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HAHN SPACES IN FRÉCHET SPACES AND APPLICATIONS TO REAL SEQUENCE SPACES

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Dedicated to Lech Drewnowski on
the occasion of his 70th birthday

Abstract: In a joint paper with Grahame Bennett and Toivo Leiger (cf. [3]) the second author introduced for real sequence spaces the Hahn property and the notion of Hahn spaces. The aim of this paper is to extend these notions and a series of results in [3] to subspaces of Fréchet spaces by replacing the space ω of all real sequences and the set χ of all sequences of 0's and 1's by any Fréchet space H and a suitable subset χ of H, respectively. Applications of the general considerations to the original case of Hahn spaces of real sequences are also a main subject matter of the paper.

Keywords: Hahn spaces, FH-spaces, inclusion theorems, barrelledness.

1. Introduction

Let χ denote the set of all sequences of 0's and 1's. In [3] Bennett, Boos and Leiger asked for any sequence space E to what extent the linear span $\chi(E)$ of $\chi \cap E$ determines E. Mainly, they dealt with the most interesting formulation of this problem: Does

$$\chi(E) \subset F \implies E \subset F$$

hold whenever F is an arbitrary FK-space? In this case they defined E to be a Hahn space (have the Hahn property). They stated that the Hahn property, in a sense, is completely understood, at least for FK-spaces:

Theorem 1.1 ([4, Theorem 1] and [3, Theorem 1.1]). *Let E be an FK-space. Then the following conditions are equivalent:*

(i) *E is a Hahn space.*
(ii) *$\chi(E)$ is dense and barrelled in E.*

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The authors explained on page 76: 'The problem with Theorem 1.1 is that its hypotheses are difficult to check. The density of $\chi(E)$ in E is obvious in the corollary, but it may be far less transparent in other cases ($bs + c$, for example[1]). The real difficulty, however, lies in checking that $\chi(E)$ is barrelled in E. This is a non–trivial task even when $E = \ell_\infty$.' So if we develop methods to detect Hahn spaces E which are FK-spaces, then we can conclude to barrelledness (and density) of $\chi(E)$. For instance, by [3, Theorem 3.4] (cf. 1.2(d)) the FK-space $bs+c$ is obviously a Hahn space, so that $\chi(bs + c)$ is dense and barrelled by 1.1.

We recall some results on Hahn spaces given in [3] and which motivate the research aiming at in this paper.

Proposition 1.2 (cf. [3]).

(a) *ℓ_∞ is a Hahn space.*
(b) *If E is a Hahn space, then $E \subset \ell_\infty$ and $\chi(E)$ is dense in $(E, \| \|_\infty)$.*
(c) *Let E be an FK-space containing φ,* the set of all finite sequences. *If E has the Hahn property, then E is non–separable.*
(d) *If E is a sequence space satisfying[2] $bs + \langle e\rangle \subset E \subset \ell_\infty$, then E is a Hahn space and, in fact, $E = bs + \chi(E)$.*

The Hahn spaces (Hahn properties) are studied extensively in various papers (e.g. [8], [27], [12] and [26]), and a 'duality' of the Nikodym property of the set of all null sets of the density defined by any nonnegative matrix (sequence of nonnegative matrices) and the Hahn property (HP) of the strong null domain of it is stated in [9], [10], and [11]. In almost all of these papers results due to Lech Drewnowski with various coauthors (e. g. [16], [15], [1], and [14]) play an essential role.

The aim of this paper is to extend the notion of Hahn spaces and results in [3] (like those in 1.2) to subspaces of Fréchet spaces by replacing the space ω of all real sequences and the set χ of all sequences of 0's and 1's by any Fréchet space H and a suitable subset χ of H, respectively. Among new and general results these considerations will give us a better appreciation of the (original) Hahn spaces.

For that we give some notation and preliminaries in Section 2 and prepare the main results of the paper by basic considerations of F- and FH-spaces in Section 3. In particular, we introduce the notions of the FH-hull $\widehat{E}$ of a subset E in an Fréchet space H, and — in the same context — of the FH-regularity of E.

Section 4, one of the main parts of the paper, is divided into four subsections. The first one contains the notion of a Hahn tuple (H, χ) in generalization of the tuple (ω, χ) where H is an Fréchet space and χ is a suitable subset of H and — for a fixed Hahn tuple (H, χ) and a linear subspace E of H — the notion of a Hahn space. In the second subsection we generalize the characterization of a Hahn space E in Theorem 1.1 to the general situation of a Hahn space where now E is assumed to be an FH-space. Moreover, further characterizing and typical properties of Hahn spaces are discussed. In the third subsection, for any Hahn space E, a family of

[1] $bs := \{(x_k) \,|\, sup_n \left|\sum_{k=1}^{n} x_k\right| < \infty\}$
[2] $e := (1, 1, \ldots)$

linear subspaces and of linear supspaces of E containing exclusively Hahn spaces are presented and are called lower χ-zones and upper χ-zones, respectively. In [3, Theorem 3.4] it has been proved that[3] $E := bs \oplus \langle e \rangle$ is a Hahn space and that each sequence space X with $E \subset X \subset \ell_\infty (= \widehat{\chi})$ is a Hahn space too. In general, such Hahn spaces are called big and the fourth subsection is devoted to big Hahn spaces.

In Section 5 we apply results of Section 4 to the original case of real Hahn spaces, that is, to Hahn spaces in the case of the Hahn tuple (ω, χ) where χ denotes especially the set of all sequences of 0's and 1's. Thereby, among other results the main aim is to show that $A(\ell_\infty) + \langle e \rangle$ is a big Hahn space (cf. Theorem 5.16) for any χ-regular matrix (cf. Definition 5.11). This result generalizes essentially [3, Theorem 3.4] (case $A := \Sigma^{-1}$) and [12, Theorem 2.6] (more general case $A := \Sigma_N^{-1}$) where $\Sigma = (\sigma_{nk})$ denotes the summation matrix defined by $\sigma_{nk} = 1$ if $k \leqslant n$ and $\sigma_{nk} = 0$ otherwise.

2. Notation and preliminaries

We start with a few preliminaries. (Otherwise, we refer to [5], [23], [24], [3], and [12]). ω denotes the space of all sequences $x = (x_k)$ in $\mathbb{K}$, $\mathbb{K} := \mathbb{R}$ or $\mathbb{K} := \mathbb{C}$, and any linear subspace of ω is called a *sequence space*. ω endowed with the topology τ_ω of coordinatewise convergence is an Fréchet space. A locally convex sequence space (E, τ) is called *FK-space* if it is an Fréchet space and the inclusion map $i : E \longrightarrow \omega$, $x \longrightarrow x$ is continuous; if the topology is even normable, then (E, τ) is called *BK-space*. More general, we consider FH-spaces: If (H, τ_H) is any given Hausdorff space, then a locally convex space (E, τ) is called *FH-space* (relative to H), if it is an Fréchet space included in H and the inclusion map $i : E \longrightarrow H$, $x \longrightarrow x$ is continuous; if the topology is even normable, then (E, τ) is called *BH-space*. Note, the topology of FH-spaces is monotone and uniquely determined. Consequently, it makes sense to use the notions of *FH-* and *FK-topology*.

Familiar examples of FK-spaces are ω (which is not a BK-space), ℓ_∞ (bounded sequences) with the supremum norm $\| \ \|_\infty$, and its closed subspaces c (convergent sequences) and c_0 (null sequences), ℓ_p for $1 \leqslant p < \infty$ (absolutely p-summable sequences) with its natural norm, and

$$bs := \left\{ x = (x_k) \;\middle|\; \|x\|_{bs} := \sup_n \left| \sum_{k=1}^{n} x_k \right| < \infty \right\} \qquad \text{(bounded series)}$$

with the norm $\| \ \|_{bs}$.

Obviously, FK-spaces (BK-spaces) are special FH-spaces (BH-spaces) if we choose $H := \omega$. For any given Fréchet space X another examples of FH-spaces are given by the consideration of $\omega(X) := X^{\mathbb{N}}$ (set of all sequences in X) and subspaces like $\ell_\infty(X)$, $c(X)$, $c_0(X)$ etc. (cf. for instance [7] and [20]). If $H := F[0, 1]$

[3]If X and Y are linear subspaces of a linear space E with $X \cap Y = \{0\}$, then $X \oplus Y$ denotes the algebraic direct sum of X and Y.

(real functions on $[0,1]$), then for example $B[0,1]$ and $C[0,1]$ (bounded and continuous real functions on $[0,1]$, respectively) endowed with the supremum norm are BH-spaces. Note that $F[0,1]$ with the topology of pointwise convergence is a locally convex non-metrizable Hausdorff space, and that $C[0,1]$ is also an BH-space if we consider the Banach space $H := B[0,1]$ endowed with the supremum norm. Further examples of FH-spaces are presented in [25].

Below we'll use the following simple properties of FH-spaces.

Proposition 2.1. *Let H be an Fréchet space and $\widetilde{H}$ be an FH-space relative to H. Then the following statements hold:*

(a) *Each FH-space relative to $\widetilde{H}$ is an FH-space relative to H.*
(b) *Let E be a linear subspace of $\widetilde{H}$. Then E is an FH-space relative to $\widetilde{H}$ if and only if there exists an FH-space F relative to H with $E = F \cap \widetilde{H}$.*

Proof. We omit the obvious proofs. ■

3. Fréchet spaces, FH-spaces, and FH-regularity

Bennett and Kalton stated in [4, Proposition 1] the following Proposition for Fréchet spaces where the proof is only given in the special case of Banach spaces. In his 'Diplomarbeit' the first author proved this result also in the general case of Fréchet spaces (cf. [19, Satz 2.2.17]).

Proposition 3.1 ([4, Proposition 1]). *Let E be an Fréchet space and E_0 a dense linear subspace of E. Then the following statements are equivalent:*

(i) *E_0 is barrelled.*
(ii) *$T(F) = E$ holds for every Fréchet space F and each continuous linear map $T : F \longrightarrow E$ with $E_0 \subset T(F)$.*

Proof. See the proof of [19, Satz 2.2.17] and note [19, Bemerkung 2.2.18]. ■

General assumption: In the following, let H be any (fixed) Fréchet space.

Theorem 3.2. *Let E be an Fréchet space and $f : E \longrightarrow H$ be linear and continuous. Then $f(E)$ is an FH-space.*

Proof. We consider the canonical partition of f illustrated by

$$\begin{tikzcd}
(E,\tau_E) \arrow[r, "f"] \arrow[d, "\pi"'] & (H,\tau_H) \\
(E/Kern(f),\tau_{E/Kern(f)}) \arrow[r, "f^*"] & (f(E),\tau_{f(E)}) \arrow[u, "i"']
\end{tikzcd}$$

Obviously, the quotient space $E/Kern(f)$ is an Fréchet space since $Kern(f)$ is a closed subspace in (E, τ_E). With the canonical isomorphism $f^* : E/Kern(f) \longrightarrow f(E)$ we transfer the F-topology of $E/Kern(f)$ to $f(E)$ so that $(f(E), \tau_{f(E)})$ is an F-space. It is even an FH-space since the inclusion map $i : (f(E), \tau_{f(E)}) \longrightarrow (H, \tau_H)$ is continuous as we can easily verify: Let U be a zero neighbourhood in (H, τ_H). Then $f^{-1}(U)$ is a zero neighbourhood in E since f is continuous. Because the continuous onto maps f^* and π are open by the open mapping theorem, $(f^* \circ \pi)\left(f^{-1}(U)\right) = f\left(f^{-1}(U)\right) = U \cap F(E)$ is a zero neighbourhood in $(f(E), \tau_{f(E)})$ contained in U. ■

The following Theorem is presented in [5, 7.3.10] for the case of FK-spaces. Applying 3.2 we get a compact proof in the more general case of FH-spaces.

