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BANACH ENVELOPES OF p-BANACH LATTICES, $0 < p < 1$, AND CESÀRO SPACES

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Dedicated to Lech Drewnowski on
the occasion of his 70th birthday

Abstract: In this note we characterize Banach envelopes of p-Banach lattices, $0 < p < 1$, such that their positive cones are 1-concave. In particular we show that the Banach envelope of Cesàro sequence space $\widehat{ces_p(v)}$, $0 < p < 1$, coincides isometrically with the weighted $\ell_1(w)$ space where $w(n) = \|e_n\|_{ces_p(v)} = \left(\sum_{i=n}^{\infty} i^{-p} v(i)\right)^{1/p}$ and e_n are the unit vectors. For Cesàro function space $Ces_p(v)$, $0 < p < 1$, its Banach envelope $\widehat{Ces_p(v)}$ is isometrically equal to $L_1(w)$ with $w(t) = \left(\int_t^{\infty} s^{-p} v(s)\, ds\right)^{1/p}$, $t \in (0, \infty)$.

Keywords: Banach envelopes, Mackey topology, p-Banach lattices for $0 < p < 1$, Cesáro function and sequence spaces.

Mackey topologies and in particular (Fréchet) Banach envelopes have been studied by several authors in the context of different spaces. Kalton in [3, 4] found Banach envelopes of separable Orlicz function and sequence spaces, Drewnowski [1] and Nawrocki [2] characterized Mackey topology and Banach envelope of separable Musielak-Orlicz function space and Orlicz sequence space in general case. Nawrocki, Ortyński, Popa [10, 11, 13] advanced these studies for Lorentz function and sequence spaces. Recently Pietsch [12] have investigated the Mackey topology of Marcinkiewicz sequence space, and in [8] this topology has been examined in certain class of Orlicz-Lorentz spaces. In [7], Mastyło and the first author found an isomorphic description of Banach envelope of a rearrangement invariant space whose cone of decreasing functions is 1-concave. The goal of this note is to study Banach envelopes of p-Banach function and sequence lattices, $0 < p < 1$. Under the assumption that the lattice E is 1-concave on the cone of non-negative elements and in function case under additional assumption that certain "uniform averaging operation" is well defined we show that the Banach envelope $\widehat{E}$ of E coincides with

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the space $\ell_1(w)$ or $L_1(w)$ respectively, for some positive weight w. We apply these results to Cesàro sequence and function spaces $\mathrm{ces}_p(v)$ and $\mathrm{Ces}_p(v)$ respectively, for $0 < p < 1$, getting isometric representations of their Banach envelopes.

Let $0 < p \leqslant 1$. A *p-norm* on a vector space X is a map $x \mapsto \|x\|$ such that:

1. $\|x\| > 0$ if $x \neq 0$.
2. $\|tx\| = |t|\|x\|$ for all $x \in X$ and all scalars t.
3. $\|x + y\|^p \leqslant \|x\|^p + \|y\|^p$.

Let $B = \{x \in X : \|x\| \leqslant 1\}$. Then the family $\{rB : r > 0\}$ is a basis of neighborhoods of 0 for a Hausdorff locally bounded vector space X. If X is complete, then we say that X is a *p-Banach space*. The *Mackey topology* τ of a locally bounded space X with separating dual is the strongest locally convex topology on X which is weaker than the original one. The Minkowski functional of the set $\overline{co}(B)$ is called the *Mackey norm* on X. The completion of the space (X, τ) is called the Mackey completion of X and is denoted by $\widehat{X}$. $\widehat{X}$ equipped with the Mackey norm $\|\cdot\|_{\widehat{X}}$ is also called a Banach envelope of X. There are well known several equivalent formulas for the Mackey norm [5, 14, 7, 10]. In this paper we shall use that for any $x \in \widehat{X}$,

$$\|x\|_{\widehat{X}} = \inf\left\{\sum_{i=1}^n \|x_i\|_X : \sum_{i=1}^n x_i = x\right\}.$$

