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A BANACH FUNCTION SPACE X FOR WHICH ALL OPERATORS FROM ℓ^p TO X ARE COMPACT

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Dedicated to Lech Drewnowski on
the occasion of his 70th birthday

Abstract: We construct a rearrangement invariant space X on $[0,1]$ with the property that all bounded linear operators from ℓ^p, $1 < p < \infty$, to X are compact, but there exists a non-compact operator from ℓ^∞ to X. The techniques used allow to give a new proof of the characterization given by Hernández, Raynaud and Semenov of the rearrangement invariant spaces on $[0,1]$ for which the canonical embedding into $L^1([0,1])$ is finitely strictly singular.

Keywords: compact operator, strictly singular operator, rearrangement invariant space, vector measure.

1. The problem and the results

Given a Banach space X, let $ca(\Sigma, X)$ denote the space of all countably additive measures defined on the σ-algebra Σ with values in X, considered with the norm given by the semivariation, and $cca(\Sigma, X)$ denote the subspace of $ca(\Sigma, X)$ consisting of those measures with relatively compact range. For Y a Banach space, let $\mathcal{L}(Y, X)$ be the space of all bounded linear operators from Y to X, and $\mathcal{K}(Y, X)$ be the subspace of all compact operators in $\mathcal{L}(Y, X)$. If Z_1, Z_2 are Banach spaces, we write $Z_1 \subset Z_2$ whenever Z_2 has a subspace isomorphic to Z_1.

In [5] and [6] Drewnowski proved the following result: *Suppose that the σ-algebra Σ admits a nonzero atomless finite positive measure. Then the following conditions are equivalent.*

(i) $ca(\Sigma, X) \supset \ell_\infty$.
(ii) $ca(\Sigma, X) \supset c_0$.

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(iii) $cca(\Sigma, X) \supset c_0$.
(iv) $\mathcal{L}(\ell^2, X) \neq \mathcal{K}(\ell^2, X)$.

In analogy with the classical fact that c_0 is not complemented in ℓ^∞, it can be deduced from the above equivalences the following result, which was first proved by Emmanuelle, [7]:

If $\mathcal{L}(\ell^2, X) \neq \mathcal{K}(\ell^2, X)$ then $cca(\Sigma, X)$ is not complemented in $ca(\Sigma, X)$.

The following question naturally arises: for a Banach space X such that $cca(\Sigma, X) \neq ca(\Sigma, X)$, Is it true that $cca(\Sigma, X)$ is not complemented in $ca(\Sigma, X)$? A possible approach to this question would be to show that the existence of an X-valued measure with non-relatively compact range implies the existence of a noncompact operador from ℓ^2 to X. Taking into account the well known equivalence

$$ca(\Sigma, X) = cca(\Sigma, X) \iff \mathcal{L}(\ell^\infty, X) = \mathcal{K}(\ell^\infty, X),$$

for an arbitrary σ-algebra Σ (see [6]), the problem is that of deciding if

$$\mathcal{L}(\ell^2, X) = \mathcal{K}(\ell^2, X) \Rightarrow \mathcal{L}(\ell^\infty, X) = \mathcal{K}(\ell^\infty, X). \tag{1}$$

The question of complementability of $cca(\Sigma, X)$ in $ca(\Sigma, X)$ has been solved by Rodríguez-Piazza using a different approach. He proved that for an arbitrary σ-algebra Σ the subspace $cca(\Sigma, X)$ is complemented in $ca(\Sigma, X)$ if and only if $cca(\Sigma, X) = ca(\Sigma, X)$; [19].

In [10] Jarchow considered, for $1 \leqslant p \leqslant \infty$, the class $\mathcal{K}_p$ consisting of all Banach spaces X such that every weakly compact linear operator from $L^p = L^p([0,1])$ to X is compact. For $p = 1$ the class $\mathcal{K}_1$ consists of the Banach spaces X which have the Schur property. In the case $1 < p \leqslant \infty$ the class $\mathcal{K}_p$ consists of the Banach spaces X for which

$$\mathcal{L}(L^p, X) = \mathcal{K}(L^p, X). \tag{2}$$

For $1 < p < \infty$, this follows directly since L^p is reflexive. For $p = \infty$, it can be deduced using the fact that the space X contains a copy of ℓ^∞ whenever there exists a non weakly compact operator $T \in \mathcal{L}(L^\infty, X)$, [4, Corollary VI.1.3].

For $1 \leqslant p \leqslant 2$, condition (2) is equivalent to requiring that $\mathcal{L}(\ell^p, X) = \mathcal{K}(\ell^p, X)$ (see the proof of [10, Fact (2), p. 226]). For $p = \infty$, condition (2) is equivalent to $\mathcal{L}(\ell^\infty, X) = \mathcal{K}(\ell^\infty, X)$ since ℓ^∞ and L^∞ are isomorphic Banach spaces. Jarchow proved that for $1 < r < p \leqslant 2$ we have $\mathcal{K}_r \subsetneq \mathcal{K}_p$, and for $2 < q < \infty$ $\mathcal{K}_\infty \subseteq \mathcal{K}_q = \mathcal{K}_2$. He posed two problems:

(P1) Is $\mathcal{K}_\infty \subsetneq \mathcal{K}_2$?
(P2) Is $\mathcal{K}_p \subseteq \mathcal{K}_\infty$ when $1 < p \leqslant 2$?

Note that (P1) is precisely the question in (1). Jarchow showed that if X is a Banach space with finite cotype, then $X \in \mathcal{K}_p$ for some $p \leqslant 2$ implies $X \in \mathcal{K}_\infty$. Other conditions related to these questions can be found in [1], [3].

Problems (P1) and (P2) have remained unsolved. The aim of the present note is answer them showing that, for $1 < p < \infty$, we have $\mathcal{K}_p \not\subset \mathcal{K}_\infty$, that is

$$\mathcal{L}(\ell^p, X) = \mathcal{K}(\ell^p, X) \not\Rightarrow \mathcal{L}(\ell^\infty, X) = \mathcal{K}(\ell^\infty, X).$$

This follows from the main result of this paper.

Theorem A. *There exists a rearrangement invariant space X on $[0,1]$ such that for every $p \in (1,\infty)$ if $T\colon \ell^p \to X$ is a bounded linear operator then T is compact.*

Since X is a rearrangement invariant space on [0,1] it satisfies $L^\infty([0,1]) \subseteq X \subseteq L^1([0,1])$ with continuous inclusions, which shows that there exists a non-compact weakly compact operator from ℓ^∞ to X.

The construction of the Banach space X satisfying the properties of Theorem A will be given in Section 3. Though this space turns out to be a Lorentz space, we have chosen to present a slightly different approach. We think this approach is more self-contained and will allow a smoother reading for those who are not familiar with Lorentz spaces and their properties.

