Functiones et Approximatio
51.1 (2014), 141–166
doi: 10.7169/facm/2014.51.1.8

EXPANDING THE APPLICABILITY OF A TWO STEP NEWTON-TYPE PROJECTION METHOD FOR ILL-POSED PROBLEMS

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Abstract: There are many classes of ill-posed problems that cannot be solved with existing iterative methods, since the usual Lipschitz-type assumptions are not satisfied. In this study, we expand the applicability of a two step Newton-type projection method considered in [10], [11], using weaker assumptions. Numerical examples for the method and examples where the old assumptions are not satisfied but the new assumptions are satisfied are provided at the end of this study.

Keywords: Discretized Two Step Newton Tikhonov method, ill-posed Hammerstein-type operator equations, balancing principle, monotone operator, regularization method, projection method.

1. Introduction

This paper deals with the finite dimensional realization of a method considered in [10] for (nonlinear) Hammerstein-type equation

$$KF(x) = f. \tag{1.1}$$

Here $F : D(F) \subseteq X \to Z$ is nonlinear, $K : Z \to Y$ is a bounded linear operator ([7],[8]) and X, Z, Y are Hilbert spaces with inner product $\langle ., . \rangle$ and norm $\|.\|$ respectively.

We will assume that the problem (1.1) is ill-posed due to the non-closedness of the linear operator K (see [9]). It is assumed that $f^\delta \in Y$ are the available noisy data with $\|f - f^\delta\| \leqslant \delta$ and F possesses a uniformly bounded Fréchet derivative for each $x \in D(F)$, i.e.,

$$\|F'(x)\| \leqslant M, \qquad x \in D(F)$$

2010 Mathematics Subject Classification: primary: 47J06, 65J15, 47H17; secondary: 47A52, 65N20, 65J20

for some M(Here and below $F'(.)$ denotes the Fréchet derivative of F). Observe that the solution x of (1.1) with f^δ in place of f can be obtained by first solving

$$Kz = f^\delta \tag{1.2}$$

for z and then solving the non-linear problem

$$F(x) = z. \tag{1.3}$$

In fact, in [10] we consider two cases of F, in the first case we assume that $F'(x)^{-1}$ exist and in the second case we assume F is monotone but $F'(x)^{-1}$ does not exist. The method in [10] was a combination of Tikhonov regularization and Two Step Newton Method.

Regularization methods for ill-posed operator equation are usually defined in an infinite dimensional setting and have to be discretized for calculating a numerical solution [12]. Since finite dimensional problem are always well-posed in the sense of stable data dependence one could think of stabilizing an ill-posed problem by discretization. Regularization of ill-posed problems by projection methods can be found in literature for eg. in [17, 18, 19]. In this paper we consider the problem of approximately solving (1.1) in the finite dimensional setting of Hilbert spaces. Our goal is to expand the applicability of this method by weakening the usual assumptions for the convergence of these methods (see Assumption 3.1 and Assumption 3.2).

Recall [20], [21], that an operator F is said to be monontone operator if $\langle F(x) - F(y), x - y\rangle \geqslant 0$ for all $x, y \in D(F)$.

The organization of this paper is as follows. Section 2 deals with Discretized Tikhonov regularization (detailed proof can be found in [11]) and Section 3 investigates the convergence of the Two Step Newton Tikhonov Projection Method (TSNTPM). Section 4 discusses the algorithm and finally the paper ends with a Numerical examples in Section 5.

2. Discretized Tikhonov regularization

This section deals with discretized Tikhonov regularized solution $z_\alpha^{h,\delta}$ of (1.2) and (an a priori and an a posteriori) error estimate for $\|F(\hat{x}) - z_\alpha^{h,\delta}\|$.

The following assumption is used as in [8] to obtain the error estimate .

Assumption 2.1. There exists a continuous, strictly monotonically increasing function $\varphi : (0, a] \to (0, \infty)$ with $a \geqslant \|K\|^2$ satisfying;

- $\lim_{\lambda\to 0}\varphi(\lambda) = 0$,
- $$\sup_{\lambda\geqslant 0} \frac{\alpha\varphi(\lambda)}{\lambda+\alpha} \leqslant \varphi(\alpha) \qquad \forall\, \lambda \in (0, a]$$

  and
- $$F(\hat{x}) - F(x_0) = \varphi(K^*K)w$$

  for some $w \in X$ such that $\|w\| \leqslant 1$.

Let $\{P_h\}_{h>0}$ be a family of orthogonal projections on X. Let

$$\varepsilon_h := \|K(I-P_h)\|,$$
$$\tau_h := \|F'(x)(I-P_h)\|, \qquad \forall x \in D(F)$$

and $\{b_h : h > 0\}$ is such that $\lim_{h\to 0}\frac{\|(I-P_h)x_0\|}{b_h} = 0$, $\lim_{h\to 0}\frac{\|(I-P_h)F(x_0)\|}{b_h} = 0$ and $\lim_{h\to 0} b_h = 0$. We assume that $\varepsilon_h \to 0$ and $\tau_h \to 0$ as $h \to 0$. The above assumption is satisfied if, $P_h \to I$ pointwise and if K and $F'(x)$ are compact operators. Further we assume that $\varepsilon_h < \varepsilon_0$, $\tau_h \leqslant \tau_0$, $b_h \leqslant b_0$ and $\delta \in (0, \delta_0]$.

The discretized Tikhonov regularization method for the regularized equation (1.2) consists of solving the equation

$$(P_hK^*KP_h + \alpha P_h)(z_{\alpha_k}^{h,\delta} - P_hF(x_0)) = P_hK^*[f^\delta - KF(x_0)]. \tag{2.1}$$

Theorem 2.2 (see [11], Theorem 2.4). *Suppose Assumption 2.1 holds. Let $z_{\alpha_k}^{h,\delta}$ be as in (2.1) and $b_h \leqslant \frac{\delta+\varepsilon_h}{\sqrt{\alpha}}$. Then*

$$\|F(\hat{x}) - z_{\alpha_k}^{h,\delta}\| \leqslant C\left(\varphi(\alpha) + \left(\frac{\delta+\varepsilon_h}{\sqrt{\alpha}}\right)\right) \tag{2.2}$$

where $C = \frac{1}{2}\max\{M\rho, 1\} + 1$.

2.1. A priori choice of the parameter

Note that the estimate $\varphi(\alpha) + \frac{\delta+\varepsilon_h}{\sqrt{\alpha}}$ in (2.2) is of optimal order for the choice $\alpha := \alpha(\delta, h)$ which satisfies $\varphi(\alpha(\delta,h)) = \frac{\delta+\varepsilon_h}{\sqrt{\alpha(\delta,h)}}$. Let $\psi(\lambda) := \lambda\sqrt{\varphi^{-1}(\lambda)}, 0 < \lambda \leqslant a$. Then we have $\delta + \varepsilon_h = \sqrt{\alpha(\delta,h)}\varphi(\alpha(\delta,h)) = \psi(\varphi(\alpha(\delta,h)))$ and

$$\alpha(\delta,h) = \varphi^{-1}(\psi^{-1}(\delta+\varepsilon_h)).$$

So the relation (2.2) leads to $\|F(\hat{x}) - z_\alpha^{h,\delta}\| \leqslant 2C\psi^{-1}(\delta+\varepsilon_h)$.

2.2. An adaptive choice of the parameter

In this subsection, we consider the balancing principle established by Pereverzev and Shock [14] for choosing the parameter α. Let

$$D_N = \{\alpha_i : 0 < \alpha_0 < \alpha_1 < \alpha_2 < \cdots < \alpha_N\}$$

be the set of possible values of the parameter α.

Let

$$l := \max\left\{i : \varphi(\alpha_i) \leqslant \frac{\delta+\varepsilon_h}{\sqrt{\alpha_i}}\right\} < N, \tag{2.3}$$

$$k = \max\{i : \alpha_i \in D_N^+\} \tag{2.4}$$

where $D_N^+ = \{\alpha_i \in D_N : \|z_{\alpha_i}^\delta - z_{\alpha_j}^\delta\| \leqslant \frac{4C(\delta+\varepsilon_h)}{\sqrt{\alpha_j}}, j = 0, 1, 2, ..., i-1\}$.

We use the following theorem, the proof of which is analogous to the proof of Theorem 4.3 in [8], for our error analysis.

Theorem 2.3 (cf. [8, Theorem 4.3]). *Let l be as in (2.3), k be as in (2.4) and $z^{h,\delta}_{\alpha_k}$ be as in (2.1) with $\alpha = \alpha_k$. Then $l \leqslant k$ and*

$$\|F(\hat{x}) - z^{h,\delta}_{\alpha_k}\| \leqslant C\Big(2 + \frac{4\mu}{\mu - 1}\Big)\mu\psi^{-1}(\delta + \varepsilon_h).$$

3. Convergence analysis of the projection method

In [11], [10], [8] the following Assumption was used, which is very difficult to verify (or does not hold) in general (see numerical examples at the last section of the paper)

Assumption 3.1 (cf.[16, Assumption 3 (A3)]). There exist a constant $k_0 \geqslant 0$ such that for every $x, u \in D(F)$ and $v \in X$ there exists an element $\Phi(x, u, v) \in X$ such that $[F'(x) - F'(u)]v = F'(u)\Phi(x, u, v), \|\Phi(x, u, v)\| \leqslant k_0\|v\|\|x - u\|$.

In the present paper we analyze the method by using a weaker Assumption than Assumption 3.1 and which is easier to verify:

Assumption 3.2. Let $x_0 \in X$ be fixed. There exists a constant $K_0 \geqslant 0$ such that for each $x, u \in D(F)$ and $v \in X$ there exists an element $\Phi(x, u, v) \in X$ depending on x_0 such that $[F'(x) - F'(u)]v = F'(u)\Phi(x, u, v), \|\Phi(x, u, v)\| \leqslant K_0\|v\|(\|x - P_hx_0\| + \|u - P_hx_0\|)$.

Note that Assumption 3.1$\Rightarrow$ Assumption 3.2 but not necessarily vice versa.

3.1. Case 1: TSNTPM when $F'(.)$ is invertible

In this section we assume that $F'(x)$ is boundedly invertible for all $x \in D(F)$ i.e.,

$$\|F'(x)^{-1}\| \leqslant \beta_1 \tag{3.1}$$

for some $\beta_1 > 0$.

For an initial guess $x_0 \in X$, the TSNTPM is defined iteratively as;

$$y^{h,\delta}_{n,\alpha_k} = x^{h,\delta}_{n,\alpha_k} - P_hF'(x^{h,\delta}_{n,\alpha_k})^{-1}P_h(F(x^{h,\delta}_{n,\alpha_k}) - z^{h,\delta}_{\alpha_k}), \tag{3.2}$$

$$x^{h,\delta}_{n+1,\alpha_k} = y^{h,\delta}_{n,\alpha_k} - P_hF'(y^{h,\delta}_{n,\alpha_k})^{-1}P_h(F(y^{h,\delta}_{n,\alpha_k}) - z^{h,\delta}_{\alpha_k}), \tag{3.3}$$

where $x^{h,\delta}_{0,\alpha_k} := P_hx_0$ and $z^{h,\delta}_{\alpha_k}$ is defined by (2.1) with $\alpha = \alpha_k$.

