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MACKEY TOPOLOGIES AND COMPACTNESS IN SPACES OF VECTOR MEASURES

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Dedicated to Lech Drewnowski on
the occasion of his 70th birthday

Abstract: Let Σ be a σ-algebra of subsets of a non-empty set Ω. Let $B(\Sigma)$ be the space of all bounded Σ-measurable scalar functions defined on Ω, equipped with the natural Mackey topology $\tau(B(\Sigma), ca(\Sigma))$. Let (E, ξ) be a quasicomplete locally convex Hausdorff space and let $ca(\Sigma, E)$ be the space of all ξ-countably additive E-valued measures on Σ, provided with the topology $\mathcal{T}_s$ of simple convergence. We characterize relative $\mathcal{T}_s$-compactness in $ca(\Sigma, E)$, in terms of the topological properties of the corresponding sets in the space $\mathcal{L}_{\tau,\xi}(B(\Sigma), E)$ of all $(\tau(B(\Sigma), ca(\Sigma)), \xi)$-continuous integration operators from $B(\Sigma)$ to E. A generalized Nikodym type convergence theorem is derived.

Keywords: spaces of bounded measurable functions, Mackey topologies, strongly Mackey space, vector measures, integration operators, topology of simple convergence.

1. Introduction and terminology

We denote by $\sigma(L, K)$ and $\tau(L, K)$ the weak topology and the Mackey topology on L with respect to a dual pair $\langle L, K\rangle$. For a locally convex space (L, η) by $(L, \eta)'$ or L'_η we denote the topological dual of (L, η). Recall that (L, η) is a strongly Mackey space if every relatively $\sigma(L'_\eta, L)$-countably compact subset of L'_η is η-equicontinuous.

We assume that Σ is a σ-algebra of subsets of a non-empty set Ω. Let $B(\Sigma)$ denote the Banach space of all bounded Σ-measurable scalar functions defined on Ω, provided with the uniform norm $\|\cdot\|$. Denote by $ba(\Sigma)$ the Banach space of all bounded finitely additive scalar measures on Σ with the norm $\|\mu\| = |\mu|(\Omega)$, where $|\mu|(A)$ denotes the variation of μ on $A \in \Sigma$. Then the Banach dual $B(\Sigma)^*$ of $B(\Sigma)$ can be identified with $ba(\Sigma)$ through the integration mapping $ba(\Sigma) \ni \mu \mapsto \Phi_\mu \in B(\Sigma)^*$, where $\Phi_\mu(f) = \int_\Omega f\, d\mu$ for $f \in B(\Sigma)$. Moreover, $\|\Phi_\mu\| = |\mu|(\Omega)$ (see [DU, Chap. 1, Theorem 13]). Let $ca(\Sigma)$ be the subspace of $ba(\Sigma)$ consisting of all countably additive measures.

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Definition 1.1. Let $\mu \in ba(\Sigma)$. A linear functional Φ_μ on $B(\Sigma)$ is said to be *σ-smooth* if $\Phi_\mu(f_n) \to 0$ for each uniformly bounded sequence (f_n) in $B(\Sigma)$ such that $f_n(\omega) \to 0$ for all $\omega \in \Omega$.

By $B(\Sigma)_c^*$ we will denote the space of all σ-smooth linear functionals on $B(\Sigma)$.

Proposition 1.1. *For $\mu \in ba(\Sigma)$ the following statements are equivalent:*

(i) Φ_μ *is σ-smooth.*
(ii) $\mu \in ca(\Sigma)$.

Proof. (i)$\Longrightarrow$(ii) Assume that Φ_μ is σ-smooth, and let $A_n \downarrow \emptyset$, $(A_n) \subset \Sigma$. Then $\mathbb{1}_{A_n}(\omega) \to 0$ for all $\omega \in \Omega$ and $\sup_n \|\mathbb{1}_{A_n}\| \leqslant 1$. Hence $\mu(A_n) = \int_\Omega \mathbb{1}_{A_n} d\,\mu \to 0$.

