) implies that

$$\begin{aligned} \sup_{P\in\mathcal{Q},|P|>1} & \left\| \left(\frac{2^{v\alpha(\cdot)}\varphi_v * f}{\|\chi_P\|_{\tau(\cdot)}} \chi_P \right)_{v \geqslant v_P^+} \right\|_{\ell^{q(\cdot)}(L^{p_1(\cdot)})} \\ & \approx \sup_{P\in\mathcal{Q},|P|>1} \left\| \left(\frac{2^{v\alpha(\cdot)}\varphi_v * f}{|P|^{1/\tau_\infty}} \chi_P \right)_{v \geqslant 0} \right\|_{\ell^{q(\cdot)}(L^{p_1(\cdot)})} \\ & \leqslant \left\| \left(2^{v\alpha(\cdot)} \varphi_v * f \right)_{v \geqslant 0} \right\|_{\ell^{q(\cdot)}(L^{p_1(\cdot)})}. \end{aligned}$$

Clearly, the last expression is bounded by $\|f\|_{B^{\alpha(\cdot)}_{p_1(\cdot),q(\cdot)}} \leqslant \|f\|_{B^{\alpha(\cdot)+n/\tau(\cdot)}_{p_1(\cdot),q(\cdot)}}$. By the property (2.6) we obtain the estimate

$$\begin{aligned}
&\sup_{P\in\mathcal{Q},|P|\leqslant 1}\left\|\left(\frac{2^{v\alpha(\cdot)}\varphi_v * f}{\|\chi_P\|_{\tau(\cdot)}}\chi_P\right)_{v\geqslant v_P^+}\right\|_{\ell^{q(\cdot)}(L^{p_1(\cdot)})}\\
&\qquad\approx \sup_{P\in\mathcal{Q},|P|\leqslant 1}\left\|\left(\frac{2^{v\alpha(\cdot)}\varphi_v * f}{|P|^{1/\tau(\cdot)}}\chi_P\right)_{v\geqslant v_P^+}\right\|_{\ell^{q(\cdot)}(L^{p_1(\cdot)})}\\
&\qquad\leqslant \sup_{P\in\mathcal{Q},|P|\leqslant 1}\left\|\left(2^{v(\alpha(\cdot)+n/\tau(\cdot))+n/\tau(\cdot)(v_P-v)}\varphi_v * f\right)_{v\geqslant v_P}\right\|_{\ell^{q(\cdot)}(L^{p_1(\cdot)})}\\
&\qquad\leqslant \sup_{P\in\mathcal{Q},|P|\leqslant 1}\left\|\left(2^{v(\alpha(\cdot)+n/\tau(\cdot))}\varphi_v * f\right)_{v\geqslant 0}\right\|_{\ell^{q(\cdot)}(L^{p_1(\cdot)})} \leqslant \|f\|_{B^{\alpha(\cdot)+n/\tau(\cdot)}_{p_1(\cdot),q(\cdot)}},
\end{aligned}$$

which completes the proof. ■

In view of Remark 3.11 we obtain

Theorem 4.8. *Let $\alpha \in C^{\log}_{\mathrm{loc}}$ and $p, q, \tau \in \mathcal{P}^{\log}_0$ with $0 < q^- \leqslant q^+ < \infty$. Then*

$$B^{\alpha(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)} \hookrightarrow B^{\alpha(\cdot)+\frac{n}{\tau(\cdot)}-\frac{n}{p(\cdot)}}_{\infty,\infty}.$$

We refer the reader to the recent paper [39] for further details, historical remarks and more references on embeddings of Besov-type spaces with fixed exponents.

Let $0 < u \leqslant p \leqslant \infty$. The Morrey space $\mathcal{M}^p_u$ is defined to be the set of all u-locally Lebesgue-integrable functions f on $\mathbb{R}^n$ such that

$$\|f\|_{\mathcal{M}^p_u} = \sup_B |B|^{\frac{1}{p}-\frac{1}{u}} \left(\int_B |f(x)|^u dx\right)^{1/u} < \infty,$$

where the supremum is taken over all balls B in $\mathbb{R}^n$. The spaces $\mathcal{M}^p_u$ are quasi-Banach spaces (Banach spaces for $u \geqslant 1$). They were introduced by Morrey in [17] and belong to the wider class of Morrey-Campanato spaces, cf. [19]. They can be considered as a complement to L^p spaces. As a matter of fact, $\mathcal{M}^p_p = L^p$. One can easily see that

$$\mathcal{M}^p_w \hookrightarrow \mathcal{M}^p_u \qquad \text{if } 0 < u \leqslant w \leqslant \infty.$$