Note. Observe that if $b_0 < \frac{1}{K_0}$ and if $x \in B_r(P_hx_0)$ where $r < \frac{1}{K_0} - b_0$, then $F'(x)^{-1}$ exists and is bounded. This can be seen as follows:

$$\begin{aligned}\|F'(x)^{-1}\| &= \sup_{\|v\|\leqslant 1} \|[I + F'(x_0)^{-1}(F'(x) - F'(x_0))]^{-1}F'(x_0)^{-1}v\| \\ &\leqslant \sup_{\|v\|\leqslant 1} \frac{\|F'(x_0)^{-1}\|}{1 - \|F'(x_0)^{-1}(F'(x) - F'(x_0))v\|}.\end{aligned} \tag{3.4}$$

Now by Assumption 3.2 and the triangle inequality;

$$\|x - x_0\| \leqslant \|x - P_h x_0\| + \|P_h x_0 - x_0\|,$$

we have

$$\|F'(x_0)^{-1}(F'(x) - F'(x_0))v\| \leqslant K_0(r + b_0).$$

Hence by (3.1) and (3.4) we have

$$\|F'(x)^{-1}\| \leqslant \frac{\beta_1}{1 - K_0(r + b_0)}.$$

Thus without loss of generality we assume that

$$\|F'(x)^{-1}\| \leqslant \beta, \ \ \forall x \in B_r(P_h x_0) \tag{3.5}$$

and for some $\beta > 0$.

Lemma 3.3. *Let $x \in B_r(P_h x_0)$, $b_0 < \frac{1}{K_0}$ and $r < \frac{1}{K_0} - b_0$. Then we have $\|P_h F'(x)^{-1} P_h F'(x)\| \leqslant 1 + \beta\tau_0$.*

Proof.

$$\begin{aligned}
\|P_h F'(x)^{-1} P_h F'(x)\| &= \sup_{\|v\| \leqslant 1} \|[P_h F'(x)^{-1} P_h F'(x)]v\| \\
&\leqslant \sup_{\|v\| \leqslant 1} \|P_h F'(x)^{-1} P_h F'(x)(P_h + I - P_h)v\| \\
&\leqslant \sup_{\|v\| \leqslant 1} \|[P_h F'(x)^{-1} P_h F'(x) P_h]v\| \\
&\quad + \sup_{\|v\| \leqslant 1} \|P_h F'(x)^{-1} P_h F'(x) \times (I - P_h)v\| \\
&\leqslant 1 + \beta\tau_h \leqslant 1 + \beta\tau_0.
\end{aligned}$$

■

Let

$$e^{h,\delta}_{n,\alpha_k} := \|y^{h,\delta}_{n,\alpha_k} - x^{h,\delta}_{n,\alpha_k}\|, \qquad \forall n \geqslant 0. \tag{3.6}$$

Suppose that

$$0 < K_0 < \frac{1}{4(1 + \beta\tau_0)} \tag{3.7}$$

and

$$\frac{\delta_0 + \varepsilon_0}{\sqrt{\alpha_0}} < \frac{2}{\beta(2M + 3)}. \tag{3.8}$$

Let $\|\hat{x} - x_0\| \leqslant \rho$, with

$$\rho < \frac{1}{M}\left(\frac{1}{\beta} - \left(\frac{3}{2} + M\right)\frac{\delta_0 + \varepsilon_0}{\sqrt{\alpha_0}}\right)$$

and

$$\gamma_\rho := \beta\left[M\rho + \left(\frac{3}{2} + M\right)\left(\frac{\varepsilon_0 + \delta_0}{\sqrt{\alpha_0}}\right)\right], \tag{3.9}$$

and let r be such that

$$r \in (r_1, r_2), \tag{3.10}$$

where

$$r_1 = \frac{1 + \sqrt{1 - 16(1+\beta\tau_0)K_0\gamma_\rho}}{8(1+\beta\tau_0)}$$

and

$$r_2 = \frac{1 - \sqrt{1 - 16(1+\beta\tau_0)K_0\gamma_\rho}}{8(1+\beta\tau_0)}$$
$$b := 4(1+\beta\tau_0)K_0 r. \tag{3.11}$$

Then, we have by (3.7)-(3.11) that

$$0 < \gamma_\rho < \frac{1}{16(1+\beta\tau_0)K_0}. \tag{3.12}$$

Theorem 3.4. *Let $e^{h,\delta}_{n,\alpha_k}$ be as in equation (3.6) with $\delta \in (0, \delta_0]$, $\alpha = \alpha_k$ and $\varepsilon_h \in (0, \varepsilon_0]$. Suppose the assumptions of Lemma 3.3 and Theorem 2.3 hold. Then, by Assumption 3.2, we have the following:*

(a)
$$\begin{aligned}\|x^{h,\delta}_{n,\alpha_k} - y^{h,\delta}_{n-1,\alpha_k}\| \leqslant \frac{K_0}{2}(1+\beta\tau_0)\Big[3\|x^{h,\delta}_{n-1,\alpha_k} - x^{h,\delta}_{0,\alpha_k}\| \\ + 5\|y^{h,\delta}_{n-1,\alpha_k} - x^{h,\delta}_{0,\alpha_k}\|\Big] e^{h,\delta}_{n-1,\alpha_k},\end{aligned} \tag{3.13}$$

(b)
$$\begin{aligned}\|x^{h,\delta}_{n,\alpha_k} - x^{h,\delta}_{n-1,\alpha_k}\| \leqslant \Big\{1 + \frac{K_0}{2}(1+\beta\tau_0)\Big[3\|x^{h,\delta}_{n-1,\alpha_k} - x^{h,\delta}_{0,\alpha_k}\| \\ + 5\|y^{h,\delta}_{n-1,\alpha_k} - x^{h,\delta}_{0,\alpha_k}\|\Big]\Big\} e^{h,\delta}_{n-1,\alpha_k},\end{aligned} \tag{3.14}$$

and

(c)
$$\begin{aligned}e^{h,\delta}_{n,\alpha_k} &\leqslant \frac{K_0}{2}(1+\beta\tau_0)[5\|x^{h,\delta}_{n,\alpha_k} - x^{h,\delta}_{0,\alpha_k}\| \\ &\quad + 3\|y^{h,\delta}_{n-1,\alpha_k} - x^{h,\delta}_{0,\alpha_k}\|]\|y^{h,\delta}_{n-1,\alpha_k} - x^{h,\delta}_{n,\alpha_k}\| \\ &\leqslant b^2 e^{h,\delta}_{n-1,\alpha_k} \leqslant b^{2n} e^{h,\delta}_{0,\alpha_k} \leqslant b^{2n}\gamma_\rho.\end{aligned}$$

Proof. Observe that

$$
\begin{aligned}
x^{h,\delta}_{n,\alpha_k} - y^{h,\delta}_{n-1,\alpha_k} &= y^{h,\delta}_{n-1,\alpha_k} - x^{h,\delta}_{n-1,\alpha_k} - P_h F'(y^{h,\delta}_{n-1,\alpha_k})^{-1} P_h (F(y^{h,\delta}_{n-1,\alpha_k}) \\
&\quad - z^{h,\delta}_{\alpha_k}) + P_h F'(x^{h,\delta}_{n-1,\alpha_k})^{-1} P_h (F(x^{h,\delta}_{n-1,\alpha_k}) - z^{h,\delta}_{\alpha_k}) \\
&= y^{h,\delta}_{n-1,\alpha_k} - x^{h,\delta}_{n-1,\alpha_k} - P_h F'(y^{h,\delta}_{n-1,\alpha_k})^{-1} P_h (F(y^{h,\delta}_{n-1,\alpha_k}) \\
&\quad - F(x^{h,\delta}_{n-1,\alpha_k})) - P_h [F'(y^{h,\delta}_{n-1,\alpha_k})^{-1} - F'(x^{h,\delta}_{n-1,\alpha_k})^{-1}] \\
&\quad \times P_h (F(x^{h,\delta}_{n-1,\alpha_k}) - z^{h,\delta}_{\alpha_k}) \\
&= P_h F'(y^{h,\delta}_{n-1,\alpha_k})^{-1} P_h \\
&\quad \times \int_0^1 \Big[F'(y^{h,\delta}_{n-1,\alpha_k}) - F'(x^{h,\delta}_{n-1,\alpha_k} + t(y^{h,\delta}_{n-1,\alpha_k} - x^{h,\delta}_{n-1,\alpha_k})) \Big] \\
&\quad \times (y^{h,\delta}_{n-1,\alpha_k} - x^{h,\delta}_{n-1,\alpha_k}) dt + P_h F'(y^{h,\delta}_{n-1,\alpha_k})^{-1} \\
&\quad \times P_h [F'(x^{h,\delta}_{n-1,\alpha_k}) - F'(y^{h,\delta}_{n-1,\alpha_k})](y^{h,\delta}_{n-1,\alpha_k} - x^{h,\delta}_{n-1,\alpha_k}) \\
&:= \Gamma_1 + \Gamma_2,
\end{aligned} \tag{3.15}
$$

where

$$
\begin{aligned}
\Gamma_1 = P_h F'(y^{h,\delta}_{n-1,\alpha_k})^{-1} P_h \int_0^1 [F'(y^{h,\delta}_{n-1,\alpha_k}) - F'(x^{h,\delta}_{n-1,\alpha_k} \\
+ t(y^{h,\delta}_{n-1,\alpha_k} - x^{h,\delta}_{n-1,\alpha_k}))](y^{h,\delta}_{n-1,\alpha_k} - x^{h,\delta}_{n-1,\alpha_k}) dt
\end{aligned}
$$

and $\Gamma_2 := P_h F'(y^{h,\delta}_{n-1,\alpha_k})^{-1} P_h [F'(x^{h,\delta}_{n-1,\alpha_k}) - F'(y^{h,\delta}_{n-1,\alpha_k})](y^{h,\delta}_{n-1,\alpha_k} - x^{h,\delta}_{n-1,\alpha_k})$. Note that by Assumption 3.2 and Lemma 3.3 we have

$$
\begin{aligned}
\|\Gamma_1\| &= \|P_h F'(y^{h,\delta}_{n-1,\alpha_k})^{-1} P_h \int_0^1 \Big[F'(y^{h,\delta}_{n-1,\alpha_k}) - F'(x^{h,\delta}_{n-1,\alpha_k} \\
&\quad + t(y^{h,\delta}_{n-1,\alpha_k} - x^{h,\delta}_{n-1,\alpha_k})) \Big] (y^{h,\delta}_{n-1,\alpha_k} - x^{h,\delta}_{n-1,\alpha_k}) dt\| \\
&\leqslant (1+\beta\tau_0) \| \int_0^1 \Phi(y^{h,\delta}_{n-1,\alpha_k}, x^{h,\delta}_{n-1,\alpha_k} + t(y^{h,\delta}_{n-1,\alpha_k} - x^{h,\delta}_{n-1,\alpha_k}), \\
&\quad \times y^{h,\delta}_{n-1,\alpha_k} - x^{h,\delta}_{n-1,\alpha_k}) dt \| \\
&\leqslant K_0 (1+\beta\tau_0) \Bigg[\int_0^1 \|x^{h,\delta}_{n-1,\alpha_k} - x^{h,\delta}_{0,\alpha_k} - t(y^{h,\delta}_{n-1,\alpha_k} - x^{h,\delta}_{n-1,\alpha_k})\| dt \\
&\quad + \|y^{h,\delta}_{n-1,\alpha_k} - x^{h,\delta}_{0,\alpha_k}\| \Bigg] \|y^{h,\delta}_{n-1,\alpha_k} - x^{h,\delta}_{n-1,\alpha_k}\| \\
&\leqslant K_0 (1+\beta\tau_0) \Bigg[\int_0^1 (1-t)\|x^{h,\delta}_{n-1,\alpha} - x^{h,\delta}_{0,\alpha_k}\| + t\|y^{h,\delta}_{n-1,\alpha_k} - x^{h,\delta}_{0,\alpha_k}\| \\
&\quad + \|y^{h,\delta}_{n-1,\alpha_k} - x^{h,\delta}_{0,\alpha_k}\| \Bigg] dt \|y^{h,\delta}_{n-1,\alpha_k} - x^{h,\delta}_{n-1,\alpha_k}\| \\
&\leqslant \frac{K_0}{2}(1+\beta\tau_0) \Big[\|x^{h,\delta}_{n-1,\alpha_k} - x^{h,\delta}_{0,\alpha_k}\| + 3\|y^{h,\delta}_{n-1,\alpha_k} - x^{h,\delta}_{0,\alpha_k}\| \Big] e^{h,\delta}_{n-1,\alpha_k}.
\end{aligned} \tag{3.16}
$$

