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SLICE CONTINUITY FOR OPERATORS AND THE DAUGAVET PROPERTY FOR BILINEAR MAPS

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Dedicated to Lech Drewnowski on the occasion of his 70th birthday

Abstract: We introduce and analyse the notion of slice continuity between operators on Banach spaces in the setting of the Daugavet property. It is shown that under the slice continuity assumption the Daugavet equation holds for weakly compact operators. As an application we define and characterise the Daugavet property for bilinear maps, and we prove that this allows us to describe some p-convexifications of the Daugavet equation for operators on Banach function spaces that have recently been introduced.

Keywords: Daugavet property, slice, bilinear maps.

1. Introduction

A Banach space X is said to satisfy the Daugavet property if the so-called Daugavet equation

$$\|\mathrm{Id} + R\| = 1 + \|R\|$$

is satisfied for every rank one operator $R\colon X \to X$. In recent years, the Daugavet property for Banach spaces has been studied by several authors, and various applications have been found (see for instance [11, 12, 13, 23, 24]).

The aim of this paper is to introduce and analyse the notion of slice continuity between operators on Banach spaces. We will show that under this assumption one can easily characterise when the Daugavet equation holds for a couple of operators T and R between Banach spaces, i.e., when

$$\|T + R\| = \|T\| + \|R\|.$$

Recently, some new ideas have been introduced in this direction. The notion of Daugavet centre has been studied in [4, 5, 6]. According to Definition 1.2 in [6],

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a nonzero operator T between (maybe different) Banach spaces is a Daugavet centre if the above Daugavet equation holds for every rank one operator R. In this paper we develop a notion that is in a sense connected to this one but provides a direct tool for analysing when a particular couple of operators satisfies the Daugavet equation. Our idea is to relate the set of slices defined by each of the two operators. Recall that the slice $S(x', \varepsilon)$ of the unit ball of a real Banach space X determined by a norm one element $x' \in X'$ and an $\varepsilon > 0$ is the set

$$S(x', \varepsilon) = \{x \in B_X \colon \langle x, x' \rangle \geqslant 1 - \varepsilon\}.$$

Let Y be a Banach space. Let $T\colon X \to Y$ be an operator. We will define the set of slices associated to T by

$$S_T := \{S(T'(y')/\|T'(y')\|, \varepsilon) \colon 0 < \varepsilon < 1,\ y' \in Y',\ T'(y') \neq 0\},$$

and we will say that an operator $R\colon X \to Y$ is slice continuous with respect to T – we will write $S_R \leqslant S_T$ – if for every $S \in S_R$ there is a slice $S_1 \in S_T$ such that $S_1 \subset S$. This notion will be used for characterising when the Daugavet equation holds by adapting some of the known results on the geometric description of the Daugavet property to our setting. From the technical point of view, we use some arguments on the Daugavet property defined by subspaces of X' that can be found in [11]. This is done in Section 2. In Section 3 we develop the framework for using our results in the setting of the bilinear maps in order to obtain the main results of the paper regarding applications. Finally in Section 4 we provide examples and applications, mainly related to a unified general point of view to understand the p-convexification of the Daugavet equation for Banach function spaces that have recently been studied in [20, 21].

Our notation is standard. Let X, Y and Z be real Banach spaces. B_X and S_X are the unit ball and the unit sphere of X, respectively. We write U_X for the open unit ball and X' for the dual space. We denote by $L(X, Y)$ the space of continuous operators and by $B(X \times Y, Z)$ the space of continuous bilinear maps from $X \times Y$ to Z. If T is an operator, we write T' for its adjoint operator. If $x' \in X'$ and $y \in Y$, we identify the tensor $x' \otimes y$ with the operator $x' \otimes y\colon X \to Y$ mapping x to $x'(x)y$. Throughout the paper all the bilinear maps are assumed to be continuous. In general, we consider the norm $\|(x, y)\| = \max\{\|x\|_X, \|y\|_Y\}$ for the direct product $X \times Y$. We will say that a bilinear map B is convex or has convex range (resp. is weakly compact) if the norm closure of $B(B_X, B_Y)$ is convex (resp. weakly compact).

Regarding Banach function spaces we also use standard notation. If $1 \leqslant p \leqslant \infty$ we write p' for the extended real number satisfying $1/p + 1/p' = 1$. Let us fix some definitions and basic results. Let (Ω, Σ, μ) be a σ-finite measure space. A Banach function space $X(\mu)$ over the measure μ is an order ideal of $L^0(\mu)$ (the space of μ-a.e. equivalence classes of integrable functions) that is a Banach space with a lattice norm $\|\,.\,\|$ such that for every $A \in \Sigma$ of finite measure, $\chi_A \in X(\mu)$ (see [14, Def. 1.b.17]). We will write X instead of $X(\mu)$ if the measure μ is clear from the context. Of course, Banach function spaces are Banach lattices, so the following

definition makes sense for these spaces. Let $0 < p < \infty$. A Banach lattice E is p-convex if there is a constant K such that for each finite sequence $(x_i)_{i=1}^n$ in E,

$$\Big\|\Big(\sum_{i=1}^n |x_i|^p\Big)^{1/p}\Big\|_E \leqslant K\Big(\sum_{i=1}^n \|x_i\|_E^p\Big)^{1/p}.$$

It is said that it is p-concave if there is a constant k such that for every finite sequence $(x_i)_{i=1}^n$ in X,

$$\Big(\sum_{i=1}^n \|x_i\|_E^p\Big)^{1/p} \leqslant k\Big\|\Big(\sum_{i=1}^n |x_i|^p\Big)^{1/p}\Big\|_E.$$

$M^{(p)}(E)$ and $M_{(p)}(E)$ are the best constants in these inequalities, respectively.

Let $0 \leqslant p < \infty$. Consider a Banach function space $X(\mu)$. Then the set

$$X(\mu)_{[p]} := \{h \in L^0(\mu)\colon |h|^{1/p} \in X(\mu)\}$$

is called the p-th power of $X(\mu)$, which is a quasi-Banach function space when endowed with the quasinorm $\|h\|_{X_{[p]}} := \||h|^{1/p}\|_X^p$, $h \in X^p$ (see [8], [15, 7] or [16, Ch. 2]; the symbols that are used there for this concept are X^p, $X^{1/p}$ and $X_{[p]}$, respectively); if X is p-convex and $M^{(p)}(X(\mu)) = 1$ – we will say that X is constant 1 p-convex –, then $X(\mu)_{[p]}$ is a Banach function space, since in this case $\|\,.\,\|_{X_{[p]}}$ is a norm; if $0 < p < 1$, the p-th power of a Banach function space is always a Banach function space. Every p-convex Banach lattice can be renormed in such a way that the new norm is a lattice norm with p-convexity constant 1 ([14, Prop. 1.d.8]). Let $f \in X$. Throughout the paper we use the notation f^p for the sign preserving p-th power of the function f, i.e., $f^p := \operatorname{sign}\{f\}|f|^p$.

Remark 1.1. The following basic facts regarding p-th powers of Banach function spaces will be used several times. Their proofs are immediate using the results in [16, Ch. 2]. Let $X(\mu)$ be a Banach function space and $0 < p < \infty$.