Definition 4.9. Let $\{\mathcal{F}\varphi_v\}_{v\in\mathbb{N}_0}$ be a resolution of unity, $\alpha : \mathbb{R}^n \to \mathbb{R}, 0 < u \leqslant p \leqslant \infty$ and $0 < q \leqslant \infty$. The Besov-Morrey space $\mathcal{N}^{\alpha(\cdot)}_{p,q,u}$ is the collection of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\|f\|_{\mathcal{N}^{\alpha(\cdot)}_{p,q,u}} = \left(\sum_{v=0}^{\infty} \left\|2^{v\alpha(\cdot)}\varphi_v * f\right\|^q_{\mathcal{M}^p_u}\right)^{1/q} < \infty.$$

Besov-Morrey spaces with fixed exponents were introduced by Kozono and Yamazaki [18]. They studied semilinear heat equations and Navier-Stokes equations with initial data belonging to Besov-Morrey spaces. The investigations were continued by Mazzucato [16], where one can find the wavelet decomposition of Besov-Morrey spaces. Further properties for these function spaces can be found in [23], [24], [25] and [28].

The Besov-Morrey spaces with variable exponents have been first introduced in [13], where are introduced equivalent quasi-norms of these new spaces, which are formulated in terms of Peetre's maximal functions. Also the authors obtain the atomic, molecular and wavelet decompositions of these new spaces.

In the next proposition we present the relations between variable Besov-Morrey spaces and variable Besov-type spaces. The proof follows from Theorem 3.12.

Proposition 4.10. *Let* $\alpha \in C^{\log}_{\mathrm{loc}}$, $0 < q < \infty$ *and* $0 < p < u < \infty$.

(i) *For* $0 < q < \infty$ *we have the continuous embeddings*

$$\mathcal{N}^{\alpha(\cdot)}_{u,q,p} \hookrightarrow B^{\alpha(\cdot),\left(\frac{1}{p}-\frac{1}{u}\right)^{-1}}_{p,q}.$$

(ii) *We have*

$$\mathcal{N}^{\alpha(\cdot)}_{u,\infty,p} = B^{\alpha(\cdot),\left(\frac{1}{p}-\frac{1}{u}\right)^{-1}}_{p,\infty}.$$

5. Appendix

Here we present more technical proofs of the Lemmas.

Proof of Lemma 2.11. (i). We can distinguish two cases as follows:

Case $v_P \leqslant 0$: in this case we obtain $|P| = 2^{-nv_P} \geqslant 1$ and then by the property (2.7) we get

$$\|\chi_P\|_{\tau(\cdot)} \approx |P|^{\frac{1}{\tau_\infty}} = |P + kl(Q)|^{\frac{1}{\tau_\infty}} \approx \left\|\chi_{P+kl(Q)}\right\|_{\tau(\cdot)}.$$

Case $v_P > 0$: here we have by Lemma 2.8

$$\begin{aligned} 2^{\frac{nv_P}{\tau(x)}} &\leqslant c(1 + 2^{v_P}|x-y|)^{c_{\log}\left(\frac{1}{\tau}\right)} 2^{\frac{nv_P}{\tau(y)}} \\ &\leqslant c\left(1 + \frac{l(Q)}{l(P)}|k|\right)^{c_{\log}\left(\frac{1}{\tau}\right)} 2^{\frac{nv_P}{\tau(y)}} \end{aligned}$$

for any $x \in P$ and $y \in P + kl(Q)$, with $y = x + kl(Q)$. Using the property (2.6) we get

$$\begin{aligned} \left\|\chi_{P+kl(Q)}\right\|_{\tau(\cdot)} \approx 2^{\frac{-nv_{P+kl(Q)}}{\tau(y)}} &\leqslant c\left(1 + \frac{l(Q)}{l(P)}|k|\right)^{c_{\log}\left(\frac{1}{\tau}\right)} 2^{\frac{-nv_P}{\tau(x)}} \\ &\approx \left(1 + \frac{l(Q)}{l(P)}|k|\right)^{c_{\log}\left(\frac{1}{\tau}\right)} \|\chi_P\|_{\tau(\cdot)}, \end{aligned}$$

in view of the fact that $v_P = v_{P+kl(Q)}$.