Similarly, we obtain

$$\|\Gamma_2\| \leqslant K_0(1+\beta\tau_0)[\|y^{h,\delta}_{n-1,\alpha_k} - x^{h,\delta}_{0,\alpha_k}\| + \|x^{h,\delta}_{0,\alpha_k} - x^{h,\delta}_{n-1,\alpha_k}\|]e^{h,\delta}_{n-1,\alpha_k}. \tag{3.17}$$

Hence from (3.15), (3.16) and (3.17), we get (a). Now (b) follows from (a) and the triangle inequality;

$$\|x^{h,\delta}_{n,\alpha_k} - x^{h,\delta}_{n-1,\alpha_k}\| \leqslant \|x^{h,\delta}_{n,\alpha_k} - y^{h,\delta}_{n-1,\alpha_k}\| + \|y^{h,\delta}_{n-1,\alpha_k} - x^{h,\delta}_{n-1,\alpha_k}\|.$$

To prove (c) we observe that

$$\begin{aligned} e^{h,\delta}_{n,\alpha_k} &= \|x^{h,\delta}_{n,\alpha_k} - y^{h,\delta}_{n-1,\alpha_k} - (P_hF'(x^{h,\delta}_{n,\alpha_k}))^{-1}P_h(F(x^{h,\delta}_{n,\alpha_k}) \\ &\quad - z^{h,\delta}_{\alpha_k}) + P_hF'(y^{h,\delta}_{n-1,\alpha_k})^{-1}P_h(F(y^{h,\delta}_{n-1,\alpha_k}) - z^{h,\delta}_{\alpha_k})\| \\ &= \|x^{h,\delta}_{n,\alpha_k} - y^{h,\delta}_{n-1,\alpha_k} - P_hF'(x^{h,\delta}_{n,\alpha_k})^{-1}P_h(F(x^{h,\delta}_{n,\alpha_k}) \\ &\quad - F(y^{h,\delta}_{n-1,\alpha_k})) + P_h[F'(y^{h,\delta}_{n-1,\alpha_k})^{-1} - F'(x^{h,\delta}_{n,\alpha_k})^{-1}] \\ &\quad \times P_h(F(y^{h,\delta}_{n-1,\alpha_k}) - z^{h,\delta}_{\alpha_k})\| \\ &\leqslant \Lambda_1 + \Lambda_2, \end{aligned} \tag{3.18}$$

where

$$\Lambda_1 := \|x^{h,\delta}_{n,\alpha_k} - y^{h,\delta}_{n-1,\alpha_k} - P_hF'(x^{h,\delta}_{n,\alpha_k})^{-1}P_h(F(x^{h,\delta}_{n,\alpha_k}) - F(y^{h,\delta}_{n-1,\alpha_k}))\|$$

and

$$\Lambda_2 := \|P_h[F'(y^{h,\delta}_{n-1,\alpha_k})^{-1} - F'(x^{h,\delta}_{n,\alpha_k})^{-1}]P_h(F(y^{h,\delta}_{n-1,\alpha_k}) - z^{h,\delta}_{\alpha_k})\|.$$

Analogous to the proof of (3.16) and (3.17), one can see that

$$\Lambda_1 \leqslant \frac{K_0}{2}(1+\beta\tau_0)[3\|x^{h,\delta}_{n,\alpha_k} - x^{h,\delta}_{0,\alpha_k}\| + \|y^{h,\delta}_{n-1,\alpha_k} - x^{h,\delta}_{0,\alpha_k}\|]\|x^{h,\delta}_{n,\alpha_k} - y^{h,\delta}_{n-1,\alpha_k}\|$$

and

$$\Lambda_2 \leqslant K_0(1+\beta\tau_0)[\|x^{h,\delta}_{n,\alpha_k} - x^{h,\delta}_{0,\alpha_k}\| + \|y^{h,\delta}_{n-1,\alpha_k} - x^{h,\delta}_{0,\alpha_k}\|]\|x^{h,\delta}_{n,\alpha_k} - y^{h,\delta}_{n-1,\alpha_k}\|.$$

Hence from (3.18) we obtain that

$$\begin{aligned} e^{h,\delta}_{n,\alpha_k} &\leqslant \frac{K_0}{2}(1+\beta\tau_0)[5\|x^{h,\delta}_{n,\alpha_k} - x^{h,\delta}_{0,\alpha_k}\| \\ &\quad + 3\|y^{h,\delta}_{n-1,\alpha_k} - x^{h,\delta}_{0,\alpha_k}\|]\|x^{h,\delta}_{n,\alpha_k} - y^{h,\delta}_{n-1,\alpha_k}\| \\ &\leqslant \frac{K_0}{2}(1+\beta\tau_0)(8r)\frac{K_0}{2}(1+\beta\tau_0)(8r)\|x^{h,\delta}_{n-1,\alpha_k} - y^{h,\delta}_{n-1,\alpha_k}\| \\ &\leqslant b^2\|x^{h,\delta}_{n-1,\alpha_k} - y^{h,\delta}_{n-1,\alpha_k}\| \\ &\leqslant b^{2n}e^{h,\delta}_{0,\alpha_k} \leqslant b^{2n}\gamma_\rho. \end{aligned} \tag{3.19}$$

This completes the proof of the Theorem. ■

Theorem 3.5. *Let r be as defined in (3.10) and the assumptions of Theorem 3.4 hold. Then $x_{n,\alpha_k}^{h,\delta}, y_{n,\alpha_k}^{h,\delta} \in B_r(P_h x_0)$, for all $n \geqslant 0$.*

Proof. Note that by (b) of Theorem 3.4 we have,

$$\begin{aligned}\|x_{1,\alpha_k}^{h,\delta} - P_h x_0\| &= \|x_{1,\alpha_k}^{h,\delta} - x_{0,\alpha_k}^{h,\delta}\| \\ &\leqslant \Big[1 + (1+\beta\tau_0)\frac{K_0}{2}(8r)\Big]\gamma_\rho \qquad (3.20)\\ &\leqslant (1+b)\gamma_\rho \\ &\leqslant \frac{\gamma_\rho}{1-b} < r,\end{aligned}$$

i.e., $x_{1,\alpha_k}^{h,\delta} \in B_r(P_h x_0)$. Again note that from (3.20) and Theorem 3.4 we get,

$$\begin{aligned}\|y_{1,\alpha_k}^{h,\delta} - P_h x_0\| &\leqslant \|y_{1,\alpha_k}^{h,\delta} - x_{1,\alpha_k}^{h,\delta}\| + \|x_{1,\alpha_k}^{h,\delta} - P_h x_0\| \\ &\leqslant [1 + (1+\beta\tau_0)4K_0 r + ((1+\beta\tau_0)4K_0 r)^2]\gamma_\rho \\ &\leqslant (1+b+b^2)\gamma_\rho \\ &\leqslant \frac{\gamma_\rho}{1-b} < r,\end{aligned}$$

i.e., $y_{1,\alpha_k}^{h,\delta} \in B_r(P_h x_0)$. Further by (3.20) and (b) of Theorem 3.4 we have,

$$\begin{aligned}\|x_{2,\alpha_k}^{h,\delta} - P_h x_0\| &\leqslant \|x_{2,\alpha_k}^{h,\delta} - x_{1,\alpha_k}^{h,\delta}\| + \|x_{1,\alpha_k}^{h,\delta} - P_h x_0\| \\ &\leqslant (1+b)e_{1,\alpha_k}^{h,\delta} + (1+b)\gamma_\rho \\ &\leqslant (1+b+b^2+b^3)\gamma_\rho \\ &\leqslant \frac{1}{1-b}\gamma_\rho < r\end{aligned}$$

and

$$\begin{aligned}\|y_{2,\alpha_k}^{h,\delta} - P_h x_0\| &\leqslant \|y_{2,\alpha_k}^{h,\delta} - x_{2,\alpha_k}^{h,\delta}\| + \|x_{2,\alpha_k}^{h,\delta} - P_h x_0\| \\ &\leqslant b^4\gamma_\rho + (1+b+b^2+b^3)\gamma_\rho \\ &\leqslant (1+b+b^2+b^3+b^4)\gamma_\rho \\ &\leqslant \frac{1}{1-b}\gamma_\rho < r\end{aligned}$$

by the choice of r, i.e., $x_{2,\alpha_k}^{h,\delta}, y_{2,\alpha_k}^{h,\delta} \in B_r(P_h x_0)$. Continuing this way one can prove that $x_{n,\alpha_k}^{h,\delta}, y_{n,\alpha_k}^{h,\delta} \in B_r(P_h x_0), \forall n \geqslant 0$. This completes the proof. ■

Theorem 3.6. *Let $y_{n,\alpha_k}^{h,\delta}$ and $x_{n,\alpha_k}^{h,\delta}$ be as in (3.2) and (3.3) respectively and hypotheses of Theorem 3.5 hold. Then $(x_{n,\alpha_k}^{h,\delta})$ is a Cauchy sequence in $B_r(P_h x_0)$ and converges to $x_{\alpha_k}^{h,\delta} \in \overline{B_r(P_h x_0)}$. Further $P_h F(x_{\alpha_k}^{h,\delta}) = z_{\alpha_k}^{h,\delta}$ and*

$$\|x_{n,\alpha_k}^{h,\delta} - x_{\alpha_k}^{h,\delta}\| \leqslant \frac{(1+b)b^{2n}\gamma_\rho}{1-b^2}$$

where γ_ρ and b are defined by (3.9) and (3.11), respectively.

