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ON THE SPECIAL VALUES OF ARTIN L-FUNCTIONS FOR DIHEDRAL EXTENSIONS

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Abstract: We study special values of Artin L-functions for dihedral extensions at negative integers. We give a relation between these values and orders of the χ-parts of certain étale cohomology groups.

Keywords: étale cohomology, K-group, class number, Iwasawa theory, Artin L-function.

1. Introduction and the main result

Let p and l be distinct odd primes. We denote by D_{2l} the dihedral group of order $2l$. Let L^+ be a dihedral extension over a number field F^+ of degree $2l$. Suppose that both L^+ and F^+ are totally real. For a totally positive algebraic number $r \in F^+$, let $L = L^+(\sqrt{-r})$ and $F = F^+(\sqrt{-r})$. Let O_L be the integer ring of L. Let χ be a character of $\mathrm{Gal}(L/F^+)$. Denote by $\mathscr{L}(L/F^+, \chi, s)$ the Artin L-function attached to χ and put $d_\chi = [\mathbb{Z}_p[\mathrm{Im}(\chi)] : \mathbb{Z}_p]$. We say that χ is even if it is the inflation of a character of $\mathrm{Gal}(L^+/F^+)$, while odd if it is the product of an even character with the inflation of the non-trivial character of $\mathrm{Gal}(F/F^+)$. Moreover, $a \sim_p b$ signifies that a and b are two p-adic numbers with the same valuation. Let $H^i_{\text{ét}}(\mathrm{Spec}\, O_L[1/p], \mathbb{Z}_p(n))$ be the étale cohomology group, which we will simply denote by $H^i(O'_L, \mathbb{Z}_p(n))$. The main result of this paper is the following theorem.

Theorem 1.1. *Let $n \geqslant 2$ be an integer and χ an irreducible character of $\mathrm{Gal}(L/F^+)$. Assume that χ is even if n is even and χ is odd if n is odd. Then*

$$\mathscr{L}\left(L/F^+, \overline{\chi}, 1-n\right)^{\chi(1)d_\chi} \sim_p \frac{\#H^2\left(O'_L, \mathbb{Z}_p(n)\right)^\chi}{\#H^1\left(O'_L, \mathbb{Z}_p(n)\right)^\chi},$$

where $H^i\left(O'_L, \mathbb{Z}_p(n)\right)^\chi$ means the χ-part of $H^i\left(O'_L, \mathbb{Z}_p(n)\right)$.

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The definition of χ-part will be given in Section 2. Theorem 1.1 is close to the following known result for an abelian extension, which will be used by our proof.

Theorem 1.2 ([3], p. 707). *Let $n \geqslant 2$ be an integer and L/K a totally complex abelian extension of the totally real base field K of degree prime to p. Let χ be a character of* $\mathrm{Gal}(L/K)$*, such that $\chi(-1) = (-1)^n$, and view χ as a p-adic character. Then*

$$\mathscr{L}\left(L/K, \chi^{-1}, 1-n\right)^{d_\chi} \sim_p \frac{\# H^2(O'_L, \mathbb{Z}_p(n))^\chi}{\# H^1(O'_L, \mathbb{Z}_p(n))^\chi}.$$

Now, we can interpret Theorem 1.1 in terms of K-groups. For $n \geqslant 2$, it is seen that the p-adic Chern maps

$$K_{2n-i}(O_L) \otimes_{\mathbb{Z}} \mathbb{Z}_p \longrightarrow H^i\left(O'_L, \mathbb{Z}_p(n)\right) \qquad (i = 1, 2)$$

are isomorphisms, which is known as the Quillen-Lichtenbaum conjecture (cf. [7], [8]). Consequently, Theorem 1.1 gives the relation

$$\mathscr{L}\left(L/F^+, \overline{\chi}, 1-n\right)^{\chi(1)d_\chi} \sim_p \frac{\# K_{2n-2}(O_L)^\chi_{\mathrm{tors}}}{\# K_{2n-1}(O_L)^\chi_{\mathrm{tors}}} \tag{1.1}$$

for χ with the same parity of $n \geqslant 2$. Further, we add the fact that (1.1) is essentially valid for $n = 1$, by

$$K_0(O_L) \simeq \mathbb{Z} \oplus \mathrm{Cl}_L, \qquad K_1(O_L) \simeq O_L^\times$$

and the main theorem of [4] (p. 1063). Here, Cl_L denotes the ideal class group of L.