(ii)$\Longrightarrow$(i) Assume that $\mu \in ca(\Sigma)$ and let $f_n(\omega) \to 0$ for all $\omega \in \Omega$ and $\sup_n \|f_n\| < \infty$. Then by the Lebesgue dominated convergence theorem, $\int_\Omega |f_n| d|\mu| \to 0$. Since $|\Phi_\mu(f_n)| \leqslant \int_\Omega |f_n| d|\mu| \to 0$, we see that Φ_μ is σ-smooth. ■

For $\omega \in \Omega$ let $\Phi_\omega(f) = f(\omega)$ for $f \in B(\Sigma)$. Then $\Phi_\omega \in B(\Sigma)_c^*$ and the set $\{\Phi_\omega : \omega \in \Omega\}$ separates the points of Ω.

Let (E, ξ) be a locally convex Hausdorff space, briefly lcHs (over the field of complex or real numbers). By $ca(\Sigma, E)$ we denote the space of all ξ-countably additive vector measure $m : \Sigma \to E$, provided with the topology $\mathcal{T}_s$ of simple convergence. By $\mathcal{S}(\Sigma)$ we denote the space of all scalar-valued Σ-simple functions defined on Ω. Then $\mathcal{S}(\Sigma)$ can be endowed with the (locally convex) universal measure topology τ of Graves [G], that is, τ is the coarsest locally convex topology on $\mathcal{S}(\Sigma)$ such that the integration map $T_m : \mathcal{S}(\Sigma) \ni s \mapsto \int_\Omega s\, dm \in E$ is continuous for every locally convex space (E, ξ) and every $m \in ca(\Sigma, E)$ (see [G, p. 5]). Let $(L(\Sigma), \hat{\tau})$ stand for the completion of $(\mathcal{S}(\Sigma), \tau)$. It is known that both $(\mathcal{S}(\Sigma), \tau)$ and $(L(\Sigma), \hat{\tau})$ are strongly Mackey spaces (see [G, Corollaries 11.7 and 11.8]). It follows that $\tau = \tau(\mathcal{S}(\Sigma), ca(\Sigma))$ and $\hat{\tau} = \tau(L(\Sigma), ca(\Sigma))$ (see [G], [GR]). We have $\mathcal{S}(\Sigma) \subset B(\Sigma) \subset L(\Sigma)$ and the restriction $\hat{\tau}$ from $L(\Sigma)$ to $B(\Sigma)$ coincides with the Mackey topology $\tau(B(\Sigma), ca(\Sigma))$ (see [GR, §4]). Thus by Proposition 1.1 we get

$$\hat{\tau}|_{B(\Sigma)} = \tau(B(\Sigma), ca(\Sigma)) = \tau(B(\Sigma), B(\Sigma)_c^*).$$

Then $(B(\Sigma), \tau(B(\Sigma), ca(\Sigma)))$ is a strongly Mackey space (see [G, Corollary 11.8]). Moreover, if E is complete in its Mackey topology $\tau(E, E'_\xi)$, then for each $m \in ca(\Sigma, E)$, the integration map T_m can be uniquely extended to a $(\hat{\tau}, \xi)$-continuous map $\widetilde{T}_m : L(\Sigma) \to E$ (see [GR]).

Graves and Ruess ([GR, Theorem 7]) derived a characterization of relative $\mathcal{T}_s$-compactness in $ca(\Sigma, E)$ in terms of the corresponding integration operators $T_m : \mathcal{S}(\Sigma) \to E$ (resp. $\widetilde{T}_m : L(\Sigma) \to E$ whenever E is complete in its Mackey topology $\tau(E, E'_\xi)$).

The aim of this paper is to characterize relative compactness in $(ca(\Sigma, E), \mathcal{T}_s)$ in terms of the corresponding integration operators from $B(\Sigma)$ to E whenever (E, ξ) is a quasicomplete lcHs (see Theorem 2.2 below). As an application, we obtain a generalized Nikodym type convergence theorem (see Theorem 2.3 below).

2. Topological properties of spaces of vector measures

We start with the following useful result.

