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HILBERT MODULAR AND QUASIMODULAR FORMS

Min Ho Lee

Abstract: Quasimodular forms generalize modular forms and have been studied actively in recent years in connection with various topics in number theory and geometry. One of their interesting properties is that they correspond to finite sequences of modular forms of certain types. We extend such a correspondence to the case of Hilbert quasimodular forms. As an application we construct Poincaré series for Hilbert quasimodular forms.

Keywords: quasimodular forms, Hilbert modular forms, Poincaré series.

1. Introduction

Quasimodular forms generalize modular forms and were introduced by Kaneko and Zagier in [4]. They have been studied actively in recent years in connection with various topics in number theory and geometry (see e.g. [2], [7], [8], [9]). Derivatives of both modular and quasimodular forms are quasimodular forms, and, in fact, each quasimodular form can be expressed as a linear combination of derivatives of a finite number of modular forms.

A holomorphic function f on the Poincaré upper half plane $\mathcal{H}$ is a quasimodular form for a discrete subgroup Γ of $SL(2,\mathbb{R})$ of weight $\lambda \in \mathbb{Z}$ and depth at most $m \geqslant 0$ if there exist holomorphic functions $f_0, f_1, \dots, f_m$ on $\mathcal{H}$ satisfying

$$\frac{1}{(cz+d)^{\lambda}} f\Big(\frac{az+b}{cz+d}\Big) = f_0(z) + \frac{cf_1(z)}{cz+d} + \cdots + \frac{c^m f_m(z)}{(cz+d)^m}$$

for all $z \in \mathcal{H}$ and $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$. Given such a quasimodular form, it is known that there are modular forms $h_0, h_1, \dots, h_m$ for Γ of certain weights such that each f_k is a linear combination of derivatives of those modular forms. On the other hand, each modular form h_j can also be written as a linear combination of derivatives of the functions f_k. These results can be used to show that there is a one-to-one correspondence between quasimodular forms (of weight greater than twice the

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depth) and certain finite sequences of modular forms (cf. [6]). This correspondence may potentially be used to investigate certain properties of quasimodular forms by studying similar properties of the modular forms in the corresponding sequences.

The goal of this paper is to establish an analog of the above-mentioned correspondence for functions of several variables. In the case of several variables, as was done by F. Pellarin in [10], we can consider Hilbert quasimodular forms. In order to study such quasimodular forms more effectively we introduce Hilbert quasimodular polynomials for a discrete subgroup of $SL(2,\mathbb{R})^n$. We also consider Hilbert modular polynomials for the same discrete group, whose coefficients are Hilbert modular forms of certain types, and construct an isomorphism between the space of such polynomials and the space of Hilbert quasimodular polynomials. This isomorphism then determines our desired correspondence between Hilbert quasimodular forms and sequences of Hilbert modular forms. As an application we construct Poincaré series for Hilbert quasimodular forms.

2. Correspondences of polynomials

Given a positive integer n, we denote by $\mathcal{F}$ the ring of holomorphic functions $f : \mathcal{H}^n \to \mathbb{C}$ with $\mathcal{H}$ being the Poincaré upper half plane. Throughout this paper we use the multi-index notation. Thus, given elements $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}^n$ and $z = (z_1, \dots, z_n) \in \mathcal{H}^n$, we have

$$z^\alpha = z_1^{\alpha_1} \cdots z_n^{\alpha_n}, \qquad |\alpha| = \alpha_1 + \cdots + \alpha_n,$$

If $\alpha \in \mathbb{Z}_+^n$ with $\mathbb{Z}_+$ denoting the set of nonnegative integers and $\beta = (\beta_1, \dots, \beta_n) \in \mathbb{Z}^n$, we also have

$$\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}, \qquad \binom{\beta}{\alpha} = \binom{\beta_1}{\alpha_1} \cdots \binom{\beta_n}{\alpha_n},$$

where $\partial_i = \partial/\partial z_i$ for $1 \leqslant i \leqslant n$. For $\alpha, \beta \in \mathbb{Z}^n$, we write $\alpha \leqslant \beta$ if $\alpha_i \leqslant \beta_i$ for each $i = 1, \dots, n$. We also write $\alpha < \beta$ if $\alpha \leqslant \beta$ and $\alpha \neq \beta$. If $c \in \mathbb{Z}$, we shall use the bold-faced symbol to denote the element $\boldsymbol{c} = (c, \dots, c) \in \mathbb{Z}^n$.

We now fix an element $\mu = (\mu_1, \dots, \mu_n) \in \mathbb{Z}_+^n$. Let $X = (X_1, \dots, X_n)$, and denote by $\mathcal{F}_\mu[X]$ the complex vector space of polynomials in $X_1, \dots, X_n$ over $\mathcal{F}$ of the form

$$\Phi(z, X) = \Phi(z_1, \dots, z_n; X_1, \dots, X_n) = \sum_{\mathbf{0} \leqslant \rho \leqslant \mu} \phi_\rho(z) X^\rho \in \mathcal{F}_\mu[X] \tag{2.1}$$

with $z \in \mathcal{H}^n$, $\phi_\rho \in \mathcal{F}$ and

$$X^\rho = X_1^{\rho_1} \cdots X_n^{\rho_n}$$

for each $\rho = (\rho_1, \dots, \rho_n) \in \mathbb{Z}_+^n$. Given such a polynomial and an element $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{Z}^n$ with $\lambda \geqslant 2\mu + \mathbf{1}$, we consider two other polynomials

$$(\Xi_\lambda^\mu \Phi)(z, X),\ (\Lambda_\lambda^\mu \Phi)(z, X) \in \mathcal{F}_\mu[X]$$

defined by

$$(\Xi_\lambda^\mu \Phi)(z,X) = \sum_{\mathbf{0}\leqslant\rho\leqslant\mu} \phi_\rho^\Xi(z)X^\rho, \qquad (\Lambda_\lambda^\mu \Phi)(z,X) = \sum_{\mathbf{0}\leqslant\rho\leqslant\mu} \phi_\rho^\Lambda(z)X^\rho \tag{2.2}$$

where

$$\phi_\rho^\Xi = \frac{1}{\rho!} \sum_{\mathbf{0}\leqslant\nu\leqslant\mu-\rho} \frac{1}{\nu!(\lambda-2\rho-\nu-\mathbf{1})!} \partial^\nu \phi_{\mu-\rho-\nu} \tag{2.3}$$

$$\begin{aligned}\phi_\rho^\Lambda = (\lambda+2\rho-2\mu-\mathbf{1})^{\mathbf{1}} &\sum_{\mathbf{0}\leqslant\nu\leqslant\rho} \frac{(-1)^{|\nu|}}{\nu!}(\mu-\rho+\nu)! \\ &\times(\lambda+2\rho-2\mu-\nu-\mathbf{2})!\partial^\nu\phi_{\mu-\rho+\nu},\end{aligned} \tag{2.4}$$

for each $\rho\in\mathbb{Z}^n$ with $\mathbf{0}\leqslant\rho\leqslant\mu$. The next proposition shows that the resulting maps are linear automorphisms of $\mathcal{F}_\mu[X]$, and its proof uses the combinatorial identity

$$\sum_{r=0}^{u}(-1)^r\binom{u}{r}\binom{v-r}{u-1} = 0 \tag{2.5}$$

for positive integers u and v with $u\leqslant v$ (see e.g. [5, Lemma 2.7]).

Proposition 2.1. *The maps* $\Xi_\lambda^\mu, \Lambda_\lambda^\mu : \mathcal{F}_\mu[X]\to\mathcal{F}_\mu[X]$ *given by* (2.2) *are complex linear isomorphisms with*

$$(\Lambda_\lambda^\mu)^{-1} = \Xi_\lambda^\mu$$

for each $\lambda\in\mathbb{Z}^n$ *with* $\lambda\geqslant 2\mu+\mathbf{1}$.