Let Ω be either $\mathbb{N}$ or $(0, \infty)$. Let L^0 be the set of all (equivalence classes of) Lebesgue-measurable real valued functions f on Ω, and ℓ^0 be the set of all real sequences. We say that $E \subset L^0$ or $E \subset \ell^0$ is a p-Banach lattice, $0 < p \leqslant 1$, whenever it is complete if equipped with a p-norm $\|\cdot\|_E$ such that if for any $f \in L^0$ or $f \in \ell^0$ and $g \in E$ with $|f| \leqslant |g|$, we have that $f \in E$ and $\|f\|_E \leqslant \|g\|_E$. Moreover we assume that there exists an element $f > 0$ which belongs to E. If $E \subset \mathbb{N}$ then we say that E is a p-Banach sequence space and if $E \subset L^0$ then it is called a p-Banach function space. The lattice $(E, \|\cdot\|_E)$ is said to be order continuous whenever for any $f_n \in E$ such that $f_n \downarrow 0$ a.e. we have $\|f_n\|_E \to 0$. Recall that the symbol $\ell_1(w)$ denotes the Banach space of all sequences $x = (x(n))$ such that $\|x\|_{\ell_1(w)} = \sum_{n=1}^\infty |x(n)|w(n) < \infty$ where $w = (w(n)) > 0$. Similarly for a positive $w \in L^0$, $L_1(w)$ consists of all $f \in L^0$ with $\|f\|_{L_1(w)} = \int |f|w < \infty$.

For any p-Banach lattice $(E, \|\cdot\|_E)$ denote its positive cone by $E^+ = \{f \in E : f \geqslant 0\}$. If Q is a cone in E then E is said to be 1-concave on Q whenever there is $C > 0$ such that for any $f_1, \ldots, f_n \in Q$, we have

$$\|f_1 + \cdots + f_n\|_E \geqslant C(\|f_1\|_E + \cdots + \|f_n\|_E).$$

In the paper [7] the Banach envelope was characterized in the quasi-Banach rearrangement invariant space whose cone of decreasing functions is 1-concave. Here, we will find Banach envelopes of sequence and function p-Banach spaces that are 1-concave on the cone of non-negative elements.

First we start with some auxiliary lemmas.

Lemma 1. *If E is a p-Banach lattice which is 1-concave on E^+ then E is order continuous.*

Proof. Let $(f_n) \subset E$ and $f_n \downarrow 0$ a.e.. We first show that (f_n) is a Cauchy sequence in E. Suppose otherwise, so for some $\epsilon > 0$ and some subsequences (n_k) and (m_k) of $\mathbb{N}$ we have that for all $k \in \mathbb{N}$,

$$\|f_{n_k} - f_{m_k}\|_E \geqslant \epsilon.$$

We can assume that $n_k < m_k < n_{k+1}$ for all $k \in \mathbb{N}$. Then all differences $f_{m_k} - f_{n_{k+1}}$ and $f_{n_k} - f_{m_k}$ are non-negative and so

$$f_{n_1} = \sum_{k=1}^{\infty}(f_{m_k} - f_{n_{k+1}}) + \sum_{k=1}^{\infty}(f_{n_k} - f_{m_k}).$$

Now by 1-concavity of E^+,

$$\|f_{n_1}\|_E \geqslant C \sum_{k=1}^{\infty} \|f_{n_k} - f_{m_k}\|_E.$$

Hence $\sum_{k=1}^{\infty} \|f_{n_k} - f_{m_k}\|_E < \infty$ which contradicts the inequality $\|f_{n_k} - f_{m_k}\|_E \geqslant \epsilon$ and thus (f_n) is a Cauchy sequence in E.

