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DUAL SPACE OF THE MINKOWSKI–RÅDSTRÖM–HÖRMANDER SPACE OVER $\mathbb{R}^2$

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Dedicated to Lech Drewnowski on
the occasion of his 70th birthday

Abstract: Let X be a real normed vector space and $\mathcal{B}(X)$ be the cone of all nonempty bounded closed convex subsets of X. For $A, B, C, D \in \mathcal{B}(X)$ we have a relation of equivalence defined by $(A, B) \sim (C, D)$ if and only if $\overline{A+D} = \overline{B+C}$. By $[A, B]$ we denote the quotient class of (A, B). The quotient space $\widetilde{X} = \mathcal{B}^2(X)/_\sim$ is a vector space called the Minkowski–Rådström–Hörmander space over X. For $\tilde{x} = [A, B] \in \widetilde{X}$ we have the Hausdorff norm $\|\tilde{x}\|_H = d_H(A, B) = \inf\{\varepsilon > 0 | \, A \subset B + \varepsilon\mathbb{B}, B \subset A + \varepsilon\mathbb{B}\}$ where $\mathbb{B}$ is the closed unit ball in X. We also define Bartels-Pallaschke norm $\|\tilde{x}\|_{BP} = \inf\{\|C\| + \|D\| \,|\, (C, D) \in [A, B]\}$, where $\|A\| = \sup_{a\in A} \|a\|$. In this paper we prove that the bilinear function $(\cdot,\cdot) : (\widetilde{\mathbb{R}^2}, \|\cdot\|_H) \times (\widetilde{\mathbb{R}^2}, \|\cdot\|_{BP}) \longrightarrow \mathbb{R}$ defined by $(\tilde{x}, \tilde{y}) = 2V(\tilde{x}, \tilde{y}) + \langle s\tilde{x}, s\tilde{y}\rangle$, where $V(\tilde{x}, \tilde{y})$ is a generalized mixed volume and $s\tilde{x}$ is a generalized Steiner's point, satisfies the inequality $|(\tilde{x}, \tilde{y})| \leqslant (2\pi + 1)\|\tilde{x}\|_H\|\tilde{y}\|_{BP}$. We also prove that this bilinear function defines an isomorphic mapping between Banach spaces $(\widetilde{\mathbb{R}^2}, \|\cdot\|_{BP})$ and the dual space to $(\widetilde{\mathbb{R}^2}, \|\cdot\|_H)$ (Theorem 2).

Keywords: Minkowski–Rådström–Hörmander spaces, extreme points, pairs of closed bounded convex sets.

Let X be a real Hausdorff topological vector space and $\mathcal{B}(X)$ be the cone of all nonempty bounded closed convex subsets of X. For $A, B, C, D \in \mathcal{B}(X)$ we have the Minkowski sum $A + B = \{a + b | a \in A, b \in B\}$ and a relation of equivalence defined by

$$(A, B) \sim (C, D) \qquad \text{if and only if} \qquad \overline{A+D} = \overline{B+C}.$$

By $[A, B]$ we denote the quotient class of (A, B). The quotient space $\widetilde{X} = \mathcal{B}^2(X)/_\sim$ is a vector space with the addition and multiplication by real numbers defined by $[A, B] + [C, D] = [\overline{A+C}, \overline{B+D}]$ and $\alpha \cdot [A, B] = [\alpha^+ A + \alpha^- B, \alpha^- A + \alpha^+ B]$. The space $\widetilde{X}$ is called the Minkowski–Rådström–Hörmander (MRH) space over X [7].

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The MRH space [11], [17] is very useful in studying bounded-valued correspondences [5], [7], in quasidifferential calculus [6], [15], [20] and in calculating Aumann–Integral [4]. The MRH spaces were studied also in a number of papers, for example [8], [18] and [21].

In 2007 A. Pełczyński posed a question what is the dual space to the MRH space over $\mathbb{R}^2$. In fact the dual space depends on the norm in the MRH space.