Proof. Given $\lambda > 2\mu+\mathbf{1}$, we first consider a polynomial $\Phi(z,X)$ and its image $(\Xi_\lambda^\mu\Phi)(z,X)$ under Ξ_λ^μ as in (2.1) and (2.2), respectively. Then, using (2.3) and (2.4), we obtain

$$((\Lambda_\lambda^\mu\circ\Xi_\lambda^\mu)\Phi)(z,X) = \sum_{\mathbf{0}\leqslant\rho\leqslant\mu}\widehat{\phi}_\rho(z)X^\rho,$$

where

$$\begin{aligned}
&(\lambda+2\rho-2\mu-\mathbf{1})^{-\mathbf{1}}\widehat{\phi}_\rho \\
&\quad= \sum_{\mathbf{0}\leqslant\nu\leqslant\rho}\frac{(-1)^{|\nu|}}{\nu!}(\mu-\rho+\nu)!(\lambda+2\rho-2\mu-\nu-\mathbf{2})!\partial^\nu\phi^\Xi_{\mu-\rho+\nu} \\
&\quad= \sum_{\mathbf{0}\leqslant\nu\leqslant\rho}\sum_{\mathbf{0}\leqslant\xi\leqslant\rho-\nu}\frac{(-1)^{|\nu|}(\lambda+2\rho-2\mu-\nu-\mathbf{2})!}{\nu!\xi!(\lambda-2\mu+2\rho-2\nu-\xi-\mathbf{1})!}\partial^{\xi+\nu}\phi_{\rho-\xi-\nu} \\
&\quad= \sum_{\mathbf{0}\leqslant\eta\leqslant\rho}\sum_{\mathbf{0}\leqslant\nu\leqslant\eta}\frac{(-1)^{|\nu|}(\lambda+2\rho-2\mu-\nu-\mathbf{2})!}{\nu!(\eta-\nu)!(\lambda+2\rho-2\mu-\nu-\eta-\mathbf{1})!}\partial^\eta\phi_{\rho-\eta} \\
&\quad= (\lambda+2\rho-2\mu-\mathbf{1})^{-\mathbf{1}}\phi_\rho \\
&\qquad+ \sum_{\mathbf{0}<\eta\leqslant\rho}\sum_{\mathbf{0}\leqslant\nu\leqslant\eta}\frac{(-1)^{|\nu|}(\lambda+2\rho-2\mu-\nu-\mathbf{2})!}{\nu!(\eta-\nu)!(\lambda+2\rho-2\mu-\nu-\eta-\mathbf{1})!}\partial^\eta\phi_{\rho-\eta}
\end{aligned} \tag{2.6}$$

for $\mathbf{0} \leqslant \rho \leqslant \mu$. If $\eta > \mathbf{0}$, we note that $\eta_i \neq 0$ for some $i \in \{1, \dots, n\}$. Denoting by $\widehat{\eta}_i = (\eta_1, \dots, 0, \dots, \eta_n)$ the element of $\mathbb{Z}^n$ with η_i replaced by 0 in the i-th entry and similarly for other elements, we have

$$\begin{aligned}
&\sum_{\mathbf{0}<\eta\leqslant\rho} \sum_{\mathbf{0}\leqslant\nu\leqslant\eta} \frac{(-1)^{|\nu|}(\lambda + 2\rho - 2\mu - \nu - \mathbf{2})!}{\nu!(\eta-\nu)!(\lambda + 2\rho - 2\mu - \nu - \eta - \mathbf{1})!} \partial^\eta \phi_{\rho-\eta} \\
&\quad = \sum_{\mathbf{0}<\widehat{\eta}_i\leqslant\rho} \sum_{\mathbf{0}\leqslant\nu\leqslant\widehat{\eta}_i} \frac{(-1)^{|\nu|}(\widehat{\lambda}_i + 2\widehat{\rho}_i - 2\widehat{\mu}_i - \nu - \widehat{\mathbf{2}}_i)!}{\nu!(\widehat{\eta}_i-\nu)!(\widehat{\lambda}_i + 2\widehat{\rho}_i - 2\widehat{\mu}_i - \nu - \widehat{\eta}_i - \widehat{\mathbf{1}}_i)!} \\
&\qquad \times \sum_{\eta_i=1}^{\rho_i} \sum_{k=0}^{\eta_i} \frac{(-1)^k(\lambda_i + 2\rho_i - 2\mu_i - k - 2)!}{k!(\eta_i-k)!(\lambda_i + 2\rho_i - 2\mu_i - k - \eta_i - 1)!} \phi^\eta \phi_{\rho-\eta} \\
&\quad = \sum_{\mathbf{0}<\widehat{\eta}_i\leqslant\rho} \sum_{\mathbf{0}\leqslant\nu\leqslant\widehat{\eta}_i} \frac{(-1)^{|\nu|}(\widehat{\lambda}_i + 2\widehat{\rho}_i - 2\widehat{\mu}_i - \nu - \widehat{\mathbf{2}}_i)!}{\nu!(\widehat{\eta}_i-\nu)!(\widehat{\lambda}_i + 2\widehat{\rho}_i - 2\widehat{\mu}_i - \nu - \widehat{\eta}_i - \widehat{\mathbf{1}}_i)!} \\
&\qquad \times \sum_{\eta_i=1}^{\rho_i} \frac{1}{\eta_i} \phi^{(\eta)}_{\rho-\eta} \sum_{k=0}^{\eta_i} (-1)^k \binom{\eta_i}{k} \binom{\lambda_i + 2\rho_i - 2\mu_i - k - 2}{\eta_i - 1} = 0,
\end{aligned}$$

where we used (2.5). From this and (2.6) it follows that $\widehat{\phi}_\rho = \phi_\rho$ for all ρ with $\mathbf{0} \leqslant \rho \leqslant \mu$; hence we obtain

$$((\Lambda^\mu_\lambda \circ \Xi^\mu_\lambda)\Phi)(z, X) = \Phi(z, X).$$

We now assume that $(\Lambda^\mu_\lambda \Phi)(z, X)$ is as in (2.2) and that

$$((\Xi^\mu_\lambda \circ \Lambda^\mu_\lambda)\Phi)(z, X) = \sum_{\nu=0}^{\mu} \widetilde{\phi}_\nu(z) X^\nu.$$

Thus, in particular, (2.4) is valid for $\mathbf{0} \leqslant \rho \leqslant \mu$. Noting that clearly $\widetilde{\phi}_\mu = \phi_\mu$, we shall verify that $\widetilde{\phi}_\rho = \phi_\rho$ for $\mathbf{0} \leqslant \rho \leqslant \mu$ by using induction. Given a vector ρ with $\mathbf{0} < \rho \leqslant \mu$, we assume that

$$\phi_\eta = \widetilde{\phi}_\eta = \frac{1}{\eta!} \sum_{\mathbf{0}\leqslant\nu\leqslant\mu-\eta} \frac{1}{\nu!(\lambda - 2\eta - \nu - \mathbf{1})!} \partial^\nu \phi^\Lambda_{\mu-\eta-\nu} \tag{2.7}$$

holds for each η with $\rho < \eta \leqslant \mu$. Thus there is an index $j \in \{1, \dots, n\}$ such that

$$\rho + \varepsilon_j = (\rho_1, \dots, \rho_j + 1, \dots, \rho_n) \leqslant \mu.$$

Then from (2.4) we obtain

$$\begin{aligned}
\phi^\Lambda_{\mu-\rho} &= (\lambda - 2\rho - \mathbf{1})^{\mathbf{1}} \sum_{\mathbf{0}\leqslant\nu\leqslant\mu-\rho} \frac{(-1)^{|\nu|}}{\nu!} (\rho+\nu)!(\lambda - 2\rho - \nu - \mathbf{2})! \partial^\nu \phi_{\rho+\nu} \\
&= (\lambda - 2\rho - \mathbf{1})^{\mathbf{1}} \rho!(\lambda - 2\rho - \mathbf{2})! \phi_\rho \\
&\quad + (\lambda - 2\rho - \mathbf{1})^{\mathbf{1}} \sum_{\mathbf{0}<\nu\leqslant\mu-\rho} \frac{(-1)^{|\nu|}}{\nu!} (\rho+\nu)!(\lambda - 2\rho - \nu - \mathbf{2})! \partial^\nu \phi_{\rho+\nu}.
\end{aligned}$$