Proof. Using the relation (b) and (c) of Theorem 3.4, we obtain

$$\begin{aligned}
\|x^{h,\delta}_{n+i+1,\alpha_k} - x^{h,\delta}_{n+i,\alpha_k}\| &\leqslant (1+b)b^0\|x^{h,\delta}_{n+i,\alpha_k} - y^{h,\delta}_{n+i,\alpha_k}\| \\
&\leqslant (1+b)b\|x^{h,\delta}_{n+i,\alpha_k} - y^{h,\delta}_{n+i-1,\alpha_k}\| \\
&\leqslant (1+b)b^2\|x^{h,\delta}_{n+i-1,\alpha_k} - y^{h,\delta}_{n+i-1,\alpha_k}\| \\
&\leqslant (1+b)b^{2(n+i)}e^{h,\delta}_{0,\alpha_k} \\
&\leqslant (1+b)b^{2(n+i)}\gamma_\rho.
\end{aligned}$$

So,

$$\begin{aligned}
\|x^{h,\delta}_{n+m,\alpha_k} - x^{h,\delta}_{n,\alpha_k}\| &\leqslant \sum_{i=0}^{m-1} \|x^{h,\delta}_{n+i+1,\alpha_k} - x^{h,\delta}_{n+i,\alpha_k}\| \\
&\leqslant (1+b)b^{2n}\sum_{i=0}^{m-1} b^{2i} \\
&= (1+b)b^{2n}\frac{1-b^{2m}}{1-b^2}\gamma_\rho \to \frac{(1+b)b^{2n}}{1-b^2}\gamma_\rho,
\end{aligned}$$

as $m \to \infty$. Observe that from (3.2)

$$\begin{aligned}
\|P_h(F(x^{h,\delta}_{n,\alpha_k}) - z^{h,\delta}_{\alpha_k})\| &= \|P_hF'(x^{h,\delta}_{n,\alpha_k})(x^{h,\delta}_{n,\alpha_k} - y^{h,\delta}_{n,\alpha_k})\| \\
&\leqslant \|F'(x^{h,\delta}_{n,\alpha_k})\|\|y^{h,\delta}_{n,\alpha_k} - x^{h,\delta}_{n,\alpha_k}\| \\
&\leqslant Me^{h,\delta}_{n,\alpha_k}.
\end{aligned} \tag{3.21}$$

Now by letting $n \to \infty$ in (3.21) we obtain $P_hF(x^{h,\delta}_{\alpha_k}) = z^{h,\delta}_{\alpha_k}$. This completes the proof.

Hereafter we assume that

$$\rho \leqslant r < \frac{1}{(1+\beta\tau_0)K_0}.$$

Theorem 3.7. *Suppose* $(1+\beta\tau_0)K_0r < 1$ *and Assumption 2.1 and 3.2 hold. Then*

$$\|\hat{x} - x^{h,\delta}_{\alpha_k}\| \leqslant \frac{\beta}{(1-(1+\beta\tau_0)K_0r)}\|F(\hat{x}) - z^{h,\delta}_{\alpha_k}\|.$$

Proof. Observe that

$$\begin{aligned}
\|\hat{x} - x_{\alpha_k}^{h,\delta}\| &= \|\hat{x} - x_{\alpha_k}^{h,\delta} + P_h F'(P_h x_0)^{-1} P_h [F(x_{\alpha_k}^{h,\delta}) - F(\hat{x}) + F(\hat{x}) - z_{\alpha_k}^{h,\delta}]\| \\
&\leqslant \|P_h F'(P_h x_0)^{-1} [P_h F'(P_h x_0)(\hat{x} - x_{\alpha_k}^{h,\delta}) - P_h(F(\hat{x}) \\
&\quad - F(x_{\alpha_k}^{h,\delta}))]\| + \|P_h F'(P_h x_0)^{-1} P_h (F(\hat{x}) - z_{\alpha_k}^{h,\delta})\| \\
&\leqslant \|P_h F'(P_h x_0)^{-1} P_h \int_0^1 [F'(P_h x_0) - F'(\hat{x} + t(x_{\alpha_k}^{h,\delta} - \hat{x}))] \\
&\quad \times (\hat{x} - x_{\alpha_k}^{h,\delta}) dt\| + \|P_h F'(P_h x_0)^{-1} P_h (F(\hat{x}) - z_{\alpha_k}^{h,\delta})\| \\
&\leqslant \|P_h F'(P_h x_0)^{-1} P_h F'(P_h x_0)\| \int_0^1 \Phi(P_h x_0, \hat{x} + t(x_{\alpha_k}^{h,\delta} - \hat{x}), \\
&\quad \hat{x} - x_{\alpha_k}^{h,\delta}) dt\| + \|P_h F'(P_h x_0)^{-1} P_h (F(\hat{x}) - z_{\alpha_k}^{h,\delta})\| \\
&\leqslant (1 + \beta\tau_0) K_0 r \|\hat{x} - x_{\alpha_k}^{h,\delta}\| + \beta \|F(\hat{x}) - z_{\alpha_k}^{h,\delta}\|.
\end{aligned}$$

The last step follows from Assumption 3.2, Lemma 3.3, (3.5) and the relation $\|P_h x_0 - \hat{x} - t(x_{\alpha_k}^{h,\delta} - \hat{x})\| \leqslant r$. This completes the proof. ■

The following Theorem is a consequence of Theorem 3.6 and Theorem 3.7.

Theorem 3.8. *Let $x_{n,\alpha_k}^{h,\delta}$ be as in (3.3), assumptions in Theorem 3.6 and Theorem 3.7 hold. Then*

$$\|\hat{x} - x_{n,\alpha_k}^{h,\delta}\| \leqslant \frac{(1+b)b^{2n}\gamma_\rho}{1-b^2} + \frac{\beta}{(1 - (1 + \beta\tau_0)K_0 r)} \|F(\hat{x}) - z_{\alpha_k}^{h,\delta}\|$$

where γ_ρ is as in Theorem 3.6.

Now since $l \leqslant k$ and $\alpha_\delta \leqslant \alpha_{l+1} \leqslant \mu\alpha_l$ we have

$$\frac{\delta + \varepsilon_h}{\sqrt{\alpha_k}} \leqslant \frac{\delta + \varepsilon_h}{\sqrt{\alpha_l}} \leqslant \mu \frac{\delta + \varepsilon_h}{\sqrt{\alpha_\delta}} = \mu\varphi(\alpha(\delta, h)) = \mu\psi^{-1}(\delta + \varepsilon_h).$$

This leads to the following theorem,

Theorem 3.9. *Let $x_{n,\alpha_k}^{h,\delta}$ be as in (3.3), assumptions in Theorem 3.8 hold. Let*

$$n_k := \min\Big\{n : b^{2n} \leqslant \frac{\delta + \varepsilon_h}{\sqrt{\alpha_k}}\Big\}.$$

Then

$$\|\hat{x} - x_{n_k,\alpha_k}^{h,\delta}\| = O(\psi^{-1}(\delta + \varepsilon_h)).$$

3.2. Case 2: TSNTPM when F is monotone and $F'(.)$ is non-invertible

Assumption 3.10. There exists a continuous, strictly monotonically increasing function $\varphi_1 : (0, b] \to (0, \infty)$ with $b \geqslant \|F'(x_0)\|$ satisfying;

- $\lim_{\lambda \to 0} \varphi_1(\lambda) = 0$,

- $$\sup_{\lambda\geqslant 0}\frac{\alpha\varphi_1(\lambda)}{\lambda+\alpha}\leqslant\varphi_1(\alpha)\qquad\forall\lambda\in(0,b]$$

  and
- there exists $v\in X$ with $\|v\|\leqslant 1$ (cf.[13]) such that

$$x_0-\hat{x}=\varphi_1(F'(x_0))v.$$

- for each $x\in B_r(x_0):=\{x:\|x-x_0\|<r\}$ there exists a bounded linear operator $G(x,x_0)$ (cf.[15]) such that

$$F'(x)=F'(x_0)G(x,x_0)$$

  with $\|G(x,x_0)\|\leqslant K_1$.

First we consider an iterative method to approximate the zero $x^{h,\delta}_{c,\alpha_k}$ of

$$P_h\Big(F(x)+\frac{\alpha_k}{c}(x-x_0)\Big)=P_h z^{h,\delta}_{\alpha_k}. \tag{3.22}$$

and then we show that $x^{h,\delta}_{c,\alpha_k}$ is an approximation to the solution $\hat{x}$ of (1.1) where $c\leqslant\alpha_k$. For an initial guess $x_0\in X$ and for $R(x):=P_hF'(x)P_h+\frac{\alpha_k}{c}P_h$, the iterative method is defined as:

$$\tilde{y}^{h,\delta}_{n,\alpha_k}=\tilde{x}^{h,\delta}_{n,\alpha_k}-R(\tilde{x}^{h,\delta}_{n,\alpha_k})^{-1}P_h\Big[F(\tilde{x}^{h,\delta}_{n,\alpha_k})-z^{h,\delta}_{\alpha_k}+\frac{\alpha_k}{c}(\tilde{x}^{h,\delta}_{n,\alpha_k}-\tilde{x}^{h,\delta}_{0,\alpha_k})\Big] \tag{3.23}$$

and

$$\tilde{x}^{h,\delta}_{n+1,\alpha_k}=\tilde{y}^{h,\delta}_{n,\alpha_k}-R(\tilde{y}^{h,\delta}_{n,\alpha_k})^{-1}P_h\Big[F(\tilde{y}^{h,\delta}_{n,\alpha_k})-z^{h,\delta}_{\alpha_k}+\frac{\alpha_k}{c}(\tilde{y}^{h,\delta}_{n,\alpha_k}-\tilde{x}^{h,\delta}_{0,\alpha_k})\Big] \tag{3.24}$$

where $\tilde{x}^{h,\delta}_{0,\alpha_k}:=P_hx_0$. Note that with the above notation

(3.25)

$$\|R(\tilde{x}^{h,\delta}_{n,\alpha_k})^{-1}P_hF'(\tilde{x}^{h,\delta}_{n,\alpha_k})\|\leqslant 1+\tau_0. \tag{3.26}$$

Let

$$\tilde{e}^{h,\delta}_{n,\alpha_k}:=\|\tilde{y}^{h,\delta}_{n,\alpha_k}-\tilde{x}^{h,\delta}_{n,\alpha_k}\|,\qquad\forall n\geqslant 0 \tag{3.27}$$

and suppose that

$$0<K_0<\frac{1}{4(1+\tau_0)} \tag{3.28}$$

and

$$\frac{\delta_0+\varepsilon_0}{\sqrt{\alpha_0}}<\frac{2}{2M+3}. \tag{3.29}$$

Let $\|\hat{x}-x_0\|\leqslant\rho$, with

$$\rho<\frac{1}{M}\left(1-\left(\frac{3}{2}+M\right)\frac{\delta_0+\varepsilon_0}{\sqrt{\alpha_0}}\right)$$

and

$$\tilde{\gamma}_\rho := M\rho + \left(\frac{3}{2} + M\right)\left(\frac{\varepsilon_0 + \delta_0}{\sqrt{\alpha_0}}\right), \tag{3.30}$$

and let r be such that

$$\tilde{r} \in (\tilde{r}_1, \tilde{r}_2) \tag{3.31}$$

where

$$\tilde{r}_1 = \frac{1 + \sqrt{1 - 16(1+\tau_0)K_0\tilde{\gamma}_\rho}}{8(1+\tau_0)}$$

and

$$\begin{aligned}\tilde{r}_2 &= \frac{1 - \sqrt{1 - 16(1+\tau_0)K_0\tilde{\gamma}_\rho}}{8(1+\tau_0)}\\ \tilde{b} &:= 4(1+\tau_0)K_0\tilde{r}.\end{aligned} \tag{3.32}$$

