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ON THE CONGRUENCE $\kappa n \equiv a (\mathrm{mod}\ \varphi(n))$

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Abstract: Lehmer's totient problem asks whether there exists a composite n such that $\varphi(n) \mid (n-1)$, where φ is the Euler's function. This problem is still open. Later, several upper bounds of the derived problem "$\varphi(n) \mid (n-a)$" were given. In this paper, we extend it to n with $\varphi(n) \mid (\kappa n - a)$ and obtain some new bounds. As an application, for any integer $\lambda > 0$ we have,

$$\#\{n \leqslant x : \varphi(n) \mid (n-1), n \not\equiv 1 \ (\mathrm{mod}\ 6^{\lambda})\} \ll x^{1/2}/(\log x)^{0.559552+o(1)},$$
$$\#\{n \leqslant x : \varphi(n) \mid (3n-1)\} \ll x^{1/2}/(\log x)^{0.559552+o(1)}.$$

Keywords: Euler function, Lehmer's totient problem.

1. Introduction

Let $\varphi(n)$ be the Euler function of n; in particular, $\varphi(p) = p-1$ for a prime p. Lehmer [5] asked if there exist composite positive integers n such that $\varphi(n) \mid (n-1)$. This is still an open question. In 1976, C. Pomerance [9] proved that if one sets

$$\mathcal{L} = \{\, n : \varphi(n) \mid (n-1) \text{ and } n \text{ is composite} \,\},$$

and denote $S(x) = \{\, n \leqslant x : n \in S \,\}$ for any set S, then the cardinality $\#\mathcal{L}(x) \ll x^{2/3}(\log\log x)^{1/3}$.

The derived problem, $\varphi(n) \mid (n-a)$, was studied in a series of papers ([10, 11, 2]). The corresponding upper bounds were $x^{1/2}(\log x)^{3/4}$, $x^{1/2}(\log x)^{1/2}(\log\log x)^{-1/2}$ and $x^{1/2}(\log\log x)^{1/2}$, respectively. In [1] the bound was improved to $x^{1/2}/(\log x)^{\Theta+o(1)}$, where $\Theta = 0.129398$ and the term "$\log x$" appeared in the denominator. In [6], another upper bound for the original case $a = 1$ was given: $x^{1/2}/(\log x)^{1/2+o(1)}$.

For fixed integer a and $\kappa \geqslant 1$, put

$$\mathcal{L}_{a,\kappa} = \{\, n : \varphi(n) \mid (\kappa n - a) \,\}.$$

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In the case $a = 0$ and $\kappa = 1$, Sierpinski ([12], p.232) showed that $\{ n : \varphi(n)|n \} = \{1\} \cup \{ 2^i 3^j : i > 0, j \geqslant 0 \}$. Thus $\#\mathcal{L}_{0,1}(x) \sim (\log x)^2/2\log 2\log 3$. Actually when a definite κ is given, it is easy to determine the form of those numbers n for which $\varphi(n) \mid \kappa n$. Both the number of prime factors and the number of choices for prime factors of n are finite . It follows that $\#\mathcal{L}_{0,\kappa}(x) \ll x^{o(1)}$. Consequently, we can assume that $a \neq 0$.

When $b \in \mathcal{L}_{0,\kappa}$, we find that $pb \in \mathcal{L}_{\kappa b,\kappa}$ for each prime $p \nmid b$. To exclude trivial solutions of this kind we put

$$\mathcal{L}'_{a,\kappa} = \{ n \in \mathcal{L}_{a,\kappa} : \kappa n \neq pa \ (p \text{ prime}) \text{ whenever } \kappa p \nmid a \},$$

and following Pomerance we also define

$$\mathcal{L}''_{a,\kappa} = \left\{ n \in \mathcal{L}'_{a,\kappa} : n \text{ is square free} \right\}.$$

We find that $\mathcal{L}(x) = \mathcal{L}'_{1,1}(x) = \mathcal{L}''_{1,1}(x)$.