Let $(X, \|\cdot\|)$ be a normed space, $\tilde{x} = [A, B] \in \widetilde{X}$. The norm in the MRH space over X can be introduce in two natural ways. Hausdorff's metric determines the following norm (see [17]): $\|\tilde{x}\|_H = d_H(A, B) = \inf\{\varepsilon > 0 | A \subset B + \varepsilon\mathbb{B}, B \subset A + \varepsilon\mathbb{B}\}$ where $\mathbb{B}$ is the closed unit ball in X. We also define Bartels-Pallaschke norm $\|\tilde{x}\|_{BP} = \inf\{\|C\| + \|D\| \,|\, (C, D) \in [A, B]\}$, where $\|A\| = \sup_{a\in A}\|a\|$. Bartels-Pallaschke norm [2] is related to the norm given in [3] defined in the space of differences of real sublinear functions by $\|f\| = \inf\{\max\{\|g\|, \|h\|\} \,|\, g - h = f,$ where g, h are sublinear$\}$. Bartels-Pallaschke norm seems almost as natural as Hausdorff norm. Notice that $\|\cdot\|_H \leqslant \|\cdot\|_{BP}$. The normed space $(\widetilde{X}, \|\cdot\|_H)$ is not complete unless $\dim X = 1$ but for Banach space X, Bartels-Pallaschke norm turns the space $(\widetilde{X}, \|\cdot\|_{BP})$ into Banach space.

Let $NBV[a, b]$ be the space of normalized real functions of bounded variation on the interval $[a, b]$, that is for $f \in NBV[a, b]$ we have $f(a) = 0$ and $f(t) = f(t^+) = \lim_{s\to t^+} f(s)$, for $t \in [a, b)$.

By f^+, f^- we denote the smallest nondecreasing functions in $NBV[a, b]$ such that $f = f^+ - f^-$. The space $NBV[a, b]$ with the norm defined by $\|f\| = \mathrm{var}_a^b f = f^+(b) + f^-(b)$ is a Banach space. Now we are ready to state our first theorem.

Theorem 1. *There exists an isomorphic mapping between Banach spaces* $(\widetilde{\mathbb{R}^2}, \|\cdot\|_{BP})$ *and* $NBV[0, 2\pi]$.

Before we point out to the isomorphic mapping we need some more notations and definitions.

For our convenience we denote $e^{it} = (\cos t, \sin t) \in \mathbb{R}^2$. For $A \in \mathcal{B}(\mathbb{R}^2), u \in \mathbb{R}^2$ we define the support function $p_A : \mathbb{R}^2 \ni x \longmapsto \max_{a\in A}\langle x, a\rangle \in \mathbb{R}$, where $\langle x, a\rangle$ is the inner product of x and a, the face $H_{\langle u,\cdot\rangle}A = \{a \in A | \langle u, a\rangle = p_A(u)\}$, the boundary function $h_A : [0, 2\pi] \longrightarrow \partial A$ by $h_A = H_{\langle e^{i(t+\frac{\pi}{2})},\cdot\rangle}(H_{\langle e^{it},\cdot\rangle}A)$. We also need the arc length function $f_A : [0, 2\pi] \longrightarrow \mathbb{R}_+$ where $f_A(t)$ is the length of the arc contained in ∂A joining $h_A(0)$ and $h_A(t)$. The function f_A is a nondecreasing function in $NBV[0, 2\pi]$. Most of these notations come from [9].