(a) For every couple of functions $f \in X$ and $g \in X_{[p/p']}$ one has $\|fg\|_{X_{[p]}} \leqslant \|f\|_X\|g\|_{X_{[p/p']}}$, and

$$\|h\|_{X_{[p]}} = \inf\{\|f\|_X\|g\|_{X_{[p/p']}}\colon fg = h,\ f \in X,\ g \in X_{[p/p']}\}.$$

(b) For every $h \in X_{[p]}$ one has $h = |h|^{1/p}h^{1/p'}$, $h^{1/p} \in X$, $h^{1/p'} \in X_{[p/p']}$, and

$$\begin{aligned}\|h\|_{X_{[p]}} &= \|h^{1/p}\|_X^p = \|h^{1/p}\|_X\|h^{1/p}\|_X^{p/p'}\\ &= \|h^{1/p}\|_X\|(h^{1/p'})^{p'/p}\|_X^{p/p'} = \|h^{1/p}\|_X\|h^{1/p'}\|_{X_{[p/p']}}.\end{aligned}$$

2. Slice continuity for couples of linear maps

Let us start by adapting some facts that are already essentially well known (see [12]).

Proposition 2.1. *Let X and Y be Banach spaces. Let $T\colon X \to Y$ be a norm one linear map, and consider a norm one linear form $x' \in X'$. Let $y \in Y \setminus \{0\}$. The following assertions are equivalent.*

(1) $\|T + x' \otimes y\| = 1 + \|x' \otimes y\| = 1 + \|y\|$.
(2) *For every $\varepsilon > 0$ there is an element $x \in S(x', \varepsilon)$ such that*

$$\left\| T(x) + \frac{y}{\|y\|} \right\| \geqslant 2 - 2\varepsilon.$$

Proof. (1)⇒(2). By Lemma 11.4 in [1] (or [24, p. 78]) we can assume that $y \in S_Y$. By hypothesis, $\|T + x' \otimes y\| = 1 + \|y\| = 2$, and then there is an element $x \in B_X$ such that

$$2 - \varepsilon \leqslant \|T(x) + \langle x, x' \rangle y\| \leqslant \|T(x)\| + |\langle x, x' \rangle| \leqslant 1 + |\langle x, x' \rangle|.$$

Note that we can assume that $\langle x, x' \rangle > 0$; otherwise take $-x$ instead of x. Since for every $\varepsilon > 0$

$$\begin{aligned} 2 - \varepsilon &\leqslant \|T(x) + \langle x, x' \rangle y\| \leqslant \|T(x) + y\| + \|\langle x, x' \rangle y - y\| \\ &\leqslant \|T(x) + y\| + (1 - \langle x, x' \rangle)\|y\| \leqslant \|T(x) + y\| + \varepsilon, \end{aligned}$$

we obtain (2).

(2)⇒(1). Let $x' \in S_{X'}$ and $y \in Y$ and consider the rank one map $x' \otimes y$. Again by Lemma 11.4 in [1] we need consider only the case $\|y\| = 1$. Let $\varepsilon > 0$. Then there is an $x \in S(x', \varepsilon)$ such that $\|y + T(x)\| \geqslant 2 - 2\varepsilon$. Thus,

$$\begin{aligned} 2 - 2\varepsilon &\leqslant \|y + T(x)\| \leqslant \|y - \langle x, x' \rangle y\| + \|\langle x, x' \rangle y + T(x)\| \\ &\leqslant (1 - \langle x, x' \rangle)\|y\| + \|\langle x, x' \rangle y + T(x)\| \leqslant \varepsilon + \|x' \otimes y + T\|. \end{aligned}$$

Consequently, $\|x' \otimes y\| + \|T\| = 2 = \|x' \otimes y + T\|$. ∎

When a subset of linear maps $V \subset L(X, Y)$ is considered, the following generalisation of the Daugavet property makes sense.

Definition 2.2. Let X, Y be Banach spaces and let $T\colon X \to Y$ be a norm one operator. The Banach space Y has the T-Daugavet property with respect to $V \subset L(X, Y)$ if for every $R \in V$,

$$\|T + R\| = 1 + \|R\|.$$

This definition encompasses the notion of Daugavet centre given in Definition 1.2 of [6].

Corollary 2.3. *Let X and Y be Banach spaces. Let $T\colon X \to Y$ be an operator, and consider a set of norm one linear forms $W \subset X'$. Let $W \cdot Y = \{x' \otimes y\colon x' \in W,\ y \in Y\}$. The following statements are equivalent.*

(1) *Y has the T-Daugavet property with respect to $W \cdot Y$.*
(2) *For every $y \in S_Y$, for every $x' \in W$ and for every $\varepsilon > 0$ there is an element $x \in S(x', \varepsilon)$ such that*

$$\|T(x) + y\| \geqslant 2 - 2\varepsilon.$$

Definition 2.4. Let $T\colon X \to Y$ be a continuous linear map. Let $y' \in Y'$. We denote by $T_{y'}\colon X \to \mathbb{R}$ the linear form given by $T_{y'}(x) := \frac{\langle x, T'(y')\rangle}{\|T'(y')\|}$ whenever $T'(y') \neq 0$. The natural set of slices defined by T is then

$$S_T = \{S(T_{y'}, \varepsilon)\colon 0 < \varepsilon < 1,\ y' \in Y',\ T'(y') \neq 0\}.$$

If $R\colon X \to Y$ is another operator, we use the symbol $S_R \leqslant S_T$ to denote that for every slice S in S_R there is a slice $S_1 \in S_T$ such that $S_1 \subset S$. We will say in this case that R is slice continuous with respect to T.

For operators T having particular properties, slice continuity allows easy geometric descriptions. Let $T\colon X \to Y$ be an operator between Banach spaces such that T' is an isometry onto its range, i.e., T is a quotient map, and let $R\colon X \to Y$ be an operator. The following assertions are equivalent.

(1) $S_R \leqslant S_T$.
(2) For every $y \in S_Y$, $y' \in S_{Y'}$ such that $R'(y') \neq 0$, and every $\varepsilon > 0$ there is an element $y'_0 \in S_{Y'}$ such that $(R_{y'} \otimes y)(S(T'(y'_0), \varepsilon)) \subset B_\varepsilon(y)$.

To see this just notice that for every $y' \in S_{Y'}$ such that $R'(y') \neq 0$ and $y \in S_Y$

$$S(R_{y'}, \varepsilon) = \{x \in B_X\colon 1 - \varepsilon \leqslant R_{y'}(x) \leqslant 1\} = \{x \in B_X\colon \|R_{y'}(x)y - y\| \leqslant \varepsilon\}.$$

For a general operator T the canonical example of when the relation $S_R \leqslant S_T$ holds is given by the case $R = P \circ T$, where $T\colon X \to Y$ and $P\colon Y \to Y$ are operators. In this case, $\langle R(x), y'\rangle = \langle x, T'(P'(y'))\rangle$, and so clearly $S_R \leqslant S_T$. So the reason is that we have the inclusion $R'(Y') \subset T'(Y')$. However, there are examples of couples of operators T, R such that R is slice continuous with respect to T but $R \neq P \circ T$ for any operator P. Let us show one of them.

Example 2.5. Let $T\colon C[0,1] \oplus_1 \mathbb{R} \to C[0,1]$, $T(f, \alpha) = f$, and $R\colon C[0,1] \oplus_1 \mathbb{R} \to C[0,1]$, $R(f, \alpha) = f + \alpha\mathbf{1}$, where $\mathbf{1}$ stands for the constant one function and $\oplus_1$ denotes the direct sum with the 1-norm. Then R and T have norm one. Since the kernel of T is not contained in the kernel of R, we do not have $R = P \circ T$ for any operator P. But the slice condition holds. A simple calculation gives that for every μ in the unit sphere of $C[0,1]^*$, $\|T'(\mu)\| = \|R'(\mu)\| = 1$. Let $S_r \in S_R$ be the slice generated by any $\mu \in C[0,1]^*$ of norm one and $\varepsilon > 0$. We claim that the slice S_t generated by the same μ and $\varepsilon/2$ is contained in S_r. Indeed, if (f, α) is in the unit ball and $\langle \mu, f\rangle \geqslant 1 - \varepsilon/2$, then $\|f\| \geqslant 1 - \varepsilon/2$ and hence $|\alpha| \leqslant \varepsilon/2$. Therefore, for such (f, α), $\langle \mu, f + \alpha\mathbf{1}\rangle \geqslant \langle \mu, f\rangle - |\alpha| \geqslant 1 - \varepsilon$, and so the inclusion $S_t \subset S_r$ holds.