(ii). We can distinguish two cases as follows:
$|P| \geqslant 1$: in this case we use (2.7) to obtain $\|\chi_P\|_{\tau(\cdot)} \approx |P|^{1/\tau_\infty}$ and $\|\chi_Q\|_{\tau(\cdot)} \approx |Q|^{1/\tau_\infty}$. Hence

$$\left(\frac{|Q|}{|P|}\right)^{1/\tau^+} \leqslant \frac{\|\chi_Q\|_{\tau(\cdot)}}{\|\chi_P\|_{\tau(\cdot)}} \approx \left(\frac{|Q|}{|P|}\right)^{1/\tau_\infty} \leqslant \left(\frac{|Q|}{|P|}\right)^{1/\tau^-}.$$

Now let us consider $|P| < 1$. If $|Q| \leqslant 1$ then from (2.6) we obtain $\|\chi_P\|_{\tau(\cdot)} \approx |P|^{1/\tau(y)}$ for $y \in P$ and $\|\chi_Q\|_{\tau(\cdot)} \approx |Q|^{1/\tau(y)}$ for $y \in P \subset Q$, which means that

$$\left(\frac{|Q|}{|P|}\right)^{1/\tau^+} \leqslant \frac{\|\chi_Q\|_{\tau(\cdot)}}{\|\chi_P\|_{\tau(\cdot)}} \approx \left(\frac{|Q|}{|P|}\right)^{1/\tau(y)} \leqslant \left(\frac{|Q|}{|P|}\right)^{1/\tau^-}.$$

If $|Q| > 1$, then an application of (2.2) we obtain $|Q|^{1/\tau^+} \lesssim \|\chi_Q\|_{\tau(\cdot)} \lesssim |Q|^{1/\tau^-}$ and $|P|^{1/\tau^-} \lesssim \|\chi_P\|_{\tau(\cdot)} \lesssim |P|^{1/\tau^+}$ which finishes the proof. ■

Proof of Lemma 2.12. We have

$$\begin{aligned}
\eta_{v,m} * f_v(x) &= 2^{vn} \int_{\mathbb{R}^n} \frac{f_v(z)}{(1 + 2^v |x-z|)^m} dz \\
&= \int_{3P} \cdots dz + \sum_{k=(k_1,\dots,k_n)\in\mathbb{Z}^n, \max_{i=1,\dots,n} |k_i| \geqslant 2} \int_{P+kl(P)} \cdots dz \\
&= J_v^1(f_v)(x) + \sum_{k=(k_1,\dots,k_n)\in\mathbb{Z}^n, \max_{i=1,\dots,n} |k_i| \geqslant 2} J_{v,k}^2(f_v)(x).
\end{aligned}$$

Let $0 < r < \frac{1}{2}\min(p^-, q^-, 2)$ and define $\tilde{p} = \frac{p}{r}$ and $\tilde{q} = \frac{q}{r}$. Then clearly, $\frac{1}{\tilde{p}} + \frac{1}{\tilde{q}} \leqslant 1$. Thus we obtain

$$\begin{aligned}
&\left\| \left(\frac{\eta_{v,m} * f_v}{\|\chi_P\|_{\tau(\cdot)}} \chi_P \right)_{v \geqslant v_P^+} \right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})}^r \\
&\quad = \left\| \left(\frac{|\eta_{v,m} * f_v|^r}{\|\chi_P\|_{\tau(\cdot)}^r} \chi_P \right)_{v \geqslant v_P^+} \right\|_{\ell^{\tilde{q}(\cdot)}(L^{\tilde{p}(\cdot)})} \leqslant \left\| \left(\frac{J_v^1(f_v)}{\|\chi_P\|_{\tau(\cdot)}} \chi_P \right)_{v \geqslant v_P^+} \right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})}^r \\
&\qquad + \sum_{k=(k_1,\dots,k_n)\in\mathbb{Z}^n, \max_{i=1,\dots,n} |k_i| \geqslant 2} \left\| \left(\frac{J_{v,k}^2(f_v)}{\|\chi_P\|_{\tau(\cdot)}} \chi_P \right)_{v \geqslant v_P^+} \right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})}^r. \qquad (5.1)
\end{aligned}$$