Proposition 2.1. *Assume that (f_n) is a uniformly bounded sequence in $B(\Sigma)$ such that $f_n(\omega) \to 0$ for all $\omega \in \Omega$. Then $f_n \to 0$ for $\tau(B(\Sigma), ca(\Sigma))$.*

Proof. Let $\mathcal{M}$ be a relatively $\sigma(ca(\Sigma), B(\Sigma))$-compact subset of $ca(\Sigma)$. Then in view of [Z, Theorem 1.1] $\mathcal{M}$ is bounded and uniformly countably additive, and hence $|\mathcal{M}|$ $(= \{|\mu| : \mu \in \mathcal{M}\})$ is uniformly countably additive (see [DU, Chap. 1, Proposition 17]). By [K, Theorem 1] we obtain that $\sup_{\mu\in\mathcal{M}} \int_\Omega |f_n| d|\mu| \to 0$; hence $\sup_{\mu\in\mathcal{M}} |\int_\Omega f_n d\,\mu| \to 0$. It follows that $f_n \to 0$ for $\tau(B(\Sigma), ca(\Sigma))$, as desired. ■

For terminology and basic results concerning the integration with respect to vector measures we refer the reader to [L], [P_1], [P_2].

Let (E, ξ) be a quasicomplete lcHs (over the field of complex or real numbers) and let $\mathcal{P}_\xi$ stand for the set of all ξ-continuous seminorms on E. Let $m : \Sigma \to E$ be a ξ-bounded measure (i.e., the range of m is ξ-bounded in E). Given $f \in B(\Sigma)$, let (s_n) be a sequence of Σ-simple scalar functions that converges uniformly to f on Ω. Following [P_1, Definition 1] we say that f is m-integrable and define

$$\int_\Omega f\,dm := \xi - \lim \int_\Omega s_n\,dm.$$

The $\int_\Omega f\,dm$ is well defined (see [P_1, Lemma 5]) and the map $T_m : B(\Sigma) \to E$ given by $T_m(f) = \int_\Omega f\,dm$ is $(\|\cdot\|, \xi)$-continuous and linear, and for each $e' \in E'_\xi$

$$e'\left(\int_\Omega f\,dm\right) = \int_\Omega f\,d(e' \circ m) \qquad \text{for} \ \ f \in B(\Sigma) \qquad (\text{see } [P_1, \text{ Lemma 5}]).$$

Conversely, let $T : B(\Sigma) \to E$ be a $(\|\cdot\|, \xi)$-continuous linear operator, and let $m(A) = T(\mathbb{1}_A)$ for $A \in \Sigma$. Then $m : \Sigma \to E$ is a ξ-bounded vector measure, called the *representing measure* of T and $T_m(f) = T(f)$ for $f \in B(\Sigma)$ (see [P_1, Definition 2]).

Definition 2.1. A linear operator $T : B(\Sigma) \to E$ is said to be *σ-smooth* if $T(f_n) \to 0$ in ξ for each uniformly bounded sequence (f_n) in $B(\Sigma)$ such that $f_n(\omega) \to 0$ for all $\omega \in \Omega$.

The following characterization of σ-smooth operators from $B(\Sigma)$ into a quasicomplete lcHs (E, ξ) displays the close connection between the Mackey topology $\tau(B(\Sigma), ca(\Sigma))$ on $B(\Sigma)$ and E-valued ξ-countably additive measures.

Proposition 2.2. *Assume that (E, ξ) is a quasicomplete lcHs. Then for a ξ-bounded measure $m : \Sigma \to E$ the following statements are equivalent:*

(i) *$e' \circ m \in ca(\Sigma)$ for each $e' \in E'_\xi$.*
(ii) *$e' \circ T_m \in B(\Sigma)^*_c$ for each $e' \in E'_\xi$.*
(iii) *T_m is $(\sigma(B(\Sigma), ca(\Sigma)), \sigma(E, E'_\xi))$-continuous.*

(iv) *T_m is $(\tau(B(\Sigma), ca(\Sigma)), \xi)$-continuous.*
(v) *T_m is $(\tau(B(\Sigma), ca(\Sigma)), \xi))$-sequentially continuous.*
(vi) *T_m is σ-smooth.*
(vii) *m is ξ-countably additive.*

Proof. (i)$\Longleftrightarrow$(ii) For each $e' \in E'_\xi$ we have

$$(e' \circ T_m)(f) = \int_\Omega f\, d(e' \circ m) \qquad \text{for all } \ f \in B(\Sigma).$$

Hence by Proposition 1.1 we get $e' \circ T_m \in B(\Sigma)^*_c$ if and only if $e' \circ m \in ca(\Sigma)$.