Let sA be the Steiner's point of A (see [19], p. 42), that is

$$sA = \frac{1}{\pi}\int_{S^1} u p_A(u) d\mathcal{H}^1(u) = \frac{1}{\pi}\int_0^{2\pi} e^{it} p_A(e^{it}) dt.$$

For $u \in \mathbb{R}^2$ we define the auxiliary function $f_u : [0, 2\pi] \longrightarrow \mathbb{R}_+$ by

$$f_u(t) = \frac{\|u\|}{\pi}(\sin(t - \operatorname{Arg} u) + \sin(\operatorname{Arg} u)) = \frac{e^{it}\bar{u} - e^{-it}u + u - \bar{u}}{2\pi i}.$$

The following lemma is essential in the proof of Theorem 1:

Lemma 1. *The mapping* $k : \widetilde{\mathbb{R}^2} \longrightarrow NBV[0, 2\pi]$*, where* $k(\tilde{y}) = f_C - f_D + f_z$*,* $\tilde{y} = [C, D], z = sC - sD$ *is an* 1-1 *isomorphic mapping between vector spaces.*

Proof. Since the functions $p_C - p_D$ and $f_C - f_D$ do not depend on the choice of representatives of $\tilde{y}$ (see [9]), the mapping k is well defined and preserves the addition and the multiplication by scalars. In order to give the inverse mapping we need more definitions and facts.

In a manner of [9] for nondecreasing function $f \in NBV[0, 2\pi]$ we define the function $h_f : [0, 2\pi] \longrightarrow \mathbb{R}^2$ by $h_f(t) = \int_0^t e^{i(s+\frac{\pi}{2})} df(s)$, where the latter is the Stieltjes integer.

If $h_f(2\pi) = 0$ then we denote $A_f = \overline{\operatorname{conv}} h_f([0, 2\pi])$. Then $A_{f_A} = A - h_A(0)$ and $f_{A_f} = f$ (see [9]). If $h_f(2\pi) \neq 0$ then there exists the unique function $g \in NBV[0, 2\pi]$ such that g is nondecreasing, g takes exactly two values and $h_g(2\pi) = -h_f(2\pi)$. Hence $h_{f+g}(2\pi) = 0$.

Let $f \in NBV[0, 2\pi]$ and $w = w(f) = \int_0^{2\pi} e^{it} df(t)$. Then

$$\begin{aligned}
\int_0^{2\pi} e^{i(t+\frac{\pi}{2})} df_w(t) &= \int_0^{2\pi} e^{i(t+\frac{\pi}{2})} \frac{\|w\|}{\pi} \cos(t - \operatorname{Arg} w) dt \\
&= \frac{\|w\|}{\pi} \int_0^{2\pi} e^{i(t+\frac{\pi}{2})} \frac{e^{i(t-\operatorname{Arg} w)} + e^{-i(t-\operatorname{Arg} w)}}{2} dt \\
&= \frac{\|w\|}{2\pi} \left(\int_0^{2\pi} e^{i(\frac{\pi}{2} - \operatorname{Arg} w)} e^{i2t} dt + \int_0^{2\pi} e^{i(\frac{\pi}{2} + \operatorname{Arg} w)} dt \right) \\
&= \frac{\|w\|}{2\pi} \left(0 + 2\pi e^{i(\frac{\pi}{2} + \operatorname{Arg} w)} \right) = e^{i\frac{\pi}{2}} w = \int_0^{2\pi} e^{i(t+\frac{\pi}{2})} df(t).
\end{aligned}$$

Hence $h_{(f-f_w)^+}(2\pi) - h_{(f-f_w)^-}(2\pi) = \int_0^{2\pi} e^{i(t+\frac{\pi}{2})} d(f - f_w)(t) = 0$ and $h_{(f-f_w)^+}(2\pi) = h_{(f-f_w)^-}(2\pi)$.

Then there exists the unique function $g \in NBV[0, 2\pi]$ such that g is nondecreasing, g takes exactly two values and $h_g(2\pi) = -h_{(f-f_w)^+}(2\pi)$. Hence $h_{(f-f_w)^+ + g}(2\pi) = 0$ and $h_{(f-f_w)^- + g}(2\pi) = 0$.

Let $E = A_{(f-f_w)^+ + g}$ and $F = A_{(f-f_w)^- + g}$. Let us define the mapping $l : NBV[0,2\pi] \longrightarrow \widetilde{\mathbb{R}^2}$ by $l(f) = [E - sE + w, F - sF]$.