Hence we have

$$\begin{aligned}\phi_\rho &= \frac{1}{\rho!(\lambda - 2\rho - \mathbf{1})!}\phi^\Lambda_{\mu-\rho} \\ &\quad - \frac{1}{\rho!(\lambda - 2\rho - \mathbf{2})!}\sum_{\mathbf{0}<\nu\leqslant\mu-\rho}\frac{(-1)^{|\nu|}}{\nu!}(\rho+\nu)!(\lambda-2\rho-\nu-\mathbf{2})!\partial^\nu\phi_{\rho+\nu}.\end{aligned}$$

Shifting the index ν to $\nu+\varepsilon_j$ and using (2.7), the sum over ν on the right hand side can be written as

$$\begin{aligned}&\sum_{\mathbf{0}<\nu\leqslant\mu-\rho}\frac{(-1)^{|\nu|}}{\nu!}(\rho+\nu)!(\lambda-2\rho-\nu-\mathbf{2})!\partial^\nu\phi_{\rho+\nu} \\ &\quad= \sum_{\mathbf{0}\leqslant\nu\leqslant\mu-\rho-\varepsilon_j}\frac{(-1)^{|\nu+\varepsilon_j|}}{(\nu+\varepsilon_j)!}(\rho+\nu+\varepsilon_j)!(\lambda-2\rho-\nu-\varepsilon_j-\mathbf{2})!\partial^{\nu+\varepsilon_j}\phi_{\rho+\nu+\varepsilon_j} \\ &\quad= \sum_{\mathbf{0}\leqslant\nu\leqslant\mu-\rho-\varepsilon_j}\ \sum_{\mathbf{0}\leqslant\beta\leqslant\mu-\rho-\nu-\varepsilon_j}\frac{1}{(\rho+\nu+\varepsilon_j)!\beta!(\lambda-2\rho-2\nu-2\varepsilon_j-\beta-\mathbf{1})!} \\ &\qquad\times\frac{(-1)^{|\nu+\varepsilon_j|}}{(\nu+\varepsilon_j)!}(\rho+\nu+\varepsilon_j)!(\lambda-2\rho-\nu-\varepsilon_j-\mathbf{2})!\partial^{\varepsilon_j+\beta+\nu}\phi^\Lambda_{\mu-\rho-\varepsilon_j-\nu-\beta} \\ &\quad= \sum_{\mathbf{0}\leqslant\nu\leqslant\mu-\rho-\varepsilon_j}\ \sum_{\mathbf{0}\leqslant\beta\leqslant\mu-\rho-\nu-\varepsilon_j}\frac{(-1)^{|\nu+\varepsilon_j|}(\lambda-2\rho-\nu-\varepsilon_j-\mathbf{2})!}{\beta!(\nu+\varepsilon_j)!(\lambda-2\rho-2\nu-2\varepsilon_j-\beta-\mathbf{1})!} \\ &\qquad\times\partial^{\varepsilon_j+\beta+\nu}\phi^\Lambda_{\mu-\rho-\varepsilon_j-\nu-\beta} \\ &\quad= \sum_{\mathbf{0}\leqslant\delta\leqslant\mu-\rho-\varepsilon_j}\ \sum_{\mathbf{0}\leqslant\nu\leqslant\delta}\frac{(-1)^{|\nu+\varepsilon_j|}(\lambda-2\rho-\nu-\varepsilon_j-\mathbf{2})!}{(\delta-\nu)!(\nu+\varepsilon_j)!(\lambda-2\rho-\nu-\delta-2\varepsilon_j-\mathbf{1})!} \\ &\qquad\times\partial^{\varepsilon_j+\delta}\phi^\Lambda_{\mu-\rho-\varepsilon_j-\delta} \\ &\quad= \sum_{\mathbf{0}\leqslant\delta\leqslant\mu-\rho-\varepsilon_j}\frac{1}{\delta+\varepsilon_j}\sum_{\mathbf{0}\leqslant\nu\leqslant\delta}(-1)^{|\nu+\varepsilon_j|}\binom{\delta+\varepsilon_j}{\nu+\varepsilon_j}\binom{\lambda-2\rho-\nu-\varepsilon_j-\mathbf{2}}{\delta+\varepsilon_j-\mathbf{1}} \\ &\qquad\times\partial^{\varepsilon_j+\delta}\phi^\Lambda_{\mu-\rho-\varepsilon_j-\delta}.\end{aligned}$$

Using (2.5), we have

$$\begin{aligned}&\sum_{\mathbf{0}\leqslant\nu\leqslant\delta}(-1)^{|\nu+\varepsilon_j|}\binom{\delta+\varepsilon_j}{\nu+\varepsilon_j}\binom{\lambda-2\rho-\nu-\varepsilon_j-\mathbf{2}}{\delta+\varepsilon_j-\mathbf{1}} \\ &\qquad= \sum_{\varepsilon_j\leqslant\nu\leqslant\delta+\varepsilon_j}(-1)^{|\nu|}\binom{\delta+\varepsilon_j}{\nu}\binom{\lambda-2\rho-\nu-\mathbf{2}}{\delta+\varepsilon_j-\mathbf{1}} \\ &\qquad= -\binom{\lambda-2\rho-\mathbf{2}}{\delta+\varepsilon_j-\mathbf{1}}.\end{aligned}$$

Thus we obtain

$$\begin{aligned}\sum_{\mathbf{0}<\nu\leqslant\mu-\rho}&\frac{(-1)^{|\nu|}}{\nu!}(\rho+\nu)!(\lambda-2\rho-\nu-\mathbf{2})!\partial^{\nu}\phi_{\rho+\nu}\\&=-\sum_{\mathbf{0}\leqslant\delta\leqslant\mu-\rho-\varepsilon_j}\frac{1}{\delta+\varepsilon_j}\binom{\lambda-2\rho-\mathbf{2}}{\delta+\varepsilon_j-\mathbf{1}}\partial^{\varepsilon_j+\delta}\phi^{\Lambda}_{\mu-\rho-\varepsilon_j-\delta}\\&=-\sum_{\varepsilon_j\leqslant\delta\leqslant\mu-\rho}\frac{1}{\delta}\binom{\lambda-2\rho-\mathbf{2}}{\delta-\mathbf{1}}\partial^{\delta}\phi^{\Lambda}_{\mu-\rho-\delta},\end{aligned}$$

which implies that

$$\begin{aligned}\phi_\rho&=\frac{1}{\rho!(\lambda-2\rho-\mathbf{1})!}\phi^{\Lambda}_{\mu-\rho}\\&\quad+\frac{1}{\rho!(\lambda-2\rho-\mathbf{2})!}\sum_{\varepsilon_j\leqslant\delta\leqslant\mu-\rho}\frac{1}{\delta}\binom{\lambda-2\rho-\mathbf{2}}{\delta-\mathbf{1}}\partial^{\delta}\phi^{\Lambda}_{\mu-\rho-\delta}\\&=\frac{1}{\rho!}\sum_{\mathbf{0}\leqslant\delta\leqslant\mu-\rho}\frac{1}{\delta!(\lambda-2\rho-\delta-\mathbf{1})!}\partial^{\delta}\phi^{\Lambda}_{\mu-\rho-\delta}=\widetilde{\phi}_\rho.\end{aligned}$$

Hence we have $\widetilde{\phi}_\rho=\phi_\rho$ for all $n\in\{0,1,\dots,\mu\}$ by induction. Thus it follows that

$$((\Xi^{\mu}_{\lambda}\circ\Lambda^{\mu}_{\lambda})\Phi)(z,X)=\Phi(z,X),$$

and the proof of the proposition is complete. ■

3. Modular and quasimodular polynomials

The usual action of the group $SL(2,\mathbb{R})$ on the Poincaré upper half plane $\mathcal{H}$ by linear fractional transformations determines an action of $SL(2,\mathbb{R})^n$ on $\mathcal{H}^n$. Thus, if $z=(z_1,\dots,z_n)\in\mathcal{H}^n$ and $\gamma=(\gamma_1,\dots,\gamma_n)\in SL(2,\mathbb{R})^n$ with

$$\gamma_i=\begin{pmatrix}a_i&b_i\\c_i&d_i\end{pmatrix}\in SL(2,\mathbb{R})$$

for $1\leqslant i\leqslant n$, then we have

$$\gamma z=(\gamma_1z_1,\dots,\gamma_nz_n)=\left(\frac{a_1z_1+b_1}{c_1z_1+d_1},\dots,\frac{a_nz_n+b_n}{c_nz_n+d_n}\right).$$