Remark 2.6. Let $T, R\colon X \to Y$ be a couple of operators, $\|T\| = 1$. Notice that Proposition 2.1 gives that for every $y \in S_Y$ and $y' \in Y'$ such that $R'(y') \neq 0$, the following are equivalent.

(1) $\|T + R_{y'} \otimes y\| = 2$.
(2) For every $\varepsilon > 0$ there is an element $x \in S(R_{y'}, \varepsilon)$ such that

$$\|T(x) + y\| \geqslant 2 - 2\varepsilon.$$

Thus for the case $R = T$ and assuming that T' is an isometry onto its range we obtain that Y has the Daugavet property if and only if Y has the T-Daugavet property with respect to the set $\{T_{y'}\colon y' \in Y' \setminus \{0\}\} \cdot Y$. This is a direct consequence of the well-known characterisation of the Daugavet property (see Lemma 2.1 in [12]) and Corollary 2.3. Consequently, for any other R, if Y has the Daugavet property and $S_R \leqslant S_T$, we obtain that for every $y \in S_Y$ and every $y' \in Y'$ such that $R'(y') \neq 0$,

$$\|T + R_{y'} \otimes y\| = 2.$$

Note that something like the slice continuity requirement $S_R \leqslant S_T$ is necessary for this to be true; indeed, a quotient map $T\colon X \to Y$ is not necessarily a Daugavet centre, even if the spaces involved have the Daugavet property. Take the operators $T, R\colon L^1[0,1] \oplus_1 L^1[1,2] \to L^1[0,1]$ given by $T((f,g)) := f$ and $R((f,g)) := (\int_1^2 g\,dx) \cdot h_0$, $(f,g) \in L^1[0,1] \oplus_1 L^1[1,2]$, where h_0 is a norm one function in $L^1[0,1]$. Clearly, $\|T\| = \|R\| = 1$, but $\|T + R\| \leqslant 1$.

Theorem 2.7. *Let Y be a Banach space with the Daugavet property. Let $T\colon X \to Y$ be an operator such that T' is an isometry onto its range and $R\colon X \to Y$ a norm one operator. Then:*

(1) *If for every $\varepsilon > 0$ there is a slice $S_0 \in S_T$ and an element $y \in S_Y$ such that $R(S_0) \subset B_\varepsilon(y)$, then*

$$\|T + R\| = 2.$$

(2) *If $S_R \leqslant S_T$ and R is weakly compact, then*

$$\|T + R\| = 2.$$

Proof. (1) Take $\varepsilon > 0$. Then there are $S_0 = S(T_{y_0'}, \delta) \in S_T$ and $y \in S_Y$ such that for every $x \in S_0$, $\|R(x) - y\| \leqslant \varepsilon$. We can assume that $\delta \leqslant \varepsilon$. Since Y has the Daugavet property, Y has the T-Daugavet property with respect to the set $\{T_{y'}\colon y' \in Y' \setminus \{0\}\} \cdot Y$ (see Remark 2.6 above). Therefore, by Corollary 2.3 we find an element $x \in S_0$ such that

$$\|T + R\| \geqslant \|T(x) + y\| - \|y - R(x)\| \geqslant 2 - \varepsilon - 2\delta \geqslant 2 - 3\varepsilon.$$

Since this holds for every $\varepsilon > 0$, the proof of (1) is complete.

The proof of (2) follows the same argument as the one for operators in spaces with the Daugavet property (see [12, Th. 2.3]), so we only sketch it. Assume that $\|R\| = 1$. Since by hypothesis $K = \overline{R(B_X)}$ is a convex weakly compact set, it is the closed convex hull of its strongly exposed points. Since this set is convex and $\|R\| = 1$, there is a strongly exposed point $y_0 \in K$ such that $\|y_0\| \leqslant 1$ and $\|y_0\| > 1 - \varepsilon$. Take a functional y_0' that strongly exposes y_0 and

satisfies $\langle y_0, y_0' \rangle = \max_{y \in K} \langle y, y_0' \rangle = 1$. It can be proved by contradiction that there is a slice $S \in S_R$ such that $R(S)$ is contained in the ball $B_\varepsilon(y_0)$ (see the proof of [12, Th. 2.3]). Since $S_R \leqslant S_T$, there is also a slice $S_0 \in S_T$ such that $R(S_0) \subset R(S) \subset B_\varepsilon(y_0)$. Then part (1) gives the result. ■

The example in Remark 2.6 makes it clear that some condition like slice continuity is necessary for (2) in Theorem 2.7 to be true. The following variation of this example gives a genuine weakly compact operator that is not of finite rank which does not satisfy the Daugavet equation. Take T defined as in Remark 2.6 and R: $L^1[0,1] \oplus_1 L^2[1,2] \to L^1[0,1]$ given by $R((f,g)) := g(x-1)$. This operator is weakly compact and $\|R\| = \|T\| = 1$, but again the norm of the sum of both operators is less than 2.

Remark 2.8. Notice that the condition in (1) on the existence of a slice $S \in S_T$ such that $R(S) \subset B_\varepsilon(y)$ can be substituted by the existence of a slice $S \in S_T$ and a $\delta > 0$ such that $R(S + \delta B_X) \subset B_\varepsilon(y)$. The argument given in the proof based on this fact makes it also clear that the relation $S_T \leqslant S_R$ can be substituted by the following weaker one and the result is still true: For every slice $S \in S_R$ and $\delta > 0$ there is a slice $S_1 \in S_T$ such that

$$S_1 \subset S + \delta B_X.$$

3. Bilinear maps and the Daugavet property

In this section we analyse the Daugavet property for bilinear maps defined on Banach spaces. Our main idea is to provide a framework for the understanding of several new Daugavet type properties and prove some general versions of the main theorems that hold for the case of the Daugavet property. We centre our attention on the extension of the Daugavet equation for weakly compact bilinear maps. Let X, Y and Z be Banach spaces. Consider a norm one continuous bilinear map B: $X \times Y \to Z$. Then we can consider the linearisation T_B: $X \hat{\otimes}_\pi Y \to Z$, where $X \hat{\otimes}_\pi Y$ is the projective tensor product with the projective norm π (see for instance [9, Sec. 3.2] or [19, Th. 2.9]). This linear operator will provide meaningful results for bilinear maps by applying the ones of Section 2. However, a genuinely geometric setting for bilinear operators – slices, isometric equations, ... – will also be defined in this section in order to provide the specific links between the (bilinear) slice continuity and the Daugavet equation.

We will consider bilinear operators B_0: $X \times Y \to Z$ satisfying that $B_0(U_X \times U_Y) = U_Z$. Obviously, such a map has always convex range, i.e., $B_0(U_X \times U_Y)$ is a convex set. We will say that a map satisfying these conditions is a norming bilinear map. If B_0 is such a bilinear operator, we will say that a Banach space Z has the B_0-Daugavet property with respect to the class of bilinear maps $V \subset B(X \times Y, Z)$ if

$$\|B_0 + B\| = 1 + \|B\|$$

for all $B \in V$. Notice that Z has the B_0-Daugavet property with respect to V if and only if it has the T_{B_0}-Daugavet property with respect to the set $\{T_B\colon X \hat{\otimes}_\pi Y \to Z\colon B \in V\}$. Let us consider some examples.