Let $\tilde{y} \in \widetilde{\mathbb{R}^2}, \tilde{y} = [C, D]$. We can assume that (C, D) is a minimal pair (see [14], [9], [20] and [15]). We have $l(k(\tilde{y})) = l(f_C - f_D + f_z)$ where $z = sC - sD$.

Notice that

$$\begin{aligned} w &= \int_0^{2\pi} e^{it} d(k(\tilde{y}))(t) = \int_0^{2\pi} e^{it} df_C(t) - \int_0^{2\pi} e^{it} df_D(t) + \int_0^{2\pi} e^{it} df_z(t) \\ &= e^{-i\frac{\pi}{2}} h_{f_C}(2\pi) - e^{-i\frac{\pi}{2}} h_{f_D}(2\pi) + \int_0^{2\pi} \frac{\|z\|}{\pi} e^{it} \cos(t - \mathrm{Arg} z) dt \\ &= 0 - 0 + z = z. \end{aligned}$$

Hence $k(\tilde{y}) - f_w = f_C - f_D$. Since the pair (C, D) is minimal, the maximal nondecreasing function $g \in NBV[0, 2\pi]$ such that the functions $f_C - g, f_D - g$ are nondecreasing takes not more than two values [9].

Then $(f_C - f_D)^+ = f_C - g$ and $(f_C - f_D)^- = f_D - g$. Hence $E = A_{(f_C - f_D)^+ + g} = A_{f_C} = C - h_C(0)$ and $F = A_{(f_C - f_D)^- + g} = A_{f_D} = D - h_D(0)$ (see [9]).

Therefore, $l(k(\tilde{y})) = [E - sE + w, F - sF] = [C - sC + z, D - sD] = [C, D] = \tilde{y}$.

Now, let $f \in NBV[0, 2\pi]$. Then $k(l(f)) = k([E - sE + w, F - sF])$, where $E = A_{((f-f_w)^+ + g}, F = A_{(f-f_w)^- + g}$ and $w = \int_0^{2\pi} e^{it} df(t)$.

Hence $k(l(f)) = f_{E-sE+w} - f_{F-sF} + f_z = f_E - f_F + f_w$ because $z = s(E - sE + w) - s(F - sF) = w$.

Therefore, $k(l(f)) = (f - f_w)^+ + g - (f - f_w)^- - g + f_w = (f - f_w) + f_w = f$. ∎

Proof of Theorem 1. By the Open Operator Theorem we need only to prove that k is continuous. For $\tilde{y} = [C, D], z = sC - sD$ we have

$$\begin{aligned} \|k(\tilde{y})\| &= \|f_C - f_D + f_z\| \leqslant f_C(2\pi) + f_D(2\pi) + \|f_z\| \\ &= |\partial C| + |\partial D| + 4\frac{\|z\|}{\pi} \leqslant 2\pi\|C\| + 2\pi\|D\| + 4\frac{\|sC\| + \|sD\|}{\pi} \\ &\leqslant \Big(2\pi + \frac{4}{\pi}\Big)(\|C\| + \|D\|), \end{aligned}$$

where $|\partial A|$ is the length of the boundary ∂A.

Since these inequalities hold true for any pair $(C, D) \in \tilde{y}$ then $\|k(\tilde{y})\| \leqslant (2\pi + \frac{4}{\pi})$ $\|\tilde{y}\|_{BP}$.