2. Proof of the main theorem

Let $D_{2l} = \langle a, b \rangle$ with $a^l = b^2 = 1$ and $bab^{-1} = a^{-1}$. It is known that D_{2l} has the two one-dimensional representations and the $(l-1)/2$ irreducible two-dimensional representations. The character table is as follows:

	$1_{D_{2l}}$	$a^i\left(1 \leqslant i \leqslant \frac{l-1}{2}\right)$	b
ε	1	1	1
η	1	1	-1
$\chi_k\left(1 \leqslant k \leqslant \frac{l-1}{2}\right)$	2	$\zeta_l^{ik} + \zeta_l^{-ik}$	0

where $\zeta_l = \exp(2\pi\sqrt{-1}/l)$.

Take $\sigma \in \mathrm{Hom}\left(\langle a \rangle, \overline{\mathbb{Q}}^\times\right)$ satisfying $\sigma(a) = \zeta_l$, and write $\sigma_i = \sigma^i$ $(0 \leqslant i \leqslant l-1)$. Then, the characters χ_k are induced from σ_k and σ_{l-k}, namely,

$$\chi_k = \mathrm{Ind}\,\sigma_k = \mathrm{Ind}\,\sigma_{l-k} \tag{2.1}$$

for all $k \in \{1, \cdots, \frac{l-1}{2}\}$.

Fix an embedding $\overline{\mathbb{Q}}^\times \hookrightarrow \overline{\mathbb{Q}_p}^\times$ and regard any character as p-adic one. Let $\mathrm{Irr}(D_{2l})$ be the set of all irreducible characters of D_{2l}. For $\chi \in \mathrm{Irr}(D_{2l})$, put $\mathscr{O}_\chi = \mathbb{Z}_p[\mathrm{Im}\chi]$ and define

$$e_\chi = \frac{\chi(1)}{2l} \sum_{g \in D_{2l}} \chi(g^{-1}) g \in \mathscr{O}_\chi[D_{2l}].$$

Let M be a module over $\mathbb{Z}_p[D_{2l}]$. We call $e_\chi(M \otimes \mathscr{O}_\chi)$ the χ-part of M and simply denote this by M^χ. Put $\mathscr{O} = \mathbb{Z}_p[\zeta_l]$. Since $\{e_\chi\}_{\chi \in \mathrm{Irr}(D_{2l})}$ is orthogonal idempotents of $\mathscr{O}[D_{2l}]$ and $1_{\mathscr{O}[D_{2l}]} = \sum_{\chi \in \mathrm{Irr}(D_{2l})} e_\chi$, we may write

$$M \otimes \mathscr{O} = \bigoplus_{\chi \in \mathrm{Irr}(D_{2l})} \tilde{M}^\chi$$

where $\tilde{M}^\chi = e_\chi(M \otimes \mathscr{O})$. On the other hand, it is well-known that

$$M \otimes \mathscr{O} = \bigoplus_{i=0}^{l-1} M^{\sigma_i}$$

as an $\mathscr{O}[\langle a \rangle]$-module where $M^{\sigma_i} = \{x \in M \otimes \mathscr{O} \mid ax = \sigma_i(a)x\}$. In particular, when M is finite, we have

$$\# \bigoplus_{k=1}^{\frac{l-1}{2}} \tilde{M}^{\chi_k} = \frac{\#(M \otimes \mathscr{O})}{\#\left(\tilde{M}^\varepsilon \oplus \tilde{M}^\eta\right)} = \frac{\#(M \otimes \mathscr{O})}{\# M^{\sigma_0}} = \# \bigoplus_{k=1}^{l-1} M^{\sigma_k}, \tag{2.2}$$

since $\tilde{M}^\varepsilon \oplus \tilde{M}^\eta = \{x \in M \otimes \mathscr{O} \mid ax = x\} = M^{\sigma_0}$.

Lemma 2.1. *Let $d_k = [\mathscr{O} : \mathscr{O}_{\chi_k}]$. If M is a finite $\mathbb{Z}_p[D_{2l}]$-module, then*

$$(\# M^{\chi_k})^{d_k} = \left(\# M^{\sigma_k}\right)^2$$

for all $k \in \{1, \cdots, \frac{l-1}{2}\}$.