The techniques used to solve the previous problem also allow to address a different question.

Recall that a bounded linear operator $T\colon X \to Y$ between two Banach spaces is said to be *strictly singular* (SS in short) if it fails to be an isomorphism when restricted to any infinite dimensional subspace of X. Equivalently, T is SS if there is no infinite dimensional subspace F of X and no $\delta > 0$ such that

$$\|Tx\| \geqslant \delta\|x\|, \qquad \text{for all } x \in F.$$

The operator T is said to be *finitely strictly singular* (FSS) or *super strictly singular* if there is no $\delta > 0$ and no sequence $\{F_n\}_n$ of finite dimensional subspaces of X with $\dim(F_n) \to \infty$ such that

$$\|Tx\| \geqslant \delta\|x\|, \qquad \text{for all } x \in F_n, \ \text{for all } n \in \mathbb{N}.$$

Of course, if T is FSS then T is SS.

Let E be a rearrangement invariant space on $[0,1]$. The inclusion $E \hookrightarrow L^1$ is strictly singular if there is no infinite dimensional subspace F of E where the norms of E and L^1 are equivalent. Let G be the closure of $L^\infty([0,1])$ in the Orlicz space L^{Ψ_2} for $\Psi_2(t) := e^{t^2} - 1$. Note that when $G \subset E$ the norms of E, G and L^1 are equivalent on the subspace spanned by the Rademacher functions, and hence in this case the inclusion of $E \hookrightarrow L^1$ is not strictly singular. In [8] Hernández, Novikov and Semenov proved the converse: if $E \hookrightarrow L^1$ is not strictly singular then $G \subset E$. This result was improved by Hernández, Raynaud and Semenov, [9], proving:

Theorem B. *Let E be a r.i. space on $[0,1]$. If the inclusion $E \hookrightarrow L^1$ is not finitely strictly singular, then $G \subset E$.*

We will prove Theorem B in Section 4. A first application of the covering numbers techniques used in Section 3 allows to give a weaker result. Namely, we will prove in Corollary 11 that if $E \hookrightarrow L^1$ is not FSS then $G_1 \subset E$, where G_1 is the Lorentz space with the same fundamental function as G (in particular $G_1 \subset G$). The complete proof of Theorem B will require a new ingredient. We need to perform a random choice using Gaussian measures on $\mathbb{R}^n$.

Though our proof of Theorem B is not elementary, it is transparent in the sense that we have tried to provide all the ingredients needed to carry it out. The proof does not depend on deep structural results such as those about spaces containing copies of ℓ_p and p-stable variables.

2. Preliminaries

A rearrangement invariant (r.i.) space X is a Banach space of classes of measurable functions on [0,1] such that if $g^* \leqslant f^*$ and $f \in X$ then $g \in X$ and $\|g\|_X \leqslant \|f\|_X$. Here f^* is the decreasing rearrangement of f, that is, the right continuous inverse of its distribution function: $m_f(\lambda) = m(\{t \in [0,1] : |f(t)| > \lambda\})$, where m is the Lebesgue measure on [0,1]. The fundamental function of X is the function $\varphi_X(t) := \|\chi_{[0,t]}\|_X$.

Important examples of r.i. spaces are the Lorentz, Marcinkiewicz and Orlicz spaces. Let $\varphi\colon [0,1] \to [0,+\infty)$ be an increasing concave function with $\varphi(0+) = 0$, the Lorentz space $\Lambda(\varphi)$ consists of all measurable functions f on [0,1] such that

$$\|f\|_{\Lambda(\varphi)} := \int_0^1 f^*(s)\,\varphi'(s)\,ds \,<\, \infty.$$

If $\varphi\colon [0,1] \to [0,+\infty)$ is a quasi–concave function (i.e., φ increases, $\varphi(t)/t$ decreases and $\varphi(0) = 0$), we will denote by $M(\varphi)$ the Marcinkiewicz space whose fundamental function is φ, that is, the space of all measurable functions f on [0,1] for which

$$\|f\|_{M(\varphi)} := \sup_{0<t\leqslant 1} \frac{\varphi(t)}{t} \int_0^t f^*(s)\,ds < \infty.$$

Let N be an Orlicz function, that is, an increasing convex function on $[0,\infty)$ with $N(0) = 0$. The norm of the Orlicz space L^N is

$$\|f\|_{L^N} := \inf\left\{\lambda > 0 : \int_0^1 N\Big(\frac{|f(s)|}{\lambda}\Big)\,ds \,\leqslant\, 1\right\}.$$

The Marcinkiewicz $M(\varphi)$ and Lorentz $\Lambda(\varphi)$ spaces are, respectively, the largest and the smallest r.i. spaces with fundamental function φ, that is, if a r.i. space X has fundamental function equal to φ, then $\Lambda(\varphi) \subset X \subset M(\varphi)$, [12, Theorems II.5.5 and II.5.7].

A relevant space in the sequel is the closure of $L^\infty([0,1])$ in the Orlicz space L^{Ψ_2}, for $\Psi_2(t) := \exp(t^2) - 1$, usually denoted by G. Its importance is due to the following general version of Khintchine's inequality: the Rademacher functions span a subspace isomorphic to ℓ^2 in an r.i. space X if and only if $G \subset X$; [18]. The fundamental function of L^{Ψ_2} is $\varphi(t) = \log^{-1/2}(1/t+1)$. Since $\Psi_2(t)$ increases very rapidly, L^{Ψ_2} coincides with the Marcinkiewicz space $M(\varphi)$; [15]. This together with [12, Theorem II.5.3], gives that $\|f\|_{L^{\Psi_2}}$ is equivalent to

$$\sup_{0<t\leqslant 1} f^*(t)\log^{-1/2}(1/t+1).$$

Note that when $\|f\|_{L^{\Psi_2}} \leqslant 1$ we have

$$m(\{t \in [0,1] : |f(t)| > \lambda\}) \leqslant 2\exp(-\lambda^2), \qquad \lambda > 0.$$

Will be denote by G_1 the Lorentz space $\Lambda(\varphi)$ for φ the fundamental function of L^{Ψ_2}.

For facts related to r.i. spaces, see [2], [12], [14].

3. The construction

We proceed to the construction of the space X appearing in Theorem A. Given an integrable function $f\colon [0,1] \to \mathbb{R}$ and $0 < \varepsilon < 1$, let

$$p_\varepsilon(f) := \int_0^\varepsilon f^*(t)\,dt = \sup\Big\{ \int_A |f|\,dm : m(A) \leqslant \varepsilon \Big\}.$$

Proposition 1. *The following properties hold.*

(i) $p_\varepsilon(f) \leqslant \varepsilon\|f\|_\infty$.
(ii) $p_\varepsilon(f) \leqslant \|f\|_1$.
(iii) $\varepsilon\|f\|_1 \leqslant p_\varepsilon(f)$.
(iv) $m\big(\{t \in [0,1] : |f(t)| > p_\varepsilon(f)/\varepsilon\}\big) < \varepsilon$.