Example 3.1. (1) Take a Banach space X and consider the bilinear form $B_0\colon X \times X' \to \mathbb{R}$ given by $B_0(x, x') = \langle x, x'\rangle$, $x \in X$, $x' \in X'$. Consider the set

$$V = \{B_T\colon X \times X' \to \mathbb{R}\colon B_T(x, x') = \langle T(x), x'\rangle,\ T\colon X \to X \text{ is weakly compact}\}.$$

Then notice that

$$\sup_{x\in B_X, x'\in B_{X'}} |B_0(x, x') + B_T(x, x')| = \sup_{x\in B_X, x'\in B_{X'}} |\langle x + T(x), x'\rangle| = \|\mathrm{Id} + T\|$$

and $\|B_0\| + \|B_T\| = 1 + \|T\|$. Therefore $\mathbb{R}$ has the B_0-Daugavet property with respect to V if and only if X has the Daugavet property (see Theorem 2.3 in [12]).

(2) Take a measure space (Ω, Σ, μ) and a couple of Banach function spaces $X(\mu) = X$ and $Z(\mu) = Z$ over μ satisfying that the space of multiplication operators X^Z is a saturated Banach function space over μ and X is Z-perfect, i.e., $(X^Z)^Z = X$, and $U_X \cdot U_{X^Z} = U_Z$ (here $\cdot$ represents the pointwise product of functions). Consider the bilinear map $B_0\colon X \times X^Z \to Z$ given by $B_0(f, g) = f \cdot g$, $f \in X$, $g \in X^Z$ (see [7] for definitions and results regarding multiplication operators on Banach function spaces). Consider the set

$$V = \{B_S\colon X \times X^Z \to Z\colon B_S(f, g) = S(f \cdot g),\ S\colon Z \to Z \text{ is weakly compact}\}.$$

Then

$$\sup_{f\in B_X,\, g\in B_{X^Z}} \|B_0(f, g) + B_S(f, g)\|_Z = \sup_{f\in B_X,\, g\in B_{X^Z}} \|f \cdot g + S(f \cdot g)\|_Z = \|\mathrm{Id} + S\|$$

and $\|B_0\| + \|B_S\| = 1 + \|S\|$. Therefore Z has the B_0-Daugavet property with respect to V if and only if Z has the Daugavet property (see again Theorem 2.3 in [12]).

(3) Take $1 < p < \infty$, its conjugate index p', a measurable space (Ω, Σ), a Banach space Z and a countably additive vector measure $m\colon \Sigma \to Z$. Consider the corresponding spaces of m-integrable functions $L^p(m)$ and $L^{p'}(m)$, and the bilinear map $B_0\colon L^p(m) \times L^{p'}(m) \to Z$ given by the composition of the multiplication and the integration map $I_m\colon L^1(m) \to Z$, i.e., $B_0(f, g) = \int fg\, dm$. This map is well defined and continuous (see [16, Chapter 3] for the main definitions and results on the spaces $L^p(m)$). Assume also that $B_0(U_{L^p(m)} \times U_{L^{p'}(m)}) = I_m(U_{L^p(m)} \cdot U_{L^{p'}(m)})$ coincides with the open unit ball of Z. Take the set

$$U = \big\{B_R\colon L^p(m) \times L^{p'}(m) \to Z\colon B_R(f, g) := R(I_m(f \cdot g)),\ R\colon Z \to Z \text{ rank one}\big\}.$$

Since

$$\|B_0 + B_R\| = \sup_{f\in B_{L^p(m)},\ g\in B_{L^{p'}(m)}} \Big\| \int_\Omega fg\, dm + R\Big(\int_\Omega fg\, dm\Big)\Big\|_Z = \|\mathrm{Id} + R\|$$

and $\|B_0\| + \|B_R\| = 1 + \|R\|$, we obtain again that Z has the B_0-Daugavet property with respect to U if and only if Z has the Daugavet property.

Remark 3.2. More examples can be given by considering the following bilinear maps:

(i) $B_{C(K)}\colon C(K)\times C(K)\to C(K)$, $B_{C(K)}(f,g)=f\cdot g$.
(ii) $B_*\colon L^1(\mathbb{R})\times L^1(\mathbb{R})\to L^1(\mathbb{R})$, $B_*(f,g)=f*g$, where $*$ is the convolution product. In this case we have $B_*(U_{L^1(\mathbb{R})},U_{L^1(\mathbb{R})})=U_{L^1(\mathbb{R})}$ as a consequence of Cohen's Factorisation Theorem (see Corollary 32.30 in [10]).
(iii) For a σ-finite μ, $B_{L^\infty}\colon L^\infty(\mu)\times L^1(\mu)\to\mathbb{R}$ given by $B_{L^\infty}(f,g)=\int fg\,d\mu$.

Bilinear operators for which the Daugavet equation will be shown to hold – together with norming bilinear maps – are weakly compact operators with convex range. Although the usual way of finding such a map is to compose a bilinear map with convex range and a weakly compact linear one, other examples can be given. Let us show one of them that is in fact not norming.

Example 3.3. Consider a constant 1 p-convex reflexive Banach function space X. In particular, X must be order continuous. Take $f_0'\in S_{X'}$ and $f_0\in S_X$ and define the bilinear map $B\colon X\times X_{[p/p']}\to X_{[p]}$ given by $B(f,g)=\langle f,f_0'\rangle f_0\cdot g$. Note that $\|B\|=1$. Let us show that the (norm) closure $K=\overline{B(B_X\times B_{X_{[p/p']}})}$ is a convex weakly compact set.

Let $z_1,z_2\in B(B_X\times B_{X_{[p/p']}})$. Let f_1,g_1,f_2 and g_2 be such that $B(f_1,g_1)=z_1$ and $B(f_2,g_2)=z_2$. Take $0<\alpha<1$ and consider the element $\alpha z_1+(1-\alpha)z_2$. Let us prove that it belongs to $B(B_X\times B_{X_{[p/p']}})$. Notice that since $-1\leqslant\langle f,f_0'\rangle\leqslant 1$ for every $f\in B_X$, $g_3=\alpha\langle f_1,f_0'\rangle g_1+(1-\alpha)\langle f_2,f_0'\rangle g_2$ belongs to $B_{X_{[p/p']}}$. Take now an element $f_3\in B_X$ such that $\langle f_3,f_0'\rangle=1$ (it exists since X is reflexive), and note that

$$B(f_3,g_3)=\alpha z_1+(1-\alpha)z_2.$$

So, K is convex. Notice that $B(B_X\times B_{X_{[p/p']}})$ is also relatively weakly compact; it is enough to observe that the set is uniformly μ-absolutely continuous (see for instance Remark 2.38 in [16] and the references therein), i.e., that

$$\lim_{\mu(A)\to 0}\sup_{z\in K}\|z\chi_A\|=0.$$

But this is a direct consequence of the fact that X is order continuous (see for instance [14, Th. 1.c.5 and Prop. 1.a.8]) and the Hölder inequality for the norms of p-th power spaces (adapt [16, Lemma 2.21] or [14, Prop. 1.d.2(i)]). For every $z=\langle f,f_0'\rangle f_0\cdot g\in B(B_X,B_{X_{[p/p']}})$ and $A\in\Sigma$,

$$\|z\chi_A\|_{X_{[p]}}=\|\langle f,f_0'\rangle f_0\cdot g\|_{X_{[p]}}\leqslant|\langle f,f_0'\rangle|\|f_0\chi_A\|_X\cdot\|g\|_{X_{[p/p']}}.$$

Since X is order continuous, $\|f_0\chi_A\|_X\to 0$ when $\mu(A)\to 0$, which gives the result.