Let us prove that the first norm on the right-hand side is bounded by

$$c \left\| \left(\frac{f_v}{\|\chi_P\|_{\tau(\cdot)}} \chi_{3P} \right)_{v \geqslant v_P^+} \right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})}. \qquad (5.2)$$

By the scaling argument, we see that it suffices to consider when the last norm is less than or equal 1 and show that for any dyadic cube P

$$\sum_{v=v_P^+}^{\infty} \left\| \left| \frac{c\ J_v^1(f_v)}{\|\chi_P\|_{\tau(\cdot)}} \right|^{q(\cdot)} \chi_P \right\|_{\frac{p(\cdot)}{q(\cdot)}} \leqslant 1.$$

Our estimate, clearly follows from the inequality

$$\left\| \left| \frac{c\ J_v^1(f_v)}{\|\chi_P\|_{\tau(\cdot)}} \right|^{q(\cdot)} \chi_P \right\|_{\frac{p(\cdot)}{q(\cdot)}} \leqslant \left\| \left| \frac{f_v}{\|\chi_P\|_{\tau(\cdot)}} \right|^{q(\cdot)} \chi_{3P} \right\|_{\frac{p(\cdot)}{q(\cdot)}} + 2^{-v} = \delta$$

which holds true for any $v \geqslant v_P^+$. This claim can be reformulated as showing that

$$\left\| \delta^{-1} \left| \frac{c\ J_v^1(f_v)}{\|\chi_P\|_{\tau(\cdot)}} \right|^{q(\cdot)} \chi_P \right\|_{\frac{p(\cdot)}{q(\cdot)}} \leqslant 1,$$

which is equivalent to

$$\left\| \delta^{-\frac{1}{q(\cdot)}} \frac{J_v^1(f_v)}{\|\chi_P\|_{\tau(\cdot)}} \chi_P \right\|_{p(\cdot)} \leqslant c. \tag{5.3}$$

Taking into account that $1/q$ is log-Hölder continuous and $\delta \in [2^{-v}, 1 + 2^{-v}]$ we obtain

$$\begin{aligned} \delta^{\left(\frac{1}{q(z)} - \frac{1}{q(x)}\right)} &= (2^v \delta)^{\left(\frac{1}{q(z)} - \frac{1}{q(x)}\right)} 2^{v\left(\frac{1}{q(z)} - \frac{1}{q(x)}\right)} \\ &\leqslant c\ (1 + 2^v |x - z|)^{c_{\log}(1/q)}. \end{aligned}$$

Hence

$$\delta^{-\frac{1}{q(x)}} |J_v^1(f_v)(x)| \leqslant C\ 2^{vn} \int_{\mathbb{R}^n} \frac{\delta^{-\frac{1}{q(z)}} |f_v(z)|\ \chi_{3P}(z)}{(1 + 2^v |x - z|)^{m - c_{\log}(1/q)}} dz.$$

Now the function $z \mapsto \frac{1}{(1+|z|)^m}$ is in L^1 (since $m - c_{\log}(1/q) > n$), then using the majorant property for the Hardy-Littlewood maximal operator $\mathcal{M}$, see E. M. Stein and G. Weiss [22, Chapter 2, (3.9)],

$$\Big(|g| * \frac{1}{(1 + |\cdot|)^m}\Big)(x) \leqslant C\ \left\| \frac{1}{(1 + |\cdot|)^m} \right\|_1 \mathcal{M}(g)(x),$$

it follows that for any $x \in P$, $\delta^{-\frac{1}{q(x)}} |J_v^1(f_v)(x)| \leqslant C\ \mathcal{M}\Big(\delta^{-\frac{1}{q(\cdot)}} f_v \chi_{3P}\Big)(x)$ where the constant $C > 0$ is independent of x and v. Hence the left-hand side of (5.3) is bounded by

$$c \left\| \mathcal{M}\Big(\frac{\delta^{-\frac{1}{q(\cdot)}} f_v}{\|\chi_P\|_{\tau(\cdot)}} \chi_{3P} \Big) \right\|_{p(\cdot)} \lesssim \left\| \frac{\delta^{-\frac{1}{q(\cdot)}} f_v}{\|\chi_P\|_{\tau(\cdot)}} \chi_{3P} \right\|_{p(\cdot)},$$

and the fact that $\mathcal{M} : L^{p(\cdot)} \to L^{p(\cdot)}$ is bounded. Now the norm on the right-hand side is less than or equal to one if and only if