In this paper ,we prove the following theorem and corollaries:

Theorem 1. *For arbitrary integers $q, \lambda > 0$, we have*

$$\#\mathcal{L}'_{a,\kappa}(x) \cap \{n : \kappa n \not\equiv a \ (\mathrm{mod}\ q^\lambda)\} \ll x^{1/2}/(\log x)^{\tau+1/\varphi(q)+o(1)}. \tag{1}$$

Here $\tau = (\xi \log 2)/2$, where ξ is the least positive solution to the equation

$$\xi \log\left(1 + \frac{1+\sqrt{4\xi+1}}{2\xi}\right) + \frac{2\xi}{1+\sqrt{4\xi+1}} = 1 - 1/\varphi(q). \tag{2}$$

Corollary 1. *For arbitrary integers $\lambda_0, \lambda_1 > 0$, we have*

$$\#\mathcal{L}'_{a,\kappa}(x) \cap \{n : \kappa n \not\equiv a \ (\mathrm{mod}\ 2^{\lambda_0}3^{\lambda_1})\} \ll x^{1/2}/(\log x)^{0.559552+o(1)}.$$

Corollary 2. *For arbitrary integer $\lambda_0, \lambda_1, \lambda_2 > 0$, we have*

$$\#\mathcal{L}'_{a,\kappa}(x) \cap \{n : \kappa n \not\equiv a \ (\mathrm{mod}\ 2^{\lambda_0}3^{\lambda_1}5^{\lambda_2})\} \ll x^{1/2}/(\log x)^{0.35767+o(1)}.$$

Corollary 3. *Suppose $\kappa/(\kappa, a) > 1$, and q is the smallest prime such that $q \mid \kappa/(\kappa, a)$. Then*

$$\#\mathcal{L}'_{a,\kappa}(x) \ll x^{1/2}/(\log x)^{\tau+1/(q-1)+o(1)}.$$

Corollary 4. *Suppose $a \neq 0, 1$, $a/(a, \kappa)$ is not squarefree, and q is the smallest prime such that $q^2 \mid a/(a, \kappa)$. Then*

$$\#\mathcal{L}''_{a,\kappa}(x) \ll x^{1/2}/(\log x)^{\tau+1/(q-1)+o(1)}.$$

2. Preparation

In the rest part of this paper, we always assume $a \neq 0$ and $\kappa > 0$ are fixed integers and write $l = \log x$, $l_2 = \log\log x$, $l_k = \log l_{k-1}$ for $k > 2$. Δ denotes the set of all the primes and p always denotes a prime. For any integers q and n, the expression $\mathrm{ord}_q(n)$ is defined to be the integer satisfying $q^{\mathrm{ord}_q(n)} \| n$. We use $P(n)$ to denote the largest prime factor of n and $p(n)$ to represent the smallest one. The function $\Omega(n)$ counts the total number of prime factors of n.

Lemma 1-5 were first proved by Pomerance([10]) in the case $k = 1$; the proofs carry through for arbitrary $\kappa \geqslant 1$ with few changes and are therefore omitted.

Lemma 1 (see [10, Lemma 1]). *For any integer $q > 0$, we have the following two inequalities:*

$$\#\mathcal{L}'_{a,\kappa}(x) \leqslant 4a^2 + \sum_{d|a} \#\mathcal{L}''_{a/d,\kappa}(x/d),$$

and

$$\#\mathcal{L}'_{a,\kappa}(x) \cap \{n : \kappa n \not\equiv a \pmod{q}\} \leqslant 4a^2 + \sum_{d|a} \#\mathcal{L}''_{a/d,\kappa}(x/d) \cap \{n : \kappa n \not\equiv a/d \pmod{q}\}.$$

Thus, in order to get the upper bound of $\#\mathcal{L}'_{a,\kappa}(x)$, we only need to prove the same bound for $\#\mathcal{L}''_{a,\kappa}(x)$.

Lemma 2 (see [10, Lemma 2]). *If $n \geqslant 16a^2$, $n \in \mathcal{L}''_{a,\kappa}$, write*

$$k = \frac{\kappa n - a}{\varphi(n)}.$$

Then:

(i) *$k > \kappa$;*
(ii) *if $m \mid n$, $m \neq n$, then $m/\varphi(m) < k/\kappa$;*
(iii) *there is a prime $q > P(n)$ with $nq/\varphi(nq) > k/\kappa$.*