Proof. (i) and (ii) follow directly from the definition of $p_\varepsilon(f)$.

(iii) Since f^* is decreasing, the averages $p_\varepsilon(f)/\varepsilon = \frac{1}{\varepsilon}\int_0^\varepsilon f^*(t)\,dt$ decrease as ε increases to 1. Hence, $p_\varepsilon(f)/\varepsilon \geqslant p_1(f) = \int_0^1 f^*(t)\,dt = \int_0^1 |f(t)|\,dt$.

(iv) Set $B := \{t \in [0,1] : |f(t)| > p_\varepsilon(f)/\varepsilon\}$ and suppose that $m(B) \geqslant \varepsilon$. Then there exists $A_0 \subseteq B$ with $m(A_0) = \varepsilon$. Then

$$\int_{A_0} |f(t)|\,dt > \int_{A_0} p_\varepsilon(f)/\varepsilon\,dt = m(A_0)p_\varepsilon(f)/\varepsilon = p_\varepsilon(f),$$

which contradicts the definition of $p_\varepsilon(f)$. Hence, $m(B) < \varepsilon$. ∎

Let $\{\varepsilon_k\}$ be a positive sequence decreasing to zero and $\{a_k\}$ a sequence of strictly positive numbers such that $\sum_1^\infty \varepsilon_k a_k = 1$. Associated to these two sequences, we define the function space

$$X := \Big\{ f \in L^1([0,1]) : \|f\|_X := \sum_{k=1}^\infty a_k p_{\varepsilon_k}(f) < \infty \Big\}. \tag{3}$$

Proposition 2. *The space X defined by* (3) *satisfies the following properties.*

(i) *X is a rearrangement invariant Banach function space satisfying $\|f\|_1 \leqslant \|f\|_X \leqslant \|f\|_\infty$.*
(ii) *If $f_n \to f$ a.e. with $|f_n| \leqslant g$ a.e., for some $g \in X$, then f_n converge to f in X.*
(iii) *Simple functions are dense in X.*
(iv) *The X-valued vector measure $A \mapsto \chi_A$ has non relatively compact range in X.*

Proof. (i) Conditions (i) and (iii) in Proposition 1 and $\sum_1^\infty \varepsilon_k a_k = 1$ show that $\|f\|_1 \leqslant \|f\|_X \leqslant \|f\|_\infty$. That X is linear and $\|\cdot\|_X$ is a norm follows easily from the definition and the fact that each p_ε is a norm. The rearrangement invariance of X and $\|\cdot\|_X$ is a consequence of the rearrangement invariance of each p_ε. In order to check completeness, let $\{f_n\} \subset X$ be such that $\sum_1^\infty \|f_n\|_X < \infty$. It is easy to see that $\sum_1^\infty |f_n(t)| < \infty$ a.e. $t \in [0,1]$, and that, if $g(t) := \sum_1^\infty f_n(t)$, then $g \in X$ and $\|g - \sum_1^N f_n\|_X \leqslant \sum_{N+1}^\infty \|f_n\|_X \to 0$ as $N \to +\infty$.

(ii) The Dominated Convergence theorem together with Proposition 1(ii) gives that $p_{\varepsilon_k}(f - f_n) \to 0$ for all $k \geqslant 1$. Since $a_k p_{\varepsilon_k}(f - f_n) \leqslant 2a_k p_{\varepsilon_k}(g)$ for all $k \geqslant 1$, using the Dominated Convergence theorem for the series summation it follows that $\|f - f_n\|_X \to 0$.

(iii) Follows from (ii).

(iv) It follows from the fact that the inclusion $L^\infty([0,1]) \hookrightarrow X$ is not compact (as so is the inclusion $L^\infty([0,1]) \hookrightarrow L^1([0,1])$; see [4, Theorem VI.2.18]. ∎

Remark 3. The space X defined above is, in fact, a Lorentz space for $\varphi' = \sum_1^\infty a_k \chi_{[0,\varepsilon_k]}$. Moreover, any Lorentz space $\Lambda(\varphi)$ can be obtained by the construction given in (3). Indeed, given $f \in L^1([0,1])$ consider the measure $\mu_{f^*}(A) := \int_A f^*(t)\,dt$, which is finite. Then, $f \in \Lambda(\varphi)$ is equivalent to

$$\int_0^1 \varphi'(t)\,d\mu_{f^*}(t) < \infty \iff \sum_{n=1}^\infty 2^n \mu_{f^*}\bigl(\{t \in [0,1] : \varphi'(t) > 2^n\}\bigr) < \infty.$$

Observe that

$$\mu_{f^*}\bigl(\{t \in [0,1] : \varphi'(t) > 2^n\}\bigr) = \int_0^{\varepsilon_n} f^*(t)\,dt,$$

where ε_n is defined in such way that $\{t \in [0,1] : \varphi'(t) > 2^n\}$ is either $[0,\varepsilon_n]$ or $[0,\varepsilon_n)$. Consequently,

$$f \in \Lambda(\varphi) \iff \sum_{n=1}^\infty 2^n p_{\varepsilon_n}(f^*) < \infty.$$

The sequence $\{\varepsilon_n\}$ decreases strictly to zero since φ is increasing, concave and $\lim_{t\to 0} \varphi'(t) = +\infty$ (for $\Lambda(\varphi) \neq L^1([0,1])$). Note that $\sum_1^\infty \varepsilon_k a_k < \infty$ since $\chi_{[0,1]} \in \Lambda(\varphi)$.

Let X be defined as in (3). The following holds.

Lemma 4. *Suppose that $\{f_n\} \subseteq X$ converges weakly in X to zero and $\|f_n\|_X \geqslant C$ for some $C > 0$ and all $n \geqslant 1$. Then, there exists $\delta > 0$ such that $\|f_n\|_{L^1} \geqslant \delta$ for all $n \geqslant 1$.*

Proof. Let us assume by way of contradiction that there exists a subsequence of (f_n), which we will still denote by (f_n) for convenience, which converges to zero in $L^1([0,1])$. Set $n_1 = 1$ and, since $\|f_{n_1}\|_X \geqslant C$, choose $m_1 \in \mathbb{N}$ such that

$$\sum_{k=1}^{m_1} a_k p_{\varepsilon_k}(f_{n_1}) > C/2.$$

Since $\|f_n\|_{L^1} \to 0$ and $p_{\varepsilon_k}(f) \leqslant \|f\|_{L^1}$, for all $k \geqslant 1$, we can choose $n_2 \in \mathbb{N}$ such that

$$\sum_{k=1}^{m_1} a_k p_{\varepsilon_k}(f_{n_2}) < C/4,$$

and choose $m_2 \in \mathbb{N}$ such that

$$\sum_{k=m_1+1}^{m_2} a_k p_{\varepsilon_k}(f_{n_2}) > C/2.$$

Iterating this process we obtain sequences of integers $\{n_j\}$ and $\{m_j\}$ such that

$$\sum_{k=m_j+1}^{m_{j+1}} a_k p_{\varepsilon_k}(f_{n_{j+1}}) > C/2, \qquad j \geqslant 1 \tag{4}$$