Proof. Since $e_{\chi_k} = e_{\sigma_k} + e_{\sigma_{l-k}}$ in $\mathscr{O}[D_{2l}]$, we have the natural homomorphism

$$f : \tilde{M}^{\chi_k} \longrightarrow M^{\sigma_k} \oplus M^{\sigma_{l-k}}, \qquad e_{\chi_k} x \mapsto \left(e_{\sigma_k} x, e_{\sigma_{l-k}} x\right)$$

as abelian groups. Take $x \in M \otimes \mathscr{O}$ with $\left(e_{\sigma_k} x, e_{\sigma_{l-k}} x\right) = (0, 0)$. This yields $e_{\chi_k} x = e_{\sigma_k} x + e_{\sigma_{l-k}} x = 0$, which implies that f is injective. Thus the equation (2.2) leads to

$$\# \tilde{M}^{\chi_k} = \#\left(M^{\sigma_k} \oplus M^{\sigma_{l-k}}\right)$$

for each k, therefore f is also surjective. Note that $be_{\sigma_k} = e_{\sigma_{l-k}} b$ and $be_{\sigma_{l-k}} = e_{\sigma_k} b$. The homomorphism

$$M^{\sigma_k} \longrightarrow M^{\sigma_{l-k}}, \qquad x \mapsto bx$$

is an isomorphism because

$$M^{\sigma_{l-k}} \longrightarrow M^{\sigma_k}, \qquad x \mapsto bx$$

is its inverse map. It follows that $\#M^{\sigma_k} = \#M^{\sigma_{l-k}}$, so $\#\tilde{M}^{\chi_k} = \left(\#M^{\sigma_k}\right)^2$. On the other hand, we know $\#\tilde{M}^{\chi_k} = (\#M^{\chi_k})^{d_k}$ by

$$M \otimes \mathscr{O} \simeq M \otimes \left(\mathscr{O}_{\chi_k}^{d_k}\right) \simeq (M \otimes \mathscr{O}_{\chi_k})^{d_k}$$

as $\mathscr{O}_{\chi_k}[D_{2l}]$-modules. This completes the proof. ■

Now we give a proof of Theorem 1.1. In the following arguments, we identify $\mathrm{Gal}(L^+/F^+)$ with $D_{2l} = \langle a, b\rangle$. Let K^+ be the fixed field of $\langle a\rangle$ in L^+ and $K = K^+(\sqrt{-r})$. For an irreducible character ψ of $\mathrm{Gal}(L^+/F^+)$, we define the characters ψ^+ and ψ^- of $\mathrm{Gal}(L/F^+)$ by

$$\psi^+(g) = \psi(g|_{L^+}), \qquad \psi^-(g) = \gamma(g|_F)\psi(g|_{L^+}),$$

respectively, where γ is the non-trivial character of $\mathrm{Gal}(F/F^+)$. In fact, we know that ψ^+ is even while ψ^- is odd. For a character σ of $\mathrm{Gal}(L^+/K^+)$, define the characters $\sigma^\pm$ of $\mathrm{Gal}(L/K^+)$ in the same manner. Using these notations and Theorem 4.21 of [2], we obtain

$$\mathrm{Irr}(\mathrm{Gal}(L/F^+)) = \left\{\varepsilon^\pm, \eta^\pm, \chi_1^\pm, \cdots, \chi_{\frac{l-1}{2}}^\pm\right\}$$

and

$$\mathrm{Hom}\left(\mathrm{Gal}(L/K^+), \overline{\mathbb{Q}_p}^\times\right) = \left\{\sigma_0^\pm, \cdots, \sigma_{l-1}^\pm\right\}.$$

First, we treat the characters of two-dimensional representations. For a finite $\mathbb{Z}_p[\mathrm{Gal}(L/F^+)]$-module M, we have

$$\left(\#M^{\chi_k^\pm}\right)^{d_{\sigma_k^\pm}/d_{\chi_k^\pm}} = \left(\#M^{\sigma_k^\pm}\right)^2,$$

by Lemma 2.1, and therefore

$$\frac{\left(\#H^2(O_L', \mathbb{Z}_p(n))^{\chi_k^\pm}\right)^{d_{\sigma_k^\pm}/d_{\chi_k^\pm}}}{\left(\#H^1(O_L', \mathbb{Z}_p(n))^{\chi_k^\pm}\right)^{d_{\sigma_k^\pm}/d_{\chi_k^\pm}}} = \frac{\left(\#H^2(O_L', \mathbb{Z}_p(n))^{\sigma_k^\pm}\right)^2}{\left(\#H^1(O_L', \mathbb{Z}_p(n))^{\sigma_k^\pm}\right)^2}. \tag{2.3}$$