Theorem 3.3. *Each finite sum of FH-spaces is an FH-space too.*

Proof. Let $E_1, E_2, \dots, E_n$ be FH-spaces (for a common space H). Then $E := \prod_{i=1}^n E_i$ (with the product topology) is an Fréchet space. For each $i \in \{1, 2, \dots, n\}$ let $\pi_i : E \longrightarrow E_i$ be the projection and $\varphi_i : E_i \longrightarrow H$ the inclusion map. Then $s := \sum_{i=1}^n \varphi_i \circ \pi_i : E \longrightarrow H$ is linear and continuous. Thus, $s(E) = \sum_{i=1}^n E_i$ is an FH-space by 3.2. ■

Lemma 3.4. *Let E and F be FH-spaces and E_0 be a subspace of $E \cap F$ endowed with the topology induced by E. If E_0 is barrelled, then the inclusion maps $i_E : E_0 \longrightarrow E$ and $i_F : E_0 \longrightarrow F$ are continuous.*

Proof. Obviously, i_E is continuous. Now, let (x_n) be a sequence in E_0 converging to an $x \in E_0$ such that $(i_F(x_n)) = (x_n)$ converges in F to an $y \in F$. Since E and F are FH-spaces we get $x = y$, that is, i_F is closed. Then i_F is continuous by the assumptions on E_0 and [22, Chap. VI, Theorem 6]. ■

Likewise, the following Theorem is a generalization of [4, Theorem 1]. In that paper it was stated for FK-spaces and proved in the special case of BK-spaces. Here we consider more generally FH-spaces.

Theorem 3.5. *Let E be an FH-space and E_0 be a dense subset of E. Then the following statements are equivalent:*

(i) *E_0 equipped with the topology induced by E is barrelled.*
(ii) *$E_0 \subset F$ implies $E \subset F$ for every FH-space F.*
(iii) *$E_0 \subset F \subset E$ implies $E = F$ for every FH-space F.*

Proof. Adapt the proof of [4, Theorem 1] and apply Lemma 3.4 and Theorem 3.2 in the proof of (i)⇒(ii) and (iii)⇒(i), respectively. ■

In the remaining part we introduce some notions which will prove to be useful for the handling of the generalized version of Hahn spaces.

Definition and Remarks 3.6. Let H be an Fréchet space and E be any subset of H. Then $\widehat{E} = {}^H\widehat{E}$ denotes the intersection of all FH-spaces containing E, and we call it *FH-hull* of E (relative to H). Obviously $\widehat{E}$ is a well-defined linear subspace of H, and, in general, it is not an FH-space (cf. 3.9). However, if this is the case, then E is called *FH-regular*. Furthermore, E is called *FH-closed*, if $\widehat{E} = E$. In the standard case $H = \omega$ we use correspondingly the notions of *FK-hull*, *FK-regular* and *FK-closed*.

In general, we supply $\widehat{E} = \bigcap\{F \mid E \subset F \text{ and } F \text{ is an FH-space}\}$ with the projective topology $\tau_{\widehat{E}}$ (of the intersection) and consider E with $\tau_{\widehat{E}}|_E$. If E is FH-regular, then $\widehat{E}$ equipped with $\tau_{\widehat{E}}$ is an FH-space and the subspace E is metrizable (but not necessarily complete).

Proposition 3.7. *If H is an Fréchet space and $E, F \subset H$, then the following statements are true:*

(i) $E \subset \widehat{E} = \widehat{\langle E \rangle} = \widehat{\widehat{E}} < H$.
(ii) $E \subset F \Rightarrow \widehat{E} \subset \widehat{F}$.
(iii) *E is FH-regular $\iff$ There exists a minimal FH-space containing E.*
(iv) *E is an FH-space $\iff$ E is FH-regular and FH-closed.*
(v) *E is barrelled $\Longrightarrow$ E is FH-regular.*

Proof. We omit the simple proofs of (i) - (iv). For a proof of (v) notice that E is a dense, barrelled subspace of the FH-space $\overline{E}$ (topological closure in H) and use Theorem 3.5. ■

Theorem 3.8. *Let $(E_i)_{i\in I}$ be a family of subspaces of H. Then:*

(i) $\widehat{\bigcap_{i\in I} E_i} \subset \bigcap_{i\in I} \widehat{E_i}$.
(ii) *If each E_i, $i \in I$, is FH-closed, then $\bigcap_{i\in I} E_i$ is also FH-closed.*
(iii) $\sum_{i\in I} \widehat{E_i} \subset \widehat{\sum_{i\in I} E_i}$.

Proof. Let $E := \bigcap_{i\in I} E_i$ and $F := \sum_{i\in I} E_i$. (i): Apply 3.7 (ii) to $E \subset E_i$.

(ii): Since E_i, $i \in I$, are FH-closed we get $\widehat{E} \subset E$ by (i). Because $E \subset \widehat{E}$ holds by 3.7 (i) the FH-closedness of E follows.

(iii): Obviously, $E_i \subset F$, thus $\widehat{E_i} \subset \widehat{F}$ for all $i \in I$ by 3.7 (ii). Therefore $\sum_{i\in I} \widehat{E_i} \subset \widehat{F}$. ■

Examples 3.9.

(a) It is well-known that the set φ of all finite sequences is a subspace of ω but not an FK-space, and that φ is the intersection of all FK-spaces containing it. Therefore φ is FK-closed but not FK-regular.
(b) G. Bennett and N. J. Kalton proved in [4, Corollary of Theorem 1], that each FK-space including $m_0 := \langle\chi\rangle$ contains even the strictly bigger space ℓ_∞. Thus m_0 is not an FK-space, so that m_0 is FK-regular, but not FK-closed. Obviously, the FK-regularity of m_0 implies also that of χ (cf. 3.7 (i)).

4. Generalized Hahn spaces

In the definition of the 'classical' Hahn spaces in [3] the space $H = \omega$ and the set χ of all sequences of 0's and 1's play a decisive role: We have $\widehat{\chi} = \ell_\infty$ and ℓ_∞ is an FH-space. As a consequence of the definition of $\widehat{\chi}$, the space $\langle\chi\rangle$ is dense in ℓ_∞. The aim of this section is to consider subspaces of suitable Fréchet spaces H and subsets χ of H which enable the study of generalized Hahn spaces.

4.1. Definition of generalized Hahn spaces and preliminary remarks

Definition 4.1. A tuple[4] (H, χ) is called a *Hahn tuple*, if H is an Fréchet space and χ is an FH-regular subset of H.

Remarks 4.2. Let (H, χ) be a Hahn tuple.

(a) $\widehat{\chi}$ is the smallest FH-space that contains χ by the FH-regularity of χ. By this way $\langle\chi\rangle$ equipped with the topology induced by the FH-topology of $\widehat{\chi}$ is a metrizable locally convex space and is dense in the FH-space $\widehat{\chi}$.
(b) If $H = \omega$ and χ is the set of all sequences of 0's and 1's, then $\widehat{\chi} = \ell_\infty$, $\widehat{\chi}$ is FK-regular and (H, χ) is a Hahn tuple.

General Assumptions: In the following we consider an arbitrarily *fixed* Hahn tuple (H, χ).

Definition 4.3. For any subspace E of H the linear hull $\chi(E) := \langle\chi \cap E\rangle$ of $\chi \cap E$ is called the χ-*part* of E. Moreover, if $\chi(E) = E$, then E is called χ-*based*.

Remarks 4.4. For subspaces E, F of H the following statements hold: $\chi(E) \subset E$, $\chi(\chi(E)) = \chi(E)$, and $E \subset F$ implies $\chi(E) \subset \chi(F)$. If E is χ-based, then $E \subset \langle\chi\rangle$. Furthermore, $\chi(E)$ is the maximal χ-based subspace in E.

Analogously to the case $H = \omega$ in [3] we define the notion of a Hahn space.

Definition and Remarks 4.5. Let (H, χ) be a Hahn tuple. Then a linear subspace E of H is a *Hahn space* (has the *Hahn property*) (relative to (H, χ)), if $\chi(E) \subset F$ implies $E \subset F$ for every FH-space F.

Clearly, this definition contains that of the 'original' Hahn spaces in [3]: Consider the Hahn tuple (ω, χ) where χ is the set of all sequences of 0's and 1's. In this special case χ is a discrete set.

In general, each χ-based space is obviously a Hahn space.

Remark 4.6. If we consider subsets χ and ψ of H satisfying $\chi(E) = \psi(E)$, for instance, if $\mathbb{K} \cdot \chi = \mathbb{K} \cdot \psi$, then E is a Hahn space relative to (H, χ) if and only if it is a Hahn space relative to (H, ψ).

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[4]Note, we use here also the notation χ since it plays a similar role in this general context as the set of all sequences of 0's and 1's in the special case of sequence spaces.

Example 4.7. In the cases $\mathbb{K}\cdot\chi = H$ or $\chi \subset \{0\}$ the Hahn space theory is trivial: In the first case for each subspace E of H we have $\chi(E) = E$, and therefore E is a Hahn space. In the second case we have $\chi(E) = \{0\}$ which is an FH-space and implies therefore that $\{0\}$ is the only Hahn space.

Now, we are considering Hahn spaces relative to 'compatible' Hahn tuples.

Theorem 4.8. *If (H,χ) and $\left(\widetilde{H},\widetilde{\chi}\right)$ are Hahn tuples with $\widetilde{H} \subset H$ and $\widetilde{\chi} \subset \chi$, then the following statements hold:*

(a) *Each Hahn space relative to $\left(\widetilde{H},\widetilde{\chi}\right)$ is also a Hahn space relative to (H,χ).*

(b) *If $\widetilde{\chi} = \chi \cap \widetilde{H}$, then the Hahn spaces relative to $\left(\widetilde{H},\widetilde{\chi}\right)$ are exactly those relative to (H,χ), which are included in $\widetilde{H}$.*

(c) *If $\chi = \widetilde{\chi}$, then both Hahn tuples have the same Hahn spaces included in $\widetilde{H} \subset H$. In particular, (H,χ) and $(\widehat{\chi},\chi)$ have the same Hahn spaces in $\widehat{\chi}$.*

Proof. (a) Let E be a Hahn space relative to $(\widetilde{H},\widetilde{\chi})$ and let F be an FH-space relative to H such that $\chi\cap E \subset F$. By 2.1(b) the space $\widetilde{F} := F\cap\widetilde{H}$ is an FH-space relative to $\widetilde{H}$. Because of

$$E\cap\widetilde{\chi} \subset E\cap\chi = \left(E\cap\widetilde{H}\right)\cap\chi = (E\cap\chi)\cap\widetilde{H} \subset F\cap\widetilde{H} = \widetilde{F}$$

we get $E \subset \widetilde{F} \subset F$ since E is a Hahn space relative to $(\widetilde{H},\widetilde{\chi})$. Therefore E is also a Hahn space relative to (H,χ).

(b) Let $E \subset \widetilde{H}$ be a Hahn space relative to (H,χ) and F be an FH-space relative to $\widetilde{H}$ with $\widetilde{\chi}\cap E \subset F$. Then

$$E\cap\chi = \left(E\cap\widetilde{H}\right)\cap\chi = E\cap\left(\chi\cap\widetilde{H}\right) = E\cap\widetilde{\chi} \subset F.$$

Thus $E \subset F$ since E is a Hahn space relative to (H,χ) and F is also an FH-space relative to H by 2.1(a). Therefore E is a Hahn space relative to $\left(\widetilde{H},\widetilde{\chi}\right)$. The inverse implication holds by (a).

(c) By the assumption we get $\chi\cap\widetilde{H} = \widetilde{\chi}\cap\widetilde{H} = \widetilde{\chi}$, so that the statement is proved by applying (a) and (b). ■

The next result contains a tool for the generation of Hahn tuples and related Hahn spaces by the consideration of continuous linear operators between Fréchet spaces and known Hahn tuples and related Hahn spaces.

Theorem 4.9. *Let H and $\widetilde{H}$ be Fréchet spaces and $A : H \longrightarrow \widetilde{H}$ be a continuous linear operator. Then the following statements hold:*

(a) *If $T \subset H$, then $A\left({}^{H}\widehat{T}\right) \subset {}^{\widetilde{H}}\widehat{A(T)}$.*

(b) *If T is FH-regular in H, then $A(T)$ is FH-regular in $\widetilde{H}$.*

(c) *Let (H,χ) be a Hahn tuple. Then for each Hahn space E relative to (H,χ) the space $A(E)$ is a Hahn space relative to $\left(\widetilde{H},A(\chi)\right)$.*

Proof. (a) Let F be an FH-space relative to $\widetilde{H}$ with $A(T)\subset F$. Then $T\subset A^{-1}(F)$. Thereby $A^{-1}(F)$ is an FH-space relative to H by [6, Proposition 1.1] (cf. [19, 3.2.4] for a detailed proof). This implies ${}^H\widehat{T}\subset A^{-1}(F)$, thus $A\left({}^H\widehat{T}\right)\subset F$.