Let us now start to adapt the results of the previous section. In order to do so, let us define the natural set of slices associated to a norm one bilinear form $b\in B(X\times Y,\mathbb{R})$. Let $0<\varepsilon<1$. Following the notation given for the linear case, we define $S(b,\varepsilon)$ by

$$S(b,\varepsilon):=\{(x,y)\colon x\in B_X,\ y\in B_Y,\ b(x,y)\geqslant 1-\varepsilon\}.$$

The following result shows the relation between slices defined by a bilinear form and the ones defined by the linearisation of this map.

Lemma 3.4. *Let $b \in B(X \times Y, \mathbb{R})$ be a norm one bilinear form (i.e., $T_b \in (X \hat{\otimes}_\pi Y)'$ with norm one) and $\varepsilon > 0$. Then:*

(1) *There is an elementary tensor $x \otimes y$ such that $\|x\| = \|y\| = 1$ and $x \otimes y \in S(T_b, \varepsilon)$.*
(2) $\overline{\mathrm{co}\{x \otimes y\colon (x,y) \in S(b,\varepsilon)\}} \subset S(T_b, \varepsilon)$.
(3) $S(T_b, \varepsilon^2) \subset \overline{\mathrm{co}\{x \otimes y\colon (x,y) \in S(b,\varepsilon)\}} + 4\varepsilon B_{X \hat{\otimes}_\pi Y}$.

Proof. (1) Take a norm one element $t \in S(T_b, \varepsilon/2)$. Then there is an element $t_0 = \sum_{i=1}^n \alpha_i x_i \otimes y_i \in X \otimes Y$ such that $\|x_i\| = \|y_i\| = 1$, $\alpha_i > 0$ for all $i = 1, \ldots, n$, $\sum_{i=1}^n \alpha_i = 1$ and $\pi(t - t_0) < \varepsilon/2$. Then

$$\langle t_0, T_b \rangle = \langle t_0 - t, T_b \rangle + \langle t, T_b \rangle \geqslant \langle t, T_b \rangle - |\langle t_0 - t, T_b \rangle| > 1 - \frac{\varepsilon}{2} - \frac{\varepsilon}{2},$$

and so $t_0 \in S(T_b, \varepsilon)$. Then

$$\sum_{i=1}^n \alpha_i b(x_i, y_i) \geqslant \sum_{i=1}^n \alpha_i (1 - \varepsilon),$$

and so there is at least one index i_0 such that $b(x_{i_0}, y_{i_0}) \geqslant 1 - \varepsilon$. Consequently, $x_{i_0} \otimes y_{i_0} \in S(T_b, \varepsilon)$.

(2) is a direct consequence of the fact that $S(T_b, \varepsilon)$ is norm closed in the projective tensor product.

(3) Let us show now that $S(T_b, \varepsilon^2) \subset \overline{\mathrm{co}\{x \otimes y\colon (x,y) \in S(b,\varepsilon)\}} + 4\varepsilon B_{X \hat{\otimes}_\pi Y}$. Let $u \in S(T_b, \varepsilon^2)$. Find v such that $\|v\| < 1$, $T_b(v) \geqslant 1 - \varepsilon^2$, and $\|v - u\| \leqslant \varepsilon$. Write $v = \sum_{i=1}^\infty \alpha_i x_i \otimes y_i$ with all $\|x_i\| = \|y_i\| = 1$, $\alpha_i \geqslant 0$ and $\alpha := \sum_{i=1}^\infty \alpha_i < 1$. Note that $\alpha \geqslant 1 - \varepsilon^2$. Now consider

$$I := \{i \in \mathbb{N}\colon b(x_i, y_i) \geqslant 1 - \varepsilon\} = \{i \in \mathbb{N}\colon (x_i, y_i) \in S(b, \varepsilon)\},$$
$$J := \{i \in \mathbb{N}\colon b(x_i, y_i) < 1 - \varepsilon\}.$$

Let $\alpha_I := \sum_{i \in I} \alpha_i$ and $\alpha_J := \sum_{i \in J} \alpha_i$. We have

$$1 - \varepsilon^2 \leqslant \sum_{i=1}^\infty \alpha_i b(x_i, y_i) \leqslant \alpha_I + \alpha_J (1 - \varepsilon) < 1 - \varepsilon \alpha_J$$

and hence $\alpha_J < \varepsilon$. Let $w = \sum_I \frac{\alpha_i}{\alpha_I} x_i \otimes y_i \in \overline{\mathrm{co}\{x \otimes y\colon (x,y) \in S(b,\varepsilon)\}}$; we then have (note that $v = \alpha_I w + \sum_J \alpha_i x_i \otimes y_i$)

$$\|v - w\| \leqslant |\alpha_I - 1| \|w\| + \alpha_J.$$

Furthermore $0 \leqslant 1 - \alpha_I = \alpha_J + 1 - \alpha \leqslant \varepsilon + \varepsilon^2$; hence

$$\|u - w\| \leqslant \|u - v\| + \|v - w\| \leqslant \varepsilon + ((\varepsilon + \varepsilon^2) + \varepsilon) \leqslant 4\varepsilon,$$

as claimed. ∎

If $z \in Z$, we define $b_z\colon X \times Y \to Z$ as the (rank one) bilinear map given by $b_z(x,y) = b(x,y)z$, $x \in X$, $y \in Y$. Let $B\colon X \times Y \to Z$ be a continuous bilinear map. In what follows we need to introduce some elements related to duality and adjoint bilinear operators. Following Ramanujan and Schock in [17], we consider the adjoint operator $B^\times\colon Z' \to B(X,Y)$ given by $B^\times(z')(x,y) = \langle B(x,y), z'\rangle$ (this definition does not coincide with the one given originally by Arens in [2], although the setting is of course the same). $B^\times$ is a linear and continuous operator, and $\|B\| = \|B^\times\|$.

Definition 3.5. Let $B\colon X \times Y \to Z$ be a continuous bilinear map. Let $z' \in S_{Z'}$ and consider the adjoint bilinear form $\langle B, z'\rangle\colon X \times Y \to \mathbb{R}$ given by $\langle B, z'\rangle(x,y) = B^\times(z')(x,y)$. We denote by $B_{z'}\colon X \times Y \to \mathbb{R}$ the bilinear form given by $B_{z'}(x,y) = \frac{\langle B(x,y), z'\rangle}{\|\langle B, z'\rangle\|}$ whenever $\|\langle B, z'\rangle\| \neq 0$ and by $\langle B, Z'\rangle$ the set of all these bilinear forms. The natural set of slices defined by B is then

$$S_B = \{S(B_{z'}, \varepsilon)\colon 0 < \varepsilon < 1,\ \|B_{z'}\| \neq 0\}.$$

If B_1 is another (continuous) bilinear map, $B_1\colon X \times Y \to Z$, we use the symbol $S_B \leqslant S_{B_1}$ to denote that for every slice S in S_B there is a slice $S_1 \in S_{B_1}$ such that $S_1 \subset S$. We can also consider the relation $S_{T_B} \leqslant S_{B_1}$ to be defined in the same way: for every $S \in S_{T_B}$ there is a slice $S_1 \in S_{B_1}$ such that the set $\{x \otimes y\colon (x,y) \in S_1\}$ is included in S. Lemma 3.4 gives an idea of how this relation works.

As in the linear case, the canonical example of the relation $S_B \leqslant S_{B_1}$ between sets of slices associated to two bilinear maps is given by bilinear maps B that are defined as a composition $T \circ B_1$, where $B_1\colon X \times Y \to Z$ is a continuous bilinear map and $T\colon Z \to Z$ is a continuous operator. In this case, $\langle B(x,y), z'\rangle = \langle B_1(x,y), T'(z')\rangle$, and so clearly $S_B \leqslant S_{B_1}$. Let us show some examples.