Then we have by (3.28)-(3.32) that

$$0 < \tilde{\gamma}_\rho < \frac{1}{16(1+\tau_0)K_0}. \tag{3.33}$$

Theorem 3.11. *Let $\tilde{e}^{h,\delta}_{n,\alpha_k}$ be as in equation (3.27) with $\delta \in (0, \delta_0]$, $\alpha = \alpha_k$ and $\varepsilon_h \in (0, \varepsilon_0]$. Then by Assumption 3.2 the following hold:*

(a)
$$\begin{aligned}\|\tilde{x}^{h,\delta}_{n,\alpha_k} - \tilde{y}^{h,\delta}_{n-1,\alpha_k}\| \leqslant \frac{K_0}{2}(1+\tau_0)\Big[&3\|\tilde{x}^{h,\delta}_{n-1,\alpha_k} - \tilde{x}^{h,\delta}_{0,\alpha_k}\|\\ &+ 5\|\tilde{y}^{h,\delta}_{n-1,\alpha_k} - \tilde{x}^{h,\delta}_{0,\alpha_k}\|\Big] e^{h,\delta}_{n-1,\alpha_k}\end{aligned} \tag{3.34}$$

(b)
$$\begin{aligned}\|\tilde{x}^{h,\delta}_{n,\alpha_k} - \tilde{x}^{h,\delta}_{n-1,\alpha_k}\| \leqslant \Bigg\{1 + \frac{K_0}{2}(1+\tau_0)\Big[&3\|\tilde{x}^{h,\delta}_{n-1,\alpha_k} - \tilde{x}^{h,\delta}_{0,\alpha_k}\|\\ &+ 5\|\tilde{y}^{h,\delta}_{n-1,\alpha_k} - \tilde{x}^{h,\delta}_{0,\alpha_k}\|\Big]\Bigg\} e^{h,\delta}_{n-1,\alpha_k}.\end{aligned} \tag{3.35}$$

(c)
$$\begin{aligned}\tilde{e}^{h,\delta}_{n,\alpha_k} &\leqslant \frac{K_0}{2}(1+\tau_0)[5\|\tilde{x}^{h,\delta}_{n,\alpha_k} - \tilde{x}^{h,\delta}_{0,\alpha_k}\|\\ &\quad + 3\|\tilde{y}^{h,\delta}_{n-1,\alpha_k} - \tilde{x}^{h,\delta}_{0,\alpha_k}\|]\|\tilde{y}^{h,\delta}_{n-1,\alpha_k} - \tilde{x}^{h,\delta}_{n,\alpha_k}\|\\ &\leqslant \tilde{b}^2\tilde{e}^{h,\delta}_{n-1,\alpha_k} \leqslant \tilde{b}^{2n}\tilde{e}^{h,\delta}_{0,\alpha_k} \leqslant \tilde{b}^{2n}\gamma_\rho.\end{aligned}$$

Proof. Observe that

$$\begin{aligned}\tilde{x}^{h,\delta}_{n,\alpha_k} - \tilde{y}^{h,\delta}_{n-1,\alpha_k} &= \tilde{y}^{h,\delta}_{n-1,\alpha_k} - \tilde{x}^{h,\delta}_{n-1,\alpha_k} - R(\tilde{y}^{h,\delta}_{n-1,\alpha_k})^{-1}P_h(F(\tilde{y}^{h,\delta}_{n-1,\alpha_k}) - z^{h,\delta}_{\alpha_k} \\ &\quad + \frac{\alpha_k}{c}(\tilde{y}^{h,\delta}_{n-1,\alpha_k} - x_0)) + R(\tilde{x}^{h,\delta}_{n-1,\alpha_k})^{-1}P_h(F(\tilde{x}^{h,\delta}_{n-1,\alpha_k}) \\ &\quad - z^{h,\delta}_{\alpha_k} + \frac{\alpha_k}{c}(\tilde{x}^{h,\delta}_{n-1,\alpha_k} - x_0)) \\ &= \tilde{y}^{h,\delta}_{n-1,\alpha_k} - \tilde{x}^{h,\delta}_{n-1,\alpha_k} - R(\tilde{y}^{h,\delta}_{n-1,\alpha_k})^{-1}P_h[F(\tilde{y}^{h,\delta}_{n-1,\alpha_k}) \\ &\quad - F(\tilde{x}^{h,\delta}_{n-1,\alpha_k}) + \frac{\alpha_k}{c}(\tilde{y}^{h,\delta}_{n-1,\alpha_k} - \tilde{x}^{h,\delta}_{n-1,\alpha_k})] + [R(\tilde{x}^{h,\delta}_{n-1,\alpha_k})^{-1} \\ &\quad - R(\tilde{y}^{h,\delta}_{n-1,\alpha_k})^{-1}](F(\tilde{x}^{h,\delta}_{n-1,\alpha_k}) - z^{h,\delta}_{\alpha_k} + \frac{\alpha_k}{c}(\tilde{x}^{h,\delta}_{n-1,\alpha_k} - x_0)) \\ &= R(\tilde{y}^{h,\delta}_{n-1,\alpha_k})^{-1}P_h[F'(\tilde{y}^{h,\delta}_{n-1,\alpha_k})(\tilde{y}^{h,\delta}_{n-1,\alpha_k} - \tilde{x}^{h,\delta}_{n-1,\alpha_k}) \\ &\quad - (F(\tilde{y}^{h,\delta}_{n-1,\alpha_k}) - F(\tilde{x}^{h,\delta}_{n-1,\alpha_k})] - R(\tilde{y}^{h,\delta}_{n-1,\alpha_k})^{-1} \\ &\quad \times [F'(\tilde{y}^{h,\delta}_{n-1,\alpha_k}) - F'(\tilde{x}^{h,\delta}_{n-1,\alpha_k})](\tilde{y}^{h,\delta}_{n-1,\alpha_k} - \tilde{x}^{h,\delta}_{n-1,\alpha_k}) \\ &= R(\tilde{y}^{h,\delta}_{n-1,\alpha_k})^{-1}P_h\int_0^1 [F'(\tilde{y}^{h,\delta}_{n-1,\alpha_k}) - F'(\tilde{x}^{h,\delta}_{n-1,\alpha_k} + t(\tilde{y}^{h,\delta}_{n-1,\alpha_k} \\ &\quad - \tilde{x}^{h,\delta}_{n-1,\alpha_k}))]P_h(\tilde{y}^{h,\delta}_{n-1,\alpha_k} - \tilde{x}^{h,\delta}_{n-1,\alpha_k})dt - R(\tilde{y}^{h,\delta}_{n-1,\alpha_k})^{-1} \\ &\quad \times [F'(\tilde{y}^{h,\delta}_{n-1,\alpha_k}) - F'(\tilde{x}^{h,\delta}_{n-1,\alpha_k})](\tilde{y}^{h,\delta}_{n-1,\alpha_k} - \tilde{x}^{h,\delta}_{n-1,\alpha_k}).\end{aligned}$$

Now by Assumption 3.2 and (3.26) we have

$$\begin{aligned}\|\tilde{x}^{h,\delta}_{n,\alpha_k} - \tilde{y}^{h,\delta}_{n-1,\alpha_k}\| \leqslant (1+\tau_0)[\| \int_0^1 & \Phi(\tilde{y}^{h,\delta}_{n-1,\alpha_k}, \tilde{x}^{h,\delta}_{n-1,\alpha_k} + t(\tilde{y}^{h,\delta}_{n-1,\alpha_k} \\ & - \tilde{x}^{h,\delta}_{n-1,\alpha_k}), \tilde{y}^{h,\delta}_{n-1,\alpha_k} - \tilde{x}^{h,\delta}_{n-1,\alpha_k})dt\| \\ & + \|\Phi(\tilde{y}^{h,\delta}_{n-1,\alpha_k}, \tilde{x}^{h,\delta}_{n-1,\alpha_k}, \tilde{y}^{h,\delta}_{n-1,\alpha_k} - \tilde{x}^{h,\delta}_{n-1,\alpha_k})\|].\end{aligned}$$

The remaining part of the proof is analogous to the proof of Theorem 3.4. ■

Theorem 3.12. *Let $\tilde{r}$ be as defined in (3.31) and the assumptions of Theorem 3.11 hold. Then $\tilde{x}^{h,\delta}_{n,\alpha_k}, \tilde{y}^{h,\delta}_{n,\alpha_k} \in B_{\tilde{r}}(P_h x_0)$, for all $n \geqslant 0$.*

Proof. The proof is analogous to the proof of Theorem 3.5. ■

Theorem 3.13. *Let $\tilde{y}^{h,\delta}_{n,\alpha_k}$ and $\tilde{x}^{h,\delta}_{n,\alpha_k}$ be as in (3.23) and (3.24) respectively and hypotheses of Theorem 3.12 hold. Then $(\tilde{x}^{h,\delta}_{n,\alpha_k})$ is a Cauchy sequence in $B_{\tilde{r}}(P_h x_0)$ and converges to $x^{h,\delta}_{c,\alpha_k} \in \overline{B_{\tilde{r}}(P_h x_0)}$. Further $P_h[F(x^{h,\delta}_{c,\alpha_k}) + \frac{\alpha_k}{c}(x^{h,\delta}_{c,\alpha_k} - x_0)] = P_h z^{h,\delta}_{\alpha_k}$ and*

$$\|\tilde{x}^{h,\delta}_{n,\alpha_k} - x^{h,\delta}_{c,\alpha_k}\| \leqslant \frac{(1+\tilde{b})\tilde{b}^{2n}\tilde{\gamma}_\rho}{1-\tilde{b}^2}$$

where $\tilde{\gamma}_\rho$ and $\tilde{b}$ are defined by (3.30) and (3.32), respectively.

Proof. Using the relation (b) and (c) of Theorem 3.11, one can show that $(\tilde{x}^{h,\delta}_{n,\alpha_k})$ is a Cauchy sequence in $B_{\tilde{r}}(P_h x_0)$. Observe that from (3.23)

$$\begin{aligned}\|P_h(F(\tilde{x}^{h,\delta}_{n,\alpha_k}) - z^{h,\delta}_{\alpha_k}) + \frac{\alpha_k}{c}(\tilde{x}^{h,\delta}_{n,\alpha_k} - P_h x_0)\| &= \|R(\tilde{x}^{h,\delta}_{n,\alpha_k})(\tilde{x}^{h,\delta}_{n,\alpha_k} - \tilde{y}^{h,\delta}_{n,\alpha_k})\| \\ &\leqslant \|R(\tilde{x}^{h,\delta}_{n,\alpha_k})\|\|\tilde{y}^{h,\delta}_{n,\alpha_k} - \tilde{x}^{h,\delta}_{n,\alpha_k}\| \\ &\leqslant (\|P_h F'(\tilde{x}^{h,\delta}_{n,\alpha_k})P_h\| + \frac{\alpha_k}{c})\tilde{e}^{h,\delta}_{n,\alpha_k} \\ &\leqslant (M + \frac{\alpha_k}{c})\tilde{e}^{h,\delta}_{n,\alpha_k}. \end{aligned} \tag{3.36}$$

Now by letting $n \to \infty$ in (3.36) we obtain $P_h F(x^{h,\delta}_{c,\alpha_k}) + \frac{\alpha_k}{c}(x^{h,\delta}_{c,\alpha_k} - P_h x_0) = P_h z^{h,\delta}_{\alpha_k}$. This completes the proof. ∎

Remark 3.14.