Now by completeness of E there exists $f \in E$ such that $\|f_n - f\|_E \to 0$. The embedding $E \hookrightarrow L^0$ is continuous in the topology of local convergence in measure considered in L^0 (compare the proof of Theorem 1 in [9] on page 96). It follows that there exists a subsequence (f_{n_k}) such that $f_{n_k} \to f$ a.e.. Thus $f = 0$ and $\|f_n\|_E \downarrow 0$, and the proof is complete. ■

Lemma 2. *If the p-Banach lattice E is order continuous then its Banach envelope $\widehat{E}$ is also order continuous.*

Proof. Let $0 \leqslant f \in \widehat{E}$. By definition of the norm $\|\cdot\|_{\widehat{E}}$ we find $f_j \in E$, $j = 1, 2, \ldots, m$, such that $f = \sum_{j=1}^{m} f_j$, $f_j \geqslant 0$, $f_j \wedge f_k = 0$ for $j \neq k$, and $\|f\|_{\widehat{E}} \leqslant \sum_{j=1}^{m} \|f_j\|_E$. Let $(h_n) \subset \widehat{E}$, $0 \leqslant h_n \leqslant f$ and $h_n \downarrow 0$ a.e.. Then $h_n \chi_{\operatorname{supp} f_j} \leqslant f_j \chi_{\operatorname{supp} f_j}$ and $h_n \chi_{\operatorname{supp} f_j} \downarrow 0$ for each $j = 1, 2, \ldots, m$. Hence $\|h_n \chi_{\operatorname{supp} f_j}\|_E \downarrow 0$ by order continuity of E, and so

$$\|h_n\|_{\widehat{E}} \leqslant \sum_{j=1}^{m} \|h_n \chi_{\operatorname{supp} f_j}\|_{\widehat{E}} \leqslant \sum_{j=1}^{m} \|h_n \chi_{\operatorname{supp} f_j}\|_E \to 0,$$

as $n \to \infty$, and the proof is finished. ■

The next result is a description of the Banach envelope $\widehat{E}$ of a sequence space E.

Theorem 3. *Let E be a p-Banach sequence space* 1*-concave on the cone* E^+*. Define a sequence $w = (w(n))$ by*

$$w(n) = \|e_n\|_E.$$

Then the Banach envelope $\widehat{E}$ of E coincides up to equivalence of norms with the the weight space $\ell_1(w)$. If the 1*-concave constant is 1, then the Banach envelope $\widehat{E}$ of E is isometrically isomorphic to $\ell_1(w)$.*

Proof. Let $x = \sum_{i=1}^n a_i e_i$ be any element in E. It is clear that

$$\|x\|_{\widehat{E}} \leqslant \sum_{i=1}^n |a_i| \|e_i\|_E = \sum_{i=1}^n |a_i| w(i).$$

For any $\epsilon > 0$ there is a finite sequence $(y_j)_{j=1}^m$ in E such that $x = \sum_{j=1}^m y_j$ and

$$\|x\|_{\widehat{E}} + \epsilon \geqslant \sum_{j=1}^m \|y_j\|_E.$$

First observe that we may assume that $x \geqslant 0$, $y_j \geqslant 0$ and $\operatorname{supp}(y_j) \subset \{1, 2, \ldots, n\}$ for $j = 1, \ldots, m$. Indeed it is easy to show that for any $x, x_j \in E$ such that $x = \sum_{j=1}^m x_j$ we can find y_j such that $\operatorname{supp}(y_j) \subset \operatorname{supp} x$, $|x| = \sum_{j=1}^m y_j$ and $0 \leqslant y_j \leqslant |x_j|$ for each $j = 1, 2, \ldots, m$. Let thus for some $b_{ij} \geqslant 0$,

$$y_j = \sum_{i=1}^n b_{ij} e_i.$$

Then

$$|a_i| = a_i = \sum_{j=1}^m b_{ij} = \sum_{j=1}^m |b_{ij}|,$$

and

$$\begin{aligned} \|x\|_{\widehat{E}} + \epsilon \geqslant & \sum_{j=1}^m \Big\| \sum_{i=1}^n b_{ij} e_i \Big\|_E \geqslant C \sum_{j=1}^m \sum_{i=1}^n |b_{ij}| \|e_i\|_E \\ = & C \sum_{i=1}^n \sum_{j=1}^m |b_{ij}| w(i) = C \sum_{i=1}^n |a_i| w(i). \end{aligned}$$

Since this is true for arbitrary $\epsilon > 0$, we showed that for any $x = \sum_{i=1}^n a_i e_i$ we have $C\|x\|_{\ell_1(w)} \leqslant \|x\|_{\widehat{E}} \leqslant \|x\|_{\ell_1(w)}$, and if the 1-concave constant $C = 1$, then $\|x\|_{\widehat{E}} = \|x\|_{\ell_1(w)}$.