Example 3.6. Let (Ω, Σ, μ) be a finite measure space and consider a rearrangement invariant (r.i.) constant 1 p-convex Banach function space $X(\mu)$ (see [14, p. 28 and Sections 1.d, 2.e] or [16, Ch. 2 and p. 202]). In this case, $(X(\mu)_{[p]})'$ is also r.i. Take a measurable bijection $\Phi\colon \Omega \to \Omega$ such that $\mu(\Phi(A)) = \mu(A)$ for every $A \in \Sigma$. Then it is possible to define the isometry $T_r\colon X_{[r]} \to X_{[r]}$, $0 \leqslant r \leqslant p$, by $T_r(f) = f \circ \Phi$.

Define the bilinear map $B\colon X \times X_{[p/p']} \to X_{[p]}$ given by $B(f,g) = T_1(f) \cdot T_{p/p'}(g)$. Let us show the relation between the slices defined by $B_0\colon X \times X_{[p/p']} \to X_{[p]}$, $B_0(f,g) = fg$, and the slices defined by B. Assume also that X is order continuous. Then $X_{[p]}$ is also order continuous and the dual of the space can be identified with the Köthe dual, which is also r.i., and so every continuous linear form is an integral. Note that in this case the property $S_B \leqslant S_{B_0}$ holds, since for every couple of functions $f \in X$ and $g \in X_{[p/p']}$, $B(f,g) = T_1(f) \cdot T_{p/p'}(g) = (f \circ \Phi) \cdot (g \circ \Phi) = (f \cdot g) \circ \Phi$. Consequently, every element $z' \in S_{(X_{[p]})'}$ satisfies

that for every pair of functions f and g as above,

$$\begin{aligned}\langle B_0(f,g), z'\rangle &= \int_\Omega fgz'\,d\mu = \int_\Omega ((fg)\circ\Phi)\cdot(z'\circ\Phi)\,d\mu \\ &= \int_\Omega B(f,g)\cdot(z'\circ\Phi)\,d\mu = \langle B(f,g), z'\circ\Phi\rangle.\end{aligned}$$

Therefore, there is a one-to-one correspondence between S_{B_0} and S_B given by identifying $S((B_0)_{z'},\varepsilon)$ and $S(B_{z'\circ\Phi},\varepsilon)$, which implies that $S_{B_0} = S_B$.

Fix a norming bilinear map $B_0\colon X\times Y\to Z$ and consider a norm one bilinear map $B\colon X\times Y\to Z$. Let us provide now geometric and topological properties for B that imply that the Daugavet equation is satisfied for B_0 and B, i.e., $\|B_0+B\| = 2$. These properties will be proved as applications of the result of the previous section.

Corollary 3.7. *Let Z be a Banach space with the Daugavet property. Let $B_0\colon X\times Y\to Z$ be a norming bilinear map and $B\colon X\times Y\to Z$ a continuous bilinear map. Then:*

(1) *If for every $\varepsilon>0$ there is a slice $S_0\in S_{B_0}$ and an element $z\in S_Z$ such that $B(S_0)\subset B_\varepsilon(z)$, then*
$$\|B_0+B\| = 1+\|B\|.$$

(2) *If $S_{T_B}\leqslant S_{B_0}$ and T_B is weakly compact (equivalently, $B(B_X\times B_Y)$ is a relatively weakly compact set), then*
$$\|B_0+B\| = 1+\|B\|.$$

Proof. (1) is just a consequence of Theorem 2.7(1): let us take $\varepsilon/5$ and apply this theorem to T_{B_0} and T_B. By hypothesis there is a slice $S_0 = S(b,\delta)\in S_{B_0}$ such that $B(S_0)\subset B_{\varepsilon/5}(z)$. We can assume without loss of generality that $\delta\leqslant\varepsilon/5$ and $\|B\|\leqslant 1$. Then by Lemma 3.4(3),

$$\begin{aligned}T_B(S(T_b,\delta^2)) &\subset T_B(\overline{\mathrm{co}(S_0)}) + \tfrac{4}{5}\varepsilon T_B(B_{X\hat\otimes_\pi Y}) \\ &\subset \overline{\mathrm{co}(B(S_0))} + \tfrac{4}{5}\varepsilon\|T_B\|B_Z \subset B_{\varepsilon/5}(z) + \tfrac{4}{5}\varepsilon B_Z \\ &\subset B_\varepsilon(z),\end{aligned}$$

and (1) is proved. For (2), apply Theorem 2.7(2) and Remark 2.8. ∎

Example 3.8. It is well known that for a purely non-atomic measure μ and a Banach space E the space of Bochner integrable functions $L^1(\mu,E)$ has the Daugavet property (see [12]). The next simple application of Corollary 3.7 provides a similar result for the Pettis norm $\|\,.\,\|_P$, i.e., for operators T from $(L^1(\mu,E),\|\,.\,\|_{L^1(\mu,E)})$ to the normed space $(L^1(\mu,E),\|\,.\,\|_P)$. Consider the bilinear map $B_0\colon L^1(\mu,E)\times E'\to L^1(\mu)$ given by

$$B_0(f,x')(w) = \langle f(w),x'\rangle,\qquad w\in\Omega.$$

Take an operator $T\colon L^1(\mu,E) \to L^1(\mu,E)$ and define the bilinear map $B_T\colon L^1(\mu,E)\times E' \to L^1(\mu)$ given by

$$B_T(f,x')(w) = \langle (T(f))(w), x'\rangle, \qquad w\in\Omega.$$

Assume that B_T is weakly compact and has convex range and suppose that $S_{B_T} \leqslant S_{B_0}$ (or that T_{B_T} is weakly compact and $S_{T_{B_T}} \leqslant S_{T_{B_0}}$). Then by Corollary 3.7(3) (or (4)),

$$\begin{aligned}\sup_{f\in B_{L^1(\mu,E)}} \|f+T(f)\|_P &= \sup_{f\in B_{L^1(\mu,E)},\ x'\in B_{E'}} \|B_0(f,x')+B_T(f,x')\|_{L^1(\mu)}\\ &= \sup_{f\in B_{L^1(\mu,E)},\ x'\in B_{E'}} \|B_0(f,x')\|\\ &\quad + \sup_{f\in B_{L^1(\mu,E)},\ x'\in B_{E'}} \|B_T(f,x')\|_{L^1(\mu)}\\ &= \sup_{f\in B_{L^1(\mu,E)}} \|f\|_P + \sup_{f\in B_{L^1(\mu,E)}} \|T(f)\|_P.\end{aligned}$$

Corollary 3.7 suggests that the natural examples of bilinear maps that satisfy the Daugavet equation with respect to B_0 are the ones defined as $B = T\circ B_0$, where $T\colon Z\to Z$ is a weakly compact operator. Corollary 3.10 generalises in a sense the idea of (2) and (3) in Example 3.1. Notice, however, that there are other simple bilinear maps that fit into the Daugavet setting, as the following example shows.