(ii)$\Longleftrightarrow$(iii) See [AB, Theorem 9.26].

(iii)$\Longrightarrow$(iv) Assume that T_m is $(\sigma(B(\Sigma), ca(\Sigma)), \sigma(E, E'_\xi))$-continuous. Then T_m is $(\tau(B(\Sigma), ca(\Sigma)), \tau(E, E'_\xi))$-continuous (see [AB, Ex. 11, p. 149]). It follows that T_m is $(\tau(B(\Sigma), ca(\Sigma)), \xi)$-continuous because $\xi \subset \tau(E, E'_\xi)$.

(iv)$\Longrightarrow$(v) It is obvious.

(v)$\Longrightarrow$(vi) Assume that T_m is $(\tau(B(\Sigma), ca(\Sigma), \xi)$-sequentially continuous, and let (f_n) be a sequence in $B(\Sigma)$ such that $f_n(\omega) \to 0$ for all $\omega \in \Omega$ and $\sup \|f_n\| < \infty$. Then by Proposition 2.1, $f_n \to 0$ for $\tau(B(\Sigma), ca(\Sigma))$. Hence $T(f_n) \to 0$ for ξ.

(vi)$\Longrightarrow$(vii) Assume that (vi) holds and let $A_n \downarrow \emptyset$, $(A_n) \subset \Sigma$. Then $\mathbb{1}_{A_n}(\omega) \downarrow 0$ for $\omega \in \Omega$ and $\sup_n \|\mathbb{1}_{A_n}\| \leqslant 1$. It follows that $m(A_n) = T_m(\mathbb{1}_{A_n}) \to 0$ for ξ, i.e., m is ξ-countably additive.

(vii)$\Longrightarrow$(i) It is obvious. ■

Let $\mathcal{L}_{\tau,\xi}(B(\Sigma), E)$ stand for the space of all $(\tau(B(\Sigma), ca(\Sigma)), \xi)$-continuous linear operators from $B(\Sigma)$ to E, equipped with the topology $\mathcal{T}_s$ of simple convergence. Then $\mathcal{T}_s$ is generated by the family $\{q_{p,u} : p \in \mathcal{P}_\xi, u \in B(\Sigma)\}$ of seminorms on $\mathcal{L}_{\tau,\xi}(B(\Sigma), E)$, where

$$q_{p,u}(T) := p(T(u)) \quad \text{for all } \ T \in \mathcal{L}_{\tau,\xi}(B(\Sigma), E).$$

Denote by $\mathcal{T}_s$ the topology of simple convergence in $ca(\Sigma, E)$. Then $\mathcal{T}_s$ is generated by the family $\{q_{p,A} : p \in \mathcal{P}_\xi, A \in \Sigma\}$ of seminorms, where

$$q_{p,A}(m) := p(m(A)) \qquad \text{for all } \ m \in ca(\Sigma, E).$$

Now we establish some terminology (see [P_1, pp. 92–93]). For $p \in \mathcal{P}_\xi$, let $E_p = (E, p)$ be the associated seminormed space. Denote by $(\widetilde{E}_p, \|\cdot\|_p^\sim)$ the completion of the quotient normed space $E/p^{-1}(0)$. Let $\Pi_p : E_p \to E/p^{-1}(0) \subset \widetilde{E}_p$ be the canonical quotient map (see [P_1, p. 92]).

Given a measure $m : \Sigma \to E$, let $m_p : \Sigma \to \widetilde{E}_p$ be given by

$$m_p(A) := (\Pi_p \circ m)(A) \qquad \text{for } A \in \Sigma.$$

Then m_p is a Banach space-valued measure on Σ. We define the p-semivariation $\|m\|_p$ of m by

$$\|m\|_p(A) := \|m_p\|(A) \qquad \text{for } A \in \Sigma,$$

where $\|m_p\|$ denotes the semivariation of $m_p : \Sigma \to \widetilde{E}_p$. Note that m is ξ-bounded if and only if $\|m\|_p(\Omega) < \infty$ for each $p \in \mathcal{P}_\xi$. Moreover, we have (see [P$_1$, Lemma 7])

$$\|m\|_p(\Omega) = \|T_m\|_p = \sup\Big\{ p\Big(\int_\Omega f\,dm \Big) : f \in B(\Sigma), \|f\| \leqslant 1 \Big\}. \qquad 2.1$$

The following result will be of importance (see [SZ, Theorem 2]).