For the same z and γ, we set

$$\begin{aligned}\mathfrak{J}(\gamma,z)&=(\mathfrak{J}_1(\gamma,z),\dots,\mathfrak{J}_n(\gamma,z)),\\\mathfrak{K}(\gamma,z)&=(\mathfrak{K}_1(\gamma,z),\dots,\mathfrak{K}_n(\gamma,z)),\end{aligned}\tag{3.1}$$

where

$$\mathfrak{J}_i(\gamma,z)=c_iz_i+d_i,\qquad\mathfrak{K}_i(\gamma,z)=\frac{c_i}{c_iz_i+d_i}$$

for $1 \leqslant i \leqslant n$. These formulas determine the maps $\mathfrak{J}, \mathfrak{K} : SL(2,\mathbb{R})^n \times \mathcal{H}^n \to \mathbb{C}^n$ whose component functions satisfy

$$\mathfrak{J}_i(\gamma\gamma', z) = \mathfrak{J}_i(\gamma, \gamma' z)\mathfrak{J}_i(\gamma', z), \tag{3.2}$$

$$\mathfrak{K}_i(\gamma\gamma', z) = \mathfrak{K}_i(\gamma', z) + \mathfrak{J}_i(\gamma', z)^{-2}\mathfrak{K}_i(\gamma, \gamma' z) \tag{3.3}$$

for $\gamma, \gamma' \in SL(2,\mathbb{R})^n$, $z \in \mathcal{H}^n$ and $1 \leqslant i \leqslant n$.

If $\gamma \in SL(2,\mathbb{R})^n$, $\lambda \in \mathbb{Z}^n$, $\mu \in \mathbb{Z}^n_+$, $f \in \mathcal{F}$ and

$$\Phi(z, X) = \sum_{\mathbf{0} \leqslant \rho \leqslant \mu} \phi_\rho(z) X^\rho \in \mathcal{F}_\mu[X], \tag{3.4}$$

we set

$$(f \mid_\lambda \gamma)(z) = \mathfrak{J}(\gamma, z)^{-\lambda} f(\gamma z), \tag{3.5}$$

$$(\Phi \mid^X_\lambda \gamma)(z, X) = \sum_{\mathbf{0} \leqslant \rho \leqslant \mu} (\phi_\rho \mid_{\lambda + 2\rho} \gamma)(z) X^\rho, \tag{3.6}$$

$$(\Phi \parallel_\lambda \gamma)(z, X) = \mathfrak{J}(\gamma, z)^{-\lambda} \Phi(\gamma z, (X - \mathfrak{K}(\gamma, z))(\operatorname{diag} \mathfrak{J}(\gamma, z))^2) \tag{3.7}$$

for all $z \in \mathcal{H}^n$, where $\operatorname{diag} \mathfrak{J}(\gamma, z)$ is the $n \times n$ diagonal matrix whose diagonal entries are the components of $\mathfrak{J}(\gamma, z)$, so that

$$\begin{aligned}
&(X - \mathfrak{K}(\gamma, z))(\operatorname{diag} \mathfrak{J}(\gamma, z))^2 \\
&\qquad = (\mathfrak{J}_1(\gamma, z)^2 (X_1 - \mathfrak{K}_1(\gamma, z)), \dots, \mathfrak{J}_n(\gamma, z)^2 (X_n - \mathfrak{K}_n(\gamma, z))).
\end{aligned}$$

If γ' is another element of $SL(2,\mathbb{R})^n$, then it can be shown that

$$\begin{aligned}
(f \mid_\lambda (\gamma\gamma'))(z) &= ((f \mid_\lambda \gamma) \mid_\lambda \gamma')(z), \\
(\Phi \mid^X_\lambda (\gamma\gamma'))(z, X) &= ((\Phi \mid^X_\lambda \gamma) \parallel_\lambda \gamma')(z, X), \\
(\Phi \parallel_\lambda (\gamma\gamma'))(z, X) &= ((\Phi \parallel_\lambda \gamma) \parallel_\lambda \gamma')(z, X).
\end{aligned}$$

Thus the above operations determine right actions of $SL(2,\mathbb{R})^n$, the first one on $\mathcal{F}$ and the other two on $\mathcal{F}_\mu[X]$.

Lemma 3.1. *If $f \in \mathcal{F}$ and $\gamma \in \Gamma$, we have*

$$\partial^\nu (f \mid_\lambda \gamma)(z) = \sum_{\mathbf{0} \leqslant \alpha \leqslant \nu} (-1)^{\nu - \alpha} \frac{\nu!}{\alpha!} \binom{\lambda + \nu - \mathbf{1}}{\nu - \alpha} \frac{\mathfrak{K}(\gamma, z)^{\nu - \alpha}}{\mathfrak{J}(\gamma, z)^{\lambda + 2\alpha}} \partial^\alpha f(\gamma z) \tag{3.8}$$

for all $\nu \in \mathbb{Z}^n_+$.

Proof. Given $\nu = (\nu_1, \dots, \nu_n) \in \mathbb{Z}^n_+$, if $1 \leqslant j \leqslant n$, from [1, (1.9)] we see that

$$\partial^{\nu_j} (f \mid_\lambda \gamma)(z) = \frac{1}{\mathfrak{J}(\gamma, z)^\lambda} \sum_{\ell=0}^{\nu_j} (-1)^{\nu_j - \ell} \frac{\nu_j!}{\ell!} \binom{\lambda_j + \nu_j - 1}{\nu_j - \ell} \frac{\mathfrak{K}_j(\gamma, z)^{\nu_j - \ell}}{\mathfrak{J}_j(\gamma, z)^{2\ell}} \partial^\ell_j f(\gamma z)$$

for $1 \leqslant j \leqslant n$. If $k \neq j$ is another index with $1 \leqslant k \leqslant n$, noting that $\mathfrak{J}_j(\gamma, z)$ and $\mathfrak{K}_j(\gamma, z)$ depend only on z_j, we obtain

$$\begin{aligned}\partial^{\nu_k}\partial^{\nu_j}(f\mid_\lambda \gamma)(z) &= \sum_{\ell=0}^{\nu_j}(-1)^{\nu_j-\ell}\frac{\nu_j!}{\ell!}\binom{\lambda_j+\nu_j-1}{\nu_j-\ell} \\ &\quad \times \frac{\mathfrak{K}_j(\gamma,z)^{\nu_j-\ell}}{\mathfrak{J}_j(\gamma,z)^{2\ell}}\partial_j^\ell\partial^{\nu_k}(\mathfrak{J}(\gamma,z)^{-\lambda}f(\gamma z)) \\ &= \sum_{\ell=0}^{\nu_j}(-1)^{\nu_j-\ell}\frac{\nu_j!}{\ell!}\binom{\lambda_j+\nu_j-1}{\nu_j-\ell}\frac{\mathfrak{K}_j(\gamma,z)^{\nu_j-\ell}}{\mathfrak{J}_j(\gamma,z)^{2\ell}}\partial_j^\ell\partial^{\nu_k}(f\mid_\lambda\gamma)(z);\end{aligned}$$

hence the lemma follows by applying $\partial^{\nu_1}, \dots, \partial^{\nu_n}$ succssessively to $f\mid_\lambda \gamma$. ■

Theorem 3.2. *Given a polynomial $\Phi(z, X) \in \mathcal{F}_\mu[X]$ and an element $\lambda \in \mathbb{Z}^n$ with $\lambda \geqslant 2\mu + \mathbf{1}$, we have*

$$((\Xi_\lambda^\mu\Phi)\parallel_\lambda\gamma)(z,X) = \Xi_\lambda^\mu(\Phi\mid^X_{\lambda-2\mu}\gamma)(z,X) \tag{3.9}$$

$$((\Lambda_\lambda^\mu\Phi)\mid^X_{\lambda-2\mu}\gamma)(z,X) = \Lambda_\lambda^\mu(\Phi\parallel_\lambda\gamma)(z,X) \tag{3.10}$$

for all $\gamma \in SL(2,\mathbb{R})^n$, where Ξ_λ^μ and Λ_λ^μ are the isomorphisms in Proposition 2.1.