Lemma 3 (see [10, Lemma 3]). *Suppose k, n, κ are natural numbers with n square-free and $n/\varphi(n) > k/\kappa$. If $m|n$ and $m/\varphi(m) < k/\kappa$, then*

$$p(n/m) < \omega(n/m) \cdot (\kappa m + 1).$$

Lemma 4 (see [10, Theorem 1]). *Suppose that $n \geqslant 16a^2$, $n \in \mathcal{L}''_{a,\kappa}$. Let the prime factorization of n be $p_1 \dots p_r$, where $p_1 > \dots > p_r$ and $r = \omega(n)$. Then, for $1 \leqslant k \leqslant r$ we have*

$$p_k < (k+1)\left(1 + \kappa \prod_{i=k+1}^{r} p_i\right).$$

Lemma 5 (see [11]). *Suppose that $\delta > 0$, $a_1 \geqslant \dots \geqslant a_t \geqslant 0$, and $a_i \leqslant \delta + \sum_{j=i+1}^{t} a_j$ for $1 \geqslant i \geqslant t-1$. Then for any real number ρ such that $0 \leqslant \rho \leqslant \sum_{i=1}^{t} a_i$, there is a subset $\mathcal{I}$ of $1, \dots, t$ such that $\rho - \delta < \sum_{i \in \mathcal{I}} a_i \leqslant \rho$.*

Lemma 6 (see [3, Proposition 3]). *Let* $\mathcal{V}_\xi = \{n : \omega(n) < \xi \log\log n\}$. *For fixed* $0 < \xi_1 < 1$,

$$\#\mathcal{V}_{\xi_1}(x) \ll \frac{x}{(\log x)^{1+\xi_1 \log(\xi_1/e)} (\log\log x)^{1/2}} \qquad (x \to \infty).$$

Let $\mathcal{W}_\xi = \{n : \omega(n) > \xi \log\log n\}$. *For fixed* $\xi_2 > 1$,

$$\#\mathcal{W}_{\xi_2}(x) \ll \frac{x}{(\log x)^{1+\xi_2 \log(\xi_2/e)} (\log\log x)^{1/2}} \qquad (x \to \infty).$$

Lemma 7 (see [6, Corollary 1]). *Given any* $\xi \in (0, 2]$, *we have the estimate*

$$\#\{n \leqslant t : \mathrm{ord}_2(\varphi(n)) \leqslant \xi \log\log t\} \ll \frac{t}{(\log t)^{e_\xi}},$$

where

$$e_\xi := 1 + \xi \log 2 - \xi \log \left(1 + \frac{1 + \sqrt{4\xi + 1}}{2\xi}\right) - \frac{2\xi}{1 + \sqrt{4\xi + 1}}. \tag{3}$$

Lemma 8 (see [7]). *For* a, q *with* $(a, q) = 1$,

$$\sum_{\substack{p \equiv a \ (\mathrm{mod}\ q) \\ p \leqslant x}} \frac{1}{p} = \frac{1}{\varphi(q)} \log\log x + A + O\left(\frac{1}{logx}\right).$$

Lemma 9 (see [8, p. 316]). *For* a, q *with* $(a, q) = 1$ *and* $q \leqslant \log x$, *we have*

$$\sum_{\substack{p \equiv a \ (\mathrm{mod}\ q) \\ p \leqslant x}} \log p = x/\varphi(q) + O\left(x e^{-c\sqrt{\log x}}\right).$$

In order to prove Theorem 1, we tend to study the number of integers whose prime factors come from a particular set.

Let $\mathcal{P}$ be a subset of $\Delta = \{2, 3, 5, \ldots\}$ and $\bar{\mathcal{P}} = \Delta \setminus \mathcal{P}$. Factorize $n = n_{\mathcal{P}} n_{\bar{\mathcal{P}}}$ where

$$n_{\mathcal{P}} = \max_{d \mid n} \{d : p \in \mathcal{P} \text{ for each prime } p \mid d\}$$

is the $\mathcal{P}$-part of n, and $n_{\bar{\mathcal{P}}}$ is the $\bar{\mathcal{P}}$-part of n, respectively.