Theorem 2.1. *Let $\mathcal{K}$ be a $\mathcal{T}_s$-compact subset of $\mathcal{L}_{\tau,\xi}(B(\Sigma), E)$. If C is a $\sigma(E'_\xi, E)$-closed and ξ-equicontinuous subset of E'_ξ, then $\{e' \circ T : T \in \mathcal{K}, e' \in C\}$ is a $\sigma(B(\Sigma)^*_c, B(\Sigma))$-compact subset of $B(\Sigma)^*_c$.*

Now using Theorem 2.1 we can state a characterization of relative $\mathcal{T}_s$-compactness in the space $\mathcal{L}_{\tau,\xi}(B(\Sigma), E)$.

For a subset $\mathcal{M}$ of $ca(\Sigma, E)$ let

$$\mathcal{K}_\mathcal{M} = \{T_m \in \mathcal{L}_{\tau,\xi}(B(\Sigma), E) : m \in \mathcal{M}\}.$$

Theorem 2.2. *Assume that (E, ξ) is a quasicomplete lcHs. Then for a subset $\mathcal{M}$ of $ca(\Sigma, E)$ the following statements are equivalent:*

- (i) *$\mathcal{K}_\mathcal{M}$ is a relatively $\mathcal{T}_s$-compact set in $\mathcal{L}_{\tau,\xi}(B(\Sigma), E)$.*
- (ii) *$\mathcal{K}_\mathcal{M}$ is $(\tau(B(\Sigma), ca(\Sigma)), \xi)$-equicontinuous; and for each $f \in B(\Sigma)$, the set $\{\int_\Omega f\,dm : m \in \mathcal{M}\}$ is relatively ξ-compact in E.*
- (iii) *$\int_\Omega f_n\,dm \to 0$ in ξ uniformly for $m \in \mathcal{M}$ whenever (f_n) is a uniformly bounded sequence in $B(\Sigma)$ such that $f_n(\omega) \to 0$ for all $\omega \in \Omega$; and for each $f \in B(\Sigma)$, the set $\{\int_\Omega f\,dm : m \in \mathcal{M}\}$ is relatively ξ-compact in E.*
- (iv) *$\mathcal{M}$ is uniformly ξ-countably additive; and for each $A \in \Sigma$, the set $\{m(A) : m \in \mathcal{M}\}$ is relatively ξ-compact in E.*
- (v) *$\mathcal{M}$ is a relatively $\mathcal{T}_s$-compact set in $ca(\Sigma, E)$.*

Proof. (i)$\Longrightarrow$(ii) Assume that $\mathcal{K}$ is relatively $\mathcal{T}_s$-compact. Let W be an absolutely convex and ξ-closed neighbourhood of 0 for ξ in E. Then the polar W^0 of W, with respect to the dual pair $\langle E, E'_\xi\rangle$, is a $\sigma(E'_\xi, E)$-closed and ξ-equicontinuous subset of E'_ξ (see [AB, Theorem 9.21]). Hence in view of Theorem 2.1 the set $H = \{e' \circ T_m : m \in \mathcal{M}, e' \in W^0\}$ in $B(\Sigma)^*_c$ is relatively $\sigma(B(\Sigma)^*_c, B(\Sigma))$-compact. Since $(B(\Sigma), \tau(B(\Sigma), ca(\Sigma)))$ is a strongly Mackey space, the set H is $\tau(B(\Sigma), ca(\Sigma))$-equicontiunous. It follows that there exists a $\tau(B(\Sigma), ca(\Sigma))$-neighborhood V of 0 in $B(\Sigma)$ such that $H \subset V^0$, where V^0 denotes the polar of V with respect to the dual pair $\langle B(\Sigma), B(\Sigma)^*_c\rangle$. Hence for each $m \in \mathcal{M}$ we have that $\{e' \circ T_m : e' \in W^0\} \subset V^0$, i.e., if $e' \in W^0$, then $|e'(T_m(f))| \leqslant 1$ for all $f \in V$. This means that for each $m \in \mathcal{M}$ we get $W^0 \subset T_m(V)^0$. Hence $T_m(V) \subset T_m(V)^{00} \subset W^{00} = W$ for each $m \in \mathcal{M}$, i.e., $\mathcal{K}_\mathcal{M}$ is $(\tau(B(\Sigma), ca(\Sigma)), \xi)$-equicontinuous. Clearly, for each $f \in B(\Sigma)$, the set $\{T_m(f) : m \in \mathcal{M}\}$ is relatively ξ-compact in E.