We remark that characters of dihedral groups take real values. Since $\overline{\chi_k^\pm} = \chi_k^\pm = \mathrm{Ind}\,(\sigma_k^\pm)^{-1}$ by (2.1), it follows from Chapter VII, Proposition 10.4 (iv) of [5] that

$$\mathscr{L}\left(L/F^+, \overline{\chi_k^\pm}, 1-n\right) = \mathscr{L}\left(L/K^+, (\sigma_k^\pm)^{-1}, 1-n\right). \tag{2.4}$$

By the way, we can apply Theorem 1.2 to L/K^+ because $\mathrm{Gal}(L/K^+)$ is the direct product of two cyclic groups of order l and 2. Hence,

$$\mathscr{L}\left(L/K^+,(\sigma_k^{(n)})^{-1},1-n\right)^{d_{\sigma_k^{(n)}}} \sim_p \frac{\#H^2(O'_L,\mathbb{Z}_p(n))^{\sigma_k^{(n)}}}{\#H^1(O'_L,\mathbb{Z}_p(n))^{\sigma_k^{(n)}}} \tag{2.5}$$

where $\sigma_k^{(n)} = \sigma_k^+$ if n is even and $\sigma_k^{(n)} = \sigma_k^-$ if n is odd. Since $\chi_k^{\pm}(1) = 2$, the relationship (2.5) is equivalent to

$$\mathscr{L}\left(L/K^+,(\sigma_k^{(n)})^{-1},1-n\right)^{\chi_k^{(n)}(1)\cdot d_{\sigma_k^{(n)}}} \sim_p \frac{\left(\#H^2(O'_L,\mathbb{Z}_p(n))^{\sigma_k^{(n)}}\right)^2}{\left(\#H^1(O'_L,\mathbb{Z}_p(n))^{\sigma_k^{(n)}}\right)^2}.$$

Combining this with (2.3) and (2.4), we deduce that

$$\mathscr{L}\left(L/F^+,\overline{\chi_k^{(n)}},1-n\right)^{\chi_k^{(n)}(1)\cdot d_{\sigma_k^{(n)}}} \sim_p \frac{\left(\#H^2(O'_L,\mathbb{Z}_p(n))^{\chi_k^{(n)}}\right)^{d_{\sigma_k^{(n)}}/d_{\chi_k^{(n)}}}}{\left(\#H^1(O'_L,\mathbb{Z}_p(n))^{\chi_k^{(n)}}\right)^{d_{\sigma_k^{(n)}}/d_{\chi_k^{(n)}}}},$$

i.e.

$$\mathscr{L}\left(L/F^+,\overline{\chi_k^{(n)}},1-n\right)^{\chi_k^{(n)}(1)\cdot d_{\chi_k^{(n)}}} \sim_p \frac{\#H^2(O'_L,\mathbb{Z}_p(n))^{\chi_k^{(n)}}}{\#H^1(O'_L,\mathbb{Z}_p(n))^{\chi_k^{(n)}}}.$$

This completes the proof for the case $\chi = \chi_k^{\pm}$.

We next explain the cases $\chi = \varepsilon^{\pm}$ that are linear characters. For this purpose we prepare the following lemma, which seems folklore for experts.

Lemma 2.2. *Let L/K be a finite Galois extension of number fields and suppose p is prime to $[L:K]$. Then the canonical homomorphism*

$$H^i\left(O'_K,\mathbb{Z}_p(n)\right) \longrightarrow H^i\left(O'_L,\mathbb{Z}_p(n)\right)^{\mathrm{Gal}\,(L/K)}$$

is bijective for any i and any n.