The proof of Theorem 1 is completed. However, we can estimate the norm of the operator l. For $f \in NBV[0, 2\pi]$ we have

$$\begin{aligned}\|l(f)\|_{BP} &\leqslant \|E - sE + w\| + \|F - sF\| \\ &\leqslant \frac{1}{2}|\partial A_{(f-f_w)^+ + g}| + \|w\| + \frac{1}{2}|\partial A_{(f-f_w)^- + g}| \\ &\leqslant \frac{1}{2}((f-f_w)^+ + g)(2\pi) + \frac{1}{2}((f-f_w)^- + g)(2\pi) + \|w\| \\ &\leqslant (f-f_w)^+(2\pi) + (f-f_w)^-(2\pi) + \|w\| \leqslant \|f - f_w\| + \|w\| \\ &\leqslant \|f\| + \|f_w\| + \|w\| = \|f\| + \frac{4\|w\|}{\pi} + \|w\| \\ &\leqslant \|f\| + \Big(\frac{4}{\pi} + 1\Big)\|\int_0^{2\pi} e^{it} df(t)\| \leqslant \|f\| + \Big(\frac{4}{\pi}+1\Big)\|f\| \\ &= \Big(2 + \frac{4}{\pi}\Big)\|f\|.\end{aligned}$$

If the norm in $\mathbb{R}^2$ is not Euclidean then the inequalities above hold true with different constants. ■

Knowing that the MRH space $(\widetilde{\mathbb{R}^2}, \|\cdot\|_{BP})$ and the space $NBV[0,2\pi]$ are isomorphic we see that the dual to MRH space $(\widetilde{\mathbb{R}^2}, \|\cdot\|_{BP})$ and the dual to the space $NBV[0,2\pi]$ are also isomorphic. The dual to the space $NBV[0,2\pi]$ is described for example in [1].

Let $V : \mathcal{B}^2(\mathbb{R}^2) \longrightarrow \mathbb{R}_+$ be the mixed volume (see Theorem 5.1.6 in [19]). The function V can be extended to the bilinear function on $(\widetilde{\mathbb{R}^2})^2$ (see [22]) by $V(\tilde{x}, \tilde{y}) = V(A,C) - V(A,D) - V(B,C) + V(B,D)$, where $\tilde{x} = [A,B], \tilde{y} = [C,D]$. Also the function $s : \mathcal{B}(\mathbb{R}^2) \longrightarrow \mathbb{R}^2$, where sA is the Steiner's point of A can be extended to the linear function on $\widetilde{\mathbb{R}^2}$ by $s\tilde{x} = sA - sB$.

Theorem 2. *The function $(\cdot,\cdot) : (\widetilde{\mathbb{R}^2}, \|\cdot\|_H) \times (\widetilde{\mathbb{R}^2}, \|\cdot\|_{BP}) \longrightarrow \mathbb{R}$ defined by $(\tilde{x}, \tilde{y}) = 2V(\tilde{x}, \tilde{y}) + \langle s\tilde{x}, s\tilde{y}\rangle$ is bilinear and continuous. Moreover, the space of linear functions $\{(\cdot, \tilde{y}) | \tilde{y} \in \widetilde{\mathbb{R}^2}\}$ is dual to $(\widetilde{\mathbb{R}^2}, \|\cdot\|_H)$.*

Proof. Let $C[0,2\pi]$ be the space of all continuous functions on $[0,2\pi]$ and let $C_0[0,2\pi] = \{q \in C[0,2\pi] | q(0) = q(2\pi)\}$. The bilinear function $(\cdot,\cdot) : C[0,2\pi] \times NBV[0,2\pi] \longrightarrow \mathbb{R}$ defined by $(q,f) = \int_0^{2\pi} q(t)df(t)$ establishes duality of $NBV[0,2\pi]$ to $C[0,2\pi]$ (see the theorem of Riesz in [12], 17.7.4) and to $C_0[0,2\pi]$.

All the functions $t \longmapsto (p_A - p_B)(e^{it})$, where $A, B \in \mathcal{B}(\mathbb{R}^2)$, form a dense subspace of $C_0[0,2\pi]$. Since k is an isomorphic mapping then the space of functions $\{\widetilde{\mathbb{R}^2} \ni \tilde{x} \longmapsto \int_0^{2\pi} (p_A - p_B)(e^{it}) d(k(\tilde{y}))(t) \in \mathbb{R} \,|\, \tilde{y} \in \widetilde{\mathbb{R}^2}\}$, $\tilde{x} = [A,B]$, is the space of all continuous linear functions on $(\widetilde{\mathbb{R}^2}, \|\cdot\|_H)$.