Given any sequence $\{A_k\}$ of measurable sets with $A_k \subseteq [0,1]$ and $m(A_k) \leqslant \varepsilon_k$, and any sequence (g_k) of measurable functions satisfying $\|g_k\|_\infty \leqslant a_k$, we can define an operator $T\colon X \to \ell^1$ by

$$T(f) := \left(\int_{A_k} g_k(t) f(t)\, dt \right)_{k=1}^{\infty}, \qquad f \in X.$$

The operator T is well defined since each function $g_k f$ is integrable, and

$$\sum_{k=1}^{\infty} \left| \int_{A_k} g_k(t) f(t)\, dt \right| \leqslant \sum_{k=1}^{\infty} a_k \int_0^{\varepsilon_k} f^*(t)\, dt = \|f\|_X.$$

The above bound shows that T is bounded.

We now make a particular choice of sets A_k and functions g_k for those values of k satisfying $m_j + 1 \leqslant k \leqslant m_{j+1}$, for some $j \geqslant 1$. In this case, $g_k := a_k \chi_{A_k} \operatorname{sign}(f_{n_{j+1}})$, where the set A_k is chosen so that

$$\int_{A_k} g_k(t) f_{n_{j+1}}(t)\, dt \geqslant \frac{1}{2} a_k \int_0^{\varepsilon_k} f^*_{n_{j+1}}(t)\, dt.$$

From (4) we deduce that, for all $j \geqslant 1$,

$$\begin{aligned} \|T(f_{n_{j+1}})\|_{\ell^1} &\geqslant \sum_{k=m_j+1}^{m_{j+1}} \left| \int_{A_k} g_k(t) f_{n_{j+1}}(t)\, dt \right| \\ &\geqslant \frac{1}{2} \sum_{k=m_j+1}^{m_{j+1}} a_k \int_0^{\varepsilon_k} f^*_{n_{j+1}}(t)\, dt > C/4. \end{aligned}$$

However this contradicts the fact that $\{T(f_n)\}$ converges in norm to zero in ℓ^1, since (f_n) converges weakly to zero in X, the operator T is continuous and ℓ^1 has the Schur property. ∎

Remark 5. In view of Remark 3, the result of Lemma 4 is a general fact for Lorentz spaces. It can be alternatively proved by using the so called *subsequence splitting property*, developed by Kadec and Pełczyński for the spaces L_p in [11], and throughly studied in [20].

Next we consider a particular space X obtained by making an specific choice of sequences $\{\varepsilon_k\}$ and $\{a_k\}$.

Lemma 6. *Set* $\varepsilon_k := \exp(-e^{2^{k+1}})$ *and* $a_k := 2^{-k}\exp(e^{2^{k+1}})$, *for* $k \geqslant 1$. *Let* X *be the corresponding space defined by* (3). *Then, for every* $f \in X$ *and every* $\lambda \geqslant 2$ *we have*

$$m\big(\{t \in [0,1] : |f(t)| > \lambda \|f\|_X\}\big) \leqslant \exp(-e^{\lambda}).$$

Proof. Since f^* is decreasing, for any $k \in \mathbb{N}$ we have

$$\|f\|_X \geqslant a_k \int_0^{\varepsilon_k} f^*(t)\,dt \geqslant a_k \varepsilon_k f^*(\varepsilon_k) = \frac{1}{2^k} f^*(\varepsilon_k).$$

It suffices to prove the claim for norm one functions, so we assume $\|f\|_X = 1$. Given $\lambda \geqslant 2$, let $k_0 \in \mathbb{N}$ be such that $2^{k_0} \leqslant \lambda < 2^{k_0+1}$. Then, since f^* is decreasing, we have

$$1 \geqslant \frac{1}{2^{k_0}} f^*(\exp(-e^{2^{k_0+1}})) \geqslant \frac{1}{\lambda} f^*(\exp(-e^{\lambda})).$$

From the definition of the decreasing rearrangement it follows that

$$m\big(\{t \in [0,1] : |f(t)| > \lambda\}\big) \leqslant \exp(-e^{\lambda}). \qquad \blacksquare$$

The following standard result will be needed for proving Theorem A, see [10].

Lemma 7. *Let* $p \in (1,\infty)$ *and* Y *be a Banach space. If there exists* $T\colon \ell^p \to Y$ *a non-compact bounded linear operator, then there exists a bounded linear operator* $S\colon \ell^p \to Y$ *such that* $\|Se_n\|_Y \geqslant 1$ *for* $n \geqslant 1$.

The next result, which will be used for proving Theorem A, is of independent interest.

Theorem 8. *Let* $p \in (1,\infty)$ *and* $1/p + 1/q = 1$. *There exists a constant* $C_p > 0$ *such that if* $T\colon \ell^p \to L^0([0,1])$ *is a linear operator satisfying* $\|T(e_n)\|_{L^q} \geqslant 1$ *for all* e_n *belonging to the canonical basis in* ℓ^p, *then for every* $\lambda_0 > 0$ *there exists* $x \in \ell^p$ *with* $\|x\|_p \leqslant 1$ *and* $\lambda > \lambda_0$ *such that*

$$m\big(\{t \in [0,1] : |Tx(t)| > \lambda\}\big) > \exp(-C_p \lambda^q).$$

Proof. We prove that the statement of the theorem is true for C_p satisfying

$$C_p > 2^{q+1} \ln 5. \tag{5}$$

Suppose that the claim is not true for an operator T satisfying the hypotheses. Then we can find $N \in \mathbb{N}$ such that for all $\alpha \geqslant N^{1/q}/2^{1+1/q}$ and all $x \in \ell^p$ with $\|x\|_p \leqslant 1$ we have

$$m\big(\{t \in [0,1] : |Tx(t)| > \alpha\}\big) \leqslant \exp(-C_p\alpha^q). \tag{6}$$