(b) If, in addition, ${}^H\widehat{T}$ is an FH-space relative to H, then $A\left({}^H\widehat{T}\right)$ is an FH-space relative to $\widetilde{H}$ by 3.2. Since $A(T)\subset A\left({}^H\widehat{T}\right)$ we get ${}^{\widetilde{H}}\widehat{A(T)}\subset A\left({}^H\widehat{T}\right)$. Together with (a) this implies that $A\left({}^H\widehat{T}\right)={}^{\widetilde{H}}\widehat{A(T)}$. In particular, $A(T)$ is FH-regular in $\widetilde{H}$.

(c) By (b), $\left(\widetilde{H},A(\chi)\right)$ is a Hahn tuple because (H,χ) is. Now, let F be an FH space relative to $\widetilde{H}$ with $A(\chi)\cap A(E)\subset F$. Then $A(\chi\cap E)\subset A(\chi)\cap A(E)\subset F$, thus $\chi\cap E\subset A^{-1}(F)$. Because $A^{-1}(F)$ is an FH-space relative to H (cf. [6, Proposition 1.1] , [19, 3.2.4]) and E is assumed to be a Hahn space relative to (H,χ) we obtain $E\subset A^{-1}(F)$, thus $A(E)\subset F$. Altogether, $A(E)$ is a Hahn space relative to $(\widetilde{H},A(\chi))$. ∎

For a further discussion of examples of Hahn tuples and related Hahn spaces see Section 6.

Definition and Remark 4.10. For any $n\in\mathbb{N}$, $n>0$, χ_n denotes the set of all members of $\langle\chi\rangle$ that may be represented by a finite linear combination of at most n members in χ. Moreover we set $\chi_0:=\emptyset$. Note that χ_n, $n\in\mathbb{N}$, is in general no linear space.

Theorem 4.11. *Let $m,n\in\mathbb{N}$ be arbitrarily given. Then:*

(i) $\mathbb{K}\cdot\chi_n=\chi_n$.
(ii) $\chi_m+\chi_n\subset\chi_{m+n}$.
(iii) $\langle\chi\rangle=\bigcup_{n\in\mathbb{N}}\chi_n$.

Proof. We omit the easy proofs. ∎

By means of the sets χ_n, $n\in\mathbb{N}$, additional topological and algebraic conditions on χ may be formulated, which are obviously satisfied in the case of the set χ of all sequences of 0's and 1's.

Theorem 4.12. *Let χ be chosen such that for all $n\in\mathbb{N}$ the following conditions are fulfilled:*

(i) *χ_n is a closed subset of $\langle\chi\rangle$ where $\langle\chi\rangle$ carries the topology induced by the FH-topology of $\widehat{\chi}$.*
(ii) *χ_n contains exclusively finite dimensional subspaces.*

Then, for every FH-space $E \subset \widehat{\chi}$, the space $\chi(E)$, provided with the topology induced by the FH-topology of E, is a Baire space if and only if $\chi(E)$ is finite dimensional.

Proof. Let $n \in \mathbb{N}$ be arbitrarily given. At first, let E and $\chi(E)$ as well as $\langle \chi \rangle$ and χ_n be topologized with the topology induced by the FH-topology of $\widehat{\chi}$. Because of (i) and $\chi(E) \subset \langle \chi \rangle$ the set $\chi(E) \cap \chi_n$ is closed in $\chi(E)$. Moreover, the FH-topology of E is stronger than the topology induced by the FH-topology of $\widehat{\chi}$ by the monotony of FH-topologies. The same is obviously true for the corresponding topology on $\chi(E)$. Therefore, $\chi(E) \cap \chi_n$ is closed in the subspace $\chi(E)$ of the FH-space E.

Moreover, we have $\chi(E) = \bigcup_{n\in\mathbb{N}} (\chi(E) \cap \chi_n)$ because of $\chi(E) \subset \langle \chi \rangle$ and 4.11(iii).

Now, we assume that $\chi(E)$ is a Baire space. Then there exists an $n \in \mathbb{N}$ such that $\chi(E) \cap \chi_n$ has an interior point a in $\chi(E)$. Consequently, $\chi(E) \cap \chi_n - a$ is a zero neighbourhood in $\chi(E)$. By that and 4.11(ii) we get

$$\chi(E) \cap \chi_n - a \ \subset \chi(E) \cap \chi_n + \chi(E) \cap \chi_n \subset \chi(E) \cap (\chi_n + \chi_n) \subset \chi(E) \cap \chi_m$$

for $m := 2n$. Therefore $\chi(E) \cap \chi_m$ is a zero neighbourhood in $\chi(E)$. In particular, $\chi(E) \cap \chi_m$ is absorbing. Thus there exists for every $b \in \chi(E)$ a positive $\lambda \in \mathbb{R}$ with $\lambda b \in \chi(E) \cap \chi_m$. Because of 4.11(i) and since $\chi(E)$ is a linear space, $b \in \chi(E) \cap \chi_m \subset \chi_m$ holds. Altogether $\chi(E) \subset \chi_m$ is proved, so that $\chi(E)$ is finite dimensional by the assumption (ii).

Conversely, a finite dimensional subspace of a separated space is always an Fréchet space, thus a Baire space. ■

4.2. Characterization of Hahn spaces

First of all we generalize the characterization of the original Hahn FK-spaces (cf. Theorem 1.1) to the general case of Hahn spaces.

Theorem 4.13 (for $H := \omega$ cf. [4, Theorem 1] and [3, Theorem 1.1]). *If E is any FH-space, then the following statements are equivalent:*

(i) *E is a Hahn space.*
(ii) *$\chi(E)$ is dense and barrelled in E.*

Proof. (i)⇒(ii): Since E is an FH-space, the closure $\overline{\chi(E)}$ of $\chi(E)$ in E is also an FH-space. Because $\chi(E) \subset \overline{\chi(E)} \subset E$ and E is a Hahn space we get $E \subset \overline{\chi(E)}$, thus $E = \overline{\chi(E)}$ and $\chi(E)$ is dense in E.

Let $E_0 := \chi(E)$ and F be any FH-space with $E_0 \subset F$. Then $E \subset F$ by the Hahn property of E. Thus statement 3.5(ii) and therefore 3.5(i) is satisfied, that is, $\chi(E)$ is barrelled in E.

(ii)⇒(i): Apply 3.5(i)⇒(ii) to $E_0 := \chi(E)$. ■

Note, in the proof of Theorem 4.13 we did not use the FH-regularity. However, the importance of this property became obvious, for instance, in the proof of Theorem 4.12 and will reveal itself in 4.17.

Remark 4.14. If $H := \omega$ and χ is the set of all sequences of 0's and 1's, then Theorem 1.1 is a special case of Theorem 4.13. Moreover, the conditions (i) and (ii) in 4.12 are satisfied (as we'll state in Theorem 5.1). Therefore, with suitable conditions on χ, the consideration of (general) Hahn spaces is useful for the identification of barrelled spaces which are not necessarily Baire spaces.

Theorem 4.15. *If E is any linear subspace of H, then the following statements are equivalent:*

(i) *E is a Hahn space.*

(ii) $E \subset \widehat{\chi(E)}$.

(iii) $\widehat{E} = \widehat{\chi(E)}$.

Proof. (i)⇒(ii): Assume $x \in E \setminus \widehat{\chi(E)}$. Then there exists an FH-space F with $\chi(E) \subset F$ and $x \notin F$. Then E is not a Hahn space.

(ii)⇒(iii): This implication is an immediate consequence of 3.7(i) and (ii).

(iii)⇒(i): Let F be an FH-space with $\chi(E) \subset F$. By the definition of the FH-hull and (iii) we may conclude $E \subset \widehat{E} = \widehat{\chi(E)} \subset \widehat{F} = F$. Consequently, E is a Hahn space. ■

Proposition 4.16 (cf. [3, Cor. 1.2, Proposition 2.2] for $H := \omega$). *$\widehat{\chi}$ is a Hahn space and each Hahn space is contained in $\widehat{\chi}$. Furthermore, $\widehat{\chi}$ is an FH-space and $< \chi >$ is a dense, barrelled subspace of $\widehat{\chi}$.*

Proof. Obviously, $\widehat{\widehat{\chi}} = \widehat{\chi} = \widehat{\langle \chi \rangle} = \widehat{\chi(\widehat{\chi})}$ by 3.7(i), thus $\widehat{\chi}$ is a Hahn space by 4.15. Now, let E be an arbitrary Hahn space. Because $\chi(E) = \langle \chi \cap E \rangle \subset \langle \chi \rangle$ we get also $E \subset \widehat{\chi(E)} \subset \widehat{\langle \chi \rangle} = \widehat{\chi}$ by 4.15 and 3.7(ii). The remaining statements follow by the FH-regularity of χ and 4.13. ■

Remark 4.17. As we mentioned above, we did not use the FH-regularity of χ in the proof of Theorem 4.13. However by the assumption of FH-regularity the above theorem guarantees the existence of a Hahn space which is also an FH-space. If in addition $\langle \chi \rangle$ has infinite dimension, then there exists the chance to deduce nontrivial results.

For example, if we drop the FH-regularity of χ in the definition of a Hahn space (cf. 4.5) and set[5] $\chi := \{e^k \mid k \in \mathbb{N}\}$ and $H := \omega$. Then $\widehat{\chi} = \varphi$ is not an FH-space, that is, χ is not FH-regular, and by 4.16 each Hahn space is a subspace of φ. In particular, each Hahn space has countable dimension and is finite dimensional, if it is even an FH-space. However in the last case barrelledness is trivial.

Theorem 4.18. *If E is any Hahn space, then each linear subspace X of H with $E \subset X \subset \widehat{E}$ is also a Hahn space. In particular, $\widehat{E}$ is a Hahn space if E is a Hahn space.*

[5] $e^k := (0, \ldots, 0, 1, 0, \ldots)$ with the '1' in the k^{th} position.

Proof. Let F be an FH-space with $\chi(X) \subset F$. Then $\chi(E) \subset F$ since $E \subset X$, thus $E \subset F$ by the Hahn property of E. Therefore $X \subset \widehat{E} \subset F$. Altogether, X is a Hahn space. ■

Proposition 4.19 (vgl. [3, Proposition 2.1]). *If $(E_i)_{i\in I}$ is a family of Hahn spaces (relative to (H,χ)) and $E := \sum_{i\in I} E_i$, then E is a Hahn space too. (Note, if $(E_i)_{i\in I}$ is totally ordered, then $E = \bigcup_{i\in I} E_i$.)*

Proof. Let F be an FH-space and $\chi(E) \subset F$. Obviously, we have $\chi(E_i) \subset \chi(E) \subset F$ for all $i \in I$. Thus $E_i \subset F$ for each $i \in I$ because E_i, $i \in I$, are Hahn spaces. Consequently $E \subset F$, that is, E is a Hahn space. ■

Definition 4.20. For any linear subspace E of H we use the notation E^h for the sum of all Hahn spaces contained in E and call it the *Hahn kernel* of E. Note, E^h is well-defined since $\chi(E)$ is a Hahn space contained in E.

Proposition 4.21. *For subspaces E and F of H the following statements hold:*

(i) *E^h is the uniquely determined maximal Hahn space contained in E.*
(ii) $\chi(E) \subset E^h \subset E$.
(iii) $E \subset F \Rightarrow E^h \subset F^h$.
(iv) $(E^h)^h = E^h$.
(v) *E is a Hahn space if and only if $E^h = E$.*

Proof. The proofs are straightforward and left to the reader. ■

4.3. χ-zones

If one knows a Hahn space E, then there are two simple ways to derive further examples for Hahn spaces. The first one provides subspaces and the second one supspaces of E.