(a) The convergence order of (TSNTM) would be four under Assumption 3.1. In Theorem 3.6 and 3.13 the error bounds are too pessimistic. That is why in practice we shall use the computational order of convergence (COC) (see eg. [6]) defined by

$$\varrho \approx \ln\left(\frac{\|x_{n+1} - x^\delta_\alpha\|}{\|x_n - x^\delta_\alpha\|}\right) / \ln\left(\frac{\|x_n - x^\delta_\alpha\|}{\|x_{n-1} - x^\delta_\alpha\|}\right).$$

The (COC) ϱ will then be close to 4 which is the order of convergence of (TSNTM).

Hereafter we assume that $\tilde{r} < \frac{1}{K_0}$ and $K_1 < \frac{1-K_0\tilde{r}}{1-c}$.

We quote the following Theorems for our further analysis, whose proofs are given in [11].

Theorem 3.15 (see [11, Theorem 3.7]). *Suppose x^δ_{c,α_k} is the solution of*

$$F(x) + \frac{\alpha_k}{c}(x - x_0) = z^\delta_{\alpha_k} \tag{3.37}$$

and Assumption 3.2 and 3.10 hold. Then

$$\|\hat{x} - x^\delta_{c,\alpha_k}\| \leqslant \frac{\varphi_1(\alpha_k) + (2 + \frac{4\mu}{\mu-1})\mu\psi^{-1}(\delta + \varepsilon_h)}{1 - (1-c)K_1 - K_0\tilde{r}}.$$

Theorem 3.16 (see [11, Theorem 3.8]). *Suppose $x^{h,\delta}_{c,\alpha_k}$ is the solution of (3.22) and Assumption 2.1 and Theorem 3.15 hold. In addition if $\tau_0 < 1$, then*

$$\|x^{h,\delta}_{c,\alpha_k} - x^\delta_{c,\alpha_k}\| \leqslant \frac{2}{1-\tau_0}\left(\frac{\delta + \varepsilon_h}{\sqrt{\alpha_k}}\right).$$

Theorem 3.17. *Let $\tilde{x}^{h,\delta}_{n,\alpha_k}$ be as in (3.24), assumptions in Theorem 3.13, Theorem 3.15 and Theorem 3.16 hold. Then*

$$\|\hat{x} - \tilde{x}^{h,\delta}_{n,\alpha_k}\| \leqslant \frac{(1+\tilde{b})\tilde{b}^{2n}\tilde{\gamma}_\rho}{1-\tilde{b}^2} + \frac{\varphi_1(\alpha_k) + (2 + \frac{4\mu}{\mu-1})\mu\psi^{-1}(\delta + \varepsilon_h)}{1 - (1-c)K_1 - K_0\tilde{r}} + \frac{2}{1-\tau_0}\left(\frac{\delta + \varepsilon_h}{\sqrt{\alpha_k}}\right).$$

Theorem 3.18. *Let $\tilde{x}^{h,\delta}_{n,\alpha_k}$ be as in (3.24) and assumptions in Theorem 3.17 hold. Further let $\varphi_1(\alpha_k) \leqslant \varphi(\alpha_k)$ and*

$$n_k := \min\left\{n : \tilde{b}^{2n} \leqslant \frac{\delta + \varepsilon_h}{\sqrt{\alpha_k}}\right\}.$$

Then

$$\|\hat{x} - \tilde{x}^{h,\delta}_{n,\alpha_k}\| = O(\psi^{-1}(\delta + \varepsilon_h)).$$

4. Algorithm

Note that for $i, j \in \{0, 1, 2, \cdots, N\}$,

$$z^{h,\delta}_{\alpha_i} - z^{h,\delta}_{\alpha_j} = (\alpha_j - \alpha_i)(P_hK^*KP_h + \alpha_jI)^{-1}(P_hK^*KP_h + \alpha_iI)^{-1}P_hK^*(f^\delta - KF(x_0)).$$

Therefore the balancing principle algorithm associated with the choice of the parameter specified in section 2 involves the following steps.

Step 1: Choose α_0 such that $\delta_0 + \varepsilon_0 < \frac{2\sqrt{\alpha_0}}{\beta(2M+3)}$, $\mu > \{1, \frac{\beta(2M+3)}{2}\}$ for Case 1 and $\delta_0 + \varepsilon_0 < \frac{2\sqrt{\alpha_0}}{2M+3}$ and $\mu > 1$ for Case 2;

Step 2: $\alpha_i = \mu^{2i}\alpha_0$;

Step 3: solve for w_i:

$$(P_hK^*KP_h + \alpha_iI)w_i = P_hK^*(f^\delta - KF(x_0)); \tag{4.1}$$

Step 4: solve for $j < i$, $z^{h,\delta}_{ij}$: $(P_hK^*KP_h + \alpha_jI)z^{h,\delta}_{ij} = (\alpha_j - \alpha_i)w_i$;

Step 5: if $\|z^{h,\delta}_{ij}\| > \frac{4C(\delta+\varepsilon_h)}{\sqrt{\alpha_j}}$, then take $k = i - 1$;

Step 6: otherwise, repeat with $i + 1$ in place of i.

Step 7: choose $n_k := \min\{n : b^{2n} \leqslant \frac{\delta+\varepsilon_h}{\sqrt{\alpha_k}}\}$ in Section 3.1 and $n_k := \min\{n : \tilde{b}^{2n} \leqslant \frac{\delta+\varepsilon_h}{\sqrt{\alpha_k}}\}$ in Section 3.2

Step 8: solve $x^{h,\delta}_{n_k,\alpha_k}$ using the iteration (3.3) or $\tilde{x}^{h,\delta}_{n_k,\alpha_k}$ using the iteration (3.24).

5. Implementation of the method

Let V_n be a sequence of finite dimensional subspaces of X and let $P_h = P_{\frac{1}{n}}$ denote the orthogonal projection on X with range $R(P_h) = V_n$. We assume that $\dim V_n = n + 1$ and $\|P_hx - x\| \to 0$ *as* $h \to 0$ for all $x \in X$. We choose the linear splines $\{v_1, v_2, \cdots, v_{n+1}\}$ in a uniform grid of $n+1$ points in $[0, 1]$ as a basis of V_n.

Since $w_i \in V_n$, w_i is of the form $\sum_{i=1}^{n+1} \lambda_i v_i$ for some scalars $\lambda_1, \lambda_2, \cdots, \lambda_{n+1}$. It can be seen that w_i is a solution of (4.1) if and only if $\bar{\lambda} = (\lambda_1, \lambda_2, \cdots, \lambda_{n+1})^T$ is the unique solution of

$$(M_n + \alpha_iB_n)\bar{\lambda} = \bar{a}$$

where

$$M_n = (\langle Kv_i, Kv_j\rangle), \qquad i,j = 1,2,\cdots,n+1$$
$$B_n = (\langle v_i, v_j\rangle), \qquad i,j = 1,2,\cdots,n+1$$

and

$$\bar{a} = (\langle P_h K^*(f^\delta - KF(x_0)), v_i\rangle)^T, \qquad i = 1,2,\cdots,n+1.$$

Observe that $z_{ij}^{h,\delta}$ in step 4 of algorithm is again in V_n and hence $z_{ij}^{h,\delta} = \sum_{k=1}^{n+1}\mu_k^{ij}v_k$ for some $\mu_k^{ij}, k = 1,2,\cdots,n+1$. One can see that for $j < i$, $z_{ij}^{h,\delta}$ is a solution of

$$(P_h K^* K P_h + \alpha_j I)z_{ij}^{h,\delta} = (\alpha_j - \alpha_i)w_i$$

if and only if $\overline{\mu^{ij}} = (\mu_1^{ij}, \mu_2^{ij}, \cdots, \mu_{n+1}^{ij})^T$ is the unique solution of

$$(M_n + \alpha_j B_n)\overline{\mu^{ij}} = \bar{b}$$

where

$$\bar{b} = (\langle(\alpha_j - \alpha_i)w_i, v_i\rangle)^T.$$

Compute $z_{ij}^{h,\delta}$ till $\|z_{ij}^{h,\delta}\| > \frac{4C(\delta+\varepsilon_h)}{\sqrt{\alpha_j}}$ and fix $k = i-1$. Now choose $n_k = \min\{n : b^{2n} \leqslant \frac{\delta+\varepsilon_h}{\sqrt{\alpha_k}}\}$ for Case 1 and $n_k = \min\{n : \tilde{b}^{2n} \leqslant \frac{\delta+\varepsilon_h}{\sqrt{\alpha_k}}\}$ for Case 2.

Case 1: Since $y_{n_k,\alpha_k}^{h,\delta}$, $x_{n_k,\alpha_k}^{h,\delta} \in V_n$, let $y_{n_k,\alpha_k}^{h,\delta} = \sum_{i=1}^{n+1}\xi_i^n v_i$ and $x_{n_k,\alpha_k}^{h,\delta} = \sum_{i=1}^{n+1}\eta_i^n v_i$, where ξ_i^n and η_i^n are some scalars. Then from (3.2) we have

$$P_h F'(x_{n_k,\alpha_k}^{h,\delta})(y_{n_k,\alpha_k}^{h,\delta} - x_{n_k,\alpha_k}^{h,\delta}) = P_h[z_{\alpha_k}^{h,\delta} - F(x_{n_k,\alpha_k}^{h,\delta})]. \tag{5.1}$$

Observe that $(y_{n_k,\alpha_k}^{h,\delta} - x_{n_k,\alpha_k}^{h,\delta})$ is a solution of (5.1) if and only if $(\overline{\xi^n - \eta^n}) = (\xi_1^n - \eta_1^n, \xi_2^n - \eta_2^n, \cdots, \xi_{n+1}^n - \eta_{n+1}^n)^T$ is the unique solution of

$$Q_n(\overline{\xi^n - \eta^n}) = B_n[\overline{\lambda^n} - F_{h1}]$$

where

$$Q_n = \langle F'(x_{n_k,\alpha_k}^{h,\delta})v_i, v_j\rangle, \qquad i,j = 1,2,\cdots,n+1,$$
$$F_{h1} = [F(x_{n_k,\alpha_k}^{h,\delta})(t_1), F(x_{n_k,\alpha_k}^{h,\delta})(t_2), \cdots, F(x_{n_k,\alpha_k}^{h,\delta})(t_{n+1})]^T,$$

where $t_1, t_2, \cdots, t_{n+1}$ are the grid points.