(ii)$\Longrightarrow$(iii) Assume that (ii) holds. Let $p \in \mathcal{P}_\xi$ and $\varepsilon > 0$ be given. Then there exists a $\tau(B(\Sigma), ca(\Sigma))$-neighborhood V of 0 in $B(\Sigma)$ such that for each $m \in \mathcal{M}$ we have $p(T_m(f)) \leqslant \varepsilon$ for all $f \in V$. Let $f_n(\omega) \to 0$ for all $\omega \in \Omega$ and $\sup_n \|f_n\| < \infty$. Then $f_n \to 0$ for $\tau(B(\Sigma), ca(\Sigma))$ (see Proposition 2.1). Hence there exists $n_\varepsilon \in \mathbb{N}$ such that $f_n \in V$ for $n \geqslant n_\varepsilon$. Then $\sup_{m\in\mathcal{M}} p(T_m(f_n)) = \sup_{m\in\mathcal{M}} p(\int_\Omega f_n\, dm) \leqslant \varepsilon$ for all $n \geqslant n_\varepsilon$, as desired.

(iii)$\Longrightarrow$(iv) Assume that (iii) holds, and let $A_n \downarrow \emptyset$, $(A_n) \subset \Sigma$. Then $\mathbb{1}_{A_n}(\omega) \to 0$ for all $\omega \in \Omega$ and $\sup_n \|\mathbb{1}_{A_n}\| \leqslant 1$. Hence for each $p \in \mathcal{P}_\xi$ we have

$$\sup_{m\in\mathcal{M}} p(m(A_n)) = \sup_{m\in\mathcal{M}} p\Big(\int_\Omega \mathbb{1}_{A_n} dm\Big) \longrightarrow 0.$$

(iv)$\Longrightarrow$(v) See [GR, Theorem 7].

(v)$\Longrightarrow$(i) Assume that $\mathcal{M}$ is relatively $\mathcal{T}_s$-compact, and let (T_{m_α}) be a net in $\mathcal{K}_\mathcal{M}$. Without loss of generality, we can assume that $m_\alpha \to m$ for $\mathcal{T}_s$, where $m \in ca(\Sigma, E)$. We shall show that $T_{m_\alpha} \to T_m$ in $(\mathcal{L}_{\tau,\xi}(B(\Sigma), E), \mathcal{T}_s)$. Indeed, let $p \in \mathcal{P}_\xi$ and fix $\varepsilon > 0$. Since $\mathcal{M}$ is a $\mathcal{T}_s$-bounded subset of $ca(\Sigma, E)$, for each $A \in \Sigma$ we have $\sup_\alpha p(m_\alpha(A)) = \sup_\alpha q_{p,A}(m_\alpha) < \infty$. Hence, since the mapping $\Pi_p : E \to \widetilde{E}_p$ is $(p, \|\cdot\|_p^\sim)$-continuous, we obtain that $\sup_\alpha \|(m_\alpha)_p(A)\|_p^\sim = \sup_\alpha \|(\Pi_p \circ m_\alpha)(A)\|_p^\sim < \infty$. In view of the Nikodym boundedness theorem (see [DU, Chap. 1, Theorem 1]) and (2.1) we get

$$c = \sup_\alpha \|T_{m_\alpha}\|_p = \sup_\alpha \|m_\alpha\|_p(\Omega) < \infty.$$