Example 3.9. Let us show an example of a bilinear map $B\colon X\times Y\to Z$ such that B_0 and B satisfy the Daugavet equation but there is no operator $T\colon Z\to Z$ such that $B = T\circ B_0$. Let (Ω,Σ,μ), $X(\mu)$ and Φ be as in Example 3.6 and consider the isometry $T_1\colon X\to X$ defined there. Assume also that $\mu(\Omega)<\infty$ and the constant 1 function satisfies $\|\chi_\Omega\|_X = 1$. Consider the bilinear map $B\colon X\times X_{[p/p']}\to X_{[p]}$ given by $B(f,g) = T_1(f)\cdot g$. Then, since $T_1(\chi_\Omega)=\chi_\Omega$,

$$\begin{aligned}2 &\geqslant \|B_0+B\| = \sup_{f\in B_X,\ g\in B_{X_{[p/p']}}} \|fg+T_1(f)g\|_{X_{[p]}}\\ &= \sup_{f\in B_X,\ g\in B_{X_{[p/p']}}} \|(f+T_1(f))g\|_{X_{[p]}} = \sup_{f\in B_X} \|f+T_1(f)\|_X\\ &\geqslant \|\chi_\Omega+\chi_\Omega\| = 2.\end{aligned}$$

Notice that in general a bilinear map defined in this way cannot be written as $T\circ B_0$ for any operator T. For instance, suppose that there is a set $B\in\Sigma$ such that $0<\mu(B)$ and $B\cap\Phi(B)=\emptyset$ and consider a couple of non-trivial functions f_1 and f_2 in X with support in $\Phi(B)$ and B, respectively, and such that $\|(f_1\circ\Phi)\cdot f_2\|>0$. Then $B_0(f_1,f_2)=0$, but $B(f_1,f_2)\neq 0$, so there is no operator $T\colon X_{[p]}\to X_{[p]}$ such that $B = T\circ B_0$.

Corollary 3.10. *Let $B_0\colon X\times Y\to Z$ be a norming bilinear map. Consider the subsets R, C and WC of $L(Z,Z)$ of rank one, compact and weakly compact operators, respectively, and the sets $R\circ B_0 = \{B = T\circ B_0\colon X\times Y\to Z\colon T\in R\}$,*

$C \circ B_0 = \{B = T \circ B_0 \colon X \times Y \to Z \colon T \in C\}$ and $WC \circ B_0 = \{B = T \circ B_0 \colon X \times Y \to Z \colon T \in WC\}$. Then the following are equivalent.

(1) *Z has the Daugavet property.*
(2) *Z has the B_0-Daugavet property with respect to $R \circ B_0$.*
(3) *Z has the B_0-Daugavet property with respect to $C \circ B_0$.*
(4) *Z has the B_0-Daugavet property with respect to $WC \circ B_0$.*
(5) *For every norm one operator $T \in R$, every $z \in Z$ and every $\varepsilon > 0$ there is an element $(x, y) \in S(T \circ B_0, \varepsilon)$ such that*

$$\|z + B_0(x, y)\| \geqslant 2 - \varepsilon.$$

Proof. The equivalence between (1) and (2) is a direct consequence of the following equalities. For every rank one operator $T \colon Z \to Z$,

$$\|\mathrm{Id} + T\| = \sup_{z \in B_Z} \|z + T(z)\| = \sup_{x \in B_X, y \in B_Y} \|B_0(x, y) + T(B_0(x, y))\|.$$

Since the norm closure of the convex hull $B(B_X \times B_Y)$ is a weakly compact set, (2) implies (4) as a consequence of Corollary 3.7(2). Obviously (4) implies (2), and so the equivalence of (2) and (3) is also clear. The equivalence of (2) and (5) holds as a direct consequence of Corollary 2.3 and the arguments used above. ■

Remark 3.11. Conditions under which a bilinear map $B \colon X \times Y \to Z$ is compact or weakly compact (i.e., the norm closure $\overline{B(B_X \times B_Y)}$ is compact or weakly compact, respectively) have been studied in several papers; see [17, 18] for compactness and [3, 22] for weak compactness. The reader can find in these papers some factorisation theorems and other characterisations of these properties, also related with the notion of Arens regularity of a bilinear map.

4. Applications. *p*-convexifications of the Daugavet property and bilinear maps

Different p-convexifications of the Daugavet property have been introduced in [20, 21]. In this section we show that in a sense they can be considered as particular cases of a Daugavet property for bilinear maps. We centre our attention on the case of Banach function spaces such that their p-th powers have the Daugavet property that have been characterised in [20]. However, more examples of applications will be given as well. Throughout this section μ is supposed to be finite.

We explain now two suitable examples of p-convexification of the Daugavet property. Let us start with one regarding p-concavity in Banach function spaces.

Example 4.1. Let $1 \leqslant p < \infty$. Consider a constant 1 p-convex Banach function space X, $Y = X_{[p/p']} \oplus_\infty X_{[p/p']}$ (the direct product with the maximum norm), $Z = X_{[p]}$, and the bilinear map $B_0 \colon X \times (X_{[p/p']} \times_\infty X_{[p/p']}) \to X_{[p]}$ given by $B_0(f, (g, h)) = f \cdot P_1(g, h) = fg$. Take an operator $T \colon X \to X$ and consider the bilinear map $B \colon X \times (X_{[p/p']} \oplus_\infty X_{[p/p']}) \to X_{[p]}$ given by $B(f, (g, h)) = T(f) \cdot P_2(g, h) = fh$ (here P_1 and P_2 denote the two natural projections in the

product space $X_{[p/p']} \oplus_\infty X_{[p/p']}$). A direct calculation shows that in this case the Daugavet equation for the pair given by B_0 and B is

$$\|B_0 + B\| = 1 + \|T\|,$$

since $\|B\| = \|T\|$. Assume that $\|T\| = 1$. Then $\|T(f)\| \leqslant 1$ for every $f \in B_X$, and so, taking $g = f^{p/p'} \in B_{X_{[p/p']}}$ and $h = T(f)^{p/p'} \in B_{X_{[p/p']}}$ for each $f \in B_X$, we obtain

$$\begin{aligned} 2 \geqslant \|B_0 + B\| &= \sup_{f \in B_X,\ g \in B_{X_{[p/p']}},\ h \in B_{X_{[p/p']}}} \|fg + T(f) \cdot h\|_{X_{[p]}} \\ &\geqslant \sup_{f \in B_X} \||f|^p + |T(f)|^p\|_{X_{[p]}} \\ &\geqslant \sup_{f \in B_X} \|(|f|^p + |T(f)|^p)^{1/p}\|_X^p. \end{aligned}$$

Thus, if X is also a constant 1 p-concave space (i.e., X is an L^p-space) we get

$$\sup_{f \in B_X} \|(|f|^p + |T(f)|^p)^{1/p}\|_X^p \geqslant \sup_{f \in B_X} (\|f\|_X^p + \|T(f)\|_X^p) = 2.$$

Therefore, in this case the Daugavet equation holds for B_0 and for every bilinear map B defined by an operator $T\colon X \to X$ in the way explained above.

The following construction shows another example of a Daugavet type property for a bilinear map that is in fact a p-convex version of the Daugavet property, in the sense that is studied in [21].

Example 4.2. Let (Ω, Σ, μ) be a measure space and consider an r.i. constant 1 p-convex Banach function space $X(\mu)$. Consider as in Example 3.6 the bilinear map B_0 given by the product and a measurable bijection $\Phi\colon \Omega \to \Omega$ satisfying that $\mu(\Phi(A)) = \mu(A)$ for every $A \in \Sigma$ and the isometries $T_r\colon X_{[r]} \to X_{[r]}$, $0 < r \leqslant p$.

Take the bilinear map $B\colon X \times X_{[p/p']} \to X_{[p]}$ given by $B(f, g) = T_1(f) \cdot T_{p/p'}(g)$. Notice that $\|B\| = 1$. Then

$$\begin{aligned} 2 \geqslant \|B_0 + B\| &\geqslant \sup_{f \in B_X,\ g \in B_{X_{[p/p']}}} \|B_0(f, g) + B(f, g)\|_{X_{[p]}} \\ &\geqslant \sup_{f \in B_X} \|f \cdot f^{p/p'} + T_1(f) \cdot T_{p/p'}(f^{p/p'})\|_{X_{[p]}} \\ &= \sup_{x \in B_X} \|f^p + T_1(f)^p\|_{X_{[p]}} \\ &= \sup_{f \in B_X} \||f^p + T_1(f)^p|^{1/p}\|_X^p. \end{aligned}$$

Now, if Φ satisfies that there is a set $A \in \Sigma$ such that $\mu(A \cap \Phi(A)) < \mu(A)$, there is a norm one function f_0 such that f_0 and $T_1(f_0)$ are disjoint and $\|T_1(f_0)\| = 1$. Assume that X is also p-concave (constant 1), i.e., X is an L^p-space. Then

$$\sup_{f \in B_X} \||f^p + T_1(f)^p|^{1/p}\|_X^p \geqslant \|f_0\|_X^p + \|T_1(f_0)\|_X^p = 2,$$

and thus the so called p-Daugavet equation is satisfied for T_1 (see Definition 1.1 in [21]), and B and B_0 satisfy the Daugavet equation.