Let $\tilde{x}, \tilde{y} \in \widetilde{\mathbb{R}^2}, \tilde{x} = [A,B], \tilde{y} = [C,D], z = s\tilde{y}$. By the formulas 1.7.3 and 5.1.15 in [19],

$$sA = \frac{1}{\pi}\int_0^{2\pi} p_A(e^{it})e^{it}dt, \qquad V(A,C) = \frac{1}{2}\int_0^{2\pi} p_A(e^{it})df_C(t).$$

Then

$$
\begin{aligned}
&\int_0^{2\pi}(p_A-p_B)(e^{it})d(k(\tilde{y}))(t)\\
&\qquad=\int_0^{2\pi}(p_A-p_B)(e^{it})df_C(t)-\int_0^{2\pi}(p_A-p_B)(e^{it})df_D(t)\\
&\qquad\quad+\int_0^{2\pi}(p_A-p_B)(e^{it})df_z(t)\\
&\qquad=2V(A,C)-2V(A,D)-2V(B,C)+2V(B,D)\\
&\qquad\quad+\int_0^{2\pi}\frac{\|z\|}{\pi}(p_A-p_B)(e^{it})\cos(t-\mathrm{Arg}z)dt\\
&\qquad=2V(\tilde{x},\tilde{y})+\int_0^{2\pi}\frac{\|z\|}{\pi}(p_A-p_B)(e^{it})\langle e^{it},e^{i\mathrm{Arg}z}\rangle dt\\
&\qquad=2V(\tilde{x},\tilde{y})+\frac{1}{\pi}\int_0^{2\pi}(p_A-p_B)(e^{it})\langle e^{it},z\rangle dt\\
&\qquad=2V(\tilde{x},\tilde{y})+\left\langle\frac{1}{\pi}\int_0^{2\pi}p_A(e^{it})e^{it}dt-\frac{1}{\pi}\int_0^{2\pi}p_B(e^{it})e^{it}dt,z\right\rangle\\
&\qquad=2V(\tilde{x},\tilde{y})+\langle sA-sB,sC-sD\rangle=2V(\tilde{x},\tilde{y})+\langle\tilde{x},\tilde{y}\rangle.
\end{aligned}
$$

Let us notice that

$$
\begin{aligned}
|V(\tilde{x},\tilde{y})|&\leqslant\left|\frac{1}{2}\int_0^{2\pi}(p_A-p_B)(e^{it})df_C(t)\right|+\left|\frac{1}{2}\int_0^{2\pi}(p_A-p_B)(e^{it})df_D(t)\right|\\
&\leqslant\frac{1}{2}\sup_{u\in S^1}|(p_A-p_B)(u)|\cdot|\partial C|+\frac{1}{2}\sup_{u\in S^1}|(p_A-p_B)(u)|\cdot|\partial D|\\
&=\frac{1}{2}\|\tilde{x}\|_H(|\partial C|+|\partial D|)\leqslant\pi\|\tilde{x}\|_H(\|C\|+\|D\|).
\end{aligned}
$$

Since the inequality holds true for any $(C,D)\in\tilde{y}$, $|V(\tilde{x},\tilde{y})|\leqslant\pi\|\tilde{x}\|_H\|\tilde{y}\|_{BP}$. By [16] we have $\|s\tilde{x}\|=\|sA-sB\|\leqslant\frac{4}{\pi}d_H(A,B)=\frac{4}{\pi}\|\tilde{x}\|_H$. Therefore, $|(\tilde{x},\tilde{y})|\leqslant|2V(\tilde{x},\tilde{y})|+|\langle s\tilde{x},s\tilde{y}\rangle|\leqslant 2\pi\|\tilde{x}\|_H\|\tilde{y}\|_{BP}+\frac{4}{\pi}\|\tilde{x}\|_H\|\tilde{y}\|_H\leqslant(2\pi+\frac{4}{\pi})\|\tilde{x}\|_H\|\tilde{y}\|_{BP}$. ■