Let $B_{\ell_p^N}$ be the unit ball of ℓ_p^N and $\mathcal{E}$ be an $1/2$–net in $B_{\ell_p^N}$, that is, for every $x \in B_{\ell_p^N}$ there exists $y \in \mathcal{E}$ with $\|x-y\|_{\ell_p^N} \leqslant 1/2$. Note that $\mathcal{E}$ can be taken to have cardinal bounded by 5^N, see, for instance, [17, Lemma 4.16]. Let $z \in \ell_q^N$ and take $x \in B_{\ell_p^N}$ such that $\|z\|_{\ell_q^N} = |\langle z, x\rangle|$. Let $y_0 \in \mathcal{E}$ with $\|x - y_0\|_{\ell_p^N} \leqslant 1/2$. Then,

$$\|z\|_{\ell_q^N} \leqslant |\langle z, x-y_0\rangle| + |\langle z, y_0\rangle| \leqslant \frac{1}{2}\|z\|_{\ell_q^N} + \max_{y\in\mathcal{E}} |\langle z, y\rangle|.$$

Consequently, for a.e. $t \in [0,1]$ we have

$$\Big(\sum_{k=1}^N \big|Te_k(t)\big|^q\Big)^{1/q} \leqslant 2\max_{x\in\mathcal{E}} \Big|\sum_{k=1}^N x_k Te_k(t)\Big| = 2\max_{x\in\mathcal{E}} \big|Tx(t)\big|,$$

from where it follows that

$$\Big\{t \in [0,1] : \sum_{k=1}^N \big|Te_k(t)\big|^q > \lambda\Big\} \subseteq \bigcup_{x\in\mathcal{E}} \big\{t\in[0,1] : 2^q|Tx(t)|^q > \lambda\big\},$$

from where we can deduce, using (6), that for $\lambda > N/2$

$$m\bigg(\Big\{t \in [0,1] : \sum_{k=1}^N \big|Te_k(t)\big|^q > \lambda\Big\}\bigg) \leqslant 5^N \exp(-C_p\lambda/2^q). \tag{7}$$

Then, since by assumption $\|Te_k\|_{L^q} \geqslant 1$, using (7) we get

$$\begin{aligned}
N &\leqslant \int_0^1 \sum_{k=1}^N \big|Te_k(t)\big|^q\,dt \\
&= \int_0^{+\infty} m\bigg(\Big\{t\in[0,1] : \sum_{k=1}^N \big|Te_k(t)\big|^q > \lambda\Big\}\bigg)\,d\lambda \\
&\leqslant \int_0^{N/2} d\lambda + 5^N \int_{N/2}^{+\infty} \exp(-C_p\lambda/2^q)\,d\lambda \\
&\leqslant \frac{N}{2} + 5^N \frac{2^q}{C_p}\exp(-C_pN/2^q2) \leqslant \frac{N}{2} + \frac{1}{3} < N,
\end{aligned}$$

since, by condition (5), we have $5^N\exp(-C_pN/2^q2) \leqslant 1$ and $2^q/C_p \leqslant 1/2\ln 5 \leqslant 1/3$. Hence, we have arrived at a contradiction. ∎

We now prove Theorem A using the previous lemmas and Theorem 8.

Proof of Theorem A. Let X be the space defined by (3) as done in Lemma 6.

Suppose that $T\colon \ell^p \to X$ is a bounded linear operator which is not compact. From Lemma 7 there exists a bounded linear operator $S\colon \ell^p \to X$ such that $\|Se_n\|_X \geqslant 1$ for $n \geqslant 1$. Since $e_n \to 0$ weakly in ℓ^p, we have $Se_n \to 0$ weakly in X. Applying Lemma 4 there exists $\delta > 0$ such that $\|Se_n\|_{L^1} \geqslant \delta$ for $n \geqslant 1$. Hence, for $J := S/\delta$ we have $J\colon \ell^p \to X \subset L^0([0,1])$ satisfying $\|Je_n\|_{L^q} \geqslant 1$ for $n \geqslant 1$ and q the conjugate index of p. Applying Theorem 8, there exists $C_p > 0$ such that for every $\lambda_0 > 0$ there exists $x \in \ell^p$ with $\|x\|_p \leqslant 1$ and $\lambda > \lambda_0$ such that

$$m\bigl(\{t \in [0,1] : |Jx(t)| > \lambda\}\bigr) > \exp(-C_p\lambda^q). \tag{8}$$

From Lemma 6, since $\|Jx\|_X \leqslant \|J\|$, we have, for all $\mu \geqslant 2$,

$$\begin{aligned} m\bigl(\{t \in [0,1] : |Jx(t)| > \mu\|J\|\}\bigr) &\leqslant m\bigl(\{t \in [0,1] : |Jx(t)| > \mu\|Jx\|_X\}\bigr) \\ &\leqslant \exp(-e^{\mu}). \end{aligned} \tag{9}$$

Setting $\mu\|J\| = \lambda$ in (8) and (9) we arrive at $C_p\mu^q\|J\|^q > e^{\mu}$. Since μ can be arbitrarily large there is a contradiction, which establishes the result. ∎

4. Finitely strictly singular embeddings

We start by establishing a sufficient condition for a r.i. space containing the space G_1 and also a sufficient condition for containing G. These conditions can also be shown to be necessary.

Proposition 9. *Let E be a r.i. space on $[0,1]$.*

(i) *The inclusion $G_1 \subset E$ holds whenever there exist a sequence $\{g_k\}$ in E and constants $a > 0$ and $C > 0$ such that, for all $k \in \mathbb{N}$, we have $\|g_k\|_E \leqslant C$ and*

$$m\bigl(\{t \in [0,1] : |g_k(t)| > 2^k\}\bigr) \geqslant \exp(-a2^{2k}).$$

(ii) *The inclusion $G \subset E$ holds whenever there exist a sequence $\{g_k\}$ in E and constants $a > 0$ and $C > 0$ such that, for all $k \in \mathbb{N}$, we have $\|g_k\|_E \leqslant C$ and*

$$m\bigl(\{|g_k| > 2^j\}\bigr) \geqslant \exp(-a2^{2j}), \qquad \textit{for } j = 0,\, 1, \ldots,\, k. \tag{10}$$

Proof. (i) Since G_1 is a Lorentz space and E a r.i. space, in order to check that $G_1 \subset E$ it suffices to prove the existence of a constant $M > 0$ such that, for all $t \in (0,1]$, if A is a set with $m(A) = t$, we have

$$\|\chi_A\|_E \leqslant M\|\chi_A\|_{L^{\Psi_2}} = M\log^{-1/2}(1/t+1). \tag{11}$$

In fact, it suffices to prove (11) for t small enough (changing M if necessary). Therefore we may assume that $0 < t < \exp(-a2^2)$.

There exists $k \in \mathbb{N}$ such that

$$\exp\big(-a2^{2(k+1)}\big) \leqslant t < \exp(-a2^{2k}). \tag{12}$$

In consequence, $m(A) = t \leqslant m\big(\{|g_k| > 2^k\}\big)$ and

$$\|\chi_A\|_E \leqslant \|\chi_{\{|g_k|>2^k\}}\|_E \leqslant 2^{-k}\|g_k\|_E \leqslant C2^{-k} \leqslant 2C\log^{-1/2}(1/t).$$

The last inequality follows directly from the first inequality in (12). We obtain (11) and $G_1 \subset E$ since $\log(1/t) \approx \log(1/t+1)$ for t small.