Definition 4.22. Let E be a subspace of H. Then, by definition, the *lower χ-zone of E* is the set of all subspaces X of H with $\chi(E) \subset X \subset E$, and the *upper χ-zone of E* is the set of all subspaces X of H with $E \subset X \subset E + \mathbb{K}\cdot\chi$. (Note, that $E + \mathbb{K}\cdot\chi$ is not a linear space in general.)

Proposition 4.23. *If E and X are subspaces of H, then the following statements are equivalent:*

(i) *X is a member of the lower χ-zone of E.*
(ii) *$X \subset E$ and $\chi(X) = \chi(E)$.*

Proof. (i)⇒(ii): If $\chi(E) \subset X \subset E$, then $\chi(E) = \chi(\chi(E)) \subset \chi(X) \subset \chi(E)$ by 4.4, thus $\chi(X) = \chi(E)$.

(ii)⇒(i): The statement (ii) implies obviously $\chi(E) = \chi(X) \subset X \subset E$. ■

Theorem 4.24. *Let E and X be linear subspaces of H with $E \subset X$. Then the following statements are equivalent:*

(i) *X is a member of the upper χ-zone of E, that is $X \subset E + \mathbb{K} \cdot \chi$.*
(ii) *$F = E + \chi(F)$ holds for every subspace F with $E \subset F \subset X$.*
(iii) *For any subspace F with $E \subset F \subset X$ there exists a χ-based subspace Y of X such that $F = E \oplus Y$.*

Proof. (i)$\Rightarrow$(ii): Let $E \subset F < X$. Then $E + \chi(F) \subset F$. Since $F \subset X \subset E + \mathbb{K} \cdot \chi$ for all $x \in F$ there exist an $a \in E$, a $t \in \chi$ and a $\lambda \in \mathbb{K}$ with $x = a + \lambda t$. In particular, $\lambda t \in F$. If $\lambda = 0$, then $x \in E \subset E + \chi(F)$. If $\lambda \neq 0$, then $t \in F$, therefore $t \in \chi(F)$, thus $x \in E + \chi(F)$. Altogether, $F = E + \chi(F)$ holds.

(ii)$\Rightarrow$(iii): Let F with $E \subset F < X$ be given and

$$\mathcal{M} := \left\{ G \subset F \cap \chi \;\middle|\; E \cap \langle G \rangle = \{0\} \right\}.$$

Obviously, $\mathcal{M} \neq \emptyset$ because $\emptyset \in \mathcal{M}$. Let $(G_i)_{i \in I}$ be a linearly ordered family of members of $\mathcal{M}$ and $G := \bigcup_{i \in I} G_i$. In particular,

$$\langle G \rangle = \sum_{i \in I} \langle G_i \rangle = \bigcup_{i \in I} \langle G_i \rangle.$$

Then $E \cap \langle G \rangle = \{0\}$ since $E \cap \langle G_i \rangle = \{0\}$ $(i \in I)$. Therefore $G \in \mathcal{M}$ and, consequently, $\mathcal{M}$ contains a maximal member by Zorn's Lemma.

Let G be a maximal member of $\mathcal{M}$. First of all, $E \cap \langle G \rangle = \{0\}$ by the definition of $\mathcal{M}$. Let $t \in F \cap \chi$ be arbitrarily chosen. If $t \notin G$, then $\widetilde{G} := G + \{t\}$ is a strict superset of G and, since G is maximal in $\mathcal{M}$, $\widetilde{G} \notin \mathcal{M}$ which implies $E \cap \langle \widetilde{G} \rangle \neq \{0\}$. Thus there exists an $a \in E$ with $a \neq 0$ and $a \in \langle \widetilde{G} \rangle$. Consequently, there exist $x \in \langle G \rangle$ and $\lambda \in \mathbb{K}$ with $a = x + \lambda t$. We have $\lambda \neq 0$, otherwise $a = x \in E \cap \langle G \rangle = \{0\}$ in contradiction to $a \neq 0$. That implies $t \in E + \langle G \rangle$. If $t \in G$, then $t \in E + \langle G \rangle$ as well. Because the drawn conclusion holds for all members of $F \cap \chi$, it is also true for all members in $\langle F \cap \chi \rangle$, thus $\chi(F) \subset E + \langle G \rangle$ is proven.

Moreover, $F = E + \chi(F)$ is assumed by (ii). Therefore

$$F = E + \chi(F) \subset E + E + \langle G \rangle = E + \langle G \rangle.$$

Since both E and $\langle G \rangle$ are contained in F we get altogether $F = E + \langle G \rangle$. Obviously $Y := \langle G \rangle$ is a χ-based subspace with $E \cap Y = \{0\}$, thus $F = E \oplus Y$.

(iii)$\Rightarrow$(i): Let $x \in X$ be arbitrarily given. If x is already contained in E, then also in $E + \mathbb{K} \cdot \chi$. Now we assume $x \notin E$ and put $F := E + \langle x \rangle$. By the assumptions there exists a χ-based subspace Y of X with $F = E \oplus Y$. Necessarily, $Y \cap \chi \neq \{0\}$ otherwise, since Y is χ-based, $Y = \{0\}$ which would imply $F = E$, thus $x \in E$ in contradiction to the assumption. Let $t \in Y \cap \chi$ with $t \neq 0$ be arbitrarily chosen. Naturally $t \in E + Y = E + \langle x \rangle$. Thus there exist $a \in E$ and $\lambda \in \mathbb{K}$ with $t = a + \lambda x$. We have necessarily $\lambda \neq 0$ since otherwise $t \in E$ in contradiction to $t \neq 0$ and $E \cap Y = \{0\}$. Therefore, $x \in E + \mathbb{K} \cdot \chi$, thus X is a member of the upper χ-zone of E. ∎

Theorem 4.25. *Each element in the lower and the upper χ-zone of a Hahn space E is also a Hahn space.*

Proof. If X is a member of the lower χ-zone of a Hahn space E, then $\chi(X) = \chi(E)$ by 4.23. Now, let F be an FH-space with $\chi(X) \subset F$. Then $\chi(E) \subset F$, and thus, since E is a Hahn space, $E \subset F$ is satisfied. This implies $X \subset F$ because of $X \subset E$, that is, X is also a Hahn space. On the other hand, if X is an element of the upper χ-zone of a Hahn space E, then $X = E + \chi(X)$ by 4.24. Thereby both E and $\chi(X)$, which is χ-based, are Hahn spaces. Therefore, by 4.19, X is also a Hahn space. ■

In particular, if E is a Hahn space, then $E \cap \langle\chi\rangle$ is a Hahn space too, because $E \cap \langle\chi\rangle$ has the same χ-part than E and is therefore a member of the lower χ-zone of E.

4.4. Big Hahn spaces

It has been proved in [3, Theorem 3.4] that $bs \oplus \langle e\rangle$ is a Hahn space and that each sequence space X with $bs \oplus \langle e\rangle \subset X \subset \ell_\infty (= \widehat{\chi})$ is a Hahn space too. This result motivates the following definition.

Definition 4.26. A Hahn space E is called *big*, if each subspace X of H with $E \subset X \subset \widehat{\chi}$ is also a Hahn space.

Example 4.27.

(a) If E is a subspace of $\widehat{\chi}$ with $\chi \subset E$, then E is obviously a big Hahn space.
(b) $E = bs \oplus \langle e\rangle$ is a big Hahn space by [3, Theorem 3.4][6]

We give sufficient and necessary conditions for the property to be a big Hahn space.

Theorem 4.28. *Let E be a Hahn space.*

(a) *If $\widehat{\chi}$ is a member of the upper χ-zone of E, then E is a big Hahn space.*
(b) *If, in addition, E is an FH-space, then E is a big Hahn space if and only if $\widehat{\chi} = E + \mathbb{K} \cdot \chi$.*

Proof. (a) Let $\widehat{\chi}$ be a member of the upper χ-zone of E. Then $\widehat{\chi} \subset E + \mathbb{K} \cdot \chi$. Moreover, since E is a Hahn space, we have $E \subset \widehat{\chi}$ by 4.16 and consequently $\widehat{\chi} = E + \mathbb{K} \cdot \chi$ because $\mathbb{K} \cdot \chi \subset \widehat{\chi}$ holds. Now, let X be a subspace of H with $E \subset X \subset \widehat{\chi}$. Then X is also a member of the upper χ-zone of the Hahn space E and therefore a Hahn space by 4.25. Thus E is a big Hahn space.

(b) Let E be an FH-space with the Hahn property.

$\Leftarrow$: This implication is already proved in part (a).

$\Rightarrow$: We assume that E is a big Hahn space, implying $E + \mathbb{K} \cdot \chi \subset \widehat{\chi}$, and that there exists an $x \in \widehat{\chi}$ with $x \notin E + \mathbb{K} \cdot \chi$. We consider the subspace $X := E + \langle x\rangle$ of $\widehat{\chi}$. The spaces E and X have the same χ-part [For all $t \in X \cap \chi$ there exist an

[6]Naturally, we consider here the case with $H := \omega$ and $\chi := \{(x_k) \in \omega \,|\, x_k \in \{0, 1\}\}$.

$a \in E$ and $\lambda \in \mathbb{K}$ such that $t = a + \lambda x$. Because $\lambda \neq 0$ would imply $x \in E + \mathbb{K} \cdot \chi$, we get $\lambda = 0$ and therefore $t \in E$.]. Moreover, since E is a big Hahn space, X is a Hahn space which implies, because E is a Hahn space and $\chi(X) = \chi(E) \subset E$, that $X \subset E$ and thus $x \in E$ in contradiction to the assumption. Altogether, $\widehat{\chi} = E + \mathbb{K} \cdot \chi$ is proved. ∎

5. Hahn spaces of real sequences relative to (ω, χ)

In this section we consider exclusively spaces of real sequences, that is, subspaces of ω, and χ is the set of all sequences of 0's and 1's.

In the first part of this section we mention some special properties of χ, the set of all sequences of 0's and 1's.

Theorem 5.1. *For each $n \in \mathbb{N}$ the set χ_n is $\| \, \|_\infty$-closed in ℓ_∞.*

Proof. For the very technical proof we refer to [19, Satz 6.1.8]. ∎

Theorem 5.2. *Let $E \subset \ell_\infty$ be any FK-space. Then $\chi(E)$ is a Baire space (as subspace of E) if and only if $\chi(E)$ is finite dimensional. In particular, $m_0 = \langle \chi \rangle = \chi(\ell_\infty)$ is not a Baire space (relative to $\| \, \|_\infty$).* Note, concerning m_0, a better result is that m_0 can be covered by a countable family of proper closed hyperplanes (cf. [21, Proposition 1.3.5]).

Proof. By 4.12 and 5.1 it is sufficient to prove, that χ_n, $n \in \mathbb{N}$, contains exclusively finite dimensional linear subspaces.

Let $n \in \mathbb{N}$ be arbitrarily given and F be a linear subspace of χ_n. The members of F may accept at most 2^n different values. Let $a = (a_k)$ be a member of F with a maximal number $\nu \in \mathbb{N}$ of different values. Then there exist pairwise unequal real numbers $\lambda_1, \lambda_2, \dots, \lambda_\nu$ and sequences $t^{(1)}, t^{(2)}, \dots, t^{(\nu)}$ of 0's and 1's with pairwise disjoint support sets such that $a = \sum_{i=1}^{\nu} \lambda_i \, t^{(i)}$. Then we may choose an $\varepsilon > 0$ such that the ε-neighbourhoods of $\lambda_1, \lambda_2, \dots, \lambda_\nu$ are pairwise disjoint.