Proof. Let $\Phi(z, X) \in \mathcal{F}_\mu[X]$ be given by (3.4), so that

$$(\Xi_\lambda^\mu\Phi)(z,X) = \sum_{\mathbf{0}\leqslant\rho\leqslant\mu}\phi_\rho^\Xi(z)X^\rho,$$

where the coefficients $\phi_\rho^\Xi(z)$ are as in (2.3). If $\gamma \in SL(2,\mathbb{R})^n$, from (3.7) we obtain

$$\begin{aligned}((\Xi_\lambda^\mu\Phi)\parallel_\lambda\gamma)(z,X) &= \mathfrak{J}(\gamma,z)^{-\lambda}\sum_{\mathbf{0}\leqslant\alpha\leqslant\mu}\phi_\alpha^\Xi(\gamma z)\mathfrak{J}(\gamma,z)^{2\alpha}(X-\mathfrak{K}(\gamma,z))^\alpha \\ &= \sum_{\mathbf{0}\leqslant\alpha\leqslant\mu}\sum_{\mathbf{0}\leqslant\rho\leqslant\alpha}\binom{\alpha}{\rho}\phi_\alpha^\Xi(\gamma z)\mathfrak{J}(\gamma,z)^{2\alpha-\lambda}(-1)^{|\alpha-\rho|}\mathfrak{K}(\gamma,z)^{\alpha-\rho}X^\rho \\ &= \sum_{\mathbf{0}\leqslant\rho\leqslant\mu}\sum_{\rho\leqslant\alpha\leqslant\mu}(-1)^{|\alpha-\rho|}\binom{\alpha}{\rho}\phi_\alpha^\Xi(\gamma z)\mathfrak{J}(\gamma,z)^{2\alpha-\lambda}\mathfrak{K}(\gamma,z)^{\alpha-\rho}X^\rho.\end{aligned}$$

Thus we may write

$$((\Xi_\lambda^\mu\Phi)\parallel_\lambda\gamma)(z,X) = \sum_{\mathbf{0}\leqslant\rho\leqslant\mu}\xi_\rho^\Xi(\gamma,z)X^\rho,$$

where

$$\xi_\rho^\Xi(\gamma,z) = \sum_{\mathbf{0}\leqslant\alpha\leqslant\mu-\rho}(-1)^{|\alpha|}\binom{\alpha+\rho}{\rho}\phi_{\alpha+\rho}^\Xi(\gamma z)\mathfrak{J}(\gamma,z)^{2\alpha+2\rho-\lambda}\mathfrak{K}(\gamma,z)^\alpha$$

for each $\rho \in \mathbb{Z}^n$ with $\mathbf{0} \leqslant \rho \leqslant \mu$. Using (2.3), we have

$$\phi^{\Xi}_{\alpha+\rho}(\gamma z) = \frac{1}{(\alpha+\rho)!} \sum_{\mathbf{0} \leqslant \beta \leqslant \mu-\rho-\alpha} \frac{1}{\beta!(\lambda - 2\rho - 2\alpha - \beta - \mathbf{1})!} \partial^{\beta} \phi_{\mu-\rho-\alpha-\beta}(\gamma z),$$

and therefore we obtain

$$\xi^{\Xi}_{\rho}(\gamma, z) = \sum_{\mathbf{0} \leqslant \alpha \leqslant \mu-\rho} \sum_{\mathbf{0} \leqslant \beta \leqslant \mu-\rho-\alpha} \frac{(-1)^{|\alpha|} \mathfrak{J}(\gamma, z)^{2\alpha+2\rho-\lambda} \mathfrak{K}(\gamma, z)^{\alpha}}{\beta!\rho!\alpha!(\lambda - 2\rho - 2\alpha - \beta - \mathbf{1})!} \partial^{\beta} \phi_{\mu-\rho-\alpha-\beta}(\gamma z). \tag{3.11}$$

On the other hand, from (2.2), (2.3) and (3.6) we see that

$$\Xi^{\mu}_{\lambda}(\Phi \mid^{X}_{\lambda-2\mu} \gamma)(z, X) = \sum_{\mathbf{0} \leqslant \rho \leqslant \mu} \eta^{\Xi}_{\rho}(\gamma, z) X^{\rho},$$

where

$$\eta^{\Xi}_{\rho}(\gamma, z) = \frac{1}{\rho!} \sum_{\mathbf{0} \leqslant \beta \leqslant \mu-\rho} \frac{1}{\beta!(\lambda - 2\rho - \beta - \mathbf{1})!} \partial^{\beta} (\phi_{\mu-\rho-\beta} \mid_{\lambda-2\rho-2\beta} \gamma)(z)$$

for $\rho \geqslant 0$. Using (3.8), we have

$$\begin{aligned} \partial^{\beta} (\phi_{\mu-\rho-\beta} \mid_{\lambda-2\rho-2\beta} \gamma)(z) = \sum_{\mathbf{0} \leqslant \alpha \leqslant \beta} & (-1)^{|\beta-\alpha|} \frac{\beta!}{\alpha!} \binom{\lambda - 2\rho - \beta - \mathbf{1}}{\beta - \alpha} \\ & \times \frac{\mathfrak{K}(\gamma, z)^{\beta-\alpha}}{\mathfrak{J}(\gamma, z)^{\lambda-2\rho-2\beta+2\alpha}} \partial^{\alpha} \phi_{\mu-\rho-\beta}(\gamma z). \end{aligned}$$

Thus we obtain

$$\begin{aligned} \eta^{\Xi}_{\rho}(\gamma, z) &= \sum_{\mathbf{0} \leqslant \beta \leqslant \mu-\rho} \sum_{\mathbf{0} \leqslant \alpha \leqslant \beta} \frac{(-1)^{|\beta-\alpha|} \mathfrak{K}(\gamma, z)^{\beta-\alpha} \mathfrak{J}(\gamma, z)^{-\lambda+2\rho+2\beta-2\alpha}}{\rho!\alpha!(\beta-\alpha)!(\lambda - 2\rho - 2\beta + \alpha - \mathbf{1})!} \partial^{\alpha} \phi_{\mu-\rho-\beta}(\gamma z) \\ &= \sum_{\mathbf{0} \leqslant \alpha \leqslant \mu-\rho} \sum_{\alpha \leqslant \beta \leqslant \mu-\rho} \frac{(-1)^{|\beta-\alpha|} \mathfrak{K}(\gamma, z)^{\beta-\alpha} \mathfrak{J}(\gamma, z)^{-\lambda+2\rho+2\beta-2\alpha}}{\rho!\alpha!(\beta-\alpha)!(\lambda - 2\rho - 2\beta + \alpha - \mathbf{1})!} \partial^{\alpha} \phi_{\mu-\rho-\beta}(\gamma z) \\ &= \sum_{\mathbf{0} \leqslant \alpha \leqslant \mu-\rho} \sum_{\mathbf{0} \leqslant \beta \leqslant \mu-\alpha-\rho} \frac{(-1)^{|\beta|} \mathfrak{K}(\gamma, z)^{\beta} \mathfrak{J}(\gamma, z)^{-\lambda+2\rho+2\beta}}{\rho!\alpha!\beta!(\lambda - 2\rho - 2\beta - \alpha - \mathbf{1})!} \partial^{\alpha} \phi_{\mu-\rho-\beta-\alpha}(\gamma z). \end{aligned}$$

Comparing this with (3.11), we have $\xi_{\rho}(\gamma, z) = \eta_{\rho}(\gamma, z)$ for each $\rho \in \mathbb{Z}^n$ with $\mathbf{0} \leqslant \rho \leqslant \mu$, which verifies (3.9). The relation (3.10) follows from this and Proposition 2.1. ■

We now choose a discrete subgroup Γ of $SL(2, \mathbb{R})^n$ and consider the restriction of the operations $|_{\lambda}$, $|^{X}_{\lambda}$ and $\|_{\lambda}$ of $SL(2, \mathbb{R})^n$ with $\lambda \in \mathbb{Z}^n$ in (3.5), (3.6) and (3.7) to Γ.