Further from (3.3) it follows that

$$P_h F'(y_{n_k,\alpha_k}^{h,\delta})(x_{n_k+1,\alpha_k}^{h,\delta} - y_{n_k,\alpha_k}^{h,\delta}) = P_h[z_{\alpha_k}^{h,\delta} - F(y_{n_k,\alpha_k}^{h,\delta})]. \tag{5.2}$$

Thus $(x_{n_k+1,\alpha_k}^{h,\delta} - y_{n_k,\alpha_k}^{h,\delta})$ is a solution of (5.2) if and only if $(\overline{\eta^{n+1} - \xi^n}) = (\eta_1^{n+1} - \xi_1^n, \eta_2^{n+1} - \xi_2^n, \cdots, \eta_{n+1}^{n+1} - \xi_{n+1}^n)^T$ is the unique solution of

$$\tilde{Q}_n(\overline{\eta^{n+1} - \xi^n}) = B_n[\overline{\lambda^n} - F_{h2}]$$

where

$$\tilde{Q}_n = \langle F'(y^{h,\delta}_{n_k,\alpha_k})v_i, v_j\rangle, \qquad i,j = 1,2,\cdots,n+1,$$
$$F_{h2} = [F(y^{h,\delta}_{n_k,\alpha_k})(t_1), F(y^{h,\delta}_{n_k,\alpha_k})(t_2), \cdots, F(y^{h,\delta}_{n_k,\alpha_k})(t_{n+1})]^T.$$

Case 2: Let $\tilde{\xi}^n = (\tilde{\xi}^n_1, \tilde{\xi}^n_2, \cdots, \tilde{\xi}^n_{n+1})$, $\tilde{\eta}^n = (\tilde{\eta}^n_1, \tilde{\eta}^n_2, \cdots, \tilde{\eta}^n_{n+1})$, $\tilde{y}^{h,\delta}_{n,\alpha_k} = \sum_{i=1}^{n+1} \tilde{\xi}^n_i v_i$ and $\tilde{x}^{h,\delta}_{n,\alpha_k} = \sum_{i=1}^{n+1} \tilde{\eta}^n_i v_i$. Then from (3.23) we have

$$(P_h F'(\tilde{x}^{h,\delta}_{n,\alpha_k}) + \frac{\alpha_k}{c}) \sum_{i=1}^{n+1} (\tilde{\xi}^n_i - \tilde{\eta}^n_i) v_i = \sum_{i=1}^{n+1} \lambda_i v_i - \sum_{i=1}^{n+1} P_h F(\tilde{x}^{h,\delta}_{n,\alpha_k}) v_i + \frac{\alpha_k}{c} \sum_{i=1}^{n+1} (x_0(t_i) - \tilde{\eta}^n_i) v_i,$$

where $t_1, t_2, \cdots, t_{n+1}$ are the grid points.

Observe that $(\tilde{y}^{h,\delta}_{n,\alpha_k} - \tilde{x}^{h,\delta}_{n,\alpha_k})$ is a solution of (3.23) if and only if $(\overline{\tilde{\xi}^n - \tilde{\eta}^n}) = (\tilde{\xi}^n_1 - \tilde{\eta}^n_1, \tilde{\xi}^n_2 - \tilde{\eta}^n_2, \cdots, \tilde{\xi}^n_{n+1} - \tilde{\eta}^n_{n+1})^T$ is the unique solution of

$$(S_n + \frac{\alpha_k}{c} B_n)(\overline{\tilde{\xi}^n - \tilde{\eta}^n}) = B_n[\overline{\lambda} - F_{h3} + \frac{\alpha_k}{c}(X_0 - \overline{\tilde{\eta}^n})],$$

where

$$S_n = \langle F'(\tilde{x}^{h,\delta}_{n,\alpha_k})v_i, v_j\rangle, \qquad i,j = 1,2,\cdots,n+1$$
$$F_{h3} = [F(\tilde{x}^{h,\delta}_{n,\alpha_k})(t_1), F(\tilde{x}^{h,\delta}_{n,\alpha_k})(t_2), \cdots, F(\tilde{x}^{h,\delta}_{n,\alpha_k})(t_{n+1})]^T$$

and $X_0 = [x_0(t_1), x_0(t_2), \cdots, x_0(t_{n+1})]^T$.

Further from (3.24) it follows that

$$(P_h F'(\tilde{y}^{h,\delta}_{n,\alpha_k}) + \frac{\alpha_k}{c})(\tilde{x}^{h,\delta}_{n+1,\alpha_k} - \tilde{y}^{h,\delta}_{n,\alpha_k}) = P_h[z^{h,\delta}_{\alpha_k} - F(\tilde{y}^{h,\delta}_{n,\alpha_k}) + \frac{\alpha_k}{c}(x^{h,\delta}_{0,\alpha_k} - \tilde{y}^{h,\delta}_{n,\alpha_k})]. \tag{5.3}$$

Thus $(\tilde{x}^{h,\delta}_{n+1,\alpha_k} - \tilde{y}^{h,\delta}_{n,\alpha_k})$ is a solution of (5.3) if and only if $(\overline{\tilde{\eta}^{n+1} - \tilde{\xi}^n}) = (\tilde{\eta}^{n+1}_1 - \tilde{\xi}^n_1, \tilde{\eta}^{n+1}_2 - \tilde{\xi}^n_2, \cdots, \tilde{\eta}^{n+1}_{n+1} - \tilde{\xi}^n_{n+1})^T$ is the unique solution of

$$(\tilde{S}_n + \frac{\alpha_k}{c} B_n)(\overline{\tilde{\eta}^{n+1} - \tilde{\xi}^n}) = B_n[\overline{\lambda} - F_{h4} + \frac{\alpha_k}{c}(X_0 - \overline{\tilde{\xi}^n})],$$

where

$$\tilde{S}_n = \langle F'(\tilde{y}^{h,\delta}_{n,\alpha_k})v_i, v_j\rangle, \qquad i,j = 1,2,\cdots,n+1,$$
$$F_{h4} = [F(\tilde{y}^{h,\delta}_{n,\alpha_k})(t_1), F(\tilde{y}^{h,\delta}_{n,\alpha_k})(t_2), \cdots, F(\tilde{y}^{h,\delta}_{n,\alpha_k})(t_{n+1})]^T.$$

Example 5.1. To illustrate the method for Case 1, we consider the operator $KF : L^2(0,1) \longrightarrow L^2(0,1)$ where $F : D(F) \subseteq L^2(0,1) \longrightarrow L^2(0,1)$ defined by

$$F(u) := u^3,$$

and $K : L^2(0,1) \longrightarrow L^2(0,1)$ defined by

$$K(x)(t) = \int_0^1 k(t,s)x(s)ds$$

where

$$k(t,s) = \begin{cases} (1-t)s, & 0 \leqslant s \leqslant t \leqslant 1 \\ (1-s)t, & 0 \leqslant t \leqslant s \leqslant 1. \end{cases}$$

The Fréchet derivative of F is given by

$$F'(u)w = 3(u^2)w,$$

$$\begin{aligned} [F'(v) - F'(u)]w &= 3(v^2 - u^2)w \\ &= 3u^2\left(\frac{v^2}{u^2} - 1\right)w \\ &= F'(u)\Phi(u,v,w), \end{aligned}$$

where $\Phi(u,v,w) = (\frac{v^2}{u^2} - 1)w = \frac{(v+u)(v-u)}{u^2}w$. Thus F satisfies the Assumption 3.2 with $k_0 \geqslant \|\frac{(v+u)}{u^2}\|$.

We take $y(t) = \frac{-1}{144\pi^2}[-54 + 63\pi^2t^2 - 220\sin(\pi t) + 16\sin(\pi t)\cos^2(\pi t) + 54\cos^2(\pi t) - 63\pi^2 t]$ and $y^\delta = y + \delta$. Then the exact solution

$$\hat{x}(t) = 1/2 + \sin\pi t.$$

We use

$$x_0(t) = \sin\pi t + 3/5$$

as our initial guess, then

$$F(x_0) - F(\hat{x}) = x_0^3 - \hat{x}^3.$$

Even though we are unable to write $F(x_0) - F(\hat{x}) = \varphi(K^*K)w$ for some function φ, we use the function $\varphi(\lambda) = \lambda$ and obtain the results as given in the last column of the Table 1. Thus we expect to have an accuracy of order at least $O(\delta^{\frac{1}{2}})$.

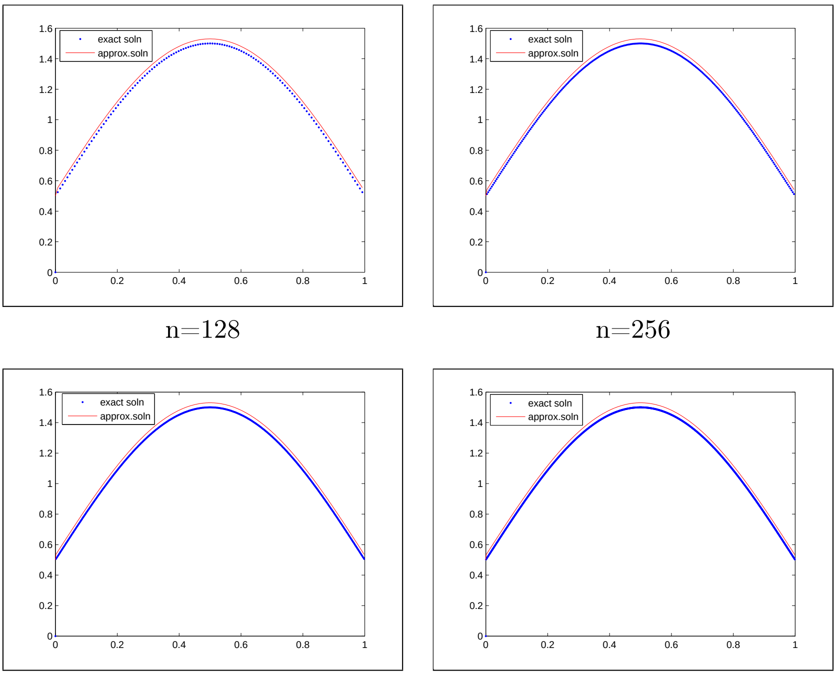
We choose $\alpha_0 = (1.5)(\delta + \varepsilon_h)^2, \mu = 1.3$, $(\delta + \varepsilon_h) = 0.1$, $g(\gamma_\rho) = 0.54$ approximately. In this example, for all n, the number of iteration $n_k = 1$. The results of the computation are presented in Table 1. The plots of the exact and the approximate solution obtained for n=256 to 1024 are given in Figure 1.