For any chosen $b \in F$ with $b \neq 0$ let $\lambda := \frac{\varepsilon}{\|b\|_\infty}$ and $a^* := a + \lambda b \in F$. Then a^* accepts also exactly ν different values by the assumptions on a. By the construction of $a^* = (a^*_n)$ the identity $a_k = a_l$ implies $a^*_k = a^*_l$ for $k, l \in \mathbb{N}$. Therefore a^* as well as b are finite linear combinations of $t^{(1)}, t^{(2)}, \dots, t^{(\nu)}$. Altogether $F \subset \left\langle \{t^{(1)}, t^{(2)}, \dots, t^{(\nu)}\} \right\rangle$, and therefore F has at most the dimension $\nu \leqslant 2^n$. ∎

In the next part of the Section we characterize for any (infinite) matrix $A = (a_{nk})$ the inclusions $A(\chi) \subset \chi$ and $A(\zeta) \subset \zeta$ where ζ denotes the set of all linear combinations of sequences of 0's and 1's with coefficients in $\mathbb{Z}$. Note that, because each subset of $\mathbb{Z}$ is bounded if and only if it is finite, ζ may be also considered as the set of all integral bounded sequences.

Theorem 5.3. *For any matrix A the following statements are equivalent:*

(i) $A(\chi) \subset \chi$.
(ii) $A \in \Gamma$ *whereby* Γ *denotes the set of all matrices of 0's and 1's such that each row contains at most one 1.*

Proof. (i)⇒(ii): The condition (i) includes that[7] $\chi \subset \omega_A$ and that $Ae^k \in \chi$ for all $k \in \mathbb{N}$, which implies that A is a matrix of 0's and 1's. Assume that there exist $k, l, n \in \mathbb{N}$ with $k \neq l$ such that the n^{th} row of A accepts the value 1 in the k^{th} and l^{th} position. We consider $t := e^k + e^l$ and $x = (x_n) := At$. Then certainly $t \in \chi$, but $x_n \notin \{0, 1\}$. Thus $At \notin \chi$ contradicting condition (i).

(ii)⇒(i): This is obvious since A contains at most one 1 in each row. ∎

Theorem 5.4. *For any matrix A the following statements are equivalent:*

(i) $A(\zeta) \subset \zeta$.
(ii) A *consists of integers and* $\|A\| < \infty$ *(row sum norm).*
(iii) $A \in \mathbb{Z}(\Gamma)$ *where* $\mathbb{Z}(\Gamma)$ *denotes the set of all finite linear combinations of members in* Γ *with integral coefficients.*

Proof. (i)⇒(ii): A consists of integers since $Ae^k \in \zeta$ for all $k \in \mathbb{N}$. Additionally $A(\chi) \subset A(\zeta) \subset \zeta \subset \ell_\infty$, thus $\chi \subset A^{-1}(\ell_\infty)$. Since $A^{-1}(\ell_\infty)$ is an FK-space and ℓ_∞ has the Hahn property we get $\ell_\infty \subset A^{-1}(\ell_\infty)$, thus $A(\ell_\infty) \subset \ell_\infty$ which is equivalent to $\|A\| < \infty$ (cf. [5, 2.3.5]).

(ii)⇒(iii): Let $A \neq 0$ (otherwise (iii) is trivially true). The condition (ii) implies that the set $\mathcal{A} := \{a_{nk} \mid n, k \in \mathbb{N}\}$ of all coefficients of $A = (a_{nk})$ is a finite subset of $\mathbb{Z}$ and that $\sup_n |\{k \in \mathbb{N} \mid a_{nk} \neq 0\}| < \infty$. Now, if $z \in \mathcal{A} \setminus \{0\}$ is given, then we define the matrix $T = (t_{nk}) \in \Gamma$ by

$$t_{nk} := \begin{cases} 1, & \text{if } a_{nk} = z \text{ and } k = \min\{r \in \mathbb{N} \mid a_{nr} = z\}, \\ 0 & \text{otherwise} \end{cases} \qquad (n, k \in \mathbb{N}).$$

Afterwards, if $A - zT \neq 0$ we may apply the same construction to the matrix $A - zT$. Repeating this procedure we get in finite many steps the zero matrix, so that $A \in \mathbb{Z}(\Gamma)$ is proven.

(iii)⇒(i): For all $B \in \Gamma$ the conclusion $B(\chi) \subset \chi \subset \zeta$ holds by 5.3 which implies $B(\zeta) \subset \zeta$. Therefore, if $A \in \mathbb{Z}(\Gamma)$, $A(\zeta) \subset \zeta$ is true. ∎

In [3, Theorem 3.4] it has been shown that $bs + \langle e \rangle = \Sigma^{-1}(\ell_\infty) + \langle e \rangle$ is a Hahn space where Σ is the summation matrix and Σ^{-1} its inverse matrix. Thereby, this property was reduced to that of ℓ_∞. Moreover, it has been also proved that it is even a big Hahn space. Now, the aim is to develop conditions so that this result remains true if we replace Σ^{-1} by certain matrices A, ℓ_∞ by certain sequence spaces E, and e by suitable, as small as possible sets T of sequences of 0's and 1's.

[7] For any matrix $A = (a_{nk})$ the set $\omega_A := \big\{(x_k) \in \omega \mid \sum_k a_{nk}x_k \text{ exists for all } n \in \mathbb{N}\big\}$ is called the *application domain of A*.

Definition 5.5. Let $E < \omega$ and $T \subset \chi$. A matrix A is called *χ-preserving on E with tolerance T*, if $E \subset \omega_A$ and $A(\chi(E)) \subset \chi(A(E) + \langle T\rangle)$.

Examples 5.6.

(a) By [3, Lemma 3.3] the matrix Σ^{-1} is χ-preserving on ℓ_∞ with tolerance $\{e\}$.

(b) Let I be a non-empty subset of $\mathbb{N}$. For simplification, we assume $I = \mathbb{N}$ if I is an infinite set, and $I = \mathbb{N}_r$ if I has exactly r $(r \in \mathbb{N})$ elements. Furthermore, let $N_i = \{n_{ij} \in \mathbb{N} | \, j \in \mathbb{N}\}$ $(i \in I)$ be infinite ordered sets such that

$$\bigcup_{i\in I} N_i = \mathbb{N}, \qquad N_i \cap N_j = \emptyset \qquad (i, j \in I, i \neq j)$$

that is, $N = (N_i |\, i \in I)$ is a partition of $\mathbb{N}$ (consisting of infinite sets). Then, generalizing bs and Σ, in [12] the sequence space

$$bs(N) := \left\{ x \in \omega \;\middle|\; \|x\|_{bs(N)} := \sup_j \|(x_k)_{k\in N_j}\|_{bs} < \infty \right\}$$

has been introduced and the matrix Σ_N defined by

$$[\Sigma_N x]_n := \sum_{j=1}^{k} x_{n_{ij}} \qquad \text{if } n = n_{ik} \quad (n, k \in \mathbb{N}, i \in I).$$

The inverse matrix of Σ_N is denoted by Σ_N^{-1}. Obviously, for every $y \in \omega$ we have

$$\left[\Sigma_N^{-1} y\right]_n = y_{n_{ik}} - y_{n_{i,k-1}} \qquad \text{when } n = n_{ik} \quad (n, k \in \mathbb{N}, i \in I, y_{n_{i0}} := 0),$$

thus $\Sigma_N^{-1}(\ell_\infty) = bs(N)$. Note that $bs = bs(N)$ and $\Sigma = \Sigma_N$ hold in the special case $N = (\mathbb{N})$.

By [12, Lemma 2.5] the matrix Σ_N^{-1} is χ-preserving on ℓ_∞ with tolerance $\{e\}$.

Theorem 5.7. *Let E be a Hahn space and A be a χ-preserving matrix on E with tolerance $T \subset \chi$. Then $A(E) + \langle T\rangle$ is a Hahn space too.*

Proof. Let $G := A(E) + \langle T\rangle$ and F be an FK-space with $\chi(G) \subset F$. By the assumptions we get $A(\chi(E)) \subset \chi(A(E) + \langle T\rangle) = \chi(G) \subset F$, thus $\chi(E) \subset A^{-1}(F)$. With F the space $A^{-1}(F)$ is also an FK-space. Then $E \subset A^{-1}(F)$, thus $A(E) \subset F$, follows by the Hahn property of E. In addition, $\langle T\rangle = \chi(\langle T\rangle) \subset \chi(G) \subset F$. Altogether $G \subset F$, and thus G is a Hahn space. ■

Let A be a given matrix and E a given sequence space. Then it is of mathematical interest to research for subsets T of χ such that T is as small as possible and A is χ-preserving with tolerance T on E. Note that, on this score, $T = \emptyset$ is not necessary a possible tolerance. The case, that $T = \emptyset$ is a tolerance, is characterized in [19]. The formulation of this result requires two more notations.

Notation 5.8. Let Γ^* be the set of all matrices which are a member of Γ or those which come from a matrix $A = (a_{nk}) \in \Gamma$ by extending A by a column (at an arbitrary position) a^* with $-a^* \in \chi$ and[8] $\mathrm{Sp}(-a^*) \supset \bigcup_k \mathrm{Sp}(a_{nk})_n$.

Moreover, Ψ denotes the set of all matrices $A = (a_{nk})$ such that $(a_{nk})_k = 0$ or $(a_{nk})_k = a$ for a fixed sequence $a \in \ell_1$, $\|a\|_1 = 1$ $(n \in \mathbb{N})$.

Theorem 5.9 ([19, p. 52, end of Section 6.3]). *If A is a given real matrix, then A is χ-preserving on every Hahn space E with tolerance $\emptyset$ if and only if $A \in (\mathbb{R} \cdot \Gamma^*) \cup (\mathbb{R} \cdot \Psi)$.*

Proof. This is an immediate corollary of [19, Satz 6.2.8][9]. ∎

In the following we consider the special case of matrices being χ-preserving motivated by the example $bs + \langle e \rangle$: We consider the sequence space $E := \ell_\infty$ and the tolerance $T := \{e\}$. For instance, the matrices Σ_N^{-1}, especially Σ^{-1}, are χ-preserving in that special case (cf. 5.6).

In the next theorem we give a sufficient condition for the matrices being in demand. Afterwards we'll give a class of matrices which satisfy that condition.

Theorem 5.10. *Let A be a matrix with $\ell_\infty \subset \omega_A$. Then $E := A(\ell_\infty) + \langle e \rangle$ is a Hahn space, if $A(\zeta) \subset \zeta$ and $\zeta \cap E \subset \chi(E)$ are satisfied.*

Proof. By the assumption $\ell_\infty \subset \omega_A$ we get $A(\chi) \subset A(\ell_\infty) \subset E$. Since $A(\zeta) \subset \zeta$ and $\zeta \cap E \subset \chi(E)$ this implies $A(\chi) = A(\chi) \cap E \subset A(\zeta) \cap E \subset \zeta \cap E \subset \chi(E)$. Therefore, A is χ-preserving on ℓ_∞ with tolerance $\{e\}$. Thus, by 5.7, E is a Hahn space. ∎

Recall, by 5.4, the condition $A(\zeta) \subset \zeta$ means exactly that A consists of integers and its row norm is finite.

Definition and Remarks 5.11. A matrix $A = (a_{nk})$ is called *χ-regular*, if the following conditions are satisfied:

(i) A is a triangle (normal matrix).
(ii) $\forall\, n \in \mathbb{N} : \sum_{k=0}^{n-1} |a_{nk}| \leqslant |a_{nn}|$.
(iii) $(a_{nn}) \in \ell_\infty$.
(iv) $\forall\, n, k \in \mathbb{N} : a_{nk} \in \mathbb{Z}$.

If A is χ-regular and D denotes the diagonal matrix with diagonal (a_{nn}), then $\|A - D\| \leqslant \|D\| < \infty$, thus $\|A\| = \|(A - D) + D\| \leqslant 2\|D\| < \infty$ and[10] $\ell_\infty \subset (\ell_\infty)_A$. Moreover, $A(\zeta) \subset \zeta$ if A is χ-regular.

The matrices Σ_N^{-1}, in particular Σ^{-1}, are χ-regular. With the following Theorem we show that [3, Lemma 3.2] and [12, Lemma 2.4] remain true when we consider more generally χ-regular matrices instead Σ_N^{-1}.

[8] $\mathrm{Sp}(x)$ denotes the support of a sequence x.

[9] We omit the presentation of the proof of this result because it does not play any role in the further research of this paper.

[10] For any matrix A and $Y < \omega$ we set $Y_A := \left\{x \in \omega_A \mid Ax := \left(\sum_k a_{nk} x_k\right)_n \in Y\right\}$.