Definition 3.3. (i) An element $f \in \mathcal{F}$ is a *Hilbert modular form for* Γ *of weight* λ if it satisfies

$$f \mid_\lambda \gamma = f$$

for all $\gamma \in \Gamma$.

(ii) An element $\Phi(z, X) \in \mathcal{F}_\mu[X]$ is a *Hilbert modular polynomial for* Γ *of weight* λ *and degree at most* μ if it satisfies

$$\Phi \mid_\lambda^X \gamma = \Phi$$

for all $\gamma \in \Gamma$, and the same element is a *Hilbert quasimodular polynomial for* Γ *of weight* λ *and degree at most* μ if it satisfies

$$\Phi \parallel_\lambda \gamma = \Phi$$

for all $\gamma \in \Gamma$.

We use $M_\lambda(\Gamma)$ to denote the space of Hilbert modular forms for Γ of weight λ. We also denote by $MP_\lambda^\mu(\Gamma)$ and $QP_\lambda^\mu(\Gamma)$ the spaces of Hilbert modular and quasimodular, respectively, polynomials for Γ of weight λ and degree at most μ.

If a polynomial $\Phi(z, X) \in \mathcal{F}_\mu[X]$ of the form

$$\Phi(z, X) = \sum_{\mathbf{0} \leqslant \rho \leqslant \mu} \phi_\rho(z) X^\rho$$

belongs to $MP_\lambda^\mu(\Gamma)$, from (3.6) and Definition 3.3(ii) we see that

$$\phi_\rho \in M_{\lambda+2\rho}(\Gamma) \tag{3.12}$$

for $\mathbf{0} \leqslant \rho \leqslant \mu$; hence a Hilbert modular polynomial determines a finite sequence of Hilbert modular forms.

Proposition 3.4. *The isomorphisms* Ξ_λ^μ *and* Λ_λ^μ *in Proposition 2.1 induce the isomorphisms*

$$\Xi_\lambda^\mu : MP_{\lambda-2\mu}^\mu(\Gamma) \to QP_\lambda^\mu(\Gamma), \qquad \Lambda_\lambda^\mu : QP_\lambda^\mu(\Gamma) \to MP_{\lambda-2\mu}^\mu(\Gamma) \tag{3.13}$$

for each $\lambda \in \mathbb{Z}^n$ *with* $\lambda \geqslant 2\mu + \mathbf{1}$.

Proof. This follows immediately from Theorem 3.2 and Definition 3.3. ∎

4. Hilbert Quasimodular forms

In this section we introduce Hilbert quasimodular forms corresponding to Hilbert quasimodular polynomials and construct Poincaré series for Hilbert quasimodular forms for congruence subgroups of $SL(2, \mathbb{R})^n$.

Let $\mathcal{F}$ and $\mathcal{F}_\mu[X]$ with $\mu \in \mathbb{Z}_+^n$ be as in Section 3, and let Γ be a discrete subgroup of $SL(2, \mathbb{R})^n$.

Definition 4.1. Given $\lambda \in \mathbb{Z}^n$, an element $\phi \in \mathcal{F}$ is a *Hilbert quasimodular form for* Γ *of weight* λ *and depth at most* μ if there are functions $\phi_\rho \in \mathcal{F}$ with $\mathbf{0} \leqslant \rho \leqslant \mu$ such that

$$(\phi \mid_\lambda \gamma)(z) = \sum_{\mathbf{0} \leqslant \rho \leqslant \mu} \phi_\rho(z) \mathfrak{K}(\gamma, z)^\rho \tag{4.1}$$

for all $z \in \mathcal{H}^n$ and $\gamma \in \Gamma$, where $\mathfrak{K}(\gamma, z)$ is as in (3.1) and $|_\lambda$ is the operation in (3.5). We denote by $QM_\lambda^\mu(\Gamma)$ the space of such Hilbert quasimodular forms.

Remark 4.2. (i) If (4.1) is satisfied for another set of functions $\{\widehat{\phi}_\rho \in \mathcal{F} \mid \mathbf{0} \leqslant \rho \leqslant \mu\}$, then we have

$$\sum_{\mathbf{0} \leqslant \rho \leqslant \mu} (\widehat{\phi}_\rho(z) - \phi_\rho(z)) \mathfrak{K}(\gamma, z)^\rho = 0$$

for all γ belonging to the infinite set Γ; hence it follows that $\widehat{\phi}_\rho = \phi_\rho$ for each ρ. Thus we see that the Hilbert quasimodular form ϕ determines the associated functions $\phi_\rho \in \mathcal{F}$ uniquely.

(ii) If γ is the identity element of Γ in (4.1), then $\mathfrak{K}(\gamma, z) = \mathbf{0}$, and therefore it follows that $\phi = \phi_{\mathbf{0}}$. On the other hand, if $\mu = \mathbf{0}$, the relation (4.1) can be written in the form

$$\phi \mid_\lambda \gamma = \phi_{\mathbf{0}} = \phi;$$

hence $QM_\lambda^{\mathbf{0}}(\Gamma)$ coincides with the space $M_\lambda(\Gamma)$ of Hilbert modular forms.

Let $\phi \in \mathcal{F}$ be a Hilbert quasimodular form belonging to $QM_\lambda^\mu(\Gamma)$ satisfying (4.1). Then we define the corresponding polynomial $(\mathcal{Q}_\lambda^\mu \phi)(z, X) \in \mathcal{F}_\mu[X]$ by

$$(\mathcal{Q}_\lambda^\mu \phi)(z, X) = \sum_{r=0}^{\mu} \phi_r(z) X^r \tag{4.2}$$

for all $z \in \mathcal{H}$. From Remark 4.2(i) we see that $\mathcal{Q}_\lambda^\mu \phi$ is well-defined, and therefore we obtain the complex linear map

$$\mathcal{Q}_\lambda^\mu : QM_\lambda^\mu(\Gamma) \to \mathcal{F}_\mu[X]$$

for each $\lambda \in \mathbb{Z}^n$.

Lemma 4.3. *An element*

$$\Phi(z, X) = \sum_{\mathbf{0} \leqslant \rho \leqslant \mu} \phi_\rho(z) X^\rho \in \mathcal{F}_\mu[X] \tag{4.3}$$

is a Hilbert quasimodular polynomial belonging to $QP_\lambda^\mu(\Gamma)$ *if and only if for each* $\rho \in \mathbb{Z}^n$ *with* $\mathbf{0} \leqslant \rho \leqslant \mu$ *the function* ϕ_ρ *satisfies*

$$(\phi_\rho \mid_{\lambda - 2\rho} \gamma)(z) = \sum_{\mathbf{0} \leqslant \alpha \leqslant \mu - \rho} \binom{\alpha + \rho}{\rho} \phi_{\alpha + \rho}(z) \mathfrak{K}(\gamma, z)^\alpha \tag{4.4}$$

for all $z \in \mathcal{H}^n$ *and* $\gamma \in \Gamma$. *In particular, if* $\Phi(z, X) \in QP_\lambda^\mu(\Gamma)$, *then* ϕ_ρ *is a Hilbert quasimodular form belonging to* $QM_{\lambda - 2\rho}^{\mu - \rho}(\Gamma)$.