Let $f \in B(\Sigma)$ be given and choose a Σ-simple function s_0 such that $\|f - s_0\| \leqslant \frac{\varepsilon}{3a}$, where $a = \max(c, \|T_m\|_p)$. Then there exists α_0 such that $p(T_{m_\alpha}(s_0) - T_m(s_0)) \leqslant \frac{\varepsilon}{3}$ for $\alpha \geqslant \alpha_0$. Hence for $\alpha \geqslant \alpha_0$ we get

$$\begin{aligned} &p(T_{m_\alpha}(f) - T_m(f)) \\ &\qquad \leqslant p(T_m(f - s_0)) + p(T_m(s_0) - T_{m_\alpha}(s_0)) + p(T_{m_\alpha}(s_0) - T_{m_\alpha}(f)) \\ &\qquad \leqslant \|T_m\|_p \cdot \|f - s_0\| + p(T_m(s_0) - T_{m_\alpha}(s_0)) + \|T_{m_\alpha}\| \cdot \|s_0 - f\| \\ &\qquad \leqslant a \cdot \frac{\varepsilon}{3a} + \frac{\varepsilon}{3} + a \cdot \frac{\varepsilon}{3a} = \varepsilon. \end{aligned}$$

This means that $T_{m_\alpha} \to T_m$ for $\mathcal{T}_s$, as desired. ■

Now we derive a generalized Nikodym type convergence theorem for integration operators $T : B(\Sigma) \to E$.

Theorem 2.3. *Assume that (E, ξ) is a quasicomplete lcHs. Let $m_k : \Sigma \to E$ be a ξ-countably additive measure for $k \in \mathbb{N}$ and assume that $m(A) = \xi - \lim m_k(A)$ exists for each $A \in \Sigma$. Then the following statements hold:*

(i) *$m : \Sigma \to E$ is a ξ-countably additive measure, and the integration operator $T_m : B(\Sigma) \to E$ is σ-smooth.*
(ii) *$\int_\Omega f\, dm = \xi - \lim \int_\Omega f\, dm_k$ for all $f \in B(\Sigma)$.*
(iii) *$\int_\Omega f_n\, dm_k \to 0$ in ξ uniformly for $k \in \mathbb{N}$ whenever (f_n) is a uniformly bounded sequence in $B(\Sigma)$ such that $f_n(\omega) \to 0$ for all $\omega \in \Omega$.*

Proof. In view of the Nikodym convergence theorem (see [GR, Theorem 9]) $m : \Sigma \to E$ is ξ-countably additive, and by Proposition 2.2 $T_m : B(\Sigma) \to E$ is σ-smooth. Arguing as in the proof of [N, Theorem 3.3], we obtain that $T_{m_k} \to T_m$ in $\mathcal{L}_{\tau,\xi}(B(\Sigma), E)$ for $\mathcal{T}_s$, i.e., $\int_\Omega f\, dm = \xi - \lim_k \int_\Omega f\, dm_k$ for all $f \in B(\Sigma)$. Since $\{T_{m_k} : k \in \mathbb{N}\} \cup \{T_m\}$ is a $\mathcal{T}_s$-compact subset of $\mathcal{L}_{\tau,\xi}(B(\Sigma), E)$, by Theorem 2.2 $\int_\Omega f_n\, dm_k \to 0$ in ξ uniformly for $k \in \mathbb{N}$ if (f_n) is a uniformly bounded sequence in $B(\Sigma)$ with $f_n(\omega) \to 0$ for all $\omega \in \Omega$. ∎

Remark 2.1. In case $B(\Sigma)$ is the Banach lattice of bounded Σ-measurable real-valued functions on Ω and (E, ξ) is a quasicomplete real lcHs that is complete in its Mackey topology, the equivalences (i)$\Longleftrightarrow$(ii)$\Longleftrightarrow$(iv) $\Longleftrightarrow$(v) in Theorem 2.2 were derived in [N, Theorem 3.2].

Remark 2.2. One can note that the equivalence (i)$\Longleftrightarrow$(iii) in Theorem 2.2 is related to a Grothedieck's characterization of relative weak compactness in the space of bounded complex Radon measures on a locally compact space (see [Gr, Theorem 2]).

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