Theorem 5.12. *Let A be a χ-regular matrix and E be a sequence space satisfying $A(\ell_\infty) \subset E$. If x is any sequence in E with coefficients exclusively in $\{0,1,2,\dots,L\}$ for a suitable $L \in \mathbb{N}_0$, then $x \in \chi(E)$.*

Proof. Let $A = (a_{nr})$, E, $x \in E$, and L be given according to the assumptions. If $L = 0$, then obviously $x \in \chi(E)$. Now, let $L > 0$ be fixed. Then we construct inductively the n^{th} coefficient of sequences $t^{(1)}, t^{(2)}, \dots, t^{(L)}$ in χ and sequences $u^{(1)}, u^{(2)}, \dots, u^{(L)} \in \omega$ with the following properties:

(i) $x = \sum_{k=1}^{L} t^{(k)}$.
(ii) $t^{(k)} = Au^{(k)}$ for $k \in \{1,2,\dots,L\}$.
(iii) $\left\| u^{(k)} - u^{(l)} \right\|_\infty \leqslant 1$ for $k,l \in \{1,2,\dots,L\}$.

For that we use the notations $t_n^{(k)}$ and $u_n^{(k)}$ ($k \in \{1,2,\dots,L\}$ and $n \in \mathbb{N}$) for the n^{th} coefficient of $t^{(k)}$ and $u^{(k)}$, respectively. Then (ii) is equivalent to

$$t_n^{(k)} = a_{nn} u_n^{(k)} + v_n^{(k)} \qquad \text{where } v_n^{(k)} := \sum_{r<n} a_{nr}\, u_r^{(k)} \qquad (k \in \{1,2,\dots,L\},\ n \in \mathbb{N}). \tag{5.1}$$

Note, the integrality together with the normality of A is here used only to guarantee $|a_{nn}| \geqslant 1$ for all $n \in \mathbb{N}$.

Basis of the induction, $n = 0$: Let $a := x_0$. Then, by the assumption, $a \in \mathbb{N}$ with $a \leqslant L$. We set

$$t_0^{(r)} := \begin{cases} 1, & \text{if } 1 \leqslant r \leqslant a, \\ 0 & \text{otherwise} \end{cases} \qquad (1 \leqslant r \leqslant L).$$

Obviously $x_0 = \sum_{k=1}^{L} t_0^{(k)}$, that is (i) for $n = 0$. In order that (ii) is true for $n = 0$ we define $u_0^{(k)} := \frac{t_0^{(k)}}{a_{00}}$ for $k \leqslant L$. Since $|a_{00}| \geqslant 1$ the condition

$$\left| u_0^{(k)} - u_0^{(l)} \right| \leqslant \left| t_0^{(k)} - t_0^{(l)} \right| \leqslant 1 \qquad (k,l \in \{1,2,\dots,L\})$$

holds, that is (iii) for $n = 0$.

Induction step $0,\dots,n-1 \to n$*:* Let $a := x_n$. Then again, by the assumption, $a \in \mathbb{N}$ with $a \leqslant L$. As arranged in (5.1) we have

$$v_n^{(k)} = \sum_{r<n} a_{nr}\, u_r^{(k)} \qquad (k \in \{1,2,\dots,L\}).$$

If we have chosen suitable, pairwise different $r_1, \dots, r_a \in \{1,2,\dots,L\}$, then we set

$$t_n^{(r)} := \begin{cases} 1, & \text{if } r \in \{r_1, \dots, r_a\}, \\ 0 & \text{otherwise} \end{cases} \qquad (1 \leqslant r \leqslant L)$$

which guarantees $x_n = \sum_{k=1}^{L} t_n^{(k)}$, that is (i) for n. The selection of pairwise different $r_1, \dots, r_a \in \{1,2,\dots,L\}$ can be done such that for all $k,l \in \{1,2,\dots,L\}$

the condition $t_n^{(k)} = 0$ and $t_n^{(l)} = 1$ implies $v_n^{(k)} \leqslant v_n^{(l)}$. Having determined $t_n^{(r)}$ we define

$$u_n^{(k)} := \frac{t_n^{(k)} - v_n^{(k)}}{a_{nn}} \qquad (1 \leqslant k \leqslant L)$$

so that (ii) holds for n. Now, we are going to prove (iii) for n, that is,

$$\left|u_n^{(k)} - u_n^{(l)}\right| \leqslant 1 \qquad (k, l \in \{1, 2, \ldots, L\}).$$

For that let $k, l \in \{1, 2, \ldots, L\}$ be arbitrarily given. By the induction hypothesis we get from (iii) for $n-1$ the estimate

$$\left|v_n^{(k)} - v_n^{(l)}\right| = \left|\sum_{r<n} a_{nr}(u_r^{(k)} - u_r^{(l)})\right| \leqslant \sum_{r<n} |a_{nr}| \leqslant |a_{nn}|\,. \tag{5.2}$$

We distinguish three cases:

Case 1, $t_n^{(k)} = t_n^{(l)}$: Then

$$\left|u_n^{(k)} - u_n^{(l)}\right| \leqslant \left|\frac{v_n^{(k)} - v_n^{(l)}}{a_{nn}}\right| \leqslant 1.$$

Case 2, $t_n^{(k)} = 1$ and $t_n^{(l)} = 0$: We get

$$\left|u_n^{(k)} - u_n^{(l)}\right| = \left|\frac{t_n^{(k)} - t_n^{(l)} - v_n^{(k)} + v_n^{(l)}}{a_{nn}}\right| = \left|\frac{1 - v_n^{(k)} + v_n^{(l)}}{a_{nn}}\right|$$

and $v_n^{(k)} - v_n^{(l)} \geqslant 0$. That leads to the estimates $0 \leqslant v_n^{(k)} - v_n^{(l)} \leqslant |a_{nn}|$ by (5.2) and $0 \leqslant 1 \leqslant |a_{nn}|$, thus to $|1 - v_n^{(k)} + v_n^{(l)}| \leqslant |a_{nn}|$. Altogether, we have again $|u_n^{(k)} - u_n^{(l)}| \leqslant 1$.

Case 3, $t_n^{(k)} = 0$ and $t_n^{(l)} = 1$: Exchange the roles of k and l in the second case.

In all, we have constructed $t^{(1)}, t^{(2)}, \ldots, t^{(L)} \in \chi$ and $u^{(1)}, u^{(2)}, \ldots, u^{(L)} \in \omega$ with the desired properties.

In particular, for arbitrary $k, l \in \{1, 2, \ldots, L\}$ we have $u^{(k)} - u^{(l)} \in \ell_\infty$ and therefore also $t^{(k)} - t^{(l)} \in A(\ell_\infty) \subset E$. Summing over l, then

$$\sum_{l=1}^{L} \left(t^{(k)} - t^{(l)}\right) = Lt^{(k)} - x \in E \qquad \text{for all } k \in \{1, 2, \ldots, L\}.$$

By the assumption $x \in E$ and $L > 0$, we get in the end $t^{(k)} \in E$, thus $t^{(k)} \in \chi \cap E$ for all $k \in \{1, 2, \ldots, L\}$ which implies $x \in \chi(E)$ by property (i). ■

Theorem 5.13. *Let A be a χ-regular matrix and $E := A(\ell_\infty) + \langle e \rangle$. Then:*

(i) *$A(\zeta) \subset \zeta$.*
(ii) *$\zeta \cap E \subset \chi(E)$.*
(iii) *E is a Hahn space.*

Proof. (i) is contained in 5.11.

(ii) Let $x \in \zeta \cap E$. Because $x \in \ell_\infty \cap \mathbb{Z}^{\mathbb{N}}$ there exists an $L \in \mathbb{N}$ such that the coefficients of $y := x + Le$ are a subset of $\{0, 1, 2, \ldots, 2L\}$. Therefore $y \in \chi(E)$ by 5.12, thus $x \in \chi(E)$ since $e \in E$.

(iii) is a simple consequence of (i) and (ii) in connection with 5.10. ■

Preconsideration 5.14. In the following we aim at a generalization of [3, Proposition 3.1]. There it has been shown that in case of $E := A(\ell_\infty) + \langle e\rangle$ with $A = \Sigma^{-1}$ the statement $F = E + \chi(F)$ *holds for each sequence space* F *with* $E \subset F \subset \ell_\infty$. This is exactly the statement in 4.24 (ii) in the case $X = \ell_\infty$, $H = \omega$ and $\mathbb{K} = \mathbb{R}$. By 4.24 this is equivalent to

$$\ell_\infty \subset E + \mathbb{R} \cdot \chi = A(\ell_\infty) + \langle e\rangle + \mathbb{R} \cdot \chi.$$

For that the condition

$$[0,1]^{\mathbb{N}} \subset A(\ell_\infty) + \chi$$

is sufficient because, as we may prove (cf. [19, 4.5.5 (ii)]), $\ell_\infty = \mathbb{R} \cdot [0,1]^{\mathbb{N}} + \langle e\rangle$. That implies $\ell_\infty \subset \mathbb{R} \cdot (A(\ell_\infty) + \chi) + \langle e\rangle = A(\ell_\infty) + \mathbb{R} \cdot \chi + \langle e\rangle$.

Theorem 5.15. *If* A *is a* χ*-regular matrix, then* $[0,1]^{\mathbb{N}} \subset A(\ell_\infty) + \chi$.

Proof. Let $A = (a_{nk})$ be χ-regular, $x = (x_n) \in [0,1]^{\mathbb{N}}$, and $M \in \mathbb{R}$ with $M \geqslant \frac{1}{2}$. We are going to construct inductively (over n) sequences $t = (t_n) \in \chi$ and $u = (u_n) \in \ell_\infty$ with the following properties:

(i) $\|u\|_\infty \leqslant M$.
(ii) $x - t = Au$.

Basis of the induction $n = 0$: By the assumption $0 \leqslant x_0 \leqslant 1$. If $x_0 > \frac{1}{2}$, then we set $t_0 := 1$ and $t_0 := 0$ otherwise. Then we have $|x_0 - t_0| \leqslant \frac{1}{2} \leqslant M$. Additionally, if we put $u_0 := \frac{x_0 - t_0}{a_{00}}$, then, because $|a_{00}| \geqslant 1$, (i) and (ii) are satisfied for $n = 0$.

Induction step $0, \ldots, n-1 \to n$: Let t_k and u_k be already constructed for $k < n$ in accordance with (i) and (ii). Then

$$[Au]_n = a_{nn}u_n + v_n \qquad \text{where } \; v_n := \sum_{k<n} a_{nk}u_k.$$

We aim to verify the existence of an $t_n \in \{0,1\}$ such that $|u_n| \leqslant M$ holds for u_n uniquely defined by $x_n - t_n = v_n + a_{nn}u_n$ $(= [Au]_n)$. For that we note that

$$|v_n| \leqslant \sum_{k<n} |a_{nk}\, u_k| \leqslant M \sum_{k<n} |a_{nk}| \leqslant M|a_{nn}|$$

by the assumptions for A. Since $x_n \in [0,1]$ this provides us

$$-M|a_{nn}| \leqslant x_n - v_n \leqslant 1 + M|a_{nn}|.$$

We consider the following two cases:

Case 1, $x_n - v_n > M|a_{nn}|$: We set $t_n := 1$. Then

$$-M|a_{nn}| \leqslant M|a_{nn}| - 1 < x_n - v_n - 1 \leqslant M|a_{nn}|$$

by $2M|a_{nn}| \geqslant 1$, thus

$$|a_{nn}u_n| = |x_n - v_n - t_n| = |x_n - v_n - 1| \leqslant M|a_{nn}|.$$

Case 2, $x_n - v_n \leqslant M|a_{nn}|$: We put $t_n := 0$. Then $|a_{nn}u_n| \leqslant M|a_{nn}|$.

In both cases we have $|u_n| \leqslant M$, so that (i) and (ii) are satisfied for n.