Proof. If a polynomial $\Phi(z, X)$ given by (4.3) belongs to $QP^{\mu}_{\lambda}(\Gamma)$, we have

$$\begin{aligned}
&\sum_{\mathbf{0}\leqslant\rho\leqslant\mu} \phi_\rho(z)X^\rho \\
&\qquad = \mathfrak{J}(\gamma,z)^{-\lambda} \sum_{\mathbf{0}\leqslant\rho\leqslant\mu} \mathfrak{J}(\gamma,z)^{2\rho}\phi_\rho(z)(X-\mathfrak{K}(\gamma,z))^\rho \\
&\qquad = \mathfrak{J}(\gamma,z)^{-\lambda} \sum_{\mathbf{0}\leqslant\rho\leqslant\mu}\sum_{\mathbf{0}\leqslant\alpha\leqslant\rho} (-1)^{|\rho-\alpha|}\binom{\rho}{\alpha}\mathfrak{J}(\gamma,z)^{2\rho}\mathfrak{K}(\gamma,z)^{\rho-\alpha}\phi_\rho(z)X^\alpha \\
&\qquad = \mathfrak{J}(\gamma,z)^{-\lambda} \sum_{\mathbf{0}\leqslant\alpha\leqslant\mu}\sum_{\alpha\leqslant\rho\leqslant\mu} (-1)^{|\rho-\alpha|}\binom{\rho}{\alpha}\mathfrak{J}(\gamma,z)^{2\rho}\mathfrak{K}(\gamma,z)^{\rho-\alpha}\phi_\rho(z)X^\alpha.
\end{aligned}$$

Replacing z by γz and γ by γ^{-1}, we obtain

$$\begin{aligned}
\sum_{\mathbf{0}\leqslant\rho\leqslant\mu} \phi_\rho(\gamma z)X^\rho &= \mathfrak{J}(\gamma^{-1},\gamma z)^{-\lambda} \sum_{\mathbf{0}\leqslant\alpha\leqslant\mu}\sum_{\alpha\leqslant\rho\leqslant\mu} (-1)^{|\rho-\alpha|}\binom{\rho}{\alpha} \\
&\qquad \times \mathfrak{J}(\gamma^{-1},\gamma z)^{2\rho}\mathfrak{K}(\gamma^{-1},\gamma z)^{\rho-\alpha}\phi_\rho(\gamma z)X^\alpha \\
&= \mathfrak{J}(\gamma^{-1},\gamma z)^{-\lambda} \sum_{\mathbf{0}\leqslant\alpha\leqslant\mu}\sum_{\mathbf{0}\leqslant\beta\leqslant\mu-\alpha} (-1)^{|\beta|}\binom{\alpha+\beta}{\alpha} \\
&\qquad \times \mathfrak{J}(\gamma^{-1},\gamma z)^{2\alpha+2\beta}\mathfrak{K}(\gamma^{-1},\gamma z)^{\beta}\phi_{\alpha+\beta}(\gamma z)X^\alpha.
\end{aligned}$$

From this and the identities

$$\mathfrak{J}(\gamma^{-1},\gamma z)^{2\alpha+2\beta} = \mathfrak{J}(\gamma,z)^{-2\alpha-2\beta}, \qquad \mathfrak{K}(\gamma^{-1},\gamma z)^{\beta} = (-\mathfrak{J}(\gamma,z))^{2\beta}\mathfrak{K}(\gamma,z)^{\beta},$$

it follows that

$$\phi_\alpha(\gamma z) = \mathfrak{J}(\gamma,z)^{\lambda} \sum_{\mathbf{0}\leqslant\beta\leqslant\mu-\alpha} \binom{\alpha+\beta}{\alpha}\mathfrak{J}(\gamma,z)^{-2\alpha}\mathfrak{K}(\gamma,z)^{-\beta}$$

for $\mathbf{0}\leqslant\alpha\leqslant\mu$, which is equivalent to (4.4). ■

If $\mathbf{0}\leqslant\alpha\leqslant\mu$, we consider the complex linear map

$$\mathfrak{S}_\alpha : \mathcal{F}_\mu[X] \to \mathcal{F}$$

defined by

$$\mathfrak{S}_\alpha\bigg(\sum_{\mathbf{0}\leqslant\rho\leqslant\mu} \phi_\alpha(z)X^\alpha\bigg) = \phi_\alpha(z)$$

for all $z \in \mathcal{H}^n$. Then from Lemma 4.3 we see that

$$\mathfrak{S}_\alpha(QP^{\mu}_{\lambda}(\Gamma)) \subset QM^{\mu-\alpha}_{\lambda-2\alpha}(\Gamma);$$

hence we obtain the map

$$\mathfrak{S}_\alpha : QP^{\mu}_{\lambda}(\Gamma) \to QM^{\mu-\alpha}_{\lambda-2\alpha}(\Gamma) \tag{4.5}$$

for each α. On the other hand, using (4.2) and (4.4), we also have

$$(\mathcal{Q}^{\mu-\rho}_{\lambda-2\rho}(\mathfrak{S}_\rho F))(z,X) = \sum_{\mathbf{0}\leqslant\alpha\leqslant\mu-\rho} \binom{\alpha+\rho}{\rho} (\mathfrak{S}_{\alpha+\rho}F)(z)X^\alpha \in QP^{\mu-\rho}_{\lambda-2\rho}(\Gamma)$$

for $F(z,X) \in QM^\mu_\lambda(\Gamma)$ and $\mathbf{0} \leqslant \rho \leqslant \mu$. In particular, the map $\mathcal{Q}^\mu_\lambda$ given by (4.2) determines the complex linear map

$$\mathcal{Q}^\mu_\lambda : QM^\mu_\lambda(\Gamma) \to QP^\mu_\lambda(\Gamma) \tag{4.6}$$

for each $\lambda \in \mathbb{Z}^n$.

Lemma 4.4. *The map $\mathfrak{S}_\mathbf{0} : QP^\mu_\lambda(\Gamma) \to QM^\mu_\lambda(\Gamma)$ in (4.5) with $\alpha = \mathbf{0}$ is an isomorphism whose inverse is the map $\mathcal{Q}^\mu_\lambda$ in (4.6).*

Proof. If $\Phi(z,X) \in QP^\mu_\lambda(\Gamma)$ is as in (4.3), we have

$$(\mathfrak{S}_\mathbf{0}\Phi)(z) = \phi_0(z)$$

for all $z \in \mathcal{H}^n$. On the other hand, from Lemma 4.3 we see that

$$(\phi_\mathbf{0} \mid_\lambda \gamma)(z) = \sum_{\mathbf{0}\leqslant\alpha\leqslant\mu} \phi_\alpha(z)\mathfrak{K}(\gamma,z)^\alpha.$$

Hence it follws that

$$((\mathcal{Q}^\mu_\lambda \circ \mathfrak{S}_\mathbf{0})\Phi)(z,X) = \Phi(z,X).$$

Since $\mathfrak{S}_\mathbf{0} \circ \mathcal{Q}^\mu_\lambda$ is clearly the identity map on $QM^\mu_\lambda(\Gamma)$, the lemma follows. ■

Let $\phi \in QM^\mu_\lambda(\Gamma)$ be a Hilbert quasimodular form satisfying

$$(\phi \mid_\lambda \gamma)(z) = \sum_{\mathbf{0}\leqslant\rho\leqslant\mu} \phi_\rho(z)\mathfrak{K}(\gamma,z)^\rho \tag{4.7}$$

for all $\gamma \in \Gamma$ and $z \in \mathcal{H}^n$, so that

$$(\mathcal{Q}^\mu_\lambda\phi)(z,X) = \sum_{\mathbf{0}\leqslant\rho\leqslant\mu} \phi_\rho(z)X^\rho \in QP^\mu_\lambda(\Gamma)$$

by (4.2). We then set

$$\widetilde{\Lambda}^\mu_\lambda(\phi) = (\phi^\Lambda_\rho)_{\mathbf{0}\leqslant\rho\leqslant\mu} \in \mathcal{F}^{|\mu|+1}. \tag{4.8}$$

where the functions $\phi^\Lambda_\rho \in \mathcal{F}$ are the coefficients of the Hilbert modular polynomial

$$((\Lambda^\mu_\lambda \circ \mathcal{Q}^\mu_\lambda)\phi)(z,X) \in MP^\mu_\lambda(\Gamma)$$

given by (2.4). Then from (3.12) we see that

$$\phi^\Lambda_r \in M_{\lambda+2r}(\Gamma)$$

for each $r \in \{0, 1, \dots, m\}$; hence the formula (4.8) determines an isomorphism

$$\widetilde{\Lambda}^{\mu}_{\lambda} : QM^{\mu}_{\lambda}(\Gamma) \to \bigoplus_{\mathbf{0} \leqslant \rho \leqslant \mu} M_{\lambda + 2k - 2\mu}(\Gamma)$$

of complex vector spaces for $\lambda \geqslant 2\mu + \mathbf{1}$.