By Theorem 15.7 in [13] the function of mixed volume V is continuous on $(\mathcal{B}(\mathbb{R}^2),d_H)\times(\mathcal{B}(\mathbb{R}^2),d_H)$. However, the following example shows that the extension of V is not continuous on $(\widetilde{\mathbb{R}^2},\|\cdot\|_H)\times(\widetilde{\mathbb{R}^2},\|\cdot\|_H)$ (compare Theorem 5.2.2 in [19]).

Example. Let A_n be a regular n-gon in $\mathbb{R}^2$ with the center in 0 and all sides of the length equal to 1. Let B_n be the n-gon A_n rotated around 0 by the angle $\frac{\pi}{n}$. The radius r_n of the circle inscribed in A_n or B_n is $\frac{1}{2\mathrm{tg}(\pi/n)}$. The radius R_n of the circumscribed circle is $\frac{1}{2\sin(\pi/n)}$. Denote $\tilde{x}_n = [A_n, B_n] \in \widetilde{\mathbb{R}^2}$. Then

$$\begin{aligned} V(\tilde{x}_n, \tilde{x}_n) &= V(A_n, A_n) - 2V(A_n, B_n) + V(B_n, B_n) \\ &= 2(V(A_n, A_n) - V(A_n, B_n)) \\ &= 2\left(\frac{1}{2}nr_n - \frac{1}{2}nR_n\right) = n(r_n - R_n). \end{aligned}$$

We have $\|\tilde{x}_n\|_H = d_H(A_n, A_n) = R_n - r_n$. Since $R_n - r_n$ tends to 0,

$$\lim_{n\to\infty} \frac{V(\tilde{x}_n, \tilde{x}_n)}{\|\tilde{x}_n\|_H^2} = \lim_{n\to\infty} \frac{-n}{R_n - r_n} = -\infty$$

and the bilinear function V is not contnuous in 0.

Theorem 2 shows that the MRH space $(\widetilde{\mathbb{R}^2}, \|\cdot\|_{BP})$ is basicly dual to the MRH space $(\widetilde{\mathbb{R}^2}, \|\cdot\|_H)$. Then the dual space to $(\widetilde{\mathbb{R}^2}, \|\cdot\|_{BP})$ is double dual space to $(\widetilde{\mathbb{R}^2}, \|\cdot\|_H)$.

We can also provide the space $\mathbb{R}^2$ with the norm of Demyanov (see [6]) which is stronger than Hausdorff norm but weaker than Bartels-Pallaschke norm. We do not know what is the dual space to that space.

The unit ball in the space $(\widetilde{X}, \|\cdot\|_H)$ has exactly two extreme points (see for example [10]). What are extreme points of the unit ball in $(\widetilde{X}, \|\cdot\|_{BP})$ is an open question, even if we know extreme points in $NBV[0,1]$.

In Theorem 2 in general we can naturally replace the bilinear function by $(\cdot,\cdot) : (\widetilde{\mathbb{R}^n}, \|\cdot\|_H) \times (\widetilde{\mathbb{R}^n}, \|\cdot\|_{BP}) \longrightarrow \mathbb{R}$ defined by

$$(\tilde{x}, \tilde{y}) = 2V(\tilde{x}, \tilde{y}, \underbrace{\mathbb{B}, \ldots, \mathbb{B}}_{n-2}) + \langle s\tilde{x}, s\tilde{y}\rangle,$$

where $\mathbb{B}$ is the Euclidean unit ball in $\mathbb{R}^n$. However, the theorem will no longer hold true.

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