Proof. We write $A = O_K[1/p]$, $B = O_L[1/p]$ and $\Gamma = \mathrm{Gal}(L/K)$. Let μ_{p^r} denote the group scheme of p^r-th root of unity over A. Then μ_{p^r} is étale and finite over A since p is invertible in A, and the Tate twist $\mu_{p^r}^{\otimes n}$ is also representable by an étale finite group scheme over A. Put $G = \mu_{p^r}^{\otimes n}$ and let $\mathrm{Res}_{B/A}G$ denote the Weil restriction with respect to the finite extension B/A. We have the natural inclusion $\iota : G \to \mathrm{Res}_{B/A}G$ and the natural norm homomorphism $\mathrm{Nr} : \mathrm{Res}_{B/A}G \to G$. Furthermore, it is readily seen that

(1) $\mathrm{Nr} \circ \iota$ is equal to the multiplication-by-$[L:K]$ map over G;
(2) $\iota \circ \mathrm{Nr}$ is equal to $\sum_{\gamma\in\Gamma} \gamma$ over $\mathrm{Res}_{B/A}G$.

Note that the Weil restriction is nothing but the direct image of the étale sheaf on $\mathrm{Spec}B$ by the morphism $\pi : \mathrm{Spec}B \to \mathrm{Spec}A$. Therefore, the canonical homomorphism

$$H^i\big(A, \mathrm{Res}_{B/A}G\big) \longrightarrow H^i(B, G)$$

is bijective since π is finite (cf. [1], Expo VIII, Cor 5.6). Moreover, the homomorphism $\iota : G \to \mathrm{Res}_{B/A}G$ gives rise to a homomorphism

$$\iota : H^i(A, G) \longrightarrow H^i\big(A, \mathrm{Res}_{B/A}G\big) \simeq H^i(B, G),$$

which is nothing but the homomorphism induced by $\pi : \mathrm{Spec}B \to \mathrm{Spec}A$. On the other hand, $\mathrm{Nr} : \mathrm{Res}_{B/A}G \to G$ gives rise to a homomorphism

$$\mathrm{Nr} : H^i(B, G) \simeq H^i\big(A, \mathrm{Res}_{B/A}G\big) \longrightarrow H^i(A, G).$$

It follows from (1) and (2) that

(1)' $\mathrm{Nr} \circ \iota$ is equal to the multiplication-by-$[L : K]$ map over $H^i(A, G)$;
(2)' $\iota \circ \mathrm{Nr}$ is equal to $\sum_{\gamma \in \Gamma} \gamma$ over $H^i(B, G)$.

Passing to the limit, we obtain homomorphisms

$$\iota : H^i(A, \mathbb{Z}_p(n)) \longrightarrow H^i(B, \mathbb{Z}_p(n))$$

and

$$\mathrm{Nr} : H^i(B, \mathbb{Z}_p(n)) \longrightarrow H^i(A, \mathbb{Z}_p(n)).$$

It follows again from (1)' and (2)' that

(1)" $\mathrm{Nr} \circ \iota$ is equal to the multiplication-by-$[L : K]$ map over $H^i(A, \mathbb{Z}(n))$;
(2)" $\iota \circ \mathrm{Nr}$ is equal to $\sum_{\gamma \in \Gamma} \gamma$ over $H^i(B, \mathbb{Z}(n))$,

and therefore $\iota \circ \mathrm{Nr}$ is equal to the multiplication-by-$[L : K]$ map over $H^i(B, \mathbb{Z}_p(n))^\Gamma$. Note that the two multiplication-by-$[L : K]$ maps $\mathrm{Nr} \circ \iota : H^i(A, \mathbb{Z}_p(n)) \to H^i(A, \mathbb{Z}_p(n))$ and $\iota \circ \mathrm{Nr} : H^i(B, \mathbb{Z}_p(n))^\Gamma \to H^i(B, \mathbb{Z}_p(n))^\Gamma$ are bijective because p does not divide $[L : K]$. This implies that $\iota : H^i(A, \mathbb{Z}_p(n)) \to H^i(B, \mathbb{Z}_p(n))^\Gamma$ is bijective. ■

Let $\gamma^+ : \mathrm{Gal}(F/F^+) \to \overline{\mathbb{Q}_p}^\times$ and $\gamma^- : \mathrm{Gal}(F/F^+) \to \overline{\mathbb{Q}_p}^\times$ be the trivial and non-trivial character, respectively. Note that $d_{\gamma^\pm} = 1$, $(\gamma^\pm)^{-1} = \gamma^\pm$, and $\overline{\varepsilon^\pm} = \varepsilon^\pm$. We can apply Theorem 1.2 to the quadratic extension F/F^+, so,

$$\mathscr{L}\left(F/F^+, (\gamma^{(n)})^{-1}, 1-n\right) \sim_p \frac{\# H^2(O'_F, \mathbb{Z}_p(n))^{\gamma^{(n)}}}{\# H^1(O'_F, \mathbb{Z}_p(n))^{\gamma^{(n)}}}. \tag{2.6}$$