In order to consider Poincaré series we now consider a totally real number field F with $[F : \mathbb{Q}] = n$ and denote its ring of integers by $\mathfrak{o}$. Thus there are n real embeddings

$$\sigma_1, \dots, \sigma_n : F \hookrightarrow \mathbb{R}$$

of F, which induce an embedding

$$SL(2, F) \hookrightarrow SL(2, \mathbb{R})^n.$$

Throughout the rest of this section we shall identify $SL(2, F)$ with its embedded image in $SL(2, \mathbb{R})^n$. Thus $SL(2, \mathfrak{o})$ is a discrete subgroup of $SL(2, \mathbb{R})^n$, which is the full Hilbert modular group. Let $\mathfrak{n}$ be an ideal of $\mathfrak{o}$, so that the corresponding principal congruence subgroup is given by

$$\Gamma(\mathfrak{n}) = \{\gamma \in SL(2, \mathfrak{o}) \mid \gamma \equiv 1 \pmod{\mathfrak{n}}\}.$$

For the rest of this section we assume that Γ is a congruence subgroup of $SL(2, F)$ containing $\Gamma(\mathfrak{n})$ with finite index, and set

$$\Gamma_\infty = \{\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right) \mid x \in F\}.$$

We denote by

$$\mathrm{Tr} : F \to \mathbb{Q}$$

the Galois trace map of F over $\mathbb{Q}$, and set

$$\mathfrak{n}^* = \{\alpha \in F \mid \mathrm{Tr}(\alpha \mathfrak{n}) \subset \mathfrak{o}\}.$$

Given $\lambda \in \mathbb{Z}^n$ and a totally positive element $\nu \in \mathfrak{n}^*$, we can consider the Poincaré series

$$\mathcal{P}_{\lambda,\nu}(z) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \mathfrak{J}(\gamma, z)^{-\lambda} \exp(2\pi i \,\mathrm{Tr}(\nu(\gamma z))), \tag{4.9}$$

which is a Hilbert modular form belonging to $M_\lambda(\Gamma)$ (cf. [3]). We now set

$$\begin{aligned} \mathcal{P}^{\mathrm{QP}}_{\lambda,\nu}(z, X) = &\sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \sum_{\mathbf{0} \leqslant \rho \leqslant \mu} \sum_{\mathbf{0} \leqslant \beta \leqslant \mu - \rho} \sum_{\mathbf{0} \leqslant \alpha \leqslant \beta} \frac{(-1)^{|\beta - \alpha|} (2\pi i)^{|\alpha|} \nu^\alpha}{\rho! \beta! (\lambda - 2\rho - \beta - \mathbf{1})!} \\ &\times \binom{\lambda - 2\rho - \beta - \mathbf{1}}{\beta - \alpha} \frac{\mathfrak{K}(\gamma, z)^{\beta - \alpha}}{\mathfrak{J}(\gamma, z)^{\lambda - 2\rho - 2\beta + 2\alpha}} \exp(2\pi i T(\nu(\gamma z))) X^\rho, \end{aligned}$$

which is a polynomial in $\mathcal{F}_\mu(X)$.

Proposition 4.5. *The polynomial $\mathcal{P}^{\mathrm{QP}}_{\lambda,\nu}(z,X)$ is a Hilbert quasimodular polynomial belonging to $QP^{\mu}_{\lambda}(\Gamma)$.*

Proof. Since the function $\mathcal{P}_{\lambda,\nu} : \mathcal{H}^n \to \mathbb{C}$ given by (4.9) belongs to $M_\lambda(\Gamma)$, we see that the polynomial

$$\Phi(z,X) = \sum_{\mathbf{0}\leqslant\rho\leqslant\mu} \mathcal{P}_{\lambda-2\mu+2\rho,\nu}(z)X^\rho$$

is a Hilbert modular polynomial belonging to $MP^{\mu}_{\lambda-2\mu}(\Gamma)$. Thus, appying the map Ξ^μ_λ in (3.13), we obtain

$$\Xi^\mu_\lambda\Phi(z,X) = \sum_{\mathbf{0}\leqslant\rho\leqslant\mu} \psi_\rho(z)X^\rho \in QP^\mu_\lambda(\Gamma),$$

where

$$\psi_\rho = \frac{1}{\rho!}\sum_{\mathbf{0}\leqslant\beta\leqslant\mu-\rho}\frac{1}{\beta!(\lambda-2\rho-\beta-\mathbf{1})!}\partial^\beta\mathcal{P}_{\lambda-2\mu-2\beta,\nu}$$

for $\mathbf{0}\leqslant\rho\leqslant\mu$. If we set

$$\eta_\nu(z) = \exp(2\pi i\,\mathrm{Tr}(\nu(z)))$$

for all $z\in\mathcal{H}^n$, we have

$$\begin{aligned}
\partial^\beta\mathcal{P}_{\lambda-2\mu-2\beta,\nu}(z) &= \sum_{\gamma\in\Gamma_\infty\backslash\Gamma}\partial^\beta(\eta_\nu\mid_{\lambda-2\rho-2\beta}\gamma)(z)\\
&= \sum_{\gamma\in\Gamma_\infty\backslash\Gamma}\sum_{\mathbf{0}\leqslant\alpha\leqslant\beta}(-1)^{|\beta-\alpha|}\binom{\lambda-2\rho-\beta-\mathbf{1}}{\beta-\alpha}\\
&\quad\times\frac{\mathfrak{K}(\gamma,z)^{\beta-\alpha}}{\mathfrak{J}(\gamma,z)^{\lambda-2\rho-2\beta+2\alpha}}(\partial^\alpha\eta_\nu)(\gamma z)\\
&= \sum_{\gamma\in\Gamma_\infty\backslash\Gamma}\sum_{\mathbf{0}\leqslant\alpha\leqslant\beta}(-1)^{|\beta-\alpha|}(2\pi i)^{|\alpha|}\nu^\alpha\binom{\lambda-2\rho-\beta-\mathbf{1}}{\beta-\alpha}\\
&\quad\times\frac{\mathfrak{K}(\gamma,z)^{\beta-\alpha}}{\mathfrak{J}(\gamma,z)^{\lambda-2\rho-2\beta+2\alpha}}\exp(2\pi i\,\mathrm{Tr}(\nu(\gamma z))).
\end{aligned}$$

Thus we see that

$$\Xi^\mu_\lambda\Phi(z,X) = \mathcal{P}^{\mathrm{QP}}_{\lambda,\nu}(z,X);$$

hence the proposition follows. ∎

From Proposition 4.5 it follows that the function $\mathcal{P}^{\mathrm{QM}}_{\lambda,\nu} \in \mathcal{F}$ given by

$$\mathcal{P}^{\mathrm{QM}}_{\lambda,\nu}(z) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \sum_{\mathbf{0} \leqslant \beta \leqslant \mu} \sum_{\mathbf{0} \leqslant \alpha \leqslant \beta} \frac{(-1)^{|\beta-\alpha|}(2\pi i)^{|\alpha|}\nu^\alpha}{\beta!(\lambda-\beta-\mathbf{1})!} \times \binom{\lambda-\beta-\mathbf{1}}{\beta-\alpha} \frac{\mathfrak{K}(\gamma,z)^{\beta-\alpha}}{\mathfrak{J}(\gamma,z)^{\lambda-2\beta+2\alpha}} \exp(2\pi i T(\nu(\gamma z)))$$

for all $z \in \mathcal{H}^n$ belongs to $QM^\nu_\lambda(\Gamma)$, an such series may be regarded as Poincaré series for Hilbert quasimodular forms.

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Address: Min Ho Lee: Department of Mathematics, University of Northern Iowa, Cedar Falls, IA 50614, U.S.A.

E-mail: lee@math.uni.edu

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