For convenience, we call n a $\mathcal{P}$-integer when $n = n_{\mathcal{P}}$ and call n an s-almost-$\mathcal{P}$-integer when $\Omega(n_{\bar{\mathcal{P}}}) \leqslant s$. Denote $\mathbb{N}_{\mathcal{P}}$ be the set of $\mathcal{P}$-integers and $\mathbb{N}_{s-\mathcal{P}}$ be the set of s-almost-$\mathcal{P}$-integers. Let $\pi_{\mathcal{P}}(x) = \{n \leqslant x : n \in \mathbb{N}_{\mathcal{P}}\}$ and $\pi_{s-\mathcal{P}}(x) = \{n \leqslant x : n \in \mathbb{N}_{s-\mathcal{P}}\}$. The next lemma can be deduced from Theorem 00 of [4] (taking $f(n)$ to be the characteristic function of elements in $\mathbb{N}_{\mathcal{P}}$).

Lemma 10. *Let* $\alpha, \beta \in (0, 1]$ *be rational numbers. Suppose* $\mathcal{P} \subseteq \Delta$ *satisfies*

$$\sum_{\substack{p \in \mathcal{P} \\ p \leqslant x}} \log p \leqslant \beta x + O\left(\frac{x}{\log^2 x}\right). \tag{4}$$

and

$$\sum_{\substack{p\in\mathcal{P}\\ p\leqslant x}} 1/p < \alpha \log\log x + B. \tag{5}$$

Then the number of $\mathcal{P}$*-integers* $n \leqslant x$ *is*

$$\pi_{\mathcal{P}}(x) \ll x(\log x)^{\alpha-1},$$

where the implied constant depends only on α, β*.*

Next we obtain a result on the number of s-almost-$\mathcal{P}$-integers.

Proposition 1. *Let* $s > 0$ *be a fixed integer, and let* $\alpha, \beta \in (0, 1]$ *be real numbers. Let* $\mathcal{P} \subseteq \{2, 3, 5, \dots\}$ *be a set of primes satisfying (4) and (5). Then*

$$\pi_{s-\mathcal{P}}(x) \ll x(\log x)^{\alpha-1}(\log\log x)^s.$$

Proof. Combining Lemma 10, one can deduce that

$$\begin{aligned}
\pi_{s-\mathcal{P}}(x) &\leqslant \sum_{i=1}^{s} \sum_{\substack{(p_1,\dots,p_i)\in\bar{\mathcal{P}}^i\\ p_1\cdots p_i\leqslant x}} \pi_{\mathcal{P}}\left(\frac{x}{p_1\dots p_i}\right)\\
&\ll \sum_{i=1}^{s} \sum_{\substack{(p_1,\dots,p_i)\in\Delta^i\\ p_1\cdots p_i\leqslant x}} (x/p_1\dots p_i)\log^{\alpha-1}(x/p_1\dots p_i)\\
&\ll xl^{\alpha-1}\sum_{i=1}^{s} \sum_{\substack{(p_1,\dots,p_i)\in\Delta^i\\ p_1\cdots p_i\leqslant x}} \frac{1}{p_1\dots p_i}\\
&\ll xl^{\alpha-1}\sum_{i=1}^{s}\left(\sum_{\substack{p\in\Delta\\ p\leqslant x}}\frac{1}{p}\right)^i \leqslant xl^{\alpha-1}\sum_{i=1}^{s}(\alpha l_2)^i \ll xl^{\alpha-1}l_2^s.
\end{aligned}$$

■

3. Proof of Theorem 1 and the corollaries

Proof of Theorem 1. In view of Lemma 1, it is sufficient to obtain a suitable bound for

$$\#\mathcal{L}''_{a,\kappa}(x) \cap \{n : \kappa n \not\equiv a \pmod{q^\lambda}\} \cap [x/2, x].$$

Denote $\mathcal{P} = \{p \in \Delta : p \not\equiv 1 \pmod q\}$ and $\bar{\mathcal{P}} = \Delta \setminus \mathcal{P}$. Now consider any $n \in \mathcal{L}''_{a,\kappa}$. For each prime factor $p \mid n$, if $p \equiv 1 \pmod q$, then

$$q \mid (p-1) \mid \varphi(n) \mid (\kappa n - a).$$

That is

$$\mathrm{ord}_q(\kappa n - a) \geqslant \#\{p \mid n : p \equiv 1 \pmod q\} = \omega(n_{\bar{\mathcal{P}}}) = \Omega(n_{\bar{\mathcal{P}}}).$$

The condition $\kappa n \not\equiv a \pmod{q^\lambda}$ leads to the fact that $\mathrm{ord}_q(\kappa n - a) < \lambda$. Hence

$$\Omega(n_{\bar{\mathcal{P}}}) < \lambda \tag{6}$$

and n belongs to $\mathbb{N}_{\lambda - \mathcal{P}}$, as does every divisor of n.