(ii) We have to prove the existence of $M > 0$ such that $\|h\|_E \leqslant M$, whenever $h \in L^\infty[0,1]$ and $\|h\|_{L^{\Psi_2}} \leqslant 1$.

Let h be such a function. The fact that $\|h\|_{L^{\Psi_2}} \leqslant 1$ yields

$$m\big(\{|h| > \lambda\}\big) \leqslant 2\exp(-\lambda^2), \qquad \text{for all } \lambda > 0.$$

Therefore

$$m\big(\{|h| > \lambda\}\big) \leqslant \exp(-\lambda^2/2), \qquad \text{for all } \lambda > 2. \tag{13}$$

Let $k \in \mathbb{N}$ and $\lambda \in [1, 2^k]$. There exists $1 \leqslant j \leqslant k$ such that $2^{j-1} \leqslant \lambda \leqslant 2^j$. Then, from (10)

$$m(\{2|g_k| > \lambda\}) \geqslant m(\{|g_k| > 2^{j-1}\}) \geqslant \exp(-a\lambda^2).$$

Therefore, for any $B > 0$ we have

$$m\big(\{2B|g_k| > \lambda\}\big) \geqslant \exp(-a\lambda^2/B^2), \qquad \text{for all } \lambda \in [B, 2^kB]. \tag{14}$$

Fix $B \geqslant 2$ sufficiently large so that $a/B^2 \leqslant 1/2$, choose $k \in \mathbb{N}$ with $2^kB \geqslant \|h\|_{L^\infty}$, and consider the function $g =: 2B|g_k| + B\chi_{[0,1]}$. By (13), (14), and the fact that $a/B^2 \leqslant 1/2$, we have

$$m\big(\{g > \lambda\}\big) \geqslant m\big(\{|h| > \lambda\}\big), \qquad \text{for all } \lambda \in [B, 2^kB].$$

Since $m\big(\{g > \lambda\}\big) = 1$, if $\lambda < B$, and $m\big(\{|h| > \lambda\}\big) = 0$, if $\lambda > 2^kB$, we deduce

$$m\big(\{g > \lambda\}\big) \geqslant m\big(\{|h| > \lambda\}\big), \qquad \text{for all } \lambda > 0.$$

Since E is a r.i. space, it follows that

$$\|h\|_E \leqslant \|g\|_E \leqslant 2B\|g_k\|_E + B\|\chi_{[0,1]}\|_E \leqslant 2BC + B\|\chi_{[0,1]}\|_E := M. \qquad \blacksquare$$

Any r.i. space E on [0,1] satisfies $L^\infty([0,1]) \subseteq E \subseteq L^1([0,1])$ with continuous inclusions. We will assume from this point on that $\|\cdot\|_{L^1} \leqslant \|\cdot\|_E$.

Proposition 10. *Let E be a r.i. space on $[0,1]$ such that $E \neq L^1$ and the inclusion $E \hookrightarrow L^1$ is not FSS. Then, there exists $C > 0$ such that for every $n \in \mathbb{N}$ there exists functions f_1, f_2, $\dots$, f_n in E satisfying*

(i) $|f_1(t)|^2 + |f_2(t)|^2 + \cdots + |f_n(t)|^2 = n$, *for all* $t \in [0,1]$.
(ii) *For all* $(\alpha_1, \alpha_2, \dots, \alpha_n) \in \mathbb{R}^n$ *we have*

$$\Big\|\sum_{j=1}^{n} \alpha_j f_j\Big\|_{L^1} \leqslant \Big\|\sum_{j=1}^{n} \alpha_j f_j\Big\|_{E} \leqslant C\Big(\sum_{j=1}^{n} |\alpha_j|^2\Big)^{1/2}.$$

Proof. Recall that we are assuming $\|\cdot\|_{L^1} \leqslant \|\cdot\|_E$. Since $E \hookrightarrow L^1$ is not FSS, there exist $K > 0$ and a sequence $\{F_n\}_n$ of finite dimensional subspaces of E with $\dim(F_n) \to +\infty$ such that

$$\|f\|_E \leqslant K\|f\|_{L^1}, \qquad \text{for all } f \in F_n, \text{ for all } n.$$

By applying Dvoretzky's Theorem and passing to a subsequence, if necessary, we may assume that $\dim(F_n) = n$ for every n and each F_n is 2-isomorphic to ℓ_2^n. Therefore, for every n there exist functions $g_1, g_2, \dots, g_n$ in E such that, for all $(\alpha_1, \alpha_2, \dots, \alpha_n) \in \mathbb{R}^n$, we have

$$\Big(\sum_{j=1}^{n} |\alpha_j|^2\Big)^{1/2} \leqslant \Big\|\sum_{j=1}^{n} \alpha_j g_j\Big\|_{L^1} \leqslant \Big\|\sum_{j=1}^{n} \alpha_j g_j\Big\|_{E} \leqslant 2K\Big(\sum_{j=1}^{n} |\alpha_j|^2\Big)^{1/2}. \tag{15}$$

Fix $n \in \mathbb{N}$ and define

$$G(t) := \Big(\sum_{j=1}^{n} |g_j(t)|^2\Big)^{1/2}, \qquad \text{for all } t \in [0,1].$$

Let $\{r_n\}$ be the Rademacher sequence. By Khintchine's inequality we have, for certain $K_1 \geqslant 1$,

$$\frac{1}{K_1} G(t) \leqslant \int_0^1 \Big|\sum_{j=1}^{n} r_j(s) g_j(t)\Big|\, ds \leqslant G(t), \qquad \text{for all } t \in [0,1].$$

Combining these inequalities with (15) and using Fubini's theorem we get

$$\sqrt{n} \leqslant \int_0^1 G(t)\, dt \leqslant \|G\|_E \leqslant 2KK_1\sqrt{n}. \tag{16}$$

Since E is a r.i. space and $E \neq L^1$, an application of the generalized Hölder inequality (see [2, Proposition I.2.4]) shows that the unit ball of E is equi-integrable. Hence, for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\|f\|_E \leqslant 1,\ m(A) < \delta \implies \int_A |f|\, dm < \varepsilon. \tag{17}$$

Set $\varepsilon = 1/8KK_1$ and choose $\delta > 0$ satisfying (17). Necessarily we have

$$m\big(\{t \in [0,1] : G(t) \geqslant \sqrt{n}/2\}\big) \geqslant \delta,$$

since, in other case, setting $A_0 := \{G \geqslant \sqrt{n}/2\}$, from (17) and the last inequality in (16) it would follow that

$$\int_{A_0} G(t)\,dt \leqslant \varepsilon \|G\|_E \leqslant \varepsilon 2KK_1\sqrt{n} = \sqrt{n}/4.$$

As $\int_{[0,1]\setminus A_0} G(t)\,dt \leqslant \sqrt{n}/2$, we would then obtain $\int_0^1 G(t)\,dt \leqslant 3\sqrt{n}/4$, which contradicts the first inequality in (16).