By Lemmas 1 and 2 the space $\widehat{E}$ is order continuous, and so the sequence (e_n) is a Schauder basis in $\widehat{E}$. Therefore by density of $\operatorname{span}(e_n)$ in both $\widehat{E}$ and $\ell_1(w)$ we finish the proof. ∎

Recall that $h \in L^0$ is called a step function whenever it assumes a finite number of values on a finite union of disjoint intervals of finite measure. It is well known that by regularity of the Lebesgue measure on $\mathbb{R}$, for every non-negative $f \in L^0$ there exists a monotone sequence of non-negative step functions h_n such that $h_n \uparrow f$ a.e.. Hence if E is order continuous then step functions are dense in E.

Theorem 4. *Let E be a p-Banach function space over $(0, \infty)$. Assume that E is 1-concave on E^+ and $\chi_{(a,b)} \in E$ for every $0 < a < b < \infty$. Suppose also that for any*

$\epsilon > 0, \alpha > 0$ there exists $\delta > 0$ such that for any two intervals $(a,b),(c,d) \subset (\alpha,\infty)$ with $|b \vee d - a \wedge c| < \delta$ we have

$$\left|\frac{1}{b-a}\|\chi_{(a,b)}\|_E - \frac{1}{d-c}\|\chi_{(c,d)}\|_E\right| < \epsilon. \tag{1}$$

Define the function w by

$$w(t) = \lim_{n\to\infty} \frac{n}{2}\|\chi_{(t-\frac{1}{n},t+\frac{1}{n})}\|_E.$$

Then the Banach envelope $\widehat{E}$ of E coincides up to equivalence of norms with the space $L_1(w)$. If the 1-concave constant $C = 1$, then the Banach envelope $\widehat{E}$ of E is isometrically isomorphic to the space $L_1(w)$.

Proof. The space E is order continuous by Lemma 1. By the assumption (1) the function w is well defined. It is also measurable since the function $h(t) := \|\chi_{(t-a,t+a)}\|_E$ is continuous on $(0,\infty)$. Indeed taking $0 < t_n < t$ and $t_n \to t$ as $n \to \infty$, we get by order continuity of E,

$$\|\chi_{(t-a,t+a)} - \chi_{(t_n-a,t_n+a)}\|_E^p \leqslant \|\chi_{(t_n-a,t-a)}\|_E^p + \|\chi_{(t_n+a,t+a)}\|_E^p \to 0,$$

as $n \to \infty$. Consequently

$$|h^p(t) - h^p(t_n)| \leqslant \|\chi_{(t-a,t+a)} - \chi_{(t_n-a,t_n+a)}\|_E^p \to 0.$$

By Lemma 2, $\widehat{E}$ is also order continuous. Let $f \in \widehat{E}$ and assume without loss of generality that $f \geqslant 0$. By order continuity of the norm $\|\cdot\|_{\widehat{E}}$ for any $\epsilon > 0$ there exists $\alpha > 0$ such that

$$\|f\chi_{(0,\alpha)}\|_{\widehat{E}} < \epsilon. \tag{2}$$

Define

$$S = \{h:\ h \text{ step function},\ 0 \leqslant h \leqslant f\chi_{(\alpha,\infty)},\ \|(f-h)\chi_{(\alpha,\infty)}\|_{\widehat{E}} < \epsilon\}. \tag{3}$$

Then

$$\int_\alpha^\infty fw = \sup_{h\in S}\int_\alpha^\infty hw. \tag{4}$$

Let $h \in S$. By the assumption (1) there is $\delta > 0$ such that if $m(B) < \delta$ and $B \subset (\alpha,\infty)$ is an interval, then for $t \in B$,

$$\left|w(t) - \frac{\|\chi_B\|_E}{m(B)}\right| < \frac{\epsilon}{\int h}\ . \tag{5}$$