For the left side of (2.6), it follows from Chapter VII, Proposition 10.4 (iii) of [5] that

$$\mathscr{L}\left(L/F^+, \overline{\varepsilon^\pm}, 1-n\right) = \mathscr{L}\left(F/F^+, (\gamma^\pm)^{-1}, 1-n\right). \tag{2.7}$$

Since $ge_{\varepsilon^\pm} = e_{\varepsilon^\pm}$ for all $g \in \mathrm{Gal}(L/F)$, we find

$$\begin{aligned} H^i(O'_F, \mathbb{Z}_p(n))^{\gamma^+} \oplus H^i(O'_F, \mathbb{Z}_p(n))^{\gamma^-} &\simeq H^i(O'_F, \mathbb{Z}_p(n)) \\ &\simeq H^i(O'_L, \mathbb{Z}_p(n))^{\mathrm{Gal}(L/F)} \\ &\simeq H^i(O'_L, \mathbb{Z}_p(n))^{\varepsilon^+} \oplus H^i(O'_L, \mathbb{Z}_p(n))^{\varepsilon^-} \end{aligned}$$

and

$$\begin{aligned} H^i(O'_F, \mathbb{Z}_p(n))^{\gamma^+} &\simeq H^i(O'_F, \mathbb{Z}_p(n))^{\mathrm{Gal}(F/F^+)} \\ &\simeq H^i(O'_{F^+}, \mathbb{Z}_p(n)) \\ &\simeq H^i(O'_L, \mathbb{Z}_p(n))^{\mathrm{Gal}(L/F^+)} \\ &\simeq H^i(O'_L, \mathbb{Z}_p(n))^{\varepsilon^+} \end{aligned}$$

by Lemma 2.2. Thus, the following equations

$$\# H^i(O'_F, \mathbb{Z}_p(n))^{\gamma^\pm} = \# H^i(O'_L, \mathbb{Z}_p(n))^{\varepsilon^\pm} \tag{2.8}$$

hold for $i = 1, 2$. These (2.6), (2.7) and (2.8) lead to

$$\mathscr{L}\left(L/F^+, \overline{\varepsilon^{(n)}}, 1-n\right) \sim_p \frac{\# H^2(O'_L, \mathbb{Z}_p(n))^{\varepsilon^{(n)}}}{\# H^1(O'_L, \mathbb{Z}_p(n))^{\varepsilon^{(n)}}}.$$

This completes the proof for the case $\chi = \varepsilon^\pm$.

Similarly, by [5, Proposition 10.4 (iii) in Ch. VII], we can apply Theorem 1.2 to K/F^+ to obtain the desired result for the case $\chi = \eta^\pm$.

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References

[1] A. Grothendieck et al, *Théorie des Topos et Cohomologie Etale des Schémas*, Lecture Notes in Mathmatics, vol. 270, 1972.

[2] I. Martin Isaacs, *Character Theory of Finite Groups*, Pure and Applied Mathematics **69**, Academic Press, 1976.

[3] M. Kolster, T. Nguyen Quang Do, V. Fleckinger, *Twisted S-Units, p-adic class number formulas, and the Lichtenbaum conjectures*, Duke Mathematical Journal **84** (1996), no.3, 679–717.

[4] Y. Konomi, *The ideal class groups of dihedral extensions over imaginary quadratic fields and the special values of the Artin L-function*, J. Number Theory **131** (2011), no.6, 1062–1069.

[5] J. Neukirch, *Algebraic Number Theory*, GdmW., vol. 322, Springer, 1999.
[6] V.P. Snaith, *Algebraic K-groups as Galois Modules*, Progress in Mathematics **206**, Birkhäuser, 2002.
[7] V. Voevodsky, *On motivic cohomology with $\mathbb{Z}/l$-coefficients*, Annals of Mathematics **174** (2011), 401–438.
[8] C. Weibel, *2007 Trieste lectures on the proof of the Bloch-Kato conjecture, some recent developments in algebraic K-theory*, ICTP Lecture Notes **23** (Abdus Salam International Centre for Theoretical Physics, Trieste, 2008), 277–305.
[9] A. Wiles, *On a conjecture of Brumer*, Annals of Mathematics **131** (1990), 555–565.

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