If $q = 2$, then $\mathcal{P} = \{2\}$ and $\bar{\mathcal{P}} = \Delta \setminus \{2\}$. It follows that $\omega(n) < \lambda + 1$ and the number of such integers no larger than x is $x^{o(1)}$. Now we assume that $q \geqslant 3$.

Similar as in [1], it can be shown that

$$\#\{n \in \mathcal{L}''_{a,\kappa} \cap \mathcal{W}_{20} : x/2 \leqslant n \leqslant x\} \ll x^{1/2}(\log x)^{-11},$$

so here we always suppose that $\omega(n) \leqslant 20 l_2$. Recall (3) for the definition of e_ξ. Let ξ be the unique solution in $(0,1)$ to the equation (2), or equivalently,

$$e_\xi = \xi \log 2 + 1/\varphi(q), \tag{7}$$

and denote $\tau = (\xi \log 2)/2$.

Now let $n \in \mathcal{L}''_a \cap [x/2, x]$ be fixed. Factor $n = p_1 p_2 \ldots p_r$ where $r = \omega(n)$ and $p_1 > \ldots > p_r$. By Lemma 4 we have

$$\log p_i < \log 2\kappa r + \sum_{j=i+1}^{r} \log p_j \qquad (1 \leqslant i \leqslant r).$$

Applying Lemma 5 with $\delta = \log 2\kappa r$, $t = r+1$, $a_i = \log p_i$ for $1 \leqslant i \leqslant r$, $a_t = 0$, and $\rho = \log(x^{1/2} l^{-\tau} l_2^2)$, we conclude that n has a positive divisor m such that $\rho - \delta < \log m \leqslant \rho$, i.e.,

$$x^{1/2} l^{-\tau} l_2 / 40\kappa \leqslant \frac{x^{1/2} l^{-\tau} l_2^2}{2\kappa r} \leqslant m \leqslant x^{1/2} l^{-\tau} l_2^2.$$

Then $d = n/m$ satisfies

$$x^{1/2} l^{\tau} l_2^{-2}/2 \leqslant d \leqslant 40\kappa x^{1/2} l^{\tau} l_2^{-1}.$$

For d with $\mathrm{ord}_2(\varphi(d)) < \xi \log\log d$, from Lemma 7 we know the number of choices for such d is

$$\begin{aligned}\#\{d \leqslant 40\kappa x^{1/2} l^{\tau} l_2^{-1} : \mathrm{ord}_2(\varphi(d)) \leqslant \xi \log\log d\} &\ll \frac{x^{1/2} l^{\tau} l_2^{-1}}{(\log(x^{1/2} l^{\tau} l_2^{-1}))^{e_\xi}} \\ &\ll x^{1/2} l^{\tau - e_\xi + o(1)}.\end{aligned}$$

Since $\kappa m d \equiv a (\mathrm{mod}\ \varphi(m)\varphi(d))$, then $\kappa m \equiv a d'/\mu_d (\mathrm{mod}\ \varphi(d)/\mu_d)$ where $\mu_d = \gcd(d, \varphi(d)) \mid a$ and d' is the inverse of d/μ_d modulo $\varphi(d)/\mu_d$. Hence m has only finite choices modulo $\varphi(d)/\mu_d$ while x is sufficiently large, because $m \ll x^{1/2} l^{-\tau} l_2^2$ and $\varphi(d)/\mu_d \gg d/\log\log d \gg x^{1/2} l^{\tau} l_2^{-3}$. It follows that d determines n up to finitely many choices when x is sufficiently large and the number of choices for n is

$$\ll x^{1/2} l^{\tau - e_\xi + o(1)}.$$

Now we consider d with $\mathrm{ord}_2(\varphi(d)) \geqslant \xi \log\log d$. Set

$$\sigma = \lfloor \xi \log\log (x^{1/2} l^{\tau} l_2^{-2}/2) \rfloor,$$

then $2^{\sigma-1} \mid \varphi(d)$. The congruence $\kappa m d \equiv a (\mathrm{mod}\ \varphi(m)\varphi(d))$ leads to

$$\kappa d \equiv a m'/\mu_m (\mathrm{mod}\ 2^{\sigma-1}\varphi(m)/\mu_m),$$

where $\mu_m = \gcd(m, 2^{\sigma-1}\varphi(m)) \mid a$ and m' is the inverse of m/μ_m modulo $2^{\sigma-1}\varphi(m)/\mu_m$. Since

$$d \leqslant 40\kappa x^{1/2} l^{\tau} l_2^{-1},$$

whereas

$$\frac{2^{\sigma-1}\varphi(m)}{\mu_m} \gg \frac{m 2^{\sigma}}{\log\log m} \gg \frac{x^{1/2} l^{-\tau+\xi \log 2} l_2}{l_2} \gg x^{1/2} l^{\tau}.$$

It follows that d (and then n) has finitely many choices modulo $2^{\sigma-1}\varphi(m)/\mu_m$ provided that x is sufficiently large. By applying Proposition 1, and in view of (6), the number of choices for such m (and hence for such n) is

$$\ll (x^{1/2} l^{-\tau} l_2^2)(\log (x^{1/2} l^{-\tau} l_2^2))^{-1/\varphi(q)} (\log\log (x^{1/2} l^{-\tau} l_2^2))^{\lambda} \ll x^{1/2} l^{-\tau-1/\varphi(q)+o(1)}.$$

From (7), $e_{\xi} - \tau = \tau + 1/\varphi(q)$, it follows that

$$\#\mathcal{L}''_{a,\kappa}(x) \cap \{\kappa n \not\equiv a \pmod{q^{\lambda}}\} \ll x^{1/2}/(\log x)^{\tau+1/\varphi(q)+o(1)}$$

Now Theorem 1 can be obtained by applying Lemma 1. ∎

Applying Theorem 1 with q such that $\varphi(q) = 2$ or $\varphi(q) = 4$, we obtain Corollary 1 and Corollary 2.

Proof of Corollary 1. For arbitrary integers λ_0, λ_1, if $n \not\equiv a \pmod{2^{\lambda_0} 3^{\lambda_1}}$, then either $n \not\equiv a \pmod{2^{\lambda_0}}$ or $n \not\equiv a \pmod{3^{\lambda_1}}$. The corollary follows by applying Theorem 1 and using the fact $\varphi(3) = 2$ to estimate $\xi = 0.171832, \tau = 0.059552$ and $\Theta = \tau + 1/2 = 0.559552$. ∎

Proof of Corollary 2. For arbitrary integer $\lambda_0, \lambda_1, \lambda_2 > 0$, if $n \not\equiv a \pmod{2^{\lambda_0} 3^{\lambda_1} 5^{\lambda_2}}$, then $n \not\equiv a \pmod{2^{\lambda_0}}$ or $n \not\equiv a \pmod{3^{\lambda_1}}$ or $n \not\equiv a \pmod{5^{\lambda_2}}$. The corollary follows by applying Theorem 1 and using the fact $\varphi(3) = 2$ and $\varphi(5) = 4$ to estimate $\xi = 0.31067, \tau = 0.10767$ and $\Theta = \tau + 1/4 = 0.35767$. ∎

Proof of Corollary 3. Suppose $\kappa/(\kappa, a) > 1$, and q is the smallest prime such that $q \mid \kappa/(\kappa, a)$. Since n is squarefree, we have $\kappa n \not\equiv a \pmod{q^{\mathrm{ord}_q(\kappa)}}$, it follows from Theorem 1 that $\#\mathcal{L}''_{a,\kappa}(x) \ll x^{1/2}/(\log x)^{\tau+1/(q-1)+o(1)}$. Since $q \mid \kappa/(\kappa, a/d)$ for each $d \mid a$, the corollary follows from Lemma 1. ∎

For example, since there is no $n > 1$ satisfying that $3n = p$ is a prime, we have

$$\#\{n \leqslant x : \varphi(n) \mid (3n-1)\} \ll x^{1/2}/(\log x)^{0.559552+o(1)}.$$

Proof of Corollary 4. Suppose $a \neq 0, 1$, $a/(a, \kappa)$ is not squarefree and q is the smallest prime such that $q^2 \mid a/(a, \kappa)$. Since n is squarefree, we have $\kappa n \not\equiv a \pmod{q^{\mathrm{ord}_q(a)}}$. The corollary follows from Theorem 1. ∎

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