We can assume that $\delta = 1/\nu$ for certain $\nu \in \mathbb{N}$. We can also assume that $A_0 = \{G \geqslant \sqrt{n}/2\}$ contains the interval $[0, 1/\nu]$. If this were not the case, we just need to consider a measure-preserving transformation $\phi\colon [0,1] \to [0,1]$ such that $\phi([0,1/\nu]) \subset A_0$ and use $g_j \circ \phi$ instead of g_j, for $j = 1, 2, \ldots, n$.

Define, for each j:

$$f_j(t) := \sqrt{n}\,\frac{g_j(t)}{G(t)}, \qquad \text{for } t \in [0, 1/\nu],$$

and

$$f_j(t + k/\nu) := f_j(t), \qquad \text{for } t \in (0, 1/\nu] \quad \text{and} \quad k = 1, \ldots, \nu - 1. \tag{18}$$

From the definition of the functions f_j it is clear that (i) holds.

Next we check (ii) with $C = 4K\nu$. Let $(\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{R}^n$ and set $f := \sum_{j=1}^n \alpha_j f_j$. From (18)

$$\|f\chi_{[0,1/\nu]}\|_E = \|f\chi_{[(k-1)/\nu, k/\nu]}\|_E, \qquad \text{for all } k = 1, 2, \ldots, \nu.$$

Therefore $\|f\|_E \leqslant \nu\|f\chi_{[0,1/\nu]}\|_E$. Observe that $\sqrt{n}/G(t) \leqslant 2$ and $|f(t)| \leqslant 2|\sum_j \alpha_j g_j(t)|$, for all $t \in [0, 1/\nu]$. Then (ii) follows since, by (15),

$$\|f\|_{L^1} \leqslant \|f\|_E \leqslant \nu\|f\chi_{[0,1/\nu]}\|_E \leqslant 2\nu\Big\|\sum_{j=1}^n \alpha_j g_j\Big\|_E \leqslant 4K\nu\Big(\sum_{j=1}^n |\alpha_j|^2\Big)^{1/2}. \qquad \blacksquare$$

As an application of the previous propositions and of the covering numbers techniques used in Section 3 we show that whenever the inclusion $E \hookrightarrow L^1$ is not FSS then $G_1 \subset E$. The corresponding version for SS was proven by Montgomery-Smith and Semenov; [16].

Corollary 11. *If the inclusion $E \hookrightarrow L^1$ is not FSS, then $G_1 \subset E$.*

Proof. We can assume $E \neq L^1$. In order to apply (i) in Proposition 9, fix $k \in \mathbb{N}$, set $n = 2^{2k+2}$, and consider $C > 0$ and $f_1, f_2, \ldots, f_n$ as given by Proposition 10.

Let $\mathcal{E}$ be a 1/2-net of minimal cardinal in the Euclidean unit sphere of $\mathbb{R}^n$. Then $\operatorname{card}(\mathcal{E}) \leqslant 5^n$, and, for every $\beta \in \mathbb{R}^n$, we have

$$\|\beta\|_2 \leqslant 2\max_{\alpha\in\mathcal{E}} |\langle \alpha, \beta\rangle|. \tag{19}$$

For $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n) \in \mathcal{E}$ consider the set

$$A_\alpha = \Big\{t \in [0,1] : \Big|\sum_{j=1}^{n} \alpha_j f_j(t)\Big| > \sqrt{n}/2 = 2^k\Big\}.$$

By Proposition 10(i) and (19), we have

$$[0,1] = \bigcup_{\alpha \in \mathcal{E}} A_\alpha.$$

Since $\operatorname{card}(\mathcal{E}) \leqslant 5^n$, there exists $\alpha^0 \in \mathcal{E}$ with $m(A_{\alpha^0}) \geqslant 5^{-n}$. Defining

$$g_k := \sum_{j=1}^{n} \alpha_j^0 f_j$$

we obtain

$$m(\{|g_k| > 2^k\}) = m(A_{\alpha^0}) \geqslant 5^{-n} = \exp(-(\log 5) \times 4 \times 2^{2k}),$$

and, from Proposition 10(ii), $\|g_k\|_E \leqslant C$.

Hence the hypotheses of Proposition 9 are satisfied with $a = 4 \log 5$. ■

The proof of Theorem B that we are going to present uses a probability argument. We shall use the standard Gaussian measure γ_n in $\mathbb{R}^n$. This is the probability with density

$$\rho(x_1 \dots, x_n) = \frac{1}{\sqrt{2\pi}} \exp\Big(-\frac{1}{2}\sum_{j=1}^{n} x_j^2\Big)$$

with respect to Lebesgue measure in $\mathbb{R}^n$. Observe that γ_n can also be viewed as the product measure of n times γ_1, where γ_1 is the standard Gaussian distribution on $\mathbb{R}$ with mean equal to 0 and variance equal to 1. The measure γ_n is rotation invariant. As a consequence, for all $\alpha \in \mathbb{R}^n$ with $\|\alpha\|_2 = 1$, the real variable

$$x \mapsto \langle x, \alpha \rangle$$

defined on the probability space $(\mathbb{R}^n, \gamma_n)$ has as distribution the standard Gaussian measure γ_1.

We will use the following result, which is a deviation inequality for Lipschitz functions on $\mathbb{R}^n$ with respect to the Gaussian measure γ_n. For its proof, see [13] (formule (1.5) on page 21) or [17, Remark 4.8].

Proposition 12. *Let $F\colon \mathbb{R}^n \to \mathbb{R}$ be a Lipschitz function with constant $\leqslant L$ (for the Euclidean distance), that is, $|F(\alpha) - F(\beta)| \leqslant L\|\alpha - \beta\|_2$, for all α, $\beta \in \mathbb{R}^n$. Then, for all $s > 0$, we have*

$$\gamma_n\Big(\Big\{\alpha \in \mathbb{R}^n : \Big|F(\alpha) - \int F\, d\gamma_n\Big| > s\Big\}\Big) \leqslant 2 \exp\Big(-\frac{2s^2}{\pi^2 L^2}\Big).$$

We can now proceed to proof Theorem B.

Proof of Theorem B. Let E be a r.i. space so that $E \hookrightarrow L^1$ is not FSS. If $E = L^1$ obviously we have $G \subset E$. Thus we can suppose that $E \neq L^1$. In order to apply Proposition 9(ii) we fix $k \in \mathbb{N}$ and look for the function g_k satisfying $\|g_k\|_E \leqslant M$ and

$$m(\{|g_k| > 2^j\}) \geqslant \exp(-a2^{2j}), \qquad \text{for } j = 0, 1, \ldots, k, \tag{20}$$

for certain M and a independent of k.