Altogether, we have constructed sequences t and u with the desired properties. Because $u \in \ell_\infty$ and $t \in \chi$, the statement $x \in A(\ell_\infty) + \chi$ is proved. ∎

Theorem 5.16. *If A is a χ-regular matrix, then $A(\ell_\infty) + \langle e \rangle$ is a big Hahn space.*

Proof. $A(\ell_\infty) + \langle e \rangle$ has the Hahn property by 5.13(iii). Moreover, with 5.15 and 5.14, we get $\ell_\infty \subset A(\ell_\infty) + \langle e \rangle + \mathbb{R} \cdot \chi$ and, noting $A(\ell_\infty) \subset \ell_\infty$, even $\ell_\infty = A(\ell_\infty) + \langle e \rangle + \mathbb{R} \cdot \chi$. Thus $A(\ell_\infty) + \langle e \rangle$ is a big Hahn space by 4.28. ∎

Corollary 5.17. *Let $A = (a_{nk})$ be a matrix with the following properties:*

(i) *A is a triangle.*
(ii) *$(a_{nn})_n = e$.*
(iii) *For each $n \in \mathbb{N}$ there exists at most one $j \in \mathbb{N}_{n-1}$ with $a_{nj} \in \{-1, 1\}$ and $a_{nk} = 0$ for $k \in \mathbb{N}_{n-1}$, $k \neq j$.*

Then $A(\ell_\infty) + \langle e \rangle$ is a big Hahn space.

Consequently, for all partitions N, the spaces $bs(N) + \langle e \rangle = \Sigma_N^{-1}(\ell_\infty) + \langle e \rangle$, in particular $bs + \langle e \rangle$, are big Hahn spaces (cf. 5.6).

Proof. Obviously, the matrix A is χ-regular. Therefore $A(\ell_\infty) + \langle e \rangle$ is a big Hahn space by 5.16. ∎

In a further corollary of Theorem 5.16 we consider convolution products.

Definition 5.18. For all $x, y \in \omega$ the sequence $x * y$ defined by $[x * y]_n := \sum_{k=0}^{n} x_k y_{n-k}$ is called *convolution product.* If $x \in \omega$ and $E \subset \omega$, then we set $x * E := \{x * y \mid y \in E\}$.

Corollary 5.19. *If $p = (p_n) \in \varphi \cap \mathbb{Z}^{\mathbb{N}}$ with $\sum_{n>0} |p_n| \leqslant |p_0|$, then $p * \ell_\infty + \langle e \rangle$ is a big Hahn space.*

Proof. Let $F_p = (f_{nk})$ be the matrix defined by

$$f_{nk} := \begin{cases} p_{n-k} & \text{for } k \leqslant n, \\ 0 & \text{otherwise} \end{cases} \qquad (k, n \in \mathbb{N}).$$

Then F_p is obviously χ-regular, thus $F_p(\ell_\infty) + \langle e \rangle = p * \ell_\infty + \langle e \rangle$ is a big Hahn space by Theorem 5.16. ∎

Example 5.20. In particular, if $p = (p_n) := e^0 - e^1 = (1, -1, 0, ...)$, then F_p is the inverse of the summation matrix Σ and $p * \ell_\infty + \langle e \rangle = bs + \langle e \rangle$ is a big Hahn space.

6. Examples of Hahn tuples (H,T) with $T \subset H < \omega$

In this section we discuss examples of Hahn tuples (H,T) where H is an FK-space and T is an FH-regular subset of H. Furthermore, χ denotes the set of all sequences of 0's and 1's.

Example 6.1 (finite sets T of sequences of 0's and 1's)**.** Let T be a finite subset of χ. Then $\langle T\rangle$ is a finite dimensional FK-space, that is, $\langle T\rangle = {}^{\omega}\widehat{T}$, $\langle T\rangle$ is FK-regular in $H := \omega$ and (ω, T) *is a Hahn tuple. Moreover, the Hahn spaces relative to* (ω, T) *are exactly the linear hulls of subsets of* T.

Examples 6.2 (iteratively generated Hahn tuples)**.** (a) Let $T_1 := \chi \cap c_{C_1} = \chi \cap c_{C_1} \cap \ell_\infty$ whereby c_{C_1} denotes the set of all C_1-summable sequences. The space $\ell_\infty \cap c_{C_1}$ has the Hahn property relative to (ω, χ) because of $bs + \langle e\rangle \subset \ell_\infty \cap c_{C_1} \subset \ell_\infty$ (cf. [3, Theorem 3.4]). Moreover, $(\ell_\infty \cap c_{C_1}, \|\ \|_\infty)$ is a BK-space, thus T_1 is FK-regular in ω and, by definition, (ω, T_1) *is a Hahn tuple.*

Now, let P be the set of all periodical sequences. On one hand side we have $P \subset \ell_\infty \cap c_{C_1}$ and on the other hand $P \subset \langle\chi\rangle$. Altogether, $\langle P \cap T_1\rangle = \langle P \cap c_{C_1} \cap \chi\rangle = \langle P \cap \chi\rangle = P$. Therefore, P is a Hahn space relative to (ω, T_1).

(b) Let ac denote the set of all almost convergent sequences. We set $T_2 := ac \cap \chi$. Then (ω, T_2) is a Hahn tuple because ac is a Hahn space relative to (ω, χ) since $bs \oplus \langle e\rangle \subset ac \subset \ell_\infty$ (cf. [3, Theorem 3.4 and (6.5)]) and ac is an FK-space.

Obviously, $P \subset ac$. Consequently, $\langle T_2 \cap P\rangle = \langle \chi \cap P\rangle = P$ implying that P is a Hahn space relative to (ω, T_2). Moreover, $\ell_\infty \cap c_{C_1}$ is not a Hahn space relative to (ω, T_2) because of $\widehat{T_2} = ac \subsetneq \ell_\infty \cap c_{C_1}$. (However it is a Hahn space relative to (ω, χ), cf. (a)).

For the next example we need a further notion.

Definition and Remark 6.3. A subset T of χ is called linearly closed, if $\langle T\rangle \cap \chi = T$. For instance, if E is any sequence space, then $T := \chi \cap E$ is linearly closed.

Proposition 6.4. *If T is a linearly closed* ***strict*** *subset of χ, then $\langle \chi \setminus T\rangle = m_0$.*

Proof. For a proof of $\langle \chi \setminus T\rangle = m_0$ it is sufficient to verify $T \subset \langle \chi \setminus T\rangle$. Let $t \in T$ be arbitrarily chosen. Then there exist sets $A, B \subset \mathbb{N}$ with $t = \chi_A$ and $\chi_B \in \chi \setminus T$, respectively. We set $C := A \cap B$. If $\chi_C \in T$, then $\chi_{B\setminus C} \in \chi \setminus T$ because otherwise $\chi_B = \chi_C + \chi_{B\setminus C} \in T$ since T is linearly closed. Consequently, $\chi_A + \chi_{B\setminus C} \in \chi \setminus T$ and $t = \chi_A$ is representable as a difference of two members of $\chi \setminus T$. On the other hand, if $\chi_C \in \chi \setminus T$, then also $\chi_{A\setminus C} \in \chi \setminus T$. Consequently, $t = \chi_{A\setminus C} + \chi_C \in \langle \chi \setminus T\rangle$. Altogether, $T \subset \langle \chi \setminus T\rangle$ is proved. ∎

Examples 6.5 (big sets T of sequences of 0's and 1's)**. (a)** We consider again $T_1 = \chi \cap c_{C_1}$ and $T_3 := \chi \setminus T_1$. Because $\chi \setminus c_{C_1} \neq \emptyset$ we have $\langle T_3\rangle = \langle \chi \setminus T_1\rangle = m_0$ by 6.4. Therefore $\widehat{T_3} = \ell_\infty$ since ℓ_∞ contains T_3 and is a Hahn space. Thus (ω, T_3) is a Hahn tuple.

The sequence space $\ell_\infty \cap c_{C_1}$ is not a Hahn space relative to (ω, T_3) because $T_3 \cap c_{C_1} = \emptyset$ and the FK-space c satisfies $\{0\} \subsetneq c \subsetneq \ell_\infty \cap c_{C_1}$.

(b) Let $T_4 := \chi \setminus P$ where P denotes the set of all periodical sequences. Then $\widehat{T_4} = \widehat{m_0} = \ell_\infty$ by 6.4 since $\chi \cap P$ is linearly closed and a strict subset of χ. Moreover $(\omega, T_4,)$ is a Hahn tuple. Obviously, P is not a Hahn space relative to this Hahn tuple.

Now, we are going to prove that

$\ell_\infty \cap c_{C_1}$ *is also a Hahn space relative to the Hahn tuple* (ω, T_4).

For that, let $t = (t_n) \in P \setminus \{0\}$ with period $p \in \mathbb{N}$, $p \geqslant 1$, be given. Because $t \neq 0$ there exists an $r \in \mathbb{N}$, $r < p$, with $t_r = 1$. Moreover, let $t^* \in \chi$ be the sequence such that $t^*_n = 1$ if $n = r + p \cdot 2^k$, $k \in \mathbb{N}$, and $t^*_n = 0$ otherwise. Obviously, t^* is C_1-summable to 0. Consequently $t, t^*, t - t^* \in \chi c_{C_1}$ and $t^*, t - t^* \notin P$. Altogether, $P \cap \chi \subset \langle T_4 \cap c_{C_1} \rangle = \langle T_4 \cap (\ell_\infty \cap c_{C_1}) \rangle$. This implies, that each FK-space F containing $T_4 \cap (\ell_\infty \cap c_{C_1})$ includes $\chi \cap c_{C_1}$ and therefore, since $\ell_\infty \cap c_{C_1}$ is a Hahn space relative to (ω, χ), necessarily $\ell_\infty \cap c_{C_1} \subset F$. Thus $\ell_\infty \cap c_{C_1}$ is also a Hahn space relative to (ω, T_4).

Examples 6.6 (Hahn tuples (H, T) with $T \subset H < \omega$ and $T \cap \chi = \emptyset$)**.** As we know, an index sequence is a strictly increasing sequence in $\mathbb{N}$, and a sequence $x = (x_n) \in \omega$ is called boundedly increasing if $(x_{n+1} - x_n) \in \ell^\infty$. We consider the set $\mathcal{I}$ of all boundedly increasing index sequences. First of all, we verify that $\mathcal{I}$ is FK-regular in ω. For that let $t \in \chi$ be arbitrarily given and Σ be the summation operator (matrix). Then both $\Sigma e = (n)$ and $\Sigma(e + t)$ are strictly boundedly increasing sequences in $\mathbb{N}$, that is, $\Sigma e \in \mathcal{I}$ and $\Sigma(e + t) \in \mathcal{I}$, thus $\Sigma t = \Sigma\,(e + t) - \Sigma e \in \langle \mathcal{I} \rangle$. Altogether, $\Sigma(\chi) \subset \langle \mathcal{I} \rangle$. Conversely, let $x \in \mathcal{I}$ be arbitrarily chosen. Then $\Sigma^{-1} x$ is a bounded sequence in $\mathbb{N}$ and therefore a member of $m_0 = \langle \chi \rangle$. In all, $\Sigma(m_0) = \langle \mathcal{I} \rangle$. By 4.9(b) the set $\langle \mathcal{I} \rangle = \Sigma(m_0)$, and therefore $\mathcal{I}$, is FK-regular because m_0 is FK-regular, whereby $\Sigma(\ell_\infty)$ is the FK-hull of $\mathcal{I}$.

Obviously, $\mathcal{I}$ is a subset of the FK-space $\Sigma(\ell_\infty)$. We consider the Hahn tuple $(\Sigma(\ell_\infty), \mathcal{I})$. Then ℓ_∞ is not a Hahn space relative to it, since $\mathcal{I} \cap \ell_\infty = \emptyset$.

On the other hand $\Sigma(\ell_\infty)$ is a Hahn space relative to this Hahn tuple, since $\Sigma(\ell_\infty)$ is the FK-hull of $\mathcal{I}$.

7. Examples of function spaces that are Hahn spaces

Searching for convenient Hahn tuples (H, χ) in the field of real functions on a nonempty set X, a first natural step is to replace sequences of 0's and 1's by functions of 0's and 1's, that is, to consider χ as a set of characteristic functions χ_A for suitable subsets A of X. Once χ can be chosen such that $\langle \chi \rangle$ is a barrelled subspace of $H := B(X)$, the space of all real bounded functions on X equipped with the topology of the supremum norm, then (H, χ) is a Hahn tuple due to 3.7(v).