Let now $h = \sum_{j=1}^m a_j\chi_{A_j}$, where $a_j > 0$, $A_j \subset (\alpha,\infty)$ are disjoint intervals with

$m(A_j) < \delta$ for all $j = 1, \ldots, m$. Thus by (2), (3) and (5) we have

$$\begin{aligned}\|f\|_{\widehat{E}} &\leqslant \|f\chi_{(\alpha,\infty)}\|_{\widehat{E}} + \epsilon \leqslant 2\epsilon + \|h\|_{\widehat{E}} \leqslant 2\epsilon + \sum_{j=1}^m a_j \|\chi_{A_j}\|_E \\ &= 2\epsilon + \sum_{j=1}^m \left(\int_{A_j} h \frac{\|\chi_{A_j}\|_E}{m(A_j)} \right) \leqslant 2\epsilon + \sum_{j=1}^m \left(\int_{A_j} h \left(w + \frac{\epsilon}{\int h} \right) \right) \\ &= 2\epsilon + \sum_{j=1}^m \int_{A_j} hw + \sum_{j=1}^m \left(\int_{A_j} h \right) \frac{\epsilon}{\int h} = 3\epsilon + \int hw \leqslant 3\epsilon + \int fw.\end{aligned}$$

It follows that

$$\|f\|_{\widehat{E}} \leqslant \int fw.$$

Now we will show the opposite inequality. Let $\epsilon > 0$ and $\alpha > 0$ be arbitrary. Take any step function $h \in S$. Applying (3) and definition of the norm $\|\cdot\|_{\widehat{E}}$ there exist $h_j \in E^+$, $j = 1, \ldots, m$ such that $h = \sum_{j=1}^m h_j$ and

$$\|f\|_{\widehat{E}} \geqslant \|f\chi_{(\alpha,\infty)}\|_{\widehat{E}} \geqslant \|h\|_{\widehat{E}} \geqslant \sum_{j=1}^m \|h_j\|_E - \epsilon.$$

Further for each j there exists a sequence $(h_j^{(n)})$ of non-negative step functions such that

$$h_j^{(n)} = \sum_i b_{ij}^{(n)} \chi_{B_{ij}^{(n)}} \qquad \text{with} \quad h_j^{(n)} \uparrow h_j \quad a.e.,$$

where $B_{ij}^{(n)} \subset (\alpha, \infty)$ are intervals of finite measure. By possible refining assume also that $m(B_{ij}^{(n)}) < \delta$ for all i, j and $n \in \mathbb{N}$. Let $\epsilon_0 = \frac{\epsilon}{\int h}$. Therefore in view of 1-concavity of E^+ and inequality (5) we get

$$\begin{aligned}\|f\|_{\widehat{E}} &\geqslant \sum_{j=1}^m \|h_j\|_E - \epsilon \geqslant \sum_{j=1}^m \|h_j^{(n)}\|_E - \epsilon = \sum_{j=1}^m \left\| \sum_i b_{ij}^{(n)} \chi_{B_{ij}^{(n)}} \right\|_E - \epsilon \\ &\geqslant C \sum_{j=1}^m \sum_i \int_{B_{ij}^{(n)}} \left(h_j^{(n)} \frac{\|\chi_{B_{ij}^{(n)}}\|_E}{m(B_{ij}^{(n)})} \right) - \epsilon \geqslant C \sum_{j=1}^m \sum_i \int_{B_{ij}^{(n)}} (h_j^{(n)}(w - \epsilon_0)) - \epsilon \\ &= C \sum_{j=1}^m \int h_j^{(n)} w - C\epsilon_0 \sum_{j=1}^m \int h_j^{(n)} - \epsilon \geqslant C \int \sum_{j=1}^m h_j^{(n)} w - C\epsilon_0 \int \sum_{j=1}^m h_j - \epsilon \\ &= C \int \sum_{j=1}^m h_j^{(n)} w - C\epsilon_0 \int h - \epsilon = C \int \sum_{j=1}^m h_j^{(n)} w - (C+1)\epsilon.\end{aligned}$$