Consider $n \in \mathbb{N}$ sufficiently large (to be determined later) and let $C > 0$ and $f_1, f_2, \ldots, f_n \in E$ given by Proposition 10. The function g_k will be of the form

$$g_k = \sum_{l=1}^{n} \beta_l f_l,$$

for adequate $\beta = (\beta_1, \beta_2, \ldots, \beta_n) \in \mathbb{R}^n$ with $\|\beta\|_2 \leqslant 2$. By Proposition 10(ii), setting $M = 2C$, we get $\|g_k\|_E \leqslant M$. The choice of β will be done via a probability argument involving the Gaussian measure γ_n. For technical reasons we will determine $\alpha = \sqrt{n}\beta$.

Define $F\colon \mathbb{R}^n \to \mathbb{R}$ by $F(\alpha) = \|\alpha\|_2$. Then

$$\int F \, d\gamma_n \leqslant \Big(\int \|\alpha\|_2^2 \, d\gamma_n(\alpha) \Big)^{1/2} = \sqrt{n}. \tag{21}$$

Applying Proposition 12 with $s = \sqrt{n}$ and $L = 1$, we have, by (21):

$$\begin{aligned} \gamma_n\big(\{\alpha \in \mathbb{R}^n : F(\alpha) > 2\sqrt{n}\}\big) &\leqslant \gamma_n\Big(\big\{\alpha \in \mathbb{R}^n : \big|F(\alpha) - \int F \, d\gamma_n\big| > s\big\}\Big) \\ &\leqslant 2\exp(-2n/\pi^2). \end{aligned} \tag{22}$$

Consider now, for $j = 0, 1, \ldots, k$, the function $\phi_j : \mathbb{R} \to \mathbb{R}$ defined by $\phi_j(t) = 0$, for $t \leqslant 2^j$; $\phi_j(t) = t - 2^j$, for $2^j \leqslant t \leqslant 2^j + 1$; and $\phi_j(t) = 1$, for $t \geqslant 2^j + 1$. Observe that $|\phi_j(t) - \phi_j(s)| \leqslant |s - t|$ for all $s, t \in \mathbb{R}$. Define $F_j : \mathbb{R}^n \to \mathbb{R}$ by

$$F_j(\alpha) := \int_0^1 \phi_j\Big(\frac{1}{\sqrt{n}} \sum_{l=1}^{n} \alpha_l f_l(t) \Big) \, dt.$$

Observe that, by Proposition 10(ii), we have, for $\alpha, \alpha' \in \mathbb{R}^n$,

$$\begin{aligned} |F_j(\alpha) - F_j(\alpha')| &\leqslant \int_0^1 \Big| \phi_j\Big(\frac{1}{\sqrt{n}} \sum_{l=1}^{n} \alpha_l f_l(x) \Big) - \phi_j\Big(\frac{1}{\sqrt{n}} \sum_{l=1}^{n} \alpha'_l f_l(x) \Big) \Big| \, dx \\ &\leqslant \int_0^1 \Big| \sum_{l=1}^{n} \frac{\alpha_l - \alpha'_l}{\sqrt{n}} f_l(t) \Big| \, dt \leqslant \frac{C\|\alpha - \alpha'\|_2}{\sqrt{n}}. \end{aligned}$$

Thus, F_j is Lipschitzian with constant $L = C/\sqrt{n}$.

Next, we evaluate the mean $\int F_j\,d\gamma_n$. Observe that, due to Proposition 10(i), the random variables $\alpha \mapsto \frac{1}{\sqrt{n}}\sum_{l=1}^{n}\alpha_l f_l(t)$, with $t \in [0,1]$, all have the same distribution with respect to γ_n, that is, the distribution of the standard Gaussian measure γ_1. Thus, by Fubini's theorem,

$$\begin{aligned}
\int F_j\,d\gamma_n &= \int_0^1 \int_{\mathbb{R}^n} \phi_j\Big(\frac{1}{\sqrt{n}}\sum_{l=1}^{n}\alpha_l f_l(t)\Big)\,d\gamma_n(\alpha)\,dt = \int_{\mathbb{R}} \phi_j\,d\gamma_1 \\
&= \int_{\mathbb{R}} \phi_j(\tau)\frac{1}{\sqrt{2\pi}}\exp(-\tau^2/2)\,d\tau \geqslant \frac{1}{\sqrt{2\pi}}\int_{2^j+1}^{2^j+2}\exp(-\tau^2/2)\,d\tau \\
&\geqslant \frac{1}{\sqrt{2\pi}}\exp(-(2^j+2)^2/2) \geqslant 2\exp(-a2^{2j}),
\end{aligned} \tag{23}$$

for $a > 0$ sufficiently large (independent of j).

Applying Proposition 12 to F_j with $s = \exp(-a2^{2j})$ and $L = C/\sqrt{n}$, by (23) we have

$$\begin{aligned}
\gamma_n\big(\{\alpha \in \mathbb{R}^n : F_j(\alpha) < \exp(-a2^{2j})\}\big) &\leqslant \gamma_n\Big(\Big\{\alpha \in \mathbb{R}^n : \Big|F_j(\alpha) - \int F_j\,d\gamma_n\Big| > s\Big\}\Big) \\
&\leqslant 2\exp\Big(-\frac{2n}{\pi^2C^2}\exp(-2a2^{2j})\Big) \\
&\leqslant 2\exp\Big(-\frac{2n}{\pi^2C^2}\exp(-a2^{2k+1})\Big).
\end{aligned} \tag{24}$$

It is clear from (22) and (24) that if n is sufficiently large we have

$$\gamma_n\big(\{F > 2\sqrt{n}\}\big) + \sum_{j=0}^{k}\gamma_n\big(\{F_j < \exp(-a2^{2j})\}\big) < 1.$$

Therefore there exists $\alpha^0 \in \mathbb{R}^n$ such that $\|\alpha^0\|_2 = F(\alpha^0) \leqslant 2\sqrt{n}$ and

$$F_j(\alpha^0) \geqslant \exp(-a2^{2j}), \quad \text{for } j = 0, 1, \ldots, k.$$

Setting $\beta = \alpha^0/\sqrt{n}$, and $g_k = \sum_{l=1}^{n}\beta_l f_l$, we have $\|g_k\|_E \leqslant M = 2C$ and, for all $j = 0, 1, \ldots, k$:

$$m(\{|g_k| > 2^j\}) \geqslant \int_0^1 \phi_j\big(g_k(x)\big)\,dx = F_j(\alpha^0) \geqslant \exp(-a2^{2j}).$$

We have proved (20) and the theorem follows. ∎

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