In the following examples, if not explicitly stated otherwise, topologies on spaces of bounded functions are always considered to be generated by the supremum norm.

Example 7.1. Let $B[0,1]$ be the space of real bounded functions on $[0,1]$ and

$$\chi[0,1] := \left\{\chi_{[a,b]} \mid 0 \leqslant a \ \leqslant b \leqslant 1\right\} \ (\subset B[0,1]).$$

Obviously, $\langle\chi[0,1]\rangle$ is the set of all step functions on $\mathbb{R}$ restricted to $[0,1]$, that is,

$$\langle\chi[0,1]\rangle = \langle\{\chi_I \mid I \text{ is an interval with } I \subset [0,1]\}\rangle .$$

We prove, that unfortunately $\langle\chi[0,1]\rangle$ is not a barrelled subspace of $B[0,1]$, which is not really surprising.

For a proof we need some preconsiderations.

Obviously, for every $f \in \langle\chi[0,1]\rangle$ there exists a unique representation

$$f = \sum_{k=1}^{r} a_k \chi_{A_k} \tag{7.1}$$

with $r \in \mathbb{N}$, (nonempty and not necessarily closed) intervals $A_1, A_2, \ldots, A_r$ forming a partition of $[0,1]$ written in the natural order and numbers $a_1, a_2, \ldots, a_r \in \mathbb{R}$ such that $a_i \neq a_{i+1}$ $(1 \leqslant i < r)$. Obviously, λf has the representation $\lambda f = \sum_{k=1}^{r} \lambda a_k \chi_{A_k}$. Now, let $f, g \in \langle\chi[0,1]\rangle$ and $f+g$ have — in the sense of (7.1) — the representations

$$f = \sum_{k=1}^{r} a_k \chi_{A_k}, \qquad g = \sum_{k=1}^{s} b_k \chi_{B_k} \qquad \text{and} \qquad f+g = \sum_{k=1}^{t} c_k \chi_{C_k}. \tag{7.2}$$

Then we get the values c_k and the partition sets C_k as follows: We consider the partition sets $A_1 \cap B_{s_1}, \ldots, A_1 \cap B_{s^1}, A_2 \cap B_{s_2}, \ldots, A_2 \cap B_{s^2}, \ldots, A_r \cap B_{s_r}, \ldots, A_r \cap B_{s^r}$ where $s_i := \min\{\nu \mid A_i \cap B_\nu \neq \emptyset \ (1 \leqslant \nu \leqslant s)\}$ and $s^i := \max\{\nu \mid A_i \cap B_\nu \neq \emptyset \ (1 \leqslant \nu \leqslant s)\}$. Obviously, the chosen sets $A_i \cap B_j$ and the accompanying numbers $a_i + b_j$ are candidates for the C_k's and c_k's.

Note $s_1 = 1$, $s_r = s$ and $a_i + b_j \neq a_i + b_{j+1}$ if $i \in \mathbb{N}_r$ and $s_i \leqslant j < s^j$. If, for instance, $a_i + b_{s^i} = a_{i+1} + b_{s_{i+1}}$, then we may consider the interval $(A_i \cap B_{b_{s^i}}) \cup (A_{i+1} \cap B_{b_{s_{i+1}}})$ as an candidate for a suitable C_k. In this way we get a partition $\widehat{C_k}$ (with natural ordering) with accompanying numbers $\widehat{c_k}$ $(k \in \mathbb{N}_{\widehat{t}})$ such that $f + g = \sum_{k=1}^{\widehat{t}} \widehat{c_k} \chi_{\widehat{C_k}}$. Finally, we unite neighboring intervals $\widehat{C_k}$ having the same value $\widehat{c_k}$ into intervals C_j with value $c_j = \widehat{c_k}$ and get the expected unique partition $C_1, \ldots, C_t$ with values $c_1, \ldots, c_t$.

Using the unique representation in (7.1), we define for any fixed nonempty interval $I \subset [0,1]$ the function

$$T := T_I : \langle\chi[0,1]\rangle \longrightarrow \mathbb{R}, \qquad T_I(f) := \sum_{k=1}^{r} a_k\, \mu(I \cap A_k) \qquad (f \in \langle\chi[0,1]\rangle),$$

whereby $\mu(X)$ denotes the length of an interval X. Let $f, g, f+g \in \langle\chi[0,1]\rangle$ with representations as in (7.2) be given. Then it follows by the above considerations

$$\begin{aligned} T\,(f) + T\,(g) &= \sum_{k=1}^{r} a_k\, \mu\,(I \cap A_k) + \sum_{l=1}^{s} b_l\, \mu\,(I \cap B_l) \\ &= \sum_{k=1}^{r} \sum_{l=s_k}^{s^k} (a_k + b_l)\; \mu\,(I \cap A_k \cap B_l) \\ &= \sum_{m=1}^{t} c_m\, \mu\,(I \cap C_m) \;=\; T(f+g). \end{aligned}$$

Obviously, $T(\lambda f) = \lambda T(f)$ for any $\lambda \in \mathbb{R}$ and $f \in \langle \chi[0,1] \rangle$. Therefore, T is linear. Moreover, T is continuous by $|a_k| \leqslant \|f\|$, $k \in \mathbb{N}_r$, and

$$|T(f)| \leqslant \sum_{k=1}^{r} |a_k|\; \mu(A_k) \leqslant \|f\| \sum_{k=1}^{r} \mu(A_k) \leqslant \|f\|\,.$$

Now, we are prepared to verify that $\langle \chi[0,1] \rangle$ is not a barrelled subspace of $B[0,1]$: Let (I_n) be a sequence of pairwise disjoint closed (non-empty) intervals contained in $[0,1]$ such that the length $\mu(I_n)$ of I_n is positive for infinite many n. Dividing every I_n in half, I_n^l and I_n^r denote respectively the left and the right half of I_n for $n \in \mathbb{N}$. For a given real sequence (α_n) let (T_n) be the sequence functions on $\langle \chi[0,1] \rangle$ defined for $n \in \mathbb{N}$, $f \in \langle \chi[0,1] \rangle$ with the representation as in (7.1), and $\alpha, \beta \in \mathbb{R}$ by

$$T_n(f) := \alpha_n \sum_{k=1}^{r} a_k \left(\mu(I_n^l \cap A_k) - \mu(I_n^r \cap A_k) \right) \qquad (n \in \mathbb{N}).$$

In particular,

$$T_n(\chi_A) = \alpha_n\; \mu(I_n^l \cap A) - \alpha_n\; \mu(I_n^r \cap A) \qquad (\chi_A \in \chi[0,1],\; n \in \mathbb{N})\,.$$

The functions T_n, $n \in \mathbb{N}$, are linear combinations of functions of the type T defined in (7.1) and are therefore linear and continuous as well.

Any such interval $A = [a,b] \subset [0,1]$ is disjoint to or contains I_n for almost every $n \in \mathbb{N}$ with at most two exceptions: Since the intervals I_n are pairwise disjoint there are at most two of them that contain a or b. Therefore $(T_n(\chi_A))_n \in \varphi$ and consequently (T_n) converges pointwise to the zero-function on $[0,1]$, and thus it is pointwise bounded. However, choosing (α_n) such that $(|\alpha_n|\, \mu(I_n)) \notin \ell_\infty$, (T_n) is not equicontinuous due to $T_n(\chi_{I_n^l}) = \frac{\alpha_n}{2} \mu(I_n)$, $n \in \mathbb{N}$. By [5, 6.8.4] the subspace $\langle \chi[0,1] \rangle$ of $B[0,1]$ is not barrelled.

Remark 7.2. Let $\mathcal{A}$ be a σ-algebra of subsets of a non-empty set I, $\chi\,(I, \mathcal{A}) := \{\chi_A \mid A \in \mathcal{A}\}$ and $m_0\,(I, \mathcal{A}) := \langle \chi(I, \mathcal{A}) \rangle$. Then $m_0\,(I, \mathcal{A})$ is a dense, barrelled subspace of the Banach space $\ell_\infty(\mathcal{A})$ of all bounded $\mathcal{A}$-measurable functions on I (cf. [17, p. 141] and [13]).

Example 7.3. Let denote $\mathcal{B}[0,1]$ the σ-algebra of all Borel subsets of $[0,1]$ and $\mathcal{M}[0,1]$ the space of all bounded Borel-measurable functions on $[0,1]$. By the above statement, $m_0([0,1],\mathcal{B}[0,1])$ is a dense, barrelled subspace of $\mathcal{M}[0,1]$. Therefore, $(B[0,1],\chi([0,1],\mathcal{B}[0,1]))$ is a Hahn tuple, and $\mathcal{M}[0,1]$ is a respective Hahn space. The space $C[0,1]$ of continuous functions on $[0,1]$ is not a Hahn space relative to this Hahn tuple: $C[0,1]\cap\chi([0,1],\mathcal{B}[0,1])$ contains only two functions and these are contained in the one-dimensional Fréchet space of constant functions on $[0,1]$ which does not contain $C[0,1]$.

Example 7.4. In the case of $I := [0,1]$ and $\mathcal{A} := \mathcal{P}([0,1])$, the power set of $[0,1]$, the barrelledness of $m_0([0,1],\mathcal{P}([0,1]))$ as a (dense) subspace of $B[0,1]$ was already observed by Grothendieck [18, p. 145, Exercise 8] (cf. [2, Introduction]). Therefore, $(B[0,1],\chi([0,1],\mathcal{P}([0,1])))$ is a Hahn tuple. Unlike $B[0,1]$, the space $C[0,1]$ of continuous functions on $[0,1]$ is, again, not a Hahn space relative to this Hahn tuple.

Example 7.5. For every $r\in\mathbb{N}$ let $P_r[0,1]$ denote the space of real polynomials on $[0,1]$ of degree r or less. Consider $\mathcal{F}_r := \{f\cdot\chi_A \mid f\in P_r[0,1], A\in\mathcal{B}[0,1]\}\subset B[0,1]$. Obviously, $m_0([0,1],\mathcal{B}[0,1]) = \langle\mathcal{F}_0\rangle\subset\langle\mathcal{F}_r\rangle\subset\mathcal{M}[0,1]$, $r\in\mathbb{N}$. As $m_0([0,1],\mathcal{B}[0,1])$ is barrelled and dense in $\langle\mathcal{F}_r\rangle$, $\langle\mathcal{F}_r\rangle$ is barrelled too ([21, Prop. 4.2.1 (ii)]), $r\in\mathbb{N}$. The same holds for $\langle\mathcal{F}\rangle$ defined by $\mathcal{F} := \bigcup_{r\in\mathbb{N}}\mathcal{F}_r$. Therefore, $(B[0,1],\mathcal{F}_r)$, $r\in\mathbb{N}$, and $(B[0,1],\mathcal{F})$ are Hahn tuples.

$P_r[0,1]$ is obviously a Hahn space relative to the Hahn tuple $(B[0,1],\mathcal{F}_r)$, $r\in\mathbb{N}$, and $P[0,1] := \bigcup_{r\in\mathbb{N}} P_r[0,1]$ is a Hahn space relative to $(B[0,1],\mathcal{F})$.

The function space $C[0,1]$ is not a Hahn space for the above Hahn tuples, and not even for $(B[0,1],\langle\mathcal{F}\rangle)$: Any $f\in C[0,1]\cap\langle\mathcal{F}\rangle$ can be described as a linear combination of p elements of $\mathcal{F}$ with a maximum degree q of the respective polynomials for some $p,q\in\mathbb{N}$. There exist at maximum $\frac{1}{2}p(p-1)q$ points in $[0,1]$ where each two of those polynomials can meet. Therefore, $[0,1]$ can be divided up into a finite number of intervals (not just Borel sets) on which f is identical to a fixed polynomial. Then a similar construction like that in 7.1 can be used to show that $C[0,1]\cap\langle\mathcal{F}\rangle$ is not barrelled (set $T_n(f\cdot\chi_A) = 0$ if f has a degree greater than 0, $n\in\mathbb{N}$). But $C[0,1]$ is an FH-space and therefore it cannot be a Hahn space.

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