Example 5.2. To illustrate the method for Case 2, we consider the operator $KF : L^2(0,1) \longrightarrow L^2(0,1)$ where $K : L^2(0,1) \longrightarrow L^2(0,1)$ defined by

$$K(x)(t) = \int_0^1 k(t,s)x(s)ds$$

Table 1.

n	k	α_k	$\|x_k - \hat{x}\|$	$\frac{\|x_k-\hat{x}\|}{\delta^{1/2}}$
8	4	0.1094	0.2199	0.6902
16	4	0.1069	0.1645	0.5192
32	4	0.1063	0.1342	0.4242
64	4	0.1061	0.1178	0.3725
128	4	0.1061	0.1091	0.3451
256	4	0.1060	0.1046	0.3308
512	4	0.1060	0.1023	0.3236
1024	4	0.1060	0.1012	0.3199



n=128 n=256

n=512 n=1024

Figure 4. Curve of the exact and approximate solutions of Case 1

and $F : D(F) \subseteq L^2(0,1) \longrightarrow L^2(0,1)$ defined by

$$F(u) := \int_0^1 k(t,s)u^3(s)ds,$$

where

$$k(t,s) = \begin{cases} (1-t)s, & 0 \leqslant s \leqslant t \leqslant 1 \\ (1-s)t, & \leqslant t \leqslant s \leqslant 1. \end{cases}$$

Then for all $x(t), y(t) : x(t) > y(t)$:

$$\langle F(x) - F(y), x - y \rangle = \int_0^1 \left[\int_0^1 k(t,s)(x^3 - y^3)(s)ds \right] \\ \times (x-y)(t)dt \geqslant 0.$$

Thus the operator F is monotone. The Fréchet derivative of F is given by

$$F'(u)w = 3\int_0^1 k(t,s)(u(s))^2 w(s)ds.$$

So for any $u \in B_r(x_0), {x_0}^2(s) \geqslant k_3 > 0, \forall s \in (0,1)$, we have

$$F'(u)w = F'(x_0)G(u,x_0)w,$$

where $G(u,x_0) = (\frac{u}{x_0})^2$.

Further observe that for $u(s) > 0$, $\forall s \in (0,1)$,

$$[F'(v) - F'(u)]w(s) = 3\int_0^1 k(t,s)u^2(s) \left[\frac{(v^2(s) - u^2(s))w(s)}{u^2(s)} \right] ds \\ := F'(u)\Phi(u,v,w),$$

where $\Phi(u,v,w) = \frac{(v^2(s) - u^2(s))w(s)}{u^2(s)}$.

Note that

$$\Phi(u,v,w) = \frac{(v(s)+u(s))(v(s)-u(s))w(s)}{u^2(s)}.$$

Thus F satisfies the Assumption 3.2 with

$$K_0 \geqslant \left\| \frac{(v(s)+u(s))}{u^2(s)} \right\|.$$

In our computation, we take

$$f(t) = \left(\frac{1}{18\pi^2}\right)(1-t)(14t - 7 + cos^3(\pi t)$$
$$+ 6cos(\pi t))t^2 - \left(\frac{1}{18\pi^2}\right) t(14t - 7 + cos^3(\pi t)$$
$$+ 6cos(\pi t))(1-t^2) + \left(\frac{1}{9\pi^2}\right) t(1-t)(14t - 7 + cos^3(\pi t) + 6cos(\pi t))$$

and $f^\delta = f + \delta$. Then the exact solution

$$\hat{x}(t) = cos\pi t.$$

We use

$$x_0(t) = cos(\pi t) + 3\Bigg[\frac{-1}{4\pi^2}(1 - t + 2\pi t^2 cos(\pi t)$$
$$\times sin(\pi t) + \pi^2 t^3 + tcos^2(\pi t) - 2\pi t cos(\pi t)$$
$$\times sin(\pi t) - \pi^2 t^2 - cos^2(\pi t)) + \frac{1}{4\pi^2} t$$
$$\times (-2cos(\pi t) sin(\pi t)\pi - 2\pi^2 t + 2\pi t cos(\pi t)$$
$$\times sin(\pi t) + \pi^2 t^2 + cos^2(\pi t) + \pi^2 - cos^2(\pi t))\Bigg]$$

as our initial guess, so that the function $x_0 - \hat{x}$ satisfies the source condition

$$x_0 - \hat{x} = \varphi_1(F'(x_0))1$$

where $\varphi_1(\lambda) = \lambda$. Thus we expect to have an accuracy of order at least $O(\delta^{\frac{1}{2}})$.

We choose $\alpha_0 = (1.3)\delta^2, \mu = 1.3,\ \delta = 0.1 = c,\ \rho = 0.19,\ \tilde{\gamma}_\rho = 0.8173$ and $g(\tilde{\gamma}_\rho) = 0.54$ approximately. For all n the number of iteration $n_k = 1$. The results of the computation are presented in Table 2. The plots of the exact and the approximate solution obtained for n=128 to 1024 are given in Figure 2.

Next we present two examples where Assumption 3.1 is not satisfied but 3.2 is satisfied.

Example 5.3. Let $X = Y = \mathbb{R}$, $D = [0, \infty), x_0 = 1$ and define function F on D by

$$F(x) = \frac{x^{1+\frac{1}{i}}}{1+\frac{1}{i}} + c_1 x + c_2, \tag{5.4}$$

where c_1, c_2 are real parameters and $i > 2$ an integer. Then $F'(x) = x^{1/i} + c_1$ is not Lipschitz on D. However central Lipschitz condition Assumption 3.2 holds for $K_0 = 2$.

Table 2.

n	k	δ	α	$\|\tilde{x}_k - \hat{x}\|$	$\frac{\|\tilde{x}_k - \hat{x}\|}{(\delta)^{1/2}}$
8	4	0.1016	0.1094	0.3652	1.1458
16	4	0.1004	0.1069	0.2664	0.8408
32	4	0.1001	0.1063	0.1994	0.6303
64	4	0.1000	0.1061	0.1554	0.4914
128	4	0.1000	0.1061	0.1278	0.4042
256	4	0.1000	0.1060	0.1115	0.3526
512	4	0.1000	0.1060	0.1024	0.3238
1024	4	0.1000	0.1060	0.0975	0.3083

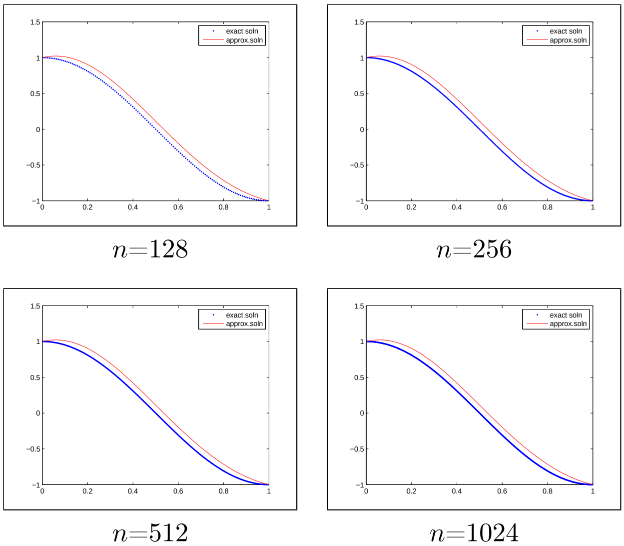


Figure 5. Curve of the exact and approximate solutions of Case 2

Indeed, we have

$$\|F'(x) - F'(x_0)\| = |x^{1/i} - x_0^{1/i}| = \frac{|x - x_0|}{x_0^{\frac{i-1}{i}} + \cdots + x^{\frac{i-1}{i}}}$$

so

$$\|F'(x) - F'(x_0)\| \leqslant 1|x - x_0|.$$

Example 5.4. We consider the integral equations

$$u(s) = f(s) + \lambda \int_a^b G(s,t)u(t)^{1+1/n}dt, \qquad n \in \mathbb{N}. \tag{5.5}$$

Here, f is a given continuous function satifying $f(s) > 0, s \in [a,b], \lambda$ is a real number, and the kernel G is continuous and positive in $[a,b] \times [a,b]$.

For example, when $G(s,t)$ is the Green kernel, the corresponding integral equation is equivalent to the boundary value problem

$$u'' = \lambda u^{1+1/n}$$
$$u(a) = f(a), u(b) = f(b).$$

These type of problems have been considered in [1]–[6].

Equation of the form (5.5) generalize equations of the form

$$u(s) = \int_a^b G(s,t)u(t)^n dt \tag{5.6}$$

studied in [1]-[6]. Instead of (5.5) we can try to solve the equation $F(u) = 0$ where

$$F : \Omega \subseteq C[a,b] \to C[a,b], \qquad \Omega = \{u \in C[a,b] : u(s) \geqslant 0, s \in [a,b]\},$$

and

$$F(u)(s) = u(s) - f(s) - \lambda \int_a^b G(s,t)u(t)^{1+1/n} dt.$$

The norm we consider is the max-norm.

The derivative F' is given by

$$F'(u)v(s) = v(s) - \lambda \left(1 + \frac{1}{n}\right) \int_a^b G(s,t)u(t)^{1/n}v(t)dt, \quad v \in \Omega.$$

First of all, we notice that F' does not satisfy a Lipschitz-type condition in Ω. Let us consider, for instance, $[a,b] = [0,1], G(s,t) = 1$ and $y(t) = 0$. Then $F'(y)v(s) = v(s)$ and

$$\|F'(x) - F'(y)\| = |\lambda| \left(1 + \frac{1}{n}\right) \int_a^b x(t)^{1/n} dt.$$

If F' were a Lipschitz function, then

$$\|F'(x) - F'(y)\| \leqslant L_1 \|x - y\|,$$

or, equivalently, the inequality

$$\int_0^1 x(t)^{1/n} dt \leqslant L_2 \max_{x \in [0,1]} x(s), \tag{5.7}$$

would hold for all $x \in \Omega$ and for a constant L_2. But this is not true. Consider, for example, the functions

$$x_j(t) = \frac{t}{j}, \qquad j \geqslant 1,\ t \in [0,1].$$

If these are substituted into (5.7)

$$\frac{1}{j^{1/n}(1 + 1/n)} \leqslant \frac{L_2}{j} \Leftrightarrow j^{1-1/n} \leqslant L_2(1 + 1/n), \qquad \forall j \geqslant 1.$$

This inequality is not true when $j \to \infty$.

Therefore, condition (5.7) is not satisfied in this case. However, condition Assumption 3.2 holds. To show this, let $x_0(t) = f(t)$ and $\gamma = \min_{s\in[a,b]} f(s), \alpha > 0$ Then for $v \in \Omega$,

$$\|[F'(x) - F'(x_0)]v\| = |\lambda| \left(1 + \frac{1}{n}\right) \max_{s\in[a,b]} \left| \int_a^b G(s,t)(x(t)^{1/n} - f(t)^{1/n})v(t)dt \right|$$
$$\leqslant |\lambda| \left(1 + \frac{1}{n}\right) \max_{s\in[a,b]} G_n(s,t)$$

where $G_n(s,t) = \frac{G(s,t)|x(t)-f(t)|}{x(t)^{(n-1)/n}+x(t)^{(n-2)/n}f(t)^{1/n}+\cdots+f(t)^{(n-1)/n}}\|v\|$.

Hence,

$$\|[F'(x) - F'(x_0)]v\| = \frac{|\lambda|(1+1/n)}{\gamma^{(n-1)/n}} \max_{s\in[a,b]} \int_a^b G(s,t)dt\|x - x_0\|$$
$$\leqslant \bar{K_0}\|x - x_0\|,$$

where $\bar{K_0} = \frac{|\lambda|(1+1/n)}{\gamma^{(n-1)/n}}N$, $K_0 = 2\bar{K_0}$ and $N = \max_{s\in[a,b]} \int_a^b G(s,t)dt$. Then condition Assumption 3.2 holds for sufficiently small λ.

Acknowledgement. Ms. Shobha, thanks National Institute of Technology Karnataka, India, for the financial support.

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Received: 27 February 2013; **revised:** 29 August 2013