Let $1 \leqslant p < \infty$. In what follows we study the p-convex spaces whose p-th powers satisfy the Daugavet property by giving some general results in the setting of the examples presented above. We analyse the case of $X = X(\mu)$, a constant 1 p-convex Banach function space, $Y = X(\mu)_{[p/p']}$, $Z = X(\mu)_{[p]}$, and $B_0\colon X \times X_{[p/p']} \to X_{[p]}$ given by $B_0(f,g) = f \cdot g$. We assume that $X_{[p]}$ has the Daugavet property. The main example we have in mind is given by $X = L^p[0,1]$, $Y = X_{[p/p']} = L^{p'}[0,1]$ and $Z = X_{[p]} = L^1[0,1]$. Recall that μ is assumed to be finite.

Definition 4.3. Let $X(\mu)$, $Y(\mu)$ and $Z(\mu)$ be three Banach function spaces over μ. We say that a continuous bilinear map $B\colon X(\mu) \times Y(\mu) \to Z(\mu)$ satisfying that for every $A, C \in \Sigma$, $B(\chi_A, \chi_C) = B(\chi_{A\cap C}, \chi_{A\cup C})$, is a symmetric bilinear map.

Proposition 4.4. *Let $X(\mu)$ be an order continuous p-convex Banach function space with p-convexity constant equal to* 1*. Then the following assertions are equivalent.*

(1) *For every rank one operator $T\colon X(\mu)_{[p]} \to X(\mu)_{[p]}$,*

$$\sup_{f\in B_X} \||f^p + T(f^p)|^{1/p}\|_X^p = 1 + \|T\|.$$

(2) *For every rank one operator $T\colon X(\mu)_{[p]} \to X(\mu)_{[p]}$,*

$$\|B_0 + T \circ B_0\| = 1 + \|T\|.$$

(3) *For every $z \in S_{X_{[p]}}$, for every $x' \in S_{(X_{[p]})'}$ and for every $\varepsilon > 0$ there is an element $(f,g) \in S((B_0)_{x'}, \varepsilon)$ such that*

$$\|z + B_0(f,g)\|_{X_{[p]}} \geqslant 2 - 2\varepsilon.$$

(4) *Each weakly compact symmetric bilinear map $B\colon X(\mu) \times X_{[p/p']} \to X_{[p]}$ satisfies the equation*

$$\|B_0 + B\| = 1 + \|B\|.$$

(5) *$X_{[p]}$ has the Daugavet property.*

Proof. For the equivalence of (1) and (2), note that the constant 1 p-convexity of X implies that $B_X \cdot B_{X_{[p/p']}} = B_{X_{[p]}}$ is the unit ball of the Banach function space $X_{[p]}$; so, using also Remark 1.1 the following inequalities are obtained:

$$\begin{aligned}\sup_{f\in B_X} \||f^p + T(f^p)|^{1/p}\|_X^p &\leqslant \sup_{f\in B_X,\ g\in B_{X_{[p/p']}}} \|fg + T(fg)\|_X \\ &\leqslant \sup_{h\in B_{X_{[p]}}} \|h + T(h)\|_{X_{[p]}} \\ &\leqslant \sup_{f\in B_X} \||f^p + T(f^p)|^{1/p}\|_X^p,\end{aligned}$$

and then both assertions are seen to be equivalent. The equivalence of (2) and (3) is obtained by applying Corollary 2.3 to the setting of bilinear maps.

Taking into account that the map $i_{[p]}\colon X \to X_{[p]}$ given by $i_{[p]}(f) = f^p$ is a bijection satisfying $\|i_{[p]}(f)\|_{X_{[p]}} = \|f\|_X^p$ for every $f \in X$, and the definition of the norm $\|\,.\,\|_{X_{[p]}}$, the equivalence of (1) and (5) is also clear using the well-known geometric characterisation of the Daugavet property in terms of slices (see for instance Lemma 2.2 in [12]).

Thus, it only remains to prove the equivalence of (2) and (4). Let us show first the following

Claim. *Let X be a p-convex (constant 1) Banach function space such that the simple functions are dense and let $B\colon X(\mu)\times X(\mu)_{[p/p']} \to X(\mu)_{[p]}$ be a continuous bilinear map. Then B is symmetric if and only if there is an operator $T\colon X_{[p]} \to X_{[p]}$ such that $B = T \circ B_0$.*

In order to prove this, note that by hypothesis the set $S(\mu)$ of simple functions is dense in $X(\mu)$ and so for every $0 \leqslant r \leqslant p$ it is also dense in $X(\mu)_{[r]}$; this can be shown by a direct computation just considering the definition of the norm in $\|\,.\,\|_{X_{[r]}}$ and the fact that if X is constant 1 p-convex then it is constant 1 r-convex for all such r, see for instance [14, Prop. 1.b.5] or [16, Prop. 2.54]. So this holds for $r = p/p'$. If B is symmetric, then for every couple of simple functions $f = \sum_{i=1}^n \alpha_i \chi_{A_i}$ and $g = \sum_{j=1}^m \beta_j \chi_{B_j}$, where $\{A_i\}_{i=1}^n$ and $\{B_i\}_{j=1}^m$ are sequences of pairwise disjoint measurable sets,

$$\begin{aligned} B(f,g) &= \sum_{i=1}^n \sum_{j=1}^m \alpha_i \beta_j B(\chi_{A_i}, \chi_{B_j}) \\ &= \sum_{j=1}^m \sum_{i=1}^n \alpha_i \beta_j B(\chi_{A_i \cap B_j}, \chi_{A_i \cup B_j}) \\ &= \sum_{j=1}^m \sum_{i=1}^n \beta_j \alpha_i B(\chi_{B_j}, \chi_{A_i}) = B(g,f). \end{aligned}$$

Therefore, because of the continuity of B and the order continuity of the spaces, $B(f,g) = B(g,f)$ for every couple of simple functions $f, g \in X \cap X_{[p/p']}$. Define now the map $T\colon X_{[p]} \to X_{[p]}$ by $T(h) = B(f,g)$ for every function $h = fg$, first for products of simple functions and then by density for the rest of the elements of $X_{[p]}$ (note that the norm closure of the set $(S(\mu)\cap B_X)\cdot(S(\mu)\cap B_{X_{[p/p']}})$ coincides with $B_{X_{[p]}}$). It can easily be proved that T is well defined since B is symmetric. For if f_1, g_1, f_2, g_2 are simple functions with $f_1 g_1 = f_2 g_2$, then $B(f_1, g_1) = B(f_2, g_2)$, and by continuity of B, $B(f,g) = B(g,f)$ for every couple $f \in X$ and $g \in X_{[p/p']}$. Further, T is continuous also by the continuity of B and Remark 1.1. Consequently, $B = T \circ B_0$ and the claim is proved.

Thus, (2) is equivalent to (4) as a consequence of Corollary 3.10, since the operator T constructed in the Claim is weakly compact if and only if B is weakly compact. ∎

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