Now in view of $h_j^{(n)} \uparrow h_j$ we obtain that $\int \sum_{j=1}^m h_j^{(n)} w \uparrow \int hw$, and so

$$\|f\|_{\widehat{E}} \geqslant C \int_\alpha^\infty hw - (C+1)\epsilon.$$

Finally since the above inequality is satisfied for every $\epsilon, \alpha > 0$ and every $h \in S$ we get by (4),

$$\|f\|_{\widehat{E}} \geqslant C \int fw.$$

Clearly if $C = 1$ then we obtain $\|f\|_{\widehat{E}} = \int fw$, which completes the proof. ■

Example 5 ([5]). Let $0 < p < 1$ and let $E = \ell_p$ with the standard p-norm $\|x\|_p = (\sum_{n=1}^{\infty} |x(n)|^p)^{1/p}$. Then ℓ_p is a 1-concave Banach lattice on E^+. Since the 1-concave constant is 1 and $\|e_n\|_E = 1$ for all $n \in \mathbb{N}$, so $\widehat{E}$ is isometrically isomorphic to ℓ_1.

Example 6. Let $0 < p < 1$ and define the operator $G : \ell^0 \to \ell^0$ for $x = (x(n))$ by

$$Gx(n) = \frac{1}{n} \sum_{i=1}^{n} |x(i)|, \qquad n \in \mathbb{N}.$$

Let $v = (v(n))$ be a weight sequence with $v(n) > 0$ for each $n \in \mathbb{N}$. Then the *Cesàro weighted sequence space* $\mathrm{ces}_p(v)$ is the set of all $x \in \ell^0$ with $G(x) \in \ell_p(v)$. This space is equipped with the p-norm

$$\|x\|_{\mathrm{ces}_p(v)} = \left(\sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{i=1}^{n} |x(i)| \right)^p v(n) \right)^{1/p}.$$

It is easy to see that $\mathrm{ces}_p(v)$ is non-trivial if and only if

$$\sum_{n=1}^{\infty} \frac{v(n)}{n^p} < \infty.$$

Thus for the constant weight sequence $v(n) = 1$, the space denoted by ces_p is trivial for any $0 < p < 1$. Clearly there exist weight sequences v that make the space non-trivial. We also observe that $\mathrm{ces}_p(v)$ is 1-concave with concavity constant equal to 1 on the cone of non-negative elements. Thus applying Theorem 3, the Banach envelope $\widehat{\mathrm{ces}_p(v)}$ is isometrically isomorphic to the space $\ell_1(w)$, where

$$w(n) = \|e_n\|_{\mathrm{ces}_p(v)} = \left(\sum_{i=n}^{\infty} \frac{v(i)}{i^p} \right)^{1/p}.$$

In particular if $v(n) = \frac{1}{n}$ then for each $n \in \mathbb{N}$,

$$w(n) = \left(\sum_{i=n}^{\infty} \frac{1}{i^{p+1}} \right)^{1/p} < \infty,$$

the space $\mathrm{ces}_p(v)$ is non-trivial and $\widehat{\mathrm{ces}_p(v)} = \ell_1(w)$.

Example 7. Similarly as for sequence spaces for $f \in L^0$ define the operator G for $s > 0$ as

$$G(f)(s) = \frac{1}{s} \int_0^s |f(t)| dt.$$

Let $0 < p < 1$ and v be a weight function that is $v \in L^0$ and $v > 0$ a.e.. Then the *Cesàro weighted function space* $\mathrm{Ces}_p(v)$ is the set of all $f \in L^0$ with $G(f) \in L_p(v)$. We shall assume further that for every $\alpha > 0$,

$$\int_\alpha^\infty \frac{v(t)}{t^p}\, dt < \infty.$$

This condition is equivalent to the fact that $\mathrm{Ces}_p(v)$ contains a positive function on $(0, \infty)$ [6]. The space $\mathrm{Ces}_p(v)$ is equipped with the p-norm

$$\|f\|_{\mathrm{Ces}_p(v)} = \left(\int G(f)^p v \right)^{1/p} = \left(\int \left(\frac{1}{t} \int_0^t |f| \right)^p v(t)\, dt \right)^{1/p}.$$

We will show that this space satisfies all assumptions of Theorem 4.