$$\left\| \delta^{-1} \left| \frac{f_v}{\|\chi_P\|_{\tau(\cdot)}} \right|^{q(\cdot)} \chi_{3P} \right\|_{\frac{p(\cdot)}{q(\cdot)}} \leqslant 1$$

which follows immediately from the definition of δ. Notice that $3P = \cup_{h=1}^{3^n} P_h$, where $\{P_h\}_{h=1}^{3^n}$ are disjoint dyadic cubes with side length $l(P_h) = l(P)$. Therefore $\chi_{3P} = \sum_{h=1}^{3^n} \chi_{P_h}$ and the expression in (5.2) can be estimated by

$$c \sum_{h=1}^{3^n} \left\| \left(\frac{f_v}{\|\chi_{P_h}\|_{\tau(\cdot)}} \chi_{P_h} \right)_{v \geqslant v_P^+} \right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} \leqslant c \, \|(f_v)_v\|_{\ell^{\tau(\cdot),q(\cdot)}(L^{p(\cdot)})} \, ,$$

where we have used the fact that $\frac{\|\chi_{P_h}\|_{\tau(\cdot)}}{\|\chi_P\|_{\tau(\cdot)}} \leqslant c$, see Lemma 2.11 (ii) and the proof of the first part is finished. The summation in (5.1) can be rewritten us

$$\sum_{k \in \mathbb{Z}^n,\, |k| \leqslant 4\sqrt{n}} \cdots + \sum_{k \in \mathbb{Z}^n,\, |k| > 4\sqrt{n}} \cdots. \tag{5.4}$$

The estimate of the first sum follows in the same manner as in the estimate of $J_v^1(f_v)$, so we need only to estimate the second sum. Let us prove that

$$\begin{aligned} &\left\| \left(|k|^{m-n-c_{\log}(1/\tau)-c_{\log}(1/q)} \frac{J_{v,k}^2(f_v)}{\|\chi_P\|_{\tau(\cdot)}} \chi_P \right)_{v \geqslant v_P^+} \right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} \\ &\qquad\qquad \lesssim \left\| \left(\frac{f_v}{\|\chi_{P+kl(P)}\|_{\tau(\cdot)}} \chi_{P+kl(P)} \right)_{v \geqslant v_P^+} \right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} . \end{aligned}$$

Again, by the scaling argument, we see that it suffices to consider when the last norm is less than or equal 1 and show that the modular of a constant times the function on the left-hand side is bounded. In particular, we will show that for any dyadic cube P

$$\sum_{v=v_P^+}^{\infty} \left\| \left| \frac{c \ |k|^{m-n-c_{\log}(1/\tau)-c_{\log}(1/q)} J_{v,k}^2(f_v)}{\|\chi_P\|_{\tau(\cdot)}} \right|^{q(\cdot)} \chi_P \right\|_{\frac{p(\cdot)}{q(\cdot)}} \leqslant 1.$$

This estimate follows from the inequality

$$\left\| \left| \frac{c\ |k|^{m-n-c_{\log}(1/\tau)-c_{\log}(1/q)} J^2_{v,k}(f_v)}{\|\chi_P\|_{\tau(\cdot)}} \right|^{q(\cdot)} \chi_P \right\|_{\frac{p(\cdot)}{q(\cdot)}} \leqslant \left\| \left| \frac{f_v}{\left\|\chi_{P+kl(P)}\right\|_{\tau(\cdot)}} \right|^{q(\cdot)} \chi_{P+kl(P)} \right\|_{\frac{p(\cdot)}{q(\cdot)}} + 2^{-v} = \delta.$$

This claim can be reformulated as showing that

$$\left\| \delta^{-1} \left| \frac{c\ |k|^{m-n-c_{\log}(1/\tau)-c_{\log}(1/q)} J^2_{v,k}(f_v)}{\|\chi_P\|_{\tau(\cdot)}} \right|^{q(\cdot)} \chi_P \right\|_{\frac{p(\cdot)}{q(\cdot)}} \leqslant 1,$$

which is equivalent to

$$\left\| \delta^{-\frac{1}{q(\cdot)}} \frac{|k|^{m-n-c_{\log}(1/\tau)-c_{\log}(1/q)} J^2_{v,k}(f_v)}{\|\chi_P\|_{\tau(\cdot)}} \chi_P \right\|_{p(\cdot)} \leqslant c. \tag{5.5}$$

Let $x \in P$ and $z \in P+kl(P)$ with $k \in \mathbb{Z}^n$ and $|k| > 4\sqrt{n}$, then $|x-z| \geqslant \frac{1}{2}|k|l(P)$. Hence for any $x \in P$ and any $v \geqslant v_P^+$, the term $\delta^{-\frac{1}{q(x)}} |J^2_{v,k}(f_v)(x)|$ is bounded by

$$\begin{aligned}
&C\ 2^{(v-v_P)(c_{\log}(1/q)-m+n)} |k|^{c_{\log}(1/q)-m} 2^{nv_P} \int_{P+kl(P)} \delta^{-\frac{1}{q(z)}} |f_v(z)|\, dz \\
&\leqslant C\ |k|^{c_{\log}(1/q)-m} 2^{nv_P} \int_{|z-x| \leqslant 2\sqrt{n}|k|2^{-v_P}} \delta^{-\frac{1}{q(z)}} |f_v(z)|\, \chi_{P+kl(P)}(z) dz \\
&\leqslant C\ |k|^{n-m+c_{\log}(1/q)} \mathcal{M}\Big(\delta^{-\frac{1}{q(\cdot)}} f_v \chi_{P+kl(P)}\Big)(x) \\
&\leqslant C\ |k|^{n-m+c_{\log}(1/q)} \mathcal{M}\Big(\delta^{-\frac{1}{q(\cdot)}} f_v \chi_{P+kl(P)}\Big)(x).
\end{aligned}$$

Hence the left-hand side of (5.5) is bounded by

$$\left\| C\ \mathcal{M}\Big(|k|^{-c_{\log}(1/\tau)} \frac{\delta^{-\frac{1}{q(\cdot)}} f_v \chi_{P+kl(P)}}{\|\chi_P\|_{\tau(\cdot)}} \Big) \right\|_{p(\cdot)} \lesssim |k|^{-c_{\log}(1/\tau)} \left\| \frac{\delta^{-\frac{1}{q(\cdot)}} f_v \chi_{P+kl(P)}}{\|\chi_P\|_{\tau(\cdot)}} \right\|_{p(\cdot)},$$

and the fact that $\mathcal{M} : L^{p(\cdot)} \to L^{p(\cdot)}$ is bounded. By Lemma 2.11 (i), we get $\frac{\|\chi_{P+kl(P)}\|_{\tau(\cdot)}}{\|\chi_P\|_{\tau(\cdot)}} \leqslant c|k|^{c_{\log}(1/\tau)}$, with $c > 0$ independent of v, h and k. Hence

$$\left\| |k|^{-c_{\log}(\frac{1}{\tau})} \frac{\delta^{-\frac{1}{q(\cdot)}} f_v \chi_{P+kl(P)}}{\|\chi_P\|_{\tau(\cdot)}} \right\|_{p(\cdot)} \leqslant c \left\| \frac{\delta^{-\frac{1}{q(\cdot)}} f_v \chi_{P+kl(P)}}{\left\|\chi_{P+kl(P)}\right\|_{\tau(\cdot)}} \right\|_{p(\cdot)} \leqslant c,$$

which follows immediately from the definition of δ. Since m can be taken large enough so that $m > 2n + c_{\log}(1/\tau) + c_{\log}(1/q)$, then the second sum in (5.4) is bounded by

$$\begin{aligned}\sum_{k\in\mathbb{Z}^n,\, |k|>4\sqrt{n}} &|k|^{n+c_{\log}(1/\tau)+c_{\log}(1/q)-m} \left\| \left(\frac{f_v}{\left\|\chi_{P+kl(P)}\right\|_{\tau(\cdot)}} \chi_{P+kl(P)} \right)_{v\geqslant v_P^+} \right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} \\ &\leqslant \sum_{k\in\mathbb{Z}^n,\, |k|>4\sqrt{n}} |k|^{n+c_{\log}(1/\tau)+c_{\log}(1/q)-m} \left\|(f_v)_v\right\|_{\ell^{\tau(\cdot),q(\cdot)}(L^{p(\cdot)})} \\ &\lesssim \left\|(f_v)_v\right\|_{\ell^{\tau(\cdot),q(\cdot)}(L^{p(\cdot)})} .\end{aligned}$$

The proof is complete. ∎

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