Let $0 < a < b < \infty$. Then

$$\begin{aligned}(b-a)^p \int_b^\infty \frac{v(t)}{t^p}\, dt &\leqslant \|\chi_{(a,b)}\|^p_{\mathrm{Ces}_p(v)} \\ &= \int_a^b \left(\frac{t-a}{t} \right)^p v(t)\, dt + (b-a)^p \int_b^\infty \frac{v(t)}{t^p}\, dt \\ &\leqslant (b-a)^p \int_a^\infty \frac{v(t)}{t^p}\, dt < \infty,\end{aligned}$$

and so every characteristic function $\chi_{(a,b)}$ belongs to $\mathrm{Ces}_p(v)$. We also have that for any $s > 0$,

$$\begin{aligned}w(s) &= \lim_{n\to\infty} \left(\left(\frac{n}{2} \right)^p \int_{s-\frac{1}{n}}^{s+\frac{1}{n}} \left(\frac{t - s + \frac{1}{n}}{t} \right)^p v(t)\, dt + \int_{s+\frac{1}{n}}^\infty \frac{v(t)}{t^p}\, dt \right)^{1/p} \\ &= \left(\int_s^\infty \frac{v(t)}{t^p}\, dt \right)^{1/p}.\end{aligned}$$

Now we will show condition (1). Let $\alpha > 0$ and $\alpha < a < b$, $\alpha < c < d$. Without loss of generality assume $a \wedge c = a$, $b \vee d = d$ and $\|\chi_{(a,b)}\|_{\mathrm{Ces}_p(v)} \geqslant \|\chi_{(c,d)}\|_{\mathrm{Ces}_p(v)}$. Let $\epsilon > 0$. There exists $A > \alpha$ such that $\int_A^\infty \frac{v(t)}{t^p}\, dt < \frac{\epsilon}{2}$. Denote $\phi(s) = \int_s^A \frac{v(t)}{t^p}\, dt$, $s \in [\alpha, A]$. Now it is enough to assume that $\alpha \leqslant a < d \leqslant A$. Then

$$\begin{aligned}L &:= \left| \frac{\|\chi_{(a,b)}\|_{\mathrm{Ces}_p(v)}}{b-a} - \frac{\|\chi_{(c,d)}\|_{\mathrm{Ces}_p(v)}}{d-c} \right| \\ &\leqslant \left(\int_a^\infty \frac{v(t)}{t^p}\, dt \right)^{1/p} - \left(\int_d^\infty \frac{v(t)}{t^p}\, dt \right)^{1/p} \leqslant \left(\phi(a) + \frac{\epsilon}{2} \right)^{1/p} - \phi(d)^{1/p}.\end{aligned}$$

Since ϕ is uniformly continuous on $[\alpha, A]$, there is $\delta > 0$ such that if $|d - a| < \delta$ then $\phi(a) \leqslant \phi(d) + \epsilon/2$. Observe also that the function $f(y) = (y + \epsilon)^{1/p} - y^{1/p}$ is increasing for $y \in [0, \phi(\alpha)]$ since $y^{1/p}$ is convex. Hence for $d \in [\alpha, A]$, $\phi(d) \leqslant \phi(\alpha)$, and so

$$L \leqslant (\phi(d) + \epsilon)^{1/p} - \phi(d)^{1/p} \leqslant (\phi(\alpha) + \epsilon)^{1/p} - \phi(\alpha)^{1/p},$$

where the right side above approaches 0 if $\epsilon \to 0$. Therefore condition (1) is satisfied.

Consequently, $\widehat{\mathrm{Ces}_p(v)}$ is isometrically isomorphic to the space $L_1(w)$ for $w(s) = \left(\int_s^\infty \frac{v(t)}{t^p}\right)^{1/p}$